Differential Invariants of Self-Dual conformal structures

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Abstract

We compute the quotient of the self-duality equation for conformal metrics by the action of the diffeomorphism group. We also determine Hilbert polynomial, counting the number of independent scalar differential invariants depending on the jet-order, and the corresponding Poincaré function. We describe the field of rational differential invariants separating generic orbits of the diffeomorphism pseudogroup action, resolving the local recognition problem for self-dual conformal structures.

Introduction

Self-duality is an important phenomenon in four-dimensional differential geometry that has numerous applications in physics, twistor theory, analysis, topology and integrability theory. A pseudo-Riemannian metric g on an oriented four-dimensional manifold M determines the Hodge operator $*:\Lambda^2TM\to\Lambda^2TM$ that satisfies the property $*^2=\mathbf{1}$ provided g has the Riemannian or split signature. In this paper we restrict to these two cases, ignoring the Lorentzian signature.

The Riemann curvature tensor splits into $O(g)$-irreducible pieces $R_g=\mathrm{Sc}_g+\mathrm{Ric}_0+W$, where the last part is the Weyl tensor [2] and $O(g)$ is the orthogonal group of g. In dimension 4, due to exceptional isomorphisms $\mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)$, $\mathfrak{so}(2,2)=\mathfrak{so}(1,2)\oplus\mathfrak{so}(1,2)$, the last component splits further $W=W_++W_-$, where $*W_\pm=\pm W_\pm$. Metric g is called self-dual if $*W=W$, i.e. $W_-=0$. This property does not depend on conformal rescalings of the metric $g\to e^{2\varphi}g$, and so is the property of the conformal structure $[g]$.

Since the space of W_- has dimension 5, and the conformal structure has 9 components in 4D, the self-duality equation appears as an underdetermined system of 5 PDE on 9 functions of 4 arguments. This is however a misleading count, since the equation is natural, and the diffeomorphism group acts as the symmetry group of the equation. Since $\mathrm{Diff}(M)$ is parametrized by 4 functions of 4 arguments, we expect to obtain a system of 5 PDE on $5 = 9-4$ functions of 4 arguments.

This 5×5 system is determined, but it has never been written explicitly. There are two approaches to eliminate the gauge freedom.

One way to fix the gauge is to pass to the quotient equation that is obtained as a system of differential relations (syzygies) on a generating set of differential invariants. By computing the latter for the self-dual conformal structures we write the quotient equation as a nonlinear 9×9 PDE system, which is determined but complicated to investigate.

Another approach is to get a cross-section or a quasi-section to the orbits of the pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M)$ action on the space $\mathcal{SD} = \{[g] : W_- = 0\}$ of self-dual conformal metric structures. This was essentially done in the recent work [5, III.A]: By choosing a convenient ansatz the authors of that work encoded all self-dual structures via a 3×3 PDE system $\mathcal{SDE}$ of the second order (this works for the neutral signature; in the Riemannian case use doubly biorthogonal coordinates to get self-duality as a 5×5 second-order PDE system [5, III.C] that can be investigated in a similar manner as the 3×3 system).

In this way almost all gauge freedom was eliminated, yet a part of symmetry remained shuffling the structures. This pseudogroup, denoted by $\mathcal{G}$, is parametrized by 5 functions of 2 arguments (and so is considerably smaller than G). We fix this freedom by computing the differential invariants of $\mathcal{G}$-action on $\mathcal{SDE}$ and passing to the quotient equation.

The differential invariants are considered in rational-polynomial form, as in [12]. This allows to describe the algebra of invariants in Lie-Tresse approach, and also using the principle of n-invariants of [1]. We count differential invariants in both approaches and organize the obtained numbers in the Hilbert polynomial and the Poincaré function.

1 Scalar invariants of self-dual structures

The first approach to compute the quotient of the self-duality equation by the local diffeomorphisms pseudogroup G action is via differential invariants of self-dual structures $\mathcal{SD}$. The signature of the metric g or conformal metric structure $[g]$ is either $(2,2)$ or $(4,0)$. In this and the following two sections we assume that g is a Riemannian metric on M for convenience. Consideration of the case $(2,2)$ is analogous.

To distinguish between metrics and conformal structures we will write $\mathcal{SD}_m$ for the former and $\mathcal{SD}_c$ for the latter. Denote the space of k-jets of such structures by $\mathcal{SD}_m^k$ and $\mathcal{SD}_c^k$ respectively. These clearly form a tower of bundles over M with projections $\pi_{k,l} : \mathcal{SD}^k_{\mathrm{x}} \to \mathcal{SD}^l_{\mathrm{x}}$, $\pi_k : \mathcal{SD}^k_{\mathrm{x}} \to M$, where x is either m or c.

1.1 Self-dual metrics: invariants

Consider the bundle $S^2_+T^*M$ of positively definite quadratic forms on TM and its space of jets $J^k(S^2_+T^*M)$. The equation $W_- = 0$ in 2-jets determines the submanifold $\mathcal{SD}^2_m \subset J^2$, and its prolongations are $\mathcal{SD}^k_m \subset J^k$ for $k > 2$.

Computation of the stabilizer of the action shows that the submanifolds $\mathcal{SD}^k_m$ are regular, meaning that generic orbits of the G-action in $\mathcal{SD}^k_m$ have the same dimension as in $J^k(S^2_+T^*M)$. This is based on a simple observation that generic self-dual metrics have no symmetry at all. Thus the differential invariants of the action on $\mathcal{SD}^k_m$ can be obtained from the differential invariants on the jet space J^k [9, 13].

These invariants can be constructed as follows. There are no invariants of order ≤ 1 due to existence of geodesic coordinates, the first invariants arise in order 2 and they are derived from the Riemann curvature tensor (as this is the only invariant of the 2-jet of g). Traces of the Ricci tensor $\mathrm{Tr}(\mathrm{Ric}^i)$, $1 \leq i \leq 4$, yield 4 invariants $I_1, \dots, I_4$ that in a Zariski open set of jets of metrics can be considered horizontally independent, meaning $\hat{d}I_1 \wedge \dots \wedge \hat{d}I_4 \neq 0$.

To get other invariants of order 2, choose an eigenbasis $e_1 \dots, e_4$ of the Ricci operator (in a Zariski open set it is simple), denote the dual coframe by $\{\theta^i\}$ and decompose $R_g = R^i_{jkl} e_i \otimes \theta^j \otimes \theta^k \wedge \theta^l$. These invariants include the previous I_i, and the totality of independent second-order invariants for self-dual metrics is

$$\dim\{R_g | W_- = 0\} - \dim O(g) = (20 - 5) - 6 = 9.$$

The invariants R^i_{jkl} are however not algebraic, but obtained as algebraic extensions via the characteristic equation. Then R^i_{jkl} (9 independent components) and e_i generate the algebra of invariants.

Alternatively, compute the basis of Tresse derivatives $\nabla_i = \hat{\partial}_{I_i}$ and express the metric in the dual coframe $\omega^j = \hat{d}I_j$: $g = G_{ij}\omega^i\omega^j$. Then the functions I_i, G_{kl} generate the space of invariants by the principle of n-invariants [1].

Remark . *There is a natural almost complex structure $\hat{J}$ on the twistor space of self-dual (M, g), i.e. on the bundle $\hat{M}$ over M whose fiber at a consists of the sphere of orthogonal complex structures on T_aM inducing the given orientation. The celebrated theorem of Penrose [15, 2] states that self-duality is equivalent to integrability of $\hat{J}$. Thus local differential invariants* *of g can be expressed through semi-global invariants of the foliation of the three-dimensional complex space $\hat{M}$ by rational curves. Similarly in the split signature one gets foliation by α-surfaces, and the geometry of this foliation of $\hat{M}$ yields the invariants on M.*

We explain how to get rid of non-algebraicity in the next subsection.

1.2 Self-dual conformal structures: invariants

Here the invariants of the second order are obtained from the Weyl tensor as the only conformally invariant part of the Riemann tensor R_g. For general conformal structures a description of the scalar invariants was given recently in [10]. In our case $W = W_+ + W_-$ the second component vanishes, and so we have only 5-dimensional space of curvature tensors $\mathcal{W}$, namely Weyl parts of R_g considered as $(3,1)$ tensors.

Let us fix a representative of the conformal structure $g_0 \in [g]$ by the requirement $\|W_+\|^2_{g_0} = 1$, this uniquely determines g_0 provided that W_+ is non-vanishing in a neighborhood (in the case of neutral signature we have to require $\|W_+\|^2_g \neq 0$ for some and hence any metric $g \in [g]$ and then we can fix g_0 up to $\pm$ by the requirement $\|W_+\|^2_{g_0} = \pm 1$). Use this representative to convert W_+ into a $(2,2)$-tensor, considered as a map $W_+ : \Lambda^2 T \to \Lambda^2 T$, where $T = T_a M$ for a fixed $a \in M$.

Recall [2] that the operator $W = W_+ + W_-$ is block-diagonal in terms of the Hodge $*$-decomposition $\Lambda^2 T = \Lambda^2_+ T \oplus \Lambda^2_- T$. Thus $W_+ : \Lambda^2_+ T \to \Lambda^2_+ T$ is a map of 3-dimensional spaces and it is traceless of norm 1. For the spectrum $\mathrm{Sp}(W_+) = \{\lambda_1, \lambda_2, \lambda_3\}$ this means $\sum \lambda_i = 0$, $\max |\lambda_i| = 1$. To conclude, we have only one scalar invariant of order 2, for which we can take $I = \mathrm{Tr}(W_+^2)$.

To obtain more differential invariants we proceed as follows. It is known that Riemannian conformal structure in 4D is equivalent to a quaternionic structure (split-quaternionic in the split-signature). In the domain, where $\mathrm{Sp}(W_+|\Lambda^2_+)$ is simple we even get a hyper-Hermitian structure (on the bundle TM pulled back to $\mathcal{SD}^2_c$, so no integrability conditions for the operators J_1, J_2, J_3) as follows.

Let $\sigma_i \in \Lambda^2_+$ be the eigenbasis of W_+ corresponding to eigenvalues λ_i, normalized by $\|\sigma_i\|^2_{g_0} = 1$ (this still leaves $\pm$ freedom for every σ_i). These 2-forms are symplectic (= nondegenerate, since again these are forms on a bundle over $\mathcal{SD}^2_c$) and g_0-orthogonal, so the operators $J_i = g_0^{-1}\sigma_i$ are anti-commuting complex operators on the space T, and they are in quaternionic relations up to the sign. We can fix one sign by requiring $J_3 = J_1 J_2$, but still have residual freedom $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now we can fix a canonical (up to above residual symmetry) frame, depending on the 3-jet of $[g]$, as follows: $e_1 = g_0^{-1}\hat{d}I/\|g_0^{-1}\hat{d}I\|_{g_0}$, $e_2 = J_1 e_1$, $e_3 = J_2 e_1$, $e_4 = J_3 e_1$. The structure functions of this frame c_{ij}^k (given by $[e_i, e_j] = c_{ij}^k e_k$) together with I constitute the fundamental invariants of the conformal structure (we can fix, for instance, $I_1 = I$, $I_2 = c_{12}^1$, $I_3 = c_{13}^1$, $I_4 = c_{14}^1$ to be the basic invariants), and together with the invariant derivations $\nabla_j = \mathcal{D}_{e_j}$ (total derivative along e_j) they generate the algebra of scalar differential invariants micro-locally.

The micro-locality comes from non-algebraicity of the invariants. Indeed, since we used eigenvalues and eigenvectors in the construction, the output depends on an algebraic extension via some additional variables y. Notice though that this involves only 2-jet coordinates, i.e. the y-variables are in algebraic relations with the fiber variables of the projection $J^2 \to J^1$, and with respect to higher jets everything is algebraic. Thus we can eliminate the y-variables, as well as the residual freedom, and obtain the algebra of global rational invariants $\mathfrak{A}_l$.

Here l is the order of jet from which only polynomial behavior of the invariants can be assumed [12]. This yields the Lie-Tresse type description of the algebra $\mathfrak{A}_l$.

It is easy to see that the rational expressions occur at most on the level of 3-jets, so the generators of the rational algebra can be chosen polynomial in the jets of order > 3. Thus we conclude:

Theorem 15. *The algebra $\mathfrak{A}_3$ of rational-polynomial invariants as well as the field $\mathfrak{F}$ of rational differential invariants of self-dual conformal metric structures are both generated by a finite number of (the indicated) differential invariants I_i and invariant derivations ∇_j, and the invariants from this algebra/field separate generic orbits in $\mathcal{SD}_c^\infty$.*

A similar statement also holds true for metric invariants of $\mathcal{SD}_m^\infty$.

2 Stabilizers of generic jets

Our method to compute the number of independent differential invariants of order k follows the approach of [13]. We will use the jet-language from the formal theory of PDE, and refer the reader to [11].

Fix a point $a \in M$. Denote by $\mathbb{D}_k$ the Lie group of k-jets of diffeomorphisms preserving the point a. This group is obtained from $\mathbb{D}_1 = \mathrm{GL}(T)$ by successive extensions according to the exact 3-sequence

$$0 \to \Delta_k \longrightarrow \mathbb{D}_k \longrightarrow \mathbb{D}_{k-1} \to \{e\},$$

where $\Delta_k = \{[\varphi]_x^k : [\varphi]_x^{k-1} = [\mathrm{id}]_x^{k-1}\} \simeq S^kT^* \otimes T$ is Abelian ($k > 1$).

Denote by $\mathrm{St}_k \subset \mathbb{D}_{k+1}$ the stabilizer of a generic point $a_k \in \mathcal{SD}^k_{\mathrm{x}}$, and by St_k^0 its connected component of unity.

2.1 Self-dual metrics: stabilizers

We refer to [13] for computations of stabilizers and note that even though the computation there is done for generic metrics, it applies to self-dual metrics as well. Thus in the metric case the stabilizers are the following: $\mathrm{St}_0 = \mathrm{St}_1 = O(g)$, and $\mathrm{St}_k^0 = 0$ for $k \geq 2$.

Consequently the action of the pseudogroup G on jets of order $k \geq 2$ is almost free, meaning that $\mathbb{D}_{k+1}$ has a discrete stabilizer on $\mathcal{SD}^k_m|_a$.

2.2 Self-dual conformal structures: stabilizers

The stabilizers for general conformal structures were computed in [10]. In the self-dual case there is a deviation from the general result. Denote by $\mathcal{C}_M = S^2_+T^*M/\mathbb{R}_+$ the bundle of conformal metric structures.

Lemma 16. **([10])** *The following is a natural isomorphism:*

$$T_{[g]}(\mathcal{C}_M) = \mathrm{End}_0^{\mathrm{sym}}(T) = \{A : T \to T \,|\, g(Au, v) = g(u, Av), \mathrm{Tr}(A) = 0\}.$$

Denote $V_M = T_{[g]}(\mathcal{C}_M)$. The differential group $\mathbb{D}_{k+1}$ acts on $\mathcal{SD}^k_c$, in particular Δ_{k+1} acts on it. The next statement is obtained by a direct computation of the symbol of Lie derivative.

Lemma 17. *The tangent to the orbit $\Delta_{k+1}(a_k)$ is the image $\mathrm{Im}(\zeta_k) \subset T\mathcal{SD}^k_c$ of the map ζ_k that is equal to the following composition*

$$S^{k+1}T^* \otimes T \xrightarrow{\delta} S^kT^* \otimes (T^* \otimes T) \xrightarrow{\mathbf{1}\otimes\Pi} S^kT^* \otimes V_M.$$

Here δ is the Spencer operator and $\Pi : T^* \otimes T \to V_M \subset T^* \otimes T$ is the projection given by

$$\langle p, \Pi(B)u\rangle = \tfrac{1}{2}\langle p, Bu\rangle + \tfrac{1}{2}\langle u_\flat, Bp^\sharp\rangle - \tfrac{1}{n}\mathrm{Tr}(B)\langle p, u\rangle,$$

where $u \in T, p \in T^*, B \in T^* \otimes T$ are arbitrary, $\langle\cdot,\cdot\rangle$ denotes the pairing between T^* and T, and $u_\flat = g(u,\cdot)$, $p^\sharp = g^{-1}(p,\cdot)$ for some representative $g \in [g]$, on which the right-hand side does not depend.

Recall that i-th prolongation of a Lie algebra $\mathfrak{h} \subset \mathrm{End}(T)$ is defined by the formula $\mathfrak{h}^{(i)} = S^{i+1}T^* \otimes T \cap S^iT^* \otimes \mathfrak{h}$. As is well-known, for the conformal algebra of $[g]$ it holds: $\mathfrak{co}(g)^{(1)} = T^*$ and $\mathfrak{co}(g)^{(i)} = 0$, $i > 1$.

Lemma 18. *We have* $\operatorname{Ker}(\zeta_k) = 0$ *for* $k > 1$*, and therefore the projectors* $\rho_{k+1,k} : \mathbb{D}_{k+1} \to \mathbb{D}_k$ *induce the injective homomorphisms* $\mathrm{St}_k \to \mathrm{St}_{k-1}$ *and* $\mathrm{St}_k^0 \to \mathrm{St}_{k-1}^0$ *for* $k > 1$.

Proof. If $\zeta_k(\Psi) = 0$, then $\delta(\Psi) \in S^k T^* \otimes \mathfrak{co}(g)$, where $\mathfrak{co}(g) \subset \operatorname{End}(T)$ is the conformal algebra. This means that $\Psi \in \mathfrak{co}(g)^{(k+1)} = 0$, if $k > 1$. Thus we conclude injectivity of ζ_k: $\Delta_{k+1} \cap \mathrm{St}_k = \{e\}$, whence the second claim. $\square$

The stabilizers of low order (for any $n \geq 3$) are the following. For any $a_0 \in \mathcal{C}_M$ its stabilizer is $\mathrm{St}_0 = CO(g) = (\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)) \times \mathbb{R}_+$.

Next, the stabilizer $\mathrm{St}_1 \subset \mathbb{D}_2$ of $a_1 \in J^1(\mathcal{C}_M)$ is the extension (by derivations) of St_0 by $\mathfrak{co}(g)^{(1)} = T^* \overset{\iota}{\hookrightarrow} \Delta_2$, where $\iota : T^* \to S^2T^* \otimes T$ is given by

$$\iota(p)(u,v) = \langle p, u\rangle v + \langle p, v\rangle u - \langle u_\flat, v\rangle p^\sharp,$$

for $p \in T^*$, $u, v \in T$. In other words, we have $\mathrm{St}_1 = CO(g) \ltimes T$.

Since for G-action on $\mathcal{SD}_c^2$ there is precisely 1 scalar differential invariant, we get $\dim \mathrm{St}_2 = (16 + 40 + 80) - (9 + 36 + 85 - 1) = 7$. This can be also seen as follows. Since $\mathrm{St}_2^0 \subset \mathrm{St}_1$ preserves the hyper-Hermitian structure determined by generic 2-jet $a_2 \in \mathcal{SD}_c^2$ (see Section 1) the $\mathbb{R}_+$ factor and one of the $\mathrm{Sp}(1)$ copies in St_0 disappears from the stabilizer of 2-jet, and we get $\mathrm{St}_2^0 \simeq \mathrm{Sp}(1) \ltimes T$.

Lemma 19. *For* $k \geq 3$ *we have:* $\mathrm{St}_k^0 = \{e\}$.

Proof. In Section 1 we constructed a canonical frame $e_1, \dots, e_4$ on T depending on (generic) jet a_3. In other words, we constructed a frame on the bundle $\pi_3^* TM$ over a Zariski open set in $\mathcal{SD}_c^3$.

The elements from St_3^0 shall preserve this frame, and so the last component $\mathrm{Sp}(1)$ from St_0 is reduced. But also the elements from St_3^0 shall preserve the 1-jet of the hyper-Hermitian structure and the invariant I determined by 2-jets, whence also the factor T is reduced, and St_3^0 is trivial (we take the connected component because of the undetermined signs $\pm$ in the normalizations). Hence the stabilizers St_k^0 for $k \geq 3$ are trivial as well. $\square$

3 The Hilbert and Poincaré function for $\mathcal{SD}$

Now we can compute the number of independent differential invariants. Since G acts transitively on M the codimension of the orbit of G in $\mathcal{SD}_{\mathrm{x}}^k$ is equal to the codimension of the orbit of $\mathbb{D}_{k+1}$ in $\mathcal{SD}_{\mathrm{x}}^k|_a$ (where $a \in M$ is a fixed point and x is either m or c). Denoting the orbit through a generic k-jet a_k by $\mathcal{O}_k \subset \mathcal{SD}^k_{\mathrm{x}}|_a$ we have:

$$\dim(\mathcal{O}_k) = \dim \mathbb{D}_{k+1} - \dim \mathrm{St}_k \,.$$

Notice that

$$\operatorname{codim}(\mathcal{O}_k) = \dim \mathcal{SD}^k_{\mathrm{x}}|_a - \dim(\mathcal{O}_k) = \operatorname{trdeg} \mathfrak{F}_k$$

is the number of (functionally independent) scalar differential invariants of order k (here $\operatorname{trdeg} \mathfrak{F}_k$ is the transcendence degree of the field of rational differential invariants on $\mathcal{SD}^k_{\mathrm{x}}$).

The Hilbert function is the number of "pure order" k differential invariants $H(k) = \operatorname{trdeg} \mathfrak{F}_k - \operatorname{trdeg} \mathfrak{F}_{k-1}$. It is known to be a polynomial for large k, so we will refer to it as the Hilbert polynomial.

The Poincaré function is the generating function for the Hilbert polynomial, defined by $P(z) = \sum_{k=0}^{\infty} H(k) z^k$. This is a rational function with the only pole $z = 1$ of order equal to the minimal number of invariant derivations in the Lie-Tresse generating set [12].

3.1 Counting differential invariants

The results of Section 2 allow to compute the Hilbert polynomial and the Poincaré function.

Theorem 20. *The Hilbert polynomial for G-action on $\mathcal{SD}_m$ is*

$$H_m(k) = \begin{cases} 0 & \text{for } k < 2, \\ 9 & \text{for } k = 2, \\ \frac{1}{6}(k-1)(k^2+25k+36) & \text{for } k > 2. \end{cases}$$

The corresponding Poincaré function is equal to

$$P_m(z) = \frac{z^2(9 + 4z - 30z^2 + 24z^3 - 6z^4)}{(1-z)^4}.$$

Notice that $H_m(k) \sim \frac{1}{3!} k^3$, meaning that the moduli of self-dual metric structures are parametrized by 1 function of 4 arguments. This function is the unavoidable rescaling factor.

Proof. As for the general metrics, there are no invariants of order < 2. Since $\mathrm{St}^0_2 = 0$, we have:

$$H_m(2) = \dim \mathcal{SD}^2_m|_a - \dim \mathbb{D}_3 = (10 + 40 + 95) - (16 + 40 + 80) = 9.$$

Alternatively, the only invariant of the 2-jet of a metric is the Riemann curvature tensor. Since $W_- = 0$, it has $20-5 = 15$ components and is acted upon effectively by the group $O(g)$ of dimension 6; hence the codimension of a generic orbit is $15 - 6 = 9$.

Starting from 2-jet we impose the self-duality constraint that, as discussed in the introduction, consist of 5 equations and is a determined system (mod gauge). In particular, there are no differential syzygies between these 5 equations, so that in "pure" order $k \geq 2$ the number of independent equations is $5 \cdot \binom{k+1}{3}$. Thus the symbol of the self-duality metric equation $W_- = 0$ on g, given by

$$\mathfrak{g}_k = \operatorname{Ker}(d\pi_{k,k-1} : T\mathcal{SD}_m^k \to T\mathcal{SD}_m^{k-1})$$

has dimension $\dim(S^kT^* \otimes S^2T^*) - \#[\text{independent equations}]$.

Since the pseudogroup G acts almost freely on jets of order $k \geq 2$ (freely from some order k), we have:

$$H_m(k) = \dim \mathfrak{g}_k - \dim \Delta_{k+1} = 10 \cdot \binom{k+3}{3} - 5 \cdot \binom{k+1}{3} - 4 \cdot \binom{k+4}{3}$$

whence the claim for the Hilbert polynomial. The formula for the Poincaré function follows. □

Theorem 21. *The Hilbert polynomial for G-action on $\mathcal{SD}_c$ is*

$$H_c(k) = \begin{cases} 0 & \text{for } k < 2, \\ 1 & \text{for } k = 2, \\ 13 & \text{for } k = 3, \\ 3k^2 - 7 & \text{for } k > 3. \end{cases}$$

The corresponding Poincaré function is equal to

$$P_c(z) = \frac{z^2(1 + 10z + 5z^2 - 17z^3 + 7z^4)}{(1-z)^3}.$$

Notice that $H_c(k) \sim 6 \cdot \frac{1}{2!} k^2$, meaning that the moduli of self-dual conformal metric structures are parametrized by 6 function of 3 arguments. This confirms the count in [6, 5].

Proof. As for the general metrics, there are no invariants of order < 2. We already counted $H_c(2) = 1$. Since $\mathrm{St}_3^0 = 0$, we have:

$$\begin{aligned} H_c(3) &= \dim \mathcal{SD}_m^3|_a - \dim \mathbb{D}_4 - H_c(2) \\ &= (9 + 36 + 85 + 160) - (16 + 40 + 80 + 140) - 1 = 13. \end{aligned}$$

Starting from 2-jet we impose the self-duality constraint, and we computed in the previous proof that this yields $5\cdot\binom{k+1}{3}$ independent equations of "pure" order $k\geq 2$. Thus the symbol of the self-duality conformal equation $W_-=0$ on $[g]$, given by

$$\mathfrak{g}_k=\mathrm{Ker}(d\pi_{k,k-1}:T\mathcal{SD}_c^k\to T\mathcal{SD}_c^{k-1}),$$

has dimension= $\dim(S^kT^*\otimes(S^2T^*/\mathbb{R}_+))-\#$[independent equations].

Since the pseudogroup G acts almost freely on jets of order $k\geq 3$ (freely from some order k), we have:

$$H_c(k)=\dim\mathfrak{g}_k-\dim\Delta_{k+1}=9\cdot\binom{k+3}{3}-5\cdot\binom{k+1}{3}-4\cdot\binom{k+4}{3}$$

whence the claim for the Hilbert polynomial. The formula for the Poincaré function follows. □

3.2 The quotient equation

Let $I_1,\dots,I_4$ be the basic differential invariants of self-dual conformal structures. For generic such structures c these invariant evaluated on c are independent. Thus we can fix the gauge by requiring $I_i=x_i$, $i=1,\dots,4$, to be the local coordinates on M. This adds 4 differential equations to 5 equations of self-duality on 9 components of c. Consequently, denoting

$$\Sigma_\infty=\{\theta\in\mathcal{SD}_c^\infty:\hat{d}I_1\wedge\hat{d}I_2\wedge\hat{d}I_3\wedge\hat{d}I_4 \text{ is not defined at } \theta \text{ or vanishes}\},$$

the moduli space $(\mathcal{SD}_c^\infty\setminus\Sigma_\infty)/G$ is given as 9×9 PDE system

$$W_-=0, I_1=x_1,\dots,I_4=x_4.$$

4 The self-duality equation

In the second approach we use a 3×3 PDE system from [5] which encodes all self-dual conformal structures. It was shown in loc.cit. that any anti-self-dual conformal structure in neutral signature $(2,2)$ locally takes the form $[g]$ where

$$g=dtdx+dzdy+p\,dt^2+2q\,dtdz+r\,dz^2. \quad (1)$$

Here p,q,r are functions of (t,x,y,z) which satisfy the following three second-order PDEs:

$$p_{xx}+2q_{xy}+r_{yy}=0,$$

$$m_x+n_y=0, \quad (2)$$

$$m_z-qm_x-rm_y+(q_x+r_y)m=n_t-pn_x-qn_y+(p_x+q_y)n,$$

where

$$m := p_z - q_t + pq_x - qp_x + qq_y - rp_y, \quad n := q_z - r_t + qr_y - rq_y + pr_x - qq_x.$$

Conversely, any such conformal structure is anti-self-dual. Therefore we can, instead of looking at arbitrary self-dual conformal structures, look at conformal structures $[g]$ where g is a metric of the Plebański-Robinson form (1) satisfying (2). So from now on we restrict to self-dual conformal structures in the neutral signature $(2,2)$.

Remark . *These equations are admittedly describing anti-self-dual metrics ($*W = -W$) instead of self-dual metrics ($*W = W$). However, in order to define the Hodge operator, one must specify an orientation. Change of orientation interchanges the equations, so from a local viewpoint self-dual and anti-self-dual structures are the same.*

Conformal structures of the form (1) are parametrized by sections of the bundle $\pi\colon \mathcal{C}_M^{\mathrm{PR}} = M \times \mathbb{R}^3(p,q,r) \to M$, where $M = \mathbb{R}^4(t,x,y,z)$. Self-dual conformal structures must, in addition, satisfy system (2), so they are described by a second-order PDE

$$\mathcal{SDE}_2 = \{\theta = [(p,q,r)]_x^2 : x \in M, \theta \text{ satisfies } (2)\} \subset J^2(\mathcal{C}_M^{\mathrm{PR}}).$$

We let $\mathcal{SDE}_k \subset J^k = J^k(\mathcal{C}_M^{\mathrm{PR}})$ denote the prolonged equation. From now on we will omit specification of the bundle over which the jet spaces are constructed, because it will always be $\mathcal{C}_M^{\mathrm{PR}}$ in what follows.

The prolonged equation $\mathcal{SDE}_k$ is given by $3\binom{k+2}{4}$ equations in J^k since the system (2) is determined. By subtracting this from the jet space dimension $\dim J^k = 4 + 3\binom{k+4}{4}$, we find

$$\dim \mathcal{SDE}_k = 4 + 3\binom{k+4}{4} - 3\binom{k+2}{4} = k^3 + \frac{9}{2}k^2 + \frac{13}{2}k + 7.$$

5 Symmetries of $\mathcal{SDE}$

Self-dual conformal structures locally correspond to sections of $\mathcal{C}_M^{\mathrm{PR}}$ that are solutions of $\mathcal{SDE}$. This correspondence is not 1-1 as there is some residual freedom left: two solutions of $\mathcal{SDE}$ can still be equivalent up to diffeomorphisms. The goal is to remove this freedom by factoring by diffeomorphisms that preserve the shape of the conformal structure $[g]$ where g is in Plebański-Robinson form (1).

These transformations form the symmetry pseudogroup $\mathcal{G}$ of the equation $\mathcal{SDE}$. We will study its Lie algebra $\mathfrak{g}$. By the Lie-Bäcklund theorem [8] for our equation all symmetries are (prolongations of) point transformations. It turns out that the Lie algebra of symmetries is the same as the Lie algebra of vector fields preserving the shape of $[g]$.

5.1 Symmetries of $\mathcal{SDE}$

A vector field X on J^0 is a symmetry of $\mathcal{SDE}$ if the prolonged vector field $X^{(2)}$ is tangent to $\mathcal{SDE}_2 \subset J^2$, i.e. if $X^{(2)}(F_i) = \lambda_i^j F_j$, where $F_1 = 0, F_2 = 0, F_3 = 0$ are the three equations (2). This gives an overdetermined system of PDEs that can be solved by the standard technique, and we obtain the following result:

Theorem 22. *The Lie algebra $\mathfrak{g}$ of symmetries of $\mathcal{SDE}$ is generated by the following five classes of vector fields $X_1(a)$, $X_2(b)$, $X_3(c)$, $X_4(d)$, $X_5(e)$, each of which depends on a function of (t, z):*

$$a\partial_t - xa_t\partial_x - xa_z\partial_y + (xa_{tt} - 2pa_t)\partial_p + (xa_{tz} - qa_t - pa_z)\partial_q + (xa_{zz} - 2qa_z)\partial_r,$$
$$b\partial_z - yb_t\partial_x - yb_z\partial_y + (yb_{tt} - 2qb_t)\partial_p + (yb_{tz} - qb_z - rb_t)\partial_q + (yb_{zz} - 2rb_z)\partial_r,$$
$$cx\partial_x + cy\partial_y + (cp - xc_t)\partial_p + (cq - \tfrac{1}{2}xc_z - \tfrac{1}{2}yc_t)\partial_q + (cr - yc_z)\partial_r,$$
$$d\partial_x - d_t\partial_p - \tfrac{1}{2}d_z\partial_q,$$
$$e\partial_y - \tfrac{1}{2}e_t\partial_q - e_z\partial_r.$$

The following table shows the commutation relations.

$[,]$	$X_1(g)$	$X_2(g)$	$X_3(g)$	$X_4(g)$	$X_5(g)$
$X_1(f)$	$X_1(fg_t - f_tg)$	$X_2(fg_t) - X_1(f_zg)$	$X_3(fg_t)$	$X_4((fg)_t) + X_5(f_zg)$	$X_5(fg_t)$
$X_2(f)$	$*$	$X_2(fg_z - f_zg)$	$X_3(fg_z)$	$X_4(fg_z)$	$X_4(f_tg) + X_5((fg)_z)$
$X_3(f)$	$*$	$*$	0	$-X_4(fg)$	$-X_5(fg)$
$X_4(f)$	$*$	$*$	$*$	0	0
$X_5(f)$	$*$	$*$	$*$	$*$	0

Notice that the Lie algebra is bi-graded $\mathfrak{g} = \oplus\mathfrak{g}_{i,j}$, meaning that we have $[\mathfrak{g}_{i_1,j_1}, \mathfrak{g}_{i_2,j_2}] \subset \mathfrak{g}_{i_1+i_2,j_1+j_2}$ with nontrivial graded pieces

$$\mathfrak{g}_{0,0} = \langle X_1, X_2\rangle, \qquad \mathfrak{g}_{0,1} = \langle X_3\rangle, \qquad \mathfrak{g}_{1,\infty} = \langle X_4, X_5\rangle.$$

5.2 Shape-preserving transformations

We say that a transformation $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ preserves the PR-shape if for every $[g] \in \Gamma(\mathcal{C}_M^{\mathrm{PR}})$ we have $[\varphi_* g] \in \Gamma(\mathcal{C}_M^{\mathrm{PR}})$. A vector field X on $\mathbb{R}^4$ preserves the PR-shape if its flow does so.

Theorem 23. *The Lie algebra of vector fields preserving the PR-shape is generated by the five classes of vector fields*

$$a\partial_t - xa_t\partial_x - xa_z\partial_y, \quad b\partial_z - yb_t\partial_x - yb_z\partial_y, \quad cx\partial_x + cy\partial_y, \quad d\partial_x, \quad e\partial_y.$$

where a, b, c, d, e are arbitrary functions of (t, z).

Proof. In order to find the Lie algebra of vector fields preserving the shape of $[g]$, we let $X = f_1\partial_t + f_2\partial_x + f_3\partial_y + f_4\partial_z$ be a general vector field and take the Lie derivative $L_X g$. The vector field preserves the PR-shape of $[g]$ if

$$L_X g = \epsilon \cdot (dtdx + dzdy) + \tilde{p}\, dt^2 + 2\tilde{q}\, dtdz + \tilde{r}\, dz^2$$

for some functions $\epsilon, \tilde{p}, \tilde{q}, \tilde{r}$. This gives an overdetermined system of 6 PDEs on 4 unknowns with the solutions parametrized by 5 functions of 2 variables as indicated. □

5.3 Unique lift to J^0

The conformal metric (1) can also be considered as a horizontal (degenerate) symmetric tensor c_{PR} on $\mathcal{C}_M^{\text{PR}}$. Namely, $c_{\text{PR}} \in \Gamma(\pi^* S^2 T^* M/\mathbb{R}_+)$ is given at the point $(t, x, y, z, p, q, r) \in \mathcal{C}_M^{\text{PR}}$ via its representative g by formula (1). The algebra of vector fields X preserving the shape of $[g]$ is naturally lifted to $\mathcal{C}_M^{\text{PR}}$ by the requirement $L_{\hat{X}} c_{\text{PR}} = 0$. This requirement algebraically restores the vertical components of the vector fields $X_1, \dots, X_5$ from Theorem 23 yielding the symmetry fields from Theorem 22. We conclude:

Theorem 24. *The Lie algebra of transformations preserving the PR-shape coincides with the Lie algebra $\mathfrak{g}$ of point symmetries of $\mathcal{SDE}$.*

Thus the conformal structure c_{PR} uniquely restores $\mathfrak{g} = \text{sym}(\mathcal{SDE})$.

5.4 Conformal tensors invariant under $\mathfrak{g}$

The goal of this subsection is to show that the simplest conformally invariant tensor with respect to $\mathfrak{g}$ is c_{PR}, so that the conformal structure (of PR-shape) is in turn uniquely determined by $\mathfrak{g}$.

We aim to describe the horizontal conformal tensors on $\mathcal{C}_M^{\text{PR}}$ that are invariant with respect to $\mathfrak{g}$. Since $\mathfrak{g}$ acts transitively on $\mathcal{C}_M^{\text{PR}}$, we consider the stabilizer $\text{St}_0 \subset \mathfrak{g}$ of the point given by $(t, x, y, z, p, q, r) = (0, 0, 0, 0, 0, 0)$ in $\mathcal{C}_M^{\text{PR}}$. Denote by St_0^k the subalgebra of $\mathfrak{g}$ consisting of fields vanishing at 0 to order k, so that $\text{St}_0 = \text{St}_0^1$.

It is easy to see from formulae of Theorem 22 that the space $\text{St}_0^1/\text{St}_0^2$ is 18-dimensional, and 12 of the generators are vertical (they belong to $\langle \partial_p, \partial_q, \partial_r \rangle$). The complimentary linear fields have the horizontal parts

$$Y_1 = t\partial_t - x\partial_x, \qquad Y_2 = z\partial_t - x\partial_y, \qquad Y_3 = t\partial_z - y\partial_x,$$
$$Y_4 = z\partial_z - y\partial_y, \qquad Y_5 = x\partial_x + y\partial_y, \qquad Y_6 = z\partial_x - t\partial_y.$$

They form a 6-dimensional Lie algebra $\mathfrak{h}$ acting on the horizontal space $\mathbb{T} = T_0M = T_0\mathcal{C}_M^{\rm PR}/\operatorname{Ker}(d\pi)$. This Lie algebra is a semi-direct product of the reductive part $\mathfrak{h}_0 = \langle Y_1, Y_2, Y_3, Y_4, Y_5\rangle$ and the nilpotent piece $\mathfrak{r} = \langle Y_6\rangle$ (the nilradical is 2-dimensional). The reductive piece splits in turn $\mathfrak{h}_0 = \mathfrak{sl}_2 \oplus \mathfrak{a}$, where the semi-simple part is $\mathfrak{sl}_2 = \langle Y_1 - Y_4, Y_2, Y_3\rangle$ and the Abelian part is $\mathfrak{a} = \langle Y_1 + Y_4, Y_5\rangle$.

It is easy to see that the space $\mathbb{T}$ is $\mathfrak{h}_0$-reducible. In fact, with respect to $\mathfrak{h}_0$ it is decomposable $\mathbb{T} = \Pi_1 \oplus \Pi_2 = \langle \partial_t, \partial_z\rangle \oplus \langle \partial_x, \partial_y\rangle$, and Π_1, Π_2 are the standard $\mathfrak{sl}_2$-representations (denoted by Π in what follows). However $\mathfrak{r}$ maps Π_1 to Π_2 and Π_2 to 0. This $\Pi_2 \subset \mathbb{T}$ is an $\mathfrak{h}$-invariant subspace, but it does not have an $\mathfrak{h}$-invariant complement.

Moreover, Π_2 is the only proper $\mathfrak{h}$-invariant subspace, so there are no conformally invariant vectors (invariant 1-space) and covectors (invariant 3-space). We sumarize this as follows.

Lemma 25. *There are no horizontal 1-tensors on $\mathcal{C}_M^{PR}$ that are conformally invariant with respect to $\mathfrak{g}$.*

Now, let's consider conformally invariant horizontal 2-tensors. Since $c_{\rm PR}$ is $\mathfrak{g}$-invariant, we can lower the indices and consider $(0,2)$-tensors. We have the splitting $\mathbb{T}^* \otimes \mathbb{T}^* = \Lambda^2\mathbb{T}^* \oplus S^2\mathbb{T}^*$.

The symmetric part further splits $S^2(\Pi_1^* \oplus \Pi_2^*) = S^2\Pi_1^* \oplus (\Pi_1^* \otimes \Pi_2^*) \oplus S^2\Pi_2^*$. As an $\mathfrak{sl}_2$-representation, this is equal to $3 \cdot S^2\Pi \oplus \Lambda^2\Pi = 3 \cdot \mathfrak{ad} \oplus \mathbf{1}$, and the only one trivial piece $\mathbf{1} \subset \Pi_1^* \otimes \Pi_2^*$ (which is also $\mathfrak{h}$-invariant) is spanned by $c_{\rm PR}$. Here $\Pi_1^* = \langle dt, dz\rangle$ and $\Pi_2^* = \langle dx, dy\rangle$. Thus there are no $\mathfrak{g}$-invariant symmetric conformal 2-tensors except $c_{\rm PR}$.

The skew-symmetric part further splits $\Lambda^2(\Pi_1^*\oplus\Pi_2^*) = \Lambda^2\Pi_1^*\oplus(\Pi_1^*\otimes\Pi_2^*)\oplus\Lambda^2\Pi_2^*$, and as an $\mathfrak{sl}_2$-representation, this is equal to $S^2\Pi\oplus 3\cdot\Lambda^2\Pi = \mathfrak{ad}\oplus 3\cdot\mathbf{1}$. Thus there are three $\mathfrak{sl}_2$-trivial pieces, and they are $\mathfrak{h}_0$-invariant. However only one of them is $\mathfrak{r}$-invariant, namely $\Lambda^2\Pi_1^*$ that is spanned by $dz \wedge dt$. Thus we have proved the following statement.

Theorem 26. *The only conformally invariant symmetric 2-tensor is c_{PR}. The only conformally invariant skew-symmetric 2-tensor is $dz \wedge dt$.*

Since $dz \wedge dt$ is degenerate and does not define a convenient geometry, $c_{\rm PR}$ is the simplest $\mathfrak{g}$-invariant conformal tensor.

5.5 Algebraicity of $\mathfrak{g}$

We say that the Lie algebra $\mathfrak{g}$ is algebraic if its sheafification is equal to the Lie algebra sheaf of some algebraic pseudo-group $\mathcal{G}$ (see definition of an algebraic pseudo-group in [12]). Algebraicity of $\mathfrak{g}$ is important because it guarantees, through the global Lie-Tresse theorem [12], existence of rational differential invariants separating generic orbits (by [16] this yields rational quotient of the action on every finite jet-level).

Let $\mathbb{D}_k \subset J^k_{(\theta,\theta)}(\mathcal{C}_M^{\text{PR}}, \mathcal{C}_M^{\text{PR}})$ denote the differential group of order k at $\theta \in \mathcal{C}_M^{\text{PR}}$. The stabilizer $\mathcal{G}_\theta \subset \mathcal{G}$ of θ can be viewed as a collection of subbundles $\mathcal{G}_\theta^k \subset \mathbb{D}_k$. The transitive Lie pseudo-group $\mathcal{G}$ is algebraic if $\mathcal{G}_\theta^k$ is an algebraic subgroup of $\mathbb{D}_k$ for every k. This is independent of the choice of θ since $\mathcal{G}$ is transitive, implying that subgroups $\mathcal{G}_\theta^k \subset \mathbb{D}_k$ are conjugate for different points $\theta \in \mathcal{C}_M^{\text{PR}}$.

When determining whether $\mathfrak{g}$ is algebraic, there are essentially two approaches. One is to try to see it from the stabilizer $\mathfrak{g}_\theta$ alone, and the other is to integrate $\mathfrak{g}$ in order to investigate the pseudo-group $\mathcal{G}_\theta$. It turns out that the latter is more efficient in our case.

Consider the following pseudo-group $\mathcal{G}$ given via its action on $\mathcal{C}_M^{\text{PR}}$.

$$t \mapsto T = A, \quad z \mapsto Z = B$$
$$x \mapsto X = C(B_z x - B_t y) + D, \quad y \mapsto Y = C(A_t y - A_z x) + E$$
$$p \mapsto P = \frac{C(B_z^2 p - 2B_t B_z q + B_t^2 r) + (CJ_{B,B_z} + B_z J_{B,C})x - (CJ_{B,B_t} + B_t J_{B,C})y + J_{B,D}}{J_{A,B}}$$
$$r \mapsto R = \frac{C(A_z^2 p - 2A_t A_z q + A_t^2 r) + (CJ_{A,A_z} + A_z J_{A,C})x - (CJ_{A,A_t} + A_t J_{A,C})y - J_{A,E}}{J_{A,B}}$$
$$q \mapsto Q = \frac{C(-A_z B_z p + (A_t B_z + A_z B_t)q - A_t B_t r) + (J_{B,E} - J_{A,D})/2}{J_{A,B}}$$
$$+ \frac{((J_{A_z,B} - J_{A,B_z})C - B_z J_{A,C} - A_z J_{B,C})x + ((J_{A,B_t} - J_{A_t,B})C + A_t J_{B,C} + B_t J_{A,C})y}{2J_{A,B}}$$

Here we use the notation $J_{F,G} = F_t G_z - F_z G_t$ for two functions F, G of (t, z). The functions A, B, C, D, E are all (locally defined) functions depending on the variables (t, z). In addition A, B satisfy the requirement that $(t, z) \mapsto (A(t, z), B(t, z))$ is a local diffeomorphism of the plane, and $C \neq 0$ wherever it is defined.[1]

It is easy to check that this is a Lie pseudo-group (one should specify the differential equations defining $\mathcal{G}$, and they are $T_x = 0, \dots, T_r = 0, \dots, X_y + Z_t = 0, \dots$). Moreover it is easy to check that the Lie algebra sheaf of $\mathcal{G}$ coincides with the sheafification of $\mathfrak{g}$.

Theorem 27. *The Lie pseudo-group $\mathcal{G}$ and consequently the Lie algebra $\mathfrak{g}$ are algebraic.*

Proof. The subgroups $\mathcal{G}_\theta^k$ of $\mathbb{D}_k$ are constructed by repeated differentiation of $T, ..., R$ by $t, ..., r$ and evaluation at θ. The formulas for the group action

[1]The formulas above are corrections of the ones from the original paper.

make it clear that $\mathcal{G}_\theta^k$ will always be an algebraic subgroup of $\mathbb{D}_k$ (they provide a rational parametrization of it as a subvariety). Thus $\mathcal{G}$ is algebraic. The statement for $\mathfrak{g}$ follows. □

Let us briefly explain how to read algebraicity from the Lie algebra $\mathfrak{g}$. Consider the Lie subalgebra $\mathfrak{f} \subset \mathfrak{gl}(T_0 J^0)$ obtained by linearization of the isotopy algebra at $0 \in J^0 = \mathcal{C}_M^{\mathrm{PR}}$. As already noticed in §5.4, this is an 18-dimensional subalgebra admitting the following exact 3-sequence

$$0 \to \mathfrak{v} \longrightarrow \mathfrak{f} \longrightarrow \mathfrak{h} \to 0,$$

where $\mathfrak{v}$ is the vertical part and $\mathfrak{h}$ – the "horizontal" (that is the quotient). The explicit form of these vector fields come from Theorem 22:

$$\begin{aligned}
\mathfrak{v} &= \langle x\partial_p, x\partial_q, x\partial_r, y\partial_p, y\partial_q, y\partial_r, t\partial_p, t\partial_q, t\partial_r, z\partial_p, z\partial_q, z\partial_r \rangle,\\
\mathfrak{h} &= \mathfrak{sl}_2 + \mathfrak{a} + \mathfrak{r}, \quad \text{where} \quad \mathfrak{r} = \langle z\partial_x - t\partial_y \rangle,\\
\mathfrak{sl}_2 &= \langle z\partial_t - x\partial_y - p\partial_q - 2q\partial_r, t\partial_z - y\partial_x - 2q\partial_p - r\partial_q,\\
&\qquad t\partial_t - z\partial_z - x\partial_x + y\partial_y - 2p\partial_p + 2r\partial_r \rangle,\\
\mathfrak{a} &= \langle t\partial_t + z\partial_z - p\partial_p - q\partial_q - r\partial_r, x\partial_x + y\partial_y + p\partial_p + q\partial_q + r\partial_r \rangle.
\end{aligned}$$

By [4] the subalgebra $[\mathfrak{f},\mathfrak{f}] \subset \mathfrak{gl}(T_0 J^0)$ is algebraic. Since $\mathfrak{f}$ is obtained from $[\mathfrak{f},\mathfrak{f}] = \mathfrak{v}+\mathfrak{sl}_2+\mathfrak{r}$ by extension by derivations $\mathfrak{a}$, and the semi-simple elements in the latter have no irrational ratio of spectral values, we conclude that $\mathfrak{f} \subset \mathfrak{gl}(T_0 J^0)$ is an algebraic Lie algebra [3]. The claim about algebraicity of $\mathfrak{g}$ follows by prolongations.

6 The Hilbert and Poincaré function for $\mathcal{SDE}$

Even though $\mathfrak{g}$ is just a PR-shape preserving Lie algebra, its prolongation to the space of 2-jets preserves $\mathcal{SDE}$ (this is an unexpected remarkable fact), and we consider the orbits of $\mathfrak{g}$ on this equation.

6.1 Dimension of generic orbits

We can compute the dimension of a generic orbit in $\mathcal{SDE}_k$ or J^k by computing the rank of the system of prolonged symmetry vector fields $X^{(k)}$ at a point in general position.

By prolonging the generators $X_1, ..., X_5$ and with the help of Maple we observe that the Lie algebra $\mathfrak{g}$ acts transitively on J^1. The dimension of a generic orbit on the Lie algebra acting on J^2 is 44, but the equation $\mathcal{SDE}_2 \subset J^2$ contains no generic orbits, and if we restrict to $\mathcal{SDE}_2$ a generic orbit of $\mathfrak{g}$ is of dimension 42. For higher jet-orders $k > 2$, the dimension of a generic orbit is the same on $\mathcal{SDE}_k$ as on J^k.

We are going to compute $\dim \mathcal{O}_k$ for $k \geq 3$ as follows. Since $\mathfrak{g}$ contains the translations ∂_t, ∂_z, all its orbits pass through the subset $S_k \subset J^k$ given by $t = 0, z = 0$. On S_k we can make the Taylor expansion of parametrizing functions a, b, c, d, e around $(t, z) = (0, 0)$.

We use $X_5(e)$ to show the idea. By varying the coefficients of the Taylor series $e(t, z) = e(0, 0) + e_t(0, 0)t + e_z(0, 0)z + \cdots$ we see that the vector fields $X_5(m, n) = z^m t^n \partial_y - \frac{n}{2} z^m t^{n-1} \partial_q - m z^{m-1} t^n \partial_r$ are contained in the symmetry algebra, with the convention that $t^{-1} = z^{-1} = 0$, and any vector field of the form $X_5(e)$ is tangent to a vector field in $\langle X_5(m, n) \rangle$. The prolongation of a vector field takes the form

$$X^{(k)} = \sum_i a_i \, \mathcal{D}_i^{(k+1)} + \sum_{|\sigma| \leq k} (\mathcal{D}_\sigma(\phi_p)\partial_{p_\sigma} + \mathcal{D}_\sigma(\phi_q)\partial_{q_\sigma} + \mathcal{D}_\sigma(\phi_r)\partial_{r_\sigma}) \tag{3}$$

where $\mathcal{D}_\sigma$ is the iterated total derivative, $\mathcal{D}_i^{(k+1)}$ the truncated total derivative (the "restriction" to the space J^{k+1}, cf. [8, 11]), $a_i = dx_i(X)$ for $(x_1, x_2, x_3, x_4) = (t, x, y, z)$, and ϕ_p, ϕ_q, ϕ_r are the generating functions for X, i.e. $\phi_p = \omega_p(X), \phi_q = \omega_q(X), \phi_r = \omega_r(X)$ where

$$\begin{aligned} \omega_p &= dp - p_t dt - p_x dx - p_y dy - p_z dz, \\ \omega_q &= dq - q_t dt - q_x dx - q_y dy - q_z dz, \\ \omega_r &= dr - r_t dt - r_x dx - r_y dy - r_z dz \end{aligned}$$

In the case of $X_5(m, n)$, the generating functions are given by

$$\phi_p = -p_y z^m t^n, \ \phi_q = -\tfrac{n}{2} z^m t^{n-1} - q_y z^m t^n, \ \phi_r = -m z^{m-1} t^n - r_y z^m t^n.$$

We see that the restriction of $X_5(m, n)^{(k)}$ to the fiber over $0 \in \mathcal{C}_M^{\mathrm{PR}}$ is nonzero only when $m+n \leq k+1$. Hence we can parametrize $\langle X_5(m, n) \rangle^{(k)}$ by $J_0^{k+1}(\mathbb{R}^2(t, z), \mathbb{R}(e))$, and by extending this argument to the whole symmetry algebra we get (the vector fields $X_k(m, n)$ for $k = 1, \ldots, 4$, are defined similarly to the vector field $X_5(m, n)$ by simply substituting $a = z^m t^n$ etc into the formulae of Theorem 22)

$$\begin{aligned} \mathfrak{g}^{(k)} &= \langle X_1(m, n), X_2(m, n), X_4(m, n), X_5(m, n) \rangle^{(k)} \oplus \langle X_3(m, n) \rangle^{(k)} \\ &= J_0^{k+1}(\mathbb{R}^2(t, z), \mathbb{R}^4(a, b, d, e)) \times J_0^k(\mathbb{R}^2(t, z), \mathbb{R}(c)). \end{aligned}$$

Using formula (3) we verify that the Lie algebra $\mathfrak{g}^{(k)}$ acts freely on $\mathcal{SDE}_k$

for $k \geq 3$, whence

$$\begin{aligned}\dim \mathcal{O}_k &= \dim \left(J_0^{k+1}(\mathbb{R}^2, \mathbb{R}^4) \times J_0^k(\mathbb{R}^2, \mathbb{R})\right) \\ &= 4 \dim \left(J_0^{k+1}(\mathbb{R}^2, \mathbb{R})\right) + \dim \left(J_0^k(\mathbb{R}^2, \mathbb{R})\right) \\ &= 4\binom{k+3}{2} + \binom{k+2}{2} = \frac{(k+2)(5k+13)}{2}.\end{aligned}$$

6.2 Counting the differential invariants

The number s_k of differential invariants of order k (as before, this is trdeg $\mathfrak{F}_k$) is equal to the codimension of a generic orbit of $\mathfrak{g}$ on $\mathcal{SDE}_k$. For the lowest orders, we have $s_0 = s_1 = 0$ and $s_2 = \dim \mathcal{SDE}_2 - \dim \mathcal{O}_2 = 46 - 42 = 4$. For higher jet-orders, the number of invariants of order k is given by

$$s_k = \text{codim}\mathcal{O}_k = \dim \mathcal{SDE}_k - \dim \mathcal{O}_k = k^3 + 2k^2 - 5k - 6, \quad k \geq 3.$$

The number of differential invariants of "pure order" k is then given by $H(k) = s_k - s_{k-1}$. The Poincaré function $P(z) = \sum_{k=0}^{\infty} H(k)z^k$ can now easily be computed, and we conclude:

Theorem 28. *The Hilbert polynomial for the action of $\mathfrak{g}$ on $\mathcal{SDE}$ is*

$$H(k) = \begin{cases} 0 & \textit{for } k < 2, \\ 4 & \textit{for } k = 2, \\ 20 & \textit{for } k = 3, \\ 3k^2 + k - 6 & \textit{for } k > 3. \end{cases}$$

The corresponding Poincaré function is equal to

$$P(z) = \frac{2z^2(2 + 4z - z^2 - 4z^3 + 2z^4)}{(1-z)^3}.$$

Notice that $H(k)$ in this statement has the same leading term as $H(k)$ in Theorem 21 for $k > 3$. The following table summarizes the counting results from the last two subsections for low order k.

k	0	1	2	3	4	5	6	7	...
$\dim \mathcal{SDE}_k$	7	19	46	94	169	277	424	616	...
$\dim \mathcal{O}_k$	7	19	42	70	99	133	172	216	...
$\text{codim}\, \mathcal{O}_k$	0	0	4	24	70	144	252	400	...
$H(k)$	0	0	4	20	46	74	108	148	...

7 The invariants of $\mathcal{SDE}$ and the quotient equation

From the global Lie-Tresse theorem [12] and Theorem 27 it follows that there exist rational differential invariants of $\mathfrak{g}$-action (or $\mathcal{G}$-action) on $\mathcal{SDE}$ that separate generic orbits.

7.1 Invariants of the second order

There are four independent differential invariants of the second order:

$$\begin{aligned}
I_1 &= \frac{1}{K}\left(p_{yy}r_{xx} - p_{xx}r_{yy} + 2\,p_{xy}q_{xx} + 4\,q_{xy}^2 + 2\,q_{yy}r_{xy}\right)\\
I_2 &= \frac{1}{K^3}\big(\left(q_{xy}r_{yy} - q_{yy}r_{xy}\right)p_{xx} + \left(p_{yy}r_{xy} - p_{xy}r_{yy}\right)q_{xx}\\
&\quad + \left(p_{xy}q_{yy} - p_{yy}q_{xy}\right)r_{xx}\big)^2\\
I_3 &= \frac{1}{K^3}\big(p_{yy}\left(q_{xx} - r_{xy}\right)^2 + r_{xx}\left(q_{yy} - p_{xy}\right)^2\\
&\quad - 2\,q_{xy}\left(p_{xy}q_{xx} + q_{yy}r_{xy} - p_{xy}r_{xy} - 2\,p_{yy}r_{xx} + 2\,q_{xy}^2 - q_{xx}q_{yy}\right)\big)^2\\
I_4 &= \frac{1}{K^2}\big(p_{xx}^2r_{yy}^2 + p_{yy}^2r_{xx}^2 - 2\,p_{xx}p_{yy}r_{xx}r_{yy} + 4\,p_{xx}p_{yy}r_{xy}^2\\
&\quad + 4\,p_{xy}^2r_{xx}r_{yy} - 4\,q_{xx}q_{yy}\left(p_{xx}r_{yy} - 4\,p_{xy}r_{xy} + p_{yy}r_{xx}\right)\\
&\quad + 4\,p_{xx}q_{xy}r_{yy}\left(p_{xx} + 4\,q_{xy} + r_{yy}\right) - 4\,p_{xy}r_{xy}\left(p_{xx}r_{yy} + p_{yy}r_{xx}\right)\\
&\quad + 4\,p_{xx}r_{xx}\left(q_{yy}^2 - p_{yy}q_{xy}\right) + 4\,p_{yy}r_{yy}\left(q_{xx}^2 - q_{xy}r_{xx}\right)\\
&\quad - 8\,p_{xy}q_{xy}\left(q_{xx}r_{yy} + q_{yy}r_{xx}\right) - 8\,q_{xy}r_{xy}\left(p_{xx}q_{yy} + p_{yy}q_{xx}\right)\big)
\end{aligned}$$

where

$$K = p_{xx}r_{yy} - 2\,p_{xy}r_{xy} + p_{yy}r_{xx} + 2\left(q_{xy}^2 - q_{xx}q_{yy}\right)$$

is a relative differential invariant.

7.2 Singular set

Let $\Sigma_2' \subset \mathcal{SDE}_2$ be the set of points θ where $\langle X_\theta^{(2)} : X \in \mathfrak{g}\rangle \subset T_\theta(\mathcal{SDE}_2)$ is of dimension less than 42. It's given by

$$\Sigma_2' = \{\theta \in \mathcal{SDE}_2 : \operatorname{rank}\left(\mathcal{A}|_\theta\right) < 4\}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & -2q_{xy}-2r_{yy} & p_{xy}+q_{yy} & 0 \\ 0 & 2p_{xy}-2q_{yy} & 2p_{yy} & p_{yy} \\ 4q_{xy}+r_{yy} & -r_{xx} & -2q_{xx} & -2q_{xx} \\ -p_{xy}+q_{yy} & q_{xx}-r_{xy} & 0 & -q_{xy} \\ -p_{yy} & 2q_{xy}-r_{yy} & q_{yy} & 0 \\ -2q_{xx}+2r_{xy} & 0 & -2r_{xx} & -3r_{xx} \\ -2q_{xy}+r_{yy} & r_{xx} & -r_{xy} & -2r_{xy} \\ -2q_{yy} & 2r_{xy} & 0 & -r_{yy} \end{pmatrix}.$$

This set contains the singular points that can be seen from a local viewpoint on $\mathcal{SDE}_2$, but there may still be some singular (non-closed) orbits of dimension 42. We use the differential invariants I_i to filter out these. Let $\Sigma_3 \subset \mathcal{SDE}_3$ be the set of points where the 4-form

$$\hat{d}I_1 \wedge \hat{d}I_2 \wedge \hat{d}I_3 \wedge \hat{d}I_4$$

is not defined or is zero. Here $\hat{d}$ is the horizontal differential

$$\hat{d}f = \mathcal{D}_t(f)dt + \mathcal{D}_x(f)dx + \mathcal{D}_y(f)dy + \mathcal{D}_z(f)dz.$$

This defines the singular sets $\Sigma_k = (\pi_{k,3}|_{\mathcal{SDE}_k})^{-1}(\Sigma_3) \subset \mathcal{SDE}_k$ and $\Sigma_2 = \pi_{3,2}(\Sigma_3)$. The set Σ_2 of all singular points in $\mathcal{SDE}_2$ contains Σ_2'.

By using Maple, we can easily verify that $\{K = K_1 = K_2 = K_3 = K_4 = 0\}$ is contained in Σ_2', where K_i is the numerator of I_i for $i = 1, 2, 3, 4$. Notice also that 2-jets of conformally flat metrics are contained in Σ_2'.

7.3 Invariants of higher orders

The 1-forms $\hat{d}I_1, \hat{d}I_2, \hat{d}I_3, \hat{d}I_4$ determine an invariant horizontal coframe on $\mathcal{SDE}_3 \backslash \Sigma_3$. The basis elements of the dual frame $\hat{\partial}_{I_1}, \hat{\partial}_{I_2}, \hat{\partial}_{I_3}, \hat{\partial}_{I_4}$ are invariant derivations, the Tresse derivatives. We can rewrite metric (1) in terms of the invariant coframe:

$$g = \sum G_{ij}\hat{d}I_i\hat{d}I_j, \qquad \text{where} \qquad G_{ij} = g(\hat{\partial}_{I_i}, \hat{\partial}_{I_j}). \tag{4}$$

Since the $\hat{d}I_i$ are invariant, and $[g]$ is invariant, the map

$$\hat{G} = [G_{11} : G_{12} : G_{13} : G_{14} : G_{22} : G_{23} : G_{24} : G_{33} : G_{34} : G_{44}] \colon J^3 \to \mathbb{R}P^9$$

is invariant. Hence the functions G_{ij}/G_{44} are rational scalar differential invariants (of third order). This has been verified in Maple by differentiation of G_{ij}/G_{44} along the elements of $\mathfrak{g}$. It was also checked that these nine invariants are independent. By the principle of n-invariants [1], I_i and G_{ij}/G_{44} generate all scalar differential invariants.

Theorem 29. *The field of rational differential invariants of $\mathfrak{g}$ on $\mathcal{SDE}$ is generated by the differential invariants $I_k, G_{ij}/G_{44}$ and invariant derivations $\hat{\partial}_{I_k}$. The differential invariants in this field separate generic orbits in $\mathcal{SDE}_\infty$.*

7.4 The quotient equation

When restricted to a section g_0 of $\mathcal{C}_M^{\rm PR}$, the functions G_{ij} can be considered as functions of I_1, I_2, I_3, I_4. Two such nonsingular sections are equivalent if they determine the same map $\hat{G}(I_1, I_2, I_3, I_4)$. The quotient equation $(\mathcal{SDE}_\infty \setminus \Sigma_\infty)/\mathfrak{g}$ is given by

$$*W_g = W_g, \qquad \text{where} \qquad g = \sum G_{ij}(I_1, I_2, I_3, I_4)\hat{d}I_i\hat{d}I_j.$$

Here we consider $I_1, \dots, I_4$ as coordinates on M. Equivalently, given local coordinates $(x_1, \dots, x_4)$ on M the quotient equation is obtained by adding to $\mathcal{SDE}$ the equations $I_i = x_i$, $1 \le i \le 4$.

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