ON AN INTERRUPTED BIVARIATE RENEWAL PROCESS AND ITS APPLICATIONS

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Abstract–A special case of the bivariate renewal process is investigated. It is supposed, that this process is considered while the second component has a positive value. The algorithm for a calculation of the corresponding time's density is presented. In addition, a case of preventive renewal is considered. Such renewal takes place when the value of the second component is positive but is less than a fixed level. The following characteristics are investigated: distribution of the number of such renewals, the density of the time of the failure, etc. Numerical examples illustrate the given presentation.

Keywords: distribution of a failure time, preventive renewals.

1. INTRODUCTION

The bivariate renewal process is a natural extension of the univariate renewal process, which is well presented in the literature [1, 2]. A general theory for renewal processes of two dimensions is developed initially in the papers [3, 4]. Bivariate generating functions and bivariate Laplace transforms of these processes are derived. However, explicit forms of analytical solutions are unknown, except in the case of the exponential distribution.

In this connection, various approximations for the computation were supposed. A simple approximation for the two-dimensional renewal function, based only on the first two moments of the variables and their correlation coefficient, is considered in paper [5]. Many investigations about various approximations are described in papers [6-10].

Generalizations on multivariate renewal processes can be found in papers [11-14].

Bivariate renewal processes have wide applicability in a variety of areas. In the paper [5] it is noted, that many warranties of a product's or service's quality "are often two dimensional, such as an automobile warranty that guarantees repair up to a certain time and mileage after the sale." Bivariate renewal processes are efficiently used in maintenance policies and reliability [6, 15-19].

This study considers a special case of the general bivariate renewal model. As is usual, it is supposed that (X, Y), (X_1 , Y_1), (X_2 , Y_2) ,..., are independent identically distributed continuous random binary vectors. The component X is positive and is interpreted as a time. The component Y has arbitrary values and is interpreted as a stock. The pair (X, Y) has the density $f(x,y),\ x \geq 0,\ -\infty < y < \infty$. It is assumed that the mean value of Y_n is negative:

$$\mu = E(Y) = \int_{-\infty}^{\infty} y \int_0^{\infty} f(x,y)dxdy < 0. \tag{1}$$

Let $\beta > 0$ be an initial stock at a zero time and

$$(T_n, R_n) = (0, \beta) + \sum_{m=1}^{n}(X_m, Y_m),\ n = 1, 2, \dots . \tag{2}$$

The sequence $(T_1, R_1), (T_2, R_2), \dots$ is considered while $R_n > 0$.

Definition 1. An interrupted bivariate renewal process is a sequence

$$(T_1, R_1), (T_2, R_2), \dots, (T_n, R_n),$$

where $R_\eta > 0,\ \eta = 1, \dots, n-1,\ R_n \leq 0$.

We will say *a failure* occurs when $R_n \leq 0$. Let $T(\beta)$ be a time till the failure:

$$T(\beta) = \min_n \{T_n : R_n \leq 0\}.$$

The first half of this paper is devoted to the investigation of the distribution of $T(\beta)$ in detail and then the following generalization is considered. A preventive level α, $0 < \alpha < \beta$, is assigned. If an accumulated stock R_n is positive but less than α, then the store is renewed up to initial level β. Here, *a preventive renewal* process is discussed.

The content of this paper is then organized into the following sections. Section 2 is dedicated to the distribution of the failure time. Some computational aspects are considered in Section 3. Section 4 contains a numerical example, a case of the preventive renewals is presented in Section 5, and Section 6 contains concluding remarks.

2. DISTRIBUTION OF THE FAILURE TIME

Let $g_n(t,r)$ be the density of (T_n , R_n) jointly with probability that $R_\eta > 0$ for $\eta = 1, \dots, n$. Then for $t \geq 0,\ n = 2, 3, \dots$

$$g_1(t,r) = \begin{cases} f(t, r-\beta), & r > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

$$g_n(t,r) = \begin{cases} \int_0^t \int_0^\infty g_{n-1}(\tau,\rho) f(t-\tau, r-\rho)\, d\rho d\tau, & r > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let for $t \geq 0$

$$F_Y(t,y) = \int_{-\infty}^{y} f(t,\rho)d\rho, \ -\infty < y < \infty, \tag{4}$$

and $g_n^*(t)$ be the density of the time, when the zero level is reached, jointly with probability that it arises for the first time on the *n*-th jump:

$$g_1^*(t) = F_Y(t,-\beta),$$

$$g_n^*(t) = \int_0^t \int_0^\infty g_{n-1}(\tau,\rho)F_Y(t-\tau,-\rho)d\rho d\tau, \ n = 2,3,.... \tag{5}$$

Now we have the following expressions for the density $h(t)$ and the cumulative distribution function $H(t)$ of the random variable $T(\beta)$:

$$h(t) = \sum_{n=1}^{\infty} g_n^*(t), \ t \geq 0, \tag{6}$$

$$H(t) = \int_0^t h(\tau)d\tau = \sum_{n=1}^{\infty} \int_0^t g_n^*(\tau)d\tau, \ t \geq 0. \tag{7}$$

These formulas allow the calculation of the expectation $E\big(T(\beta)\big)$ and the variance $Var\big(T(\beta)\big)$ of the time until the failure $T(\beta)$.

The probabilities that the *n*-th jump takes place without the failure are calculated as follows:

$$Pr_n = \int_0^\infty \int_0^\infty g_n(t,r)drdt. \tag{8}$$

The probability that the failure takes place on the *n*-th jump:

$$Pr_n^* = \int_0^\infty g_n^*(t)dt. \tag{9}$$

3. SOME ASPECTS OF COMPUTATIONS

The computational difficulties increase as the jump's number rises. This is caused by multiple convolutions in the formulas. Consequently, a special approach is adopted which will be described for the *n*-th jump.

First, the densities $g_{n-1}(t,r), \ t \geq 0, \ r \geq 0$, for the previous jump are presented by the matrix $G = \big(G_{\eta,\theta}\big)$ of the dimension $nmax \times rmax$. Continuous values t and r are replaced by the lattice points (η,θ). Next, a mesh width $\Delta > 0$ is chosen, and let $\omega_t = \Delta \times nmax$, $\omega_r = \Delta \times rmax$. The point (η,θ) corresponds to the two-dimensional interval $(\eta\Delta, (\eta+1)\Delta) \times (\theta\Delta, (\theta+1)\Delta)$. The value $G_{\eta,\theta}$ in this point is defined as follows:

$$G_{\eta,\theta} = \frac{1}{2}\big(g_{n-1}(\eta\Delta,\theta\Delta) + g_{n-1}(\eta\Delta,(\theta+1)\Delta)\big),$$
$$\eta = 0,\dots,nmax-1, \ \theta = 0,\dots,rmax-1. \tag{10}$$

The matrix G is stored in computer's memory and is used instead of the function $g_{n-1}(t,r)$. An essential value $g_{n-1}(t,r)$ is defined as follows:

$$\tilde{g}_{n-1}(t,r) = G_{\eta,\theta}, \text{ if } t \in (\eta\Delta, (\eta+1)\Delta], \; r \in (\theta\Delta, (\theta+1)\Delta\,], \; t,r \geq 0. \tag{11}$$

The correctness of such a change is verified by comparison of two integrals:

$$\int_0^{\omega_t}\int_0^{\omega_r} \tilde{g}_{n-1}(t,r)drdt \text{ and } \int_0^{\omega_t}\int_0^{\omega_r} g_{n-1}(t,r)drdt, \tag{12}$$

whose values must be very close.

Now the density $g_n(t,r)$ can be calculated using the density $\tilde{g}_{n-1}(t,r)$ instead of $g_{n-1}(t,r)$. The probability (9) that the critical level is reached on the *n*-th jump is calculated as follows:

$$\widetilde{Pr}_n^* = \int_0^{\omega_t}\int_0^{t}\int_0^{\omega_r} \tilde{g}_{n-1}(\tau,r)F_Y(t-\tau,-r)drd\tau dt.$$

Analogously, calculation of the cumulative distribution function $H(t)$ by the formula (7) requires considerable computation time. This time can be decreased if the density $h(t)$ is presented by a linear combination of two easily calculated densities $d_1(t)$ and $d_2(t)$ with a coefficient $\chi,\; 0 < \chi < 1$, namely:

$$hApp(t) = \chi d_1(t) + (1-\chi)d_2(t), \; t \geq 0. \tag{13}$$

The coefficient χ is chosen thus to minimize the criterion:

$$\int_0^{\infty}\Big(h(t) - \big(\chi d_1(t) + (1-\chi)d_2(t)\big)\Big)^2 dt. \tag{14}$$

It is easy to demonstrate that the optimal value is the following:

$$\chi = \frac{\int_0^{\infty}\big(h(t)-d_2(t)\big)\big(d_1(t)-d_2(t)\big)dt}{\int_0^{\infty}\big(d_1(t)-d_2(t)\big)^2 dt}. \tag{15}$$

Now the cumulative distribution function $H(t)$ can be calculated by formula (7) using $hApp(t)$ instead of $h(t)$.

Some additional details regarding calculations are as follows. The density $h(t)$ is defined as an infinite sum (6). Obviously, a finite number *nmax* of addends is only used. The result is a lower border (frontier) for the density $h(t)$ and the cumulative distribution function $H(t)$ (see formulas (6) and (7)).

The cumulative distribution function of the failure's time jointly with the probability that the failure arises on the *n*-th jump is defined as follows:

$$H_n(t) = \int_0^t g_n^*(\tau)\,d\tau. \quad (16)$$

Note that

$$H(t) = \sum_{n=1}^{\infty} H_n(t),\ t \geq 0. \quad (17)$$

The expectation $E\big(T(\beta)\big)$ and variance $E\big(T(\beta)\big)$ are calculated by means of (6) or (7):

$$E\big(T(\beta)\big) = \int_0^{\infty} th(t)dt = \int_0^{\infty} tdH(t) = -\int_0^{\infty} td\big(1 - H(t)\big) = \int_0^{\infty}\big(1 - H(t)\big)\,dt,$$

$$Var\big(T(\beta)\big) = \int_0^{\infty}\Big(t - E\big(T(\beta)\big)\Big)^2 h(t)dt.$$

The infinite upper limits of a sum and an integral are substituted by the finite numbers *nmax* and ωt, which gives the following result for $E\big(T(\beta)\big)$, for example:

$$E\big(T(\beta)\big) = \int_0^{\omega t} \sum_{n=1}^{nmax}\big(H_n(\omega t) - H_n(t)\big)\,dt. \quad (18)$$

4. EXAMPLE

A case is considered, where $c < 0$, $\sigma > 0$, $\lambda > 0$ and

$$f(x,y) = \lambda^2\, xe^{-\lambda x}\frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{y-cx}{\sigma}\right)^2\right),\ \ t \geq 0,\ -\infty < y < \infty. \quad (19)$$

Further with respect to formula (4)

$$F_Y(t,y) = \int_{-\infty}^{y} f(t,\rho)d\rho = \int_{-\infty}^{y} \lambda^2\, te^{-\lambda t}\frac{1}{\sqrt{2\pi}\,\sigma} exp\left(-\frac{1}{2}\left(\frac{\rho - ct}{\sigma}\right)^2\right)d\rho =$$

$$= \lambda^2\, te^{-\lambda t}\phi\left(\frac{y-ct}{\sigma}\right),\ \ t \geq 0,\ \ -\infty < r < \infty, \quad (20)$$

where $\phi(\ldots)$ is the cumulative distribution function of the standard normal distribution.

Let $GF(\,y)$ be a cumulative distribution function of the accumulated stock during an interval between two jumps:

$$GF(\,y) = \int_0^{\infty} F_Y(t,y)dt,\ \ -\infty < y < \infty.$$

Numerical results are then presented for the following data: $\lambda = 1$, $c = -1$, $\sigma = 1$, $\beta = 2$. The infinite limit ∞ of sums is replaced by *nmax* = 22 for a time and by *rmax* = 12 for a stock, and the mesh width Δ = 0.5 is chosen. Therefore $\omega t = \Delta \times nmax = 11$ and $\omega r = \Delta \times rmax = 6$. Firstly, note that the mean value of Y_n is negative (see (1)):

$$\mu = E(Y_n) = \int_{-\infty}^{\infty} y \int_0^{\infty} f(x,y)dxdy = -1.986.$$

The graph of the function $GF(\,r)$ is presented in Fig. 1.

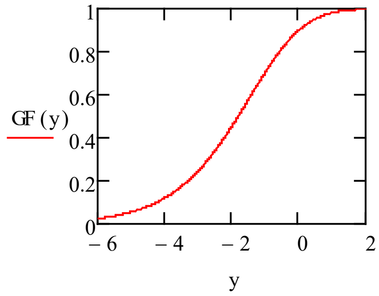


Figure 1. Cumulative distribution function $gRn(t,1) = g_n^*(t \mid 1)$.

The density $g_1(t,r)$ of the random pair (T_1, R_1) for the first jump is calculated by formula (3). The probability that the critical level is not reached is the following:

$$Pr_1 = \int_0^{\omega_t} \int_0^{\omega_r} g_1(t,r)drdt = 0.548.$$

The probability of the contrary event equals 0.452, which can be found as follows:

$$Pr_1^* = \int_0^{\omega_t} F_Y(t,-\beta)dt = \int_0^{\omega_t} \int_{-\omega}^{-\beta} f(t,r)drdt = 0.452.$$

The following results have place for the second jump:

$$Pr_2 = \int_0^{\omega_t} \int_0^{\omega_r} g_2(t,r)drdt = 0.195,$$

$$Pr_2^* = \int_0^{\omega_t} g_2^*(t)dt = 0.352.$$

Therefore, the probability that the second jump occurs equals 0.195 + 0.352 = 0.547. The true value is 0.548. The difference 0.548 – 0.547 = 0.001 is a computational error. For the third jump, we then have:

$$Pr_3 = \int_0^{\omega_t} \int_0^{\omega_r} g_3(t,r)drdt = 0.066,$$

$$Pr_3^* = \int_0^{\omega_t} g_3^*(t)dt = 0.129.$$

Now the computational error no longer exists: 0.195 – (0.066 + 0.129) = 0.

As mentioned above, the computational difficulties increase as the jump's number *n* rises. Consequently, for the fourth jump, the approach adopted has been described in Section 3. First, the density $g_3(t,r)$ is replaced by the two-dimensional matrix of the dimension *nmax* × *rmax* = 22 × 12. Table 1 contains a main sub-matrix *G3* of this matrix, having the dimension 11 × 9.

Table 1. Sub-matrix *G3*.

G3 :=

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
4.646×10^{-5}	5.64×10^{-5}	6.015×10^{-5}	5.855×10^{-5}	5.269×10^{-5}	4.3×10^{-5}	3.226×10^{-5}	2.228×10^{-5}	1.415×10^{-5}
3.405×10^{-4}	3.786×10^{-4}	3.831×10^{-4}	3.539×10^{-4}	2.959×10^{-4}	2.24×10^{-4}	1.555×10^{-4}	9.919×10^{-5}	5.845×10^{-5}
1.094×10^{-3}	1.144×10^{-3}	1.072×10^{-3}	9.18×10^{-4}	7.107×10^{-4}	4.977×10^{-4}	3.198×10^{-4}	1.884×10^{-4}	1.022×10^{-4}
2.123×10^{-3}	2.082×10^{-3}	1.823×10^{-3}	1.429×10^{-3}	1.018×10^{-3}	6.641×10^{-4}	3.971×10^{-4}	2.178×10^{-4}	1.1×10^{-4}
3.032×10^{-3}	2.766×10^{-3}	2.243×10^{-3}	1.637×10^{-3}	1.081×10^{-3}	6.519×10^{-4}	3.617×10^{-4}	1.832×10^{-4}	8.583×10^{-5}
3.392×10^{-3}	2.864×10^{-3}	2.165×10^{-3}	1.472×10^{-3}	9.011×10^{-4}	5.005×10^{-4}	2.547×10^{-4}	1.202×10^{-4}	5.197×10^{-5}
3.118×10^{-3}	2.443×10^{-3}	1.712×10^{-3}	1.066×10^{-3}	5.941×10^{-4}	3.036×10^{-4}	1.441×10^{-4}	6.264×10^{-5}	2.495×10^{-5}
2.43×10^{-3}	1.761×10^{-3}	1.137×10^{-3}	6.487×10^{-4}	3.292×10^{-4}	1.58×10^{-4}	6.934×10^{-5}	2.785×10^{-5}	1.025×10^{-5}
1.61×10^{-3}	1.069×10^{-3}	6.367×10^{-4}	3.448×10^{-4}	1.691×10^{-4}	7.552×10^{-5}	3.081×10^{-5}	1.151×10^{-5}	3.926×10^{-6}

The matrix *G3* is stored in computer's memory and is used instead of the function $g_3(t,r)$ calculation. Also, the function $g_3(t,r)$ is replaced by the function $\tilde{g}_3(t,r)$, which is calculated by formula (11). The correctness of such a change is verified by comparison of two integrals from (12). The first integral

$$\int_0^{\omega_t}\int_0^{\omega_r} \tilde{g}_3(t,r)\, drdt = 0.0659$$

is very close to above obtained value 0.066 of the second integral.

Now the density $g_4(t,r)$ can be calculated using the density $\tilde{g}_3(t,r)$ instead of the density $g_3(t,r)$. The probability that the critical level is reached on the fourth jump is as follows:

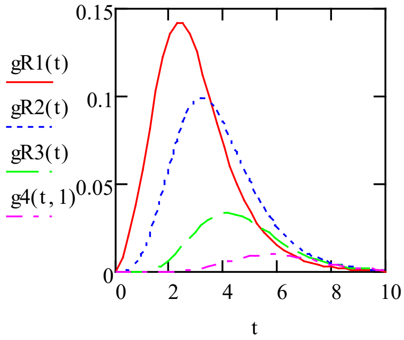
$$Pr_4^* = \int_0^{\omega_t}\int_0^{\omega_t}\int_0^{\omega_r} \tilde{g}_3(\tau,r)F_Y(t-\tau,-r)drd\tau dt = 0.043.$$

The probability Pr_4 that the critical level is not reached equals 0.023.

We act analogously for the fifth step and the following results are achieved: $Pr_5^* = 0.014$, $Pr_5 = 0.008$.

The probability that the critical level will be reached during five jumps equals 0.452 + 0.352 + 0.129 +0.043 +0.014 = 0.990. This probability is close to one and thus we can do no more than 5 steps.

Graphs of the densities $g_1^*(t)$, $g_2^*(t)$, $g_3^*(t)$ and $g_4^*(t)$ are presented in Fig. 2, where $gRi(\mathrm{t}) = g_i^*(t)$. The densities $h(t) = \sum_{n=1}^{5} g_n^*(t),\ t \geq 0$, of the time $T(\beta)$ until the failure, are also presented.



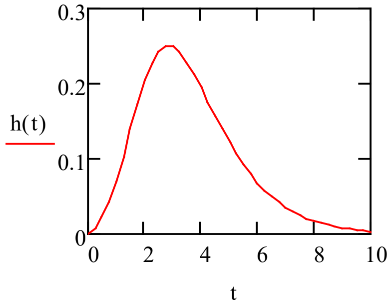


Figure 2. Densities $gRi(t)$ and h (t).

The expectation and the variance of the time $T(\beta)$ until the failure is $E(T(\beta)) = 3.624$ and $var(T(\beta)) = 3.107$.

At this stage, approximation of the cumulative distribution function $H(t)$ of the time $T(\beta)$ is necessary. To achieve this, the density $h(t)$ is represented by means of easily calculated functions. In the case considered here, a linear combination of a normal density with a coefficient χ, $0 < \chi < 1$, and a gamma density with a coefficient $1 - \chi$, was used, namely:

$$hApp(t) = \chi \frac{1}{s\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{t-\tilde{\mu}}{s}\right)^2\right) + (1-\chi)\left(\tilde{\lambda}t\right)^{\gamma-1} \frac{\tilde{\lambda}}{\Gamma(\gamma)} e^{-\tilde{\lambda}t}, \qquad t \geq 0.$$

The parameters $\tilde{\lambda}$, γ, $\tilde{\mu}$ and s are determined by such a way that both densities have the same expectation and variance. It is known that $\tilde{\mu} = E(T(\beta)) = 3.624$, $Var(T(\beta)) = 3.107$, $s = \sqrt{Var(T(\beta))} = \sqrt{3.107} = 1.763$. Further

$$\tilde{\lambda} = \frac{E(T(\beta))}{var(T(\beta))} = \frac{3.624}{3.107} = 1.166, \qquad \gamma = E(T(\beta))^2 \frac{1}{var(T(\beta))} = 4.226.$$

The coefficient χ is calculated by formula (15) and equals 0.899. Fig. 3 demonstrates graphs of the density $h(t)$ and of the approximated density $hApp(t)$. We can see the close approximation.

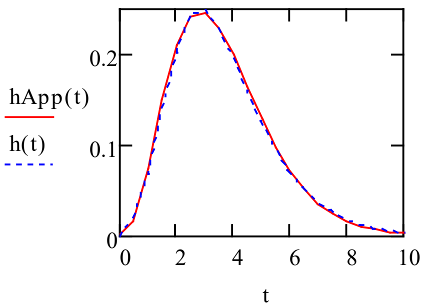


Figure 3. Density $h(t)$ and its approximation $hApp(t)$.

Using the resulting approximation, the cumulative distribution function of the time $T(\beta)$ can be calculated as follows:

$$Happ(t) = \int_0^t hApp(\theta)d\theta =$$

$$= \int_0^t \left(\chi \frac{1}{s\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{\theta-\tilde{\mu}}{s}\right)^2\right) + (1-\chi)\left(\tilde{\lambda}\theta\right)^{\gamma-1} \frac{\tilde{\lambda}}{\Gamma(\gamma)} e^{-\theta\tilde{\lambda}}\right) d\theta =$$

$$= \chi\Phi\left(\frac{t-\tilde{\mu}}{s}\right) + (1-\chi)\frac{1}{\Gamma(\gamma)}\int_0^t \left(\tilde{\lambda}\theta\right)^{\gamma-1} e^{-\theta\tilde{\lambda}}\, d\theta\tilde{\lambda} =$$

$$= \chi\Phi\left(\frac{t-\tilde{\mu}}{s}\right) + (1-\chi)\frac{1}{\Gamma(\gamma)}\int_0^{\tilde{\lambda}t} u^{\gamma-1}e^{-u}\, du, \qquad t \geq 0.$$

The corresponding graph is presented in Fig. 4.

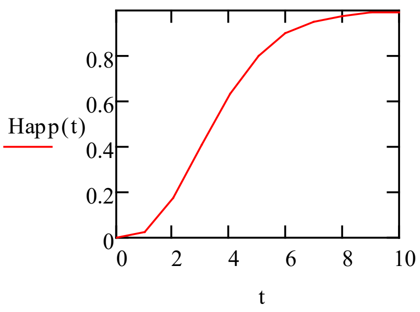


Figure 4. Approximation of the cumulative distribution function *Happ*(*t*).

5. PREVENTIVE RENEWALS

Now consideration is given to the above described modification of the presented model. A preventive level α, $0 < \alpha < \beta$, is assigned. If the current jump gives an accumulated stock R_n which is positive but less than α, then the stock is renewed up to initial level β. Here, we consider *a preventive jump* or *a preventive renewal* and the desire to calculate the earlier considered characteristics for such conditions.

Further, the following random variables are considered for the initial level β and the preventive level α:

- $T(\alpha, \beta)$ is the time of reaching the zero-level, that is to say a failure time;
- $T^{**}(\alpha, \beta)$ is the time of first reaching the α-level without the failure, $P\{T^{**}(\alpha, \beta) < \infty\} < 1$;
- $N(\alpha, \beta)$ is the number of preventive renewals until the zero-level is reached.

The above presented methods of an approximation for a calculation of corresponding distributions will be used.

As defined earlier, let $g_n(t, r \mid \alpha), r > 0,$ be the density of (T_n, R_n) jointly with probability $R_\eta > 0$ for $\eta = 1, \dots, n$. In addition, the density $g_n^{**}(t \mid \alpha)$ with respect to the time t for the n-th jump, when $R_\eta > 0$, $\eta = 1, \dots, n-1,$ and $0 < R_n \leq \alpha$ is introduced. In this case the component R_n only reaches the value α and initial value β is renewed (i.e., a preventive renewal takes place).

Then for $t \geq 0$

$$g_1(t, r \mid \alpha) = \begin{cases} f(t, r-\beta), r > 0\,, \\ 0, otherwise, \end{cases} \tag{21}$$

$$g_1^{**}(t \mid \alpha) = \int_0^\alpha g_1(t, \rho \mid \alpha) d\rho = \int_0^\alpha f(t, \rho - \beta) d\rho =$$
$$= \int_0^\alpha f(t, \rho - \beta) d(\rho - \beta) = F_Y(t, \alpha - \beta) - F_Y(t, -\beta), \tag{22}$$

$$g_n(t, r \mid \alpha) = \int_0^t \int_\alpha^\infty g_{n-1}(\tau, \rho \mid \alpha) f(t-\tau, r-\rho) d\rho d\tau +$$
$$+ \int_0^t g_{n-1}^{**}(\tau \mid \alpha) \big(f(t-\tau, r-\beta)\big) d\tau, n = 2, 3, \dots; r > 0, n = 2, 3, \dots, \tag{23}$$

$$g_n^{**}(t \mid \alpha) = \int_0^\alpha g_n(t, \rho \mid \alpha) d\rho =$$
$$= \int_0^t \int_\alpha^\infty g_{n-1}(\tau, \rho \mid \alpha) \big(F_Y(t-\tau, \alpha-\rho) - F_Y(t-\tau, -\rho)\big) d\rho d\tau + \tag{24}$$
$$+ \int_0^t g_{n-1}^{**}(\tau \mid \alpha) \big(F_Y(t-\tau, \alpha-\beta) - F_Y(t-\tau, -\beta)\big) d\tau, \qquad n = 2, 3, \dots .$$

As defined earlier, $g_n^*(t \mid \alpha)$ is the density of the time when the zero level is reached, jointly with probability that it arises on the n-th jump, is as follows for $t \geq 0$:

$$g_1^*(t \mid \alpha) = F_Y(t, -\beta),$$

$$g_n^*(t \mid \alpha) = \int_0^t \int_\alpha^\infty g_{n-1}(\tau, \rho \mid \alpha) F_Y(t-\tau, -\rho) d\rho d\tau +$$
$$+ \int_0^t g_{n-1}^{**}(\tau \mid \alpha)\, F_Y(t-\tau, -\beta) d\tau, \ n = 2, 3, \dots . \tag{25}$$

It is necessary to also introduce some computational aspects. The procedure described in Section 3 is applied to the density $g_n(t, r, \alpha)$ and the density $g_n^{**}(t \mid \alpha)$. The matrix G of the values $g_n(t, r, \alpha)$ for $r > \alpha$ is stored analogously, but the vector G^{**} is used for values $g_n^{**}(t \mid \alpha)$.

Let $Pr_n, Pr_n^{**}, Pr_n^{*}$ be the probabilities that the *n*-th jump does not have a renewal, has a preventive renewal, and gives a failure, correspondingly:

$$Pr_n = \int_0^\infty \int_\alpha^\infty g_n(t, r \mid \alpha) dr dt, \quad (26)$$

$$Pr_n^{**} = \int_0^\infty g_n^{**}(t \mid \alpha) dt, \quad (27)$$

$$Pr_n^{*} = \int_0^\infty g_n^{*}(t \mid \alpha) dt. \quad (28)$$

The obvious identity

$$Pr_n + Pr_n^{**} + Pr_n^{*} = Pr_{n-1} + Pr_{n-1}^{**}, \; n = 2, 3, \dots, \quad (29)$$

allows the control of the precision of calculations.

Now we have the following expressions for the density $h(t \mid \alpha)$ and its cumulative distribution function $H(t \mid \alpha)$ of the random variable $T(\alpha, \beta)$:

$$h(t \mid \alpha) = \sum_{n=1}^{\infty} g_n^{*}(t \mid \alpha), \; t \geq 0, \quad (30)$$

$$H(t \mid \alpha) = \int_0^t h(\tau \mid \alpha) d\tau, \; t \geq 0. \quad (31)$$

The renewal function $h^{**}(t \mid \alpha)$ for *preventive jumps* is the following:

$$h^{**}(t \mid \alpha) = \sum_{n=1}^{\infty} g_n^{**}(t \mid \alpha), \; t \geq 0. \quad (32)$$

Let us calculate expectations of the random variables $T(\alpha, \beta)$ and $N(\alpha, \beta)$:

$$E\big(T(\alpha, \beta)\big) = \int_0^\infty t h(t \mid \alpha) dt = \int_0^\infty (1 - H(t \mid \alpha)) dt, \quad (33)$$

$$E\big(N(\alpha, \beta)\big) = \int_0^\infty h^{**}(t \mid \alpha) dt. \quad (34)$$

Now the following *optimization problem* can be formulated. Let a cost of preventive renewal equal $c^{**} > 0$. On the other hand, a reward of the size φ is assigned for a unit time until a failure (a reaching of zero level). A mean reward until the failure is calculated as follows:

$$R(\alpha) = \varphi\, E\big(T(\alpha, \beta)\big) - c^{**} E\big(N(\alpha, \beta)\big), \; 0 < \alpha < \beta. \quad (35)$$

The criterion $R(\alpha)$ is calculated by means of the formulas (33) and (34). The preventive level α_1 is better than the level α_2 , if $R(\alpha_1) > R(\alpha_2)$.

6. EXAMPLE (CONTINUE)

Results of calculations for $\alpha = 1$ are now presented. Two-dimensional densities (19) give the following probability (26) that the level α is not reached for the first jump: $Pr_1 = 0.296$. The probability (27) that the first jump ends with a renewal is $Pr_1^{**} = 0.252$. The probability (28) that the first jump ends with a failure is $Pr_1^{*} = 0.452$. This coincides with earlier presented results. Table 2 contains these probabilities for many jumps. The last two rows allow a verification of the identity (29).

Table 2. Probabilities Pr_n, Pr_n^{**}, Pr_n^{*} for $\alpha = 1$

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
Pr_n	0.296	0.154	0.083	0.045	0.025	0.015
Pr_n^{**}	0.252	0.131	0.068	0.036	0.020	0.012
Pr_n^{*}	0.452	0.262	0.134	0.070	0.036	0.018
$Pr_n + Pr_n^{**}$	0.548	0.285	0.151	0.081	0.045	0.027
$Pr_n + Pr_n^{**} + Pr_n^{*}$	1.000	0.547	0.285	0.151	0.081	0.045

The probability that the critical level will be reached during six jumps equals 0.452 + 0.262 + 0.134 + 0.070 + 0.036 +0.018 = 0.972. The mean number of renewals equals 0.252 + 0.131 + 0.068 + 0.036 + 0.020 + 0.012 = 0.519.

Calculations for the second jump were performed by main formulas (23)–(25). An approximated approach is applied for calculations beginning from the third jump. As described earlier, we set $\Delta = 0.5$, $nmax = 22$, $rmax = 12$, $\omega_t = \Delta \times nmax$, and $\omega_r = \Delta \times rmax = 6$. The previous (second) density $g_2(t, r \mid 1)$ is presented by the matrix of the dimension $nmax \times rmax = 22 \times 12$. Table 3 contains a main sub-matrix *G2* of this matrix, having the dimension 11×9.

Table 3. Sub-matrix *G2* of the density $g_2(t, r \mid 1)$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 9\times10^{-4} & 9.913\times10^{-4} & 9.206\times10^{-4} & 7.299\times10^{-4} & 4.984\times10^{-4} & 2.945\times10^{-4} & 1.537\times10^{-4} & 7.156\times10^{-5} \\
0 & 0 & 4.68\times10^{-3} & 4.622\times10^{-3} & 3.83\times10^{-3} & 2.683\times10^{-3} & 1.606\times10^{-3} & 8.362\times10^{-4} & 3.844\times10^{-4} & 1.555\times10^{-4} \\
0 & 0 & 9.259\times10^{-3} & 8.22\times10^{-3} & 6.123\times10^{-3} & 3.842\times10^{-3} & 2.045\times10^{-3} & 9.297\times10^{-4} & 3.654\times10^{-4} & 1.282\times10^{-4} \\
0 & 0 & 0.012 & 9.327\times10^{-3} & 6.299\times10^{-3} & 3.579\times10^{-3} & 1.716\times10^{-3} & 6.97\times10^{-4} & 2.42\times10^{-4} & 7.332\times10^{-5} \\
0 & 0 & 0.011 & 7.94\times10^{-3} & 4.9\times10^{-3} & 2.55\times10^{-3} & 1.121\times10^{-3} & 4.185\times10^{-4} & 1.307\times10^{-4} & 3.474\times10^{-5} \\
0 & 0 & 8.179\times10^{-3} & 5.446\times10^{-3} & 3.087\times10^{-3} & 1.483\times10^{-3} & 6.001\times10^{-4} & 2.044\times10^{-4} & 5.878\times10^{-5} & 1.403\times10^{-5} \\
0 & 0 & 5.135\times10^{-3} & 3.122\times10^{-3} & 1.628\times10^{-3} & 7.24\times10^{-4} & 2.721\times10^{-4} & 8.576\times10^{-5} & 2.258\times10^{-5} & 4.938\times10^{-6} \\
0 & 0 & 2.753\times10^{-3} & 1.527\times10^{-3} & 7.32\times10^{-4} & 3.009\times10^{-4} & 1.052\times10^{-4} & 3.111\times10^{-5} & 7.704\times10^{-6} & 1.587\times10^{-6} \\
0 & 0 & 1.279\times10^{-3} & 6.454\times10^{-4} & 2.832\times10^{-4} & 1.073\times10^{-4} & 3.489\times10^{-5} & 9.647\times10^{-6} & 2.249\times10^{-6} & 4.386\times10^{-7} \\
0 & 0 & 5.185\times10^{-4} & 2.373\times10^{-4} & 9.483\times10^{-5} & 3.291\times10^{-5} & 9.853\times10^{-6} & 2.525\times10^{-6} & 5.492\times10^{-7} & 1.005\times10^{-7} \\
0 & 0 & 1.845\times10^{-4} & 7.63\times10^{-5} & 2.764\times10^{-5} & 8.733\times10^{-6} & 2.388\times10^{-6} & 5.6\times10^{-7} & 1.137\times10^{-7} & 1.942\times10^{-8}
\end{pmatrix}$$

Now we can calculate function $\tilde{g}_2(t, r \mid \alpha)$ and use one instead of the function $g_2(t, r \mid \alpha)$, analogously to formula (23). The correctness of such a change is verified by comparison of two integrals:

$$\int_0^{\omega_t}\int_\alpha^{\omega_r} g_2(t, r \mid \alpha)\,dr\,dt = 0.154 \quad \text{and} \quad \int_0^{\omega_t}\int_\alpha^{\omega_r} \tilde{g}_2(t, r \mid \alpha)\,dr\,dt = 0.156.$$

It can be seen that the difference is not large.

Analogously we act with respect to the density $g_2^{**}(t \mid \alpha)$. It is stored as the vector G^{**}, whose 14 elements are presented in Table 4.

Table 4. Vector $10^4 \times G^{**}$ of the density $g_2^{**}(t \mid 1)$

i	0	1	2	3	4	5	6
G_i^{**}	5.6	39.5	110	200	240	240	190
i	7	8	9	10	11	12	13
G_i^{**}	130	79	41.	19	7.6	2.7	0.8

As

$$\int_0^{\omega_t} g_2^{**}(t \mid \alpha)dt = 0.131 \text{ and } \int_0^{\omega_t} \tilde{g}_2^{**}(t \mid \alpha)dt = 0.13,$$

the function $\tilde{g}_2^{**}(t \mid \alpha)$ instead of function $g_2^{**}(t \mid \alpha)$ can be used. The same procedure is used for the density $g_2^{*}(t \mid \alpha)$. Its values are stored as the vector G^*, which is presented in Table 5.

Table 5. Vector $10^4 \times G^*$ of the density $g_2^*(t \mid \alpha)$

i	0	1	2	3	4	5	6
G_i^*	1.72	15.5	56.5	130	220	229	340
i	7	8	9	10	11	12	13
G_i^*	310	260	200	150	100	72	48
i	14	15	16	17	18	19	20
G_i^*	32	21	14	9.1	5.9	3.8	2.4

Using functions $\tilde{g}_2(t, r \mid \alpha)$, $g_2^{**}(t \mid \alpha)$, and $g_2^{*}(t \mid \alpha)$, all functions, related to the third jump, can be calculated. This procedure is applied also for the remaining jumps. Fig. 5 contains graphs of the densities $gRn(t,1) = g_n^*(t \mid 1)$ of failure's time on the *n*-th jump, *n* =1,…, 5. The density $h(t \mid 1)$ of the failure's time $T(1,2)$ is also presented.

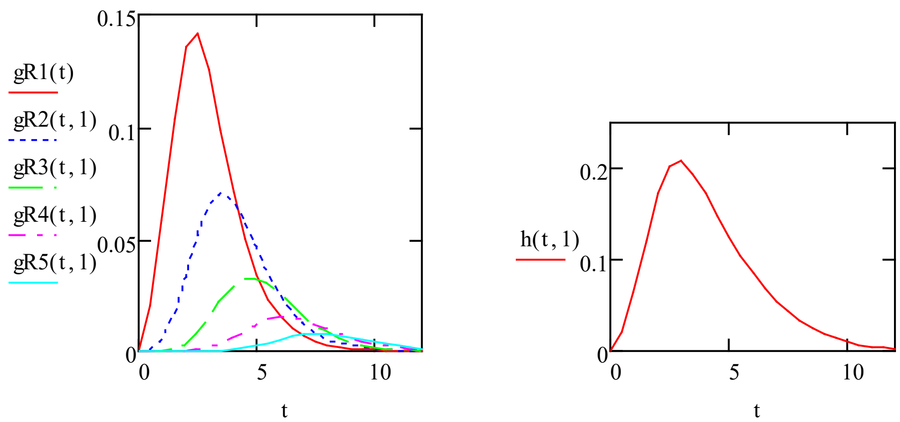


Figure 5. Functions $gRn(t,1) = g_n^*(t \mid 1)$ and $h(t,1)$.

The expectation, the variance, and the standard deviation of this time are $E(T(1,2)) = 3.993$, $Var(T(1,2)) = 4.356$, and $\sigma(T(1,2)) = 2.087$. As established earlier, for the case with $\alpha = 0$ we have $\tilde{E}(T(\beta)) = 3.624$. The difference between the expectations expressed in percent is $\frac{3.993-3.624}{3.624}100\% = 10.2\%$.

An approximation of the density $h(t \mid 1)$ is performed as earlier and gives the following result:

$$hApp(t) = 0.908\,(0.917t)^{2.66}\,\frac{0.917}{\Gamma(3.66)}e^{-0.917t} +$$

$$+(1-0.908)\frac{1}{\sqrt{2\pi}\,2.087}exp\left(-\frac{1}{2}\left(\frac{t-3.993}{2.087}\right)^2\right), \qquad t \geq 0.$$

Graphs of the densities $h(t \mid 1)$ and $hApp(t)$ are presented in Fig. 6.

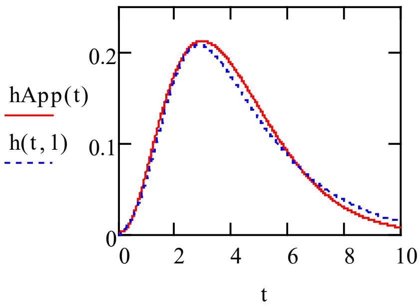


Figure 6. Densities $h(t \mid 1)$ and its approximation $hApp(t)$.

Now it is possible to perform a calculation by formula (31). Corresponding graph is presented in Fig.7.

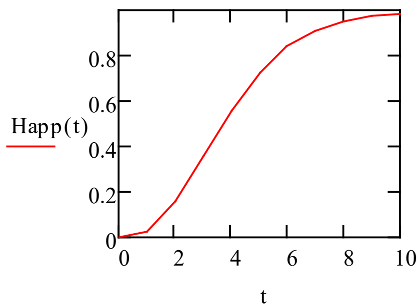


Figure 7. Cumulative distribution function $H(t)$.

In addition, some results for the case $\alpha = 0.5$ are presented, which allows comparison of the results for three cases: $\alpha_1 = 0$, $\alpha_2 = 0.5$, and $\alpha_3 = 1$. Table 2 presented earlier is replaced by Table 6.

Table 6. Probabilities Pr_n, Pr_n^{**}, Pr_n^{*} for $\alpha_2 = 0.5$

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
Pr_n	0.422	0.183	0.085	0.040	0.019	0.009

Pr_n^{**}	0.126	0.063	0.028	0.013	0.006	0.003
Pr_n^{*}	0.452	0.301	0.134	0.060	0.027	0.012
$Pr_n + Pr_n^{**}$	0.548	0.246	0.113	0.053	0.025	0.012
$Pr_n + Pr_n^{**} + Pr_n^{*}$	1.000	0.547	0.247	0.113	0.052	0.024

The probability that the critical level will be reached during five jumps equals 0.452 + 0.301 + 0.134 + 0.060 + 0.027 = 0.974. This probability for six jumps equals 0.974 + 0.012 = =0.986. The expectation of failure's time $\tilde{E}(T(\alpha_2, \beta)) = 3.865$. As a reminder, $\tilde{E}(T(\beta)) = 3.624$ for the case α = 0 and $\tilde{E}(T(\beta)) = 3.993$ for the case α = 1. The mean number of renewals $E\big(N(\alpha_2, \beta)\big) = 0.239$.

The above three preventive levels: $\alpha_1 = 0$, $\alpha_2 = 0.5$, and $\alpha_3 = 1$ were considered. Let us now consider the optimization criterion (35), using $\varphi = 1$, and various values of c^{**} . The following expressions occur:

$$R(\alpha_1) = E\big(T(\alpha_1, \beta)\big) = 3.624,$$

$$R(\alpha_2) = E\big(T(\alpha_2, \beta)\big) - c^{**} E\big(N(\alpha_2, \beta)\big) = 3.865 - 0.239\, c^{**},$$

$$R(\alpha_3) = E\big(T(\alpha_3, \beta)\big) - c^{**} E\big(N(\alpha_3, \beta)\big) = 3.993 - 0.519\, c^{**}.$$

The level α_1 is better than the level α_2 , if

$$c^{**} > \frac{1}{0.239}(3.865 - 3.624) = 1.008.$$

The level α_3 is better than the level α_2 , if

$$c^{**} < \infty \frac{1}{0.519 - 0.239}(3.993 - 3.865) = 0.457.$$

Also, (0, 0.457), (0.457, 1.008), and (1.008, ∞) are the intervals of c^{**}, for which the optimal levels are α_3, α_2, and α_1.

7. CONCLUSION

The bivariate renewal process (T_n, R_n) is investigated for a special case, where this process is considered while the second component (a stock) has a positive value. A failure occurs if a nonpositive value of the stock takes place. A calculation of a density of failure's time is performed.

Further, the following generalization is considered. A preventive level α, $0 < \alpha < \beta$, is assigned. If an accumulated stock R_n is positive but less than α, then the stock is renewed up to initial level β. We talk about preventive renewal here. Characteristics investigated for the presented case include: a distribution of a number of preventive renewals, a density of the failure's time, etc. The problem of the optimal level α is also discussed.

Some computational techniques, supported by MathCAD software, are used for a realization of calculations. Numerical examples illustrate the results obtained and reported.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. D.R. Cox, *Renewal Theory*, (Methuen & Co. Ltd., London, 1962).
2. W.L. Smith, "Renewal Theory and its Ramifications", *Journal of the Royal Statistical Society,* Series B, **B20**(2), pp.243-284, 1958.
3. P.J. Bickel and J.A. Yahav, "Renewal theorem in the plane", *Ann. Probab*. 11, pp.143-157, 1985.
4. J.J. Hunter, "Renewal Theory in Two Dimensions: Basic Results", *Advances in Applied Probability*, **6**, pp.376-391, 1974a.
5. V. Arunachalam and A. Calvache, "Approximation of the Bivariate Renewal Function", *Communications in Statistics – Simulation and Computation*, **44**(1), pp.154-167, 2015.
6. J. Eliashberg, N.D. Singpurwalla, and S.P. Wilson, "Calculating the Reserve for a Time and Usage Indexed Warranty", *Management Science*, **43**(7), pp.966-975, 1997.
7. E.M. Hadji, N.S. Kambo, and A. Rangan, "Two-dimensional Renewal Function Approximation", *Communications in Statistics – Theory and Methods*, **44**(15), pp.3107-3124, 2015.
8. J.J. Hunter, "Renewal Theory in Two Dimensions: Asymptotic Results", *Advances in Applied Probability*, **6**, pp.546-562, 1974b.
9. K.V. Mitov and E. Omey, "Intuitive approximation for the renewal function", *Stat. Probab. Letter*, 4, pp.72-80, 2015.
10. E. Omey, K.V. Mitov, and R. Vesilo, "Approximations in bivariate renewal theory", *Publications de l'Institut Mathématique, Nouvelle série*, **104**(118), pp.69-88, 2018. DOI: https://doi.org/10.2298/PIM1818069O
11. K.L. Chung, "On the renewal theorem in higher dimension", *Skand. Aktuarietidiskr*. 35, pp.188-194, 1952.
12. V. Kaniskauskas and L. Dronova-Plartbardze, "The renewal equation for multivariate renewal processes", *Siauliai Math. Semin*., **5**(13), pp.47-53, 2010.

13. F. Spitzer, “A Multidimensional Renewal Theorem in Probability, Statistical Mechanics, and Number Theory”, *Advances in Mathematics Supplemental Studies, ed. by Gian-Carlo Rota*, Academic Press, **9**, pp.147-155, 1986.
14. J. Steinebach and V.R. Eastwood, “Extreme Value Asymptotics for Multivariate Renewal Processes”, *Journal of Multivariate Analysis*, **56**, pp.284-302, 1996.
15. I.B. Gertsbakh, *Models of Preventive Maintenance,* Studies in Mathematical and Managerial Economics, V. 23, (Oxford: North-Holland Publishing, Amsterdam, 1977).
16. W. Stadje and D. Zuckerman, “Optimal Maintenance Strategies for Repairable Systems with General Degree of Repair”, *Journal of Applied Probability*, **28**, pp.384-396, 1991. https://doi.org/10.2307/3214874
17. W. Stadje and D. Zuckerman, “Optimal Strategies for Some Repair Replacement Models”, *Advances in Applied Probability*, **22**, pp.641-656, 1990. https://doi.org/10.2307/1427462
18. S.-C. Yang, *A bivariate renewal process and its applications in maintenance policies* – Ph.D. Dissertation. (Virginia polytechnic Institute and State University, Blacksburg, 1999).
19. S.-C. Yang and J.A. Nachlass, “Bivariate reliability and availability modeling”, *IEEE Trans. Reliab.*, **50**(1), pp.26–35, 2001.