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Some combinatorial invariants determined by Betti numbers of Stanley-Reisner ideals

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Jan Nyquist Roksvold

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Chapter 1

Introduction

This thesis is comprised of three papers. Each of these papers aim to contribute to the field commonly referred to as *combinatorial commutative algebra* –

> "...a fascinating new branch of commutative algebra created by Hochster and Stanley in the mid-seventies"([5, p. 207]).

Characteristic of this field is the application of algebraic and/or homological methods to problems concerning combinatorial objects – such as e.g. simplicial complexes. In the sub-branch occasionally referred to as *Stanley-Reisner theory* this is achieved by constructing from the simplicial complex an ideal generated by monomials corresponding to certain sets of edges, followed by a study of the algebraic properties of this ideal. In Paper 1 we look at the ideal whose generators correspond to the bases of a matroid, while in Papers 2 and 3 we consider the so-called Stanley-Reisner ideal, whose generators correspond to minimal nonfaces (circuits) of the simplicial complex (matroid).

A recurring theme of this thesis is the connection between certain invariants of a simplicial complex or matroid, and the Betti numbers associated to a graded minimal free resolution of the derived ideal. Examples of such invariants are the higher weights, the matroidal polynomials/weight enumerators, or even the isomorphism class of the complex itself. Also common to all three papers is that we look at constructions such as elongations, substructures (skeletons or truncations) or components (blocks) of a complex, and see how the Betti numbers of these structures relate to those of the original complex. Although each of the papers are well within the realm of pure mathematics, there is at least for Papers 1 and 2 a certain connection to the theory of error-correcting linear codes.

As is common, this introduction contains a discussion of each of the three papers. However, since these involve concepts and terminology from very different branches of mathematics, such as graph theory, linear codes, homological algebra, and of course simplicial complex/matroid theory, we shall begin by giving a brief introduction to the relevant parts of each of these topics. The intention is that these introductory sections will be sufficient for a reader with a general background in pure mathematics to fully comprehend our subsequent discussion of each of the three papers. In other words, what follows is aimed at establishing terminology and setting the stage, before we move on to discuss the papers individually.

Throughout the introduction, and throughout much of the papers, we shall let E be a finite set with n elements. The set E will typically serve as the ground set for our simplicial complexes and matroids. In most of the examples, and, indeed, for all practical purposes, E is assumed to be $\{1,\dots,n\}$. Furthermore, let $\Bbbk$ be a field, and let $S = \Bbbk[x_1,\dots,x_n]$ be the polynomial ring in $n = |E|$ variables. Crucial to all three papers is the concept of grading. It therefore seems natural to begin by recalling the properties of S as a graded $\Bbbk$-algebra. Because this concept is so central to each of the papers, we shall elaborate a bit more on it than on certain other topics. Same goes for our introduction of matroids. Also throughout, let $\mathbb{N}_0 = \{0,1,2,\dots\}$, and $\mathbb{N}_0^n = \{(a_1,a_2,\dots,a_n) : a_i \in \mathbb{N}_0\}$.

1.1 Graded rings and modules

Let R be a commutative ring with unity, and let G be an additive group. We say that R is *G-graded* if $R \cong \bigoplus_{g\in G} R_g$, where each R_g is an additive subgroup of R and with the property that $R_{g_1}R_{g_2} \subseteq R_{g_1+g_2}$.

Similarly, if R is G-graded, a *G-graded R-module* is an R-module M with the property that $M \cong \bigoplus_{g\in G} M_g$, where each M_g is a subgroup of M and $R_{g_1}M_{g_2} \subseteq M_{g_1+g_2}$. An R-module homomorphism $\phi : M \to M'$ between G-graded R-modules M and M' is said to be *homogeneous* if $\phi[M_g] \subseteq M'_g$.

Let M be a G-graded R-module. A submodule $N \subseteq M$ is said to be a *G-graded submodule* of M if $N_g = M_g \cap N$ induces a G-grading on N.

For $g \in G$ we may transform M into the *shifted* R-module $M(g)$, which is isomorphic to M when seen as an ungraded module but which has homogeneous parts $M(g)_h = M_{g+h}$.

In the particular case where $G = \mathbb{Z}$ and both R_i and M_i is 0 for all $i < 0$ we shall say that R and M are $\mathbb{N}_0$-*graded* (although $\mathbb{N}_0$ is of course not a group under addition). Similarly, we refer to $\mathbb{Z}^n$-graded rings and modules with nonzero parts in $\mathbb{N}_0^n$ only, as $\mathbb{N}_0^n$-*graded*. These two cases are the important ones to us.

1.1.1 Two gradings on S

For $t \in \mathbb{N}_0$, let S_t denote the $\Bbbk$-vector space with basis the homogeneous polynomials of degree t. The direct sum decomposition

$$S \cong \bigoplus_{t\in\mathbb{Z}} S_t$$

of S as a $\Bbbk$-vector space gives S the structure of an $\mathbb{N}_0$-graded $\Bbbk$-algebra. This is the so-called *standard* $\mathbb{N}_0$-grading of S, and whenever we refer to S as an $\mathbb{N}_0$-graded ring or module over itself this is the grading we shall be referring to.

Also, S is an $\mathbb{N}_0$-graded module over itself. In particular, we have that for each homogeneous polynomial h of degree d the S-linear map from $S(-d)$ to $\langle h\rangle_S$ defined by $1 \mapsto h$ is a homogeneous S-module isomorphism.

For monomials of S, we shall employ the standard, abbreviated notation

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$. In addition to S being $\mathbb{N}_0$-graded, we may also endow it with an $\mathbb{N}_0^n$-grading: For $\mathbf{a} \in \mathbb{N}_0^n$, let $S_{\mathbf{a}}$ denote the $\Bbbk$-vector space of rank 1 with basis $\mathbf{x}^{\mathbf{a}}$. The direct sum decomposition $S \cong \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} S_{\mathbf{a}}$ gives S the structure of an $\mathbb{N}_0^n$-graded $\Bbbk$-algebra. This is the so-called *standard* $\mathbb{N}_0^n$-grading of S, and whenever we refer to S as an $\mathbb{N}_0^n$-graded ring or module over itself this is the grading we shall be referring to. An element of $S_{\mathbf{a}}$ is called a *monomial* of *degree* $\mathbf{a}$. For each monomial $\mathbf{x}^{\mathbf{a}}$, we have that $S(-\mathbf{a})$ and $\langle \mathbf{x}^{\mathbf{a}}\rangle_S$ are isomorphic as $\mathbb{N}_0^n$-graded S-modules under the isomorphism $1 \mapsto \mathbf{x}^{\mathbf{a}}$.

By definition, an ideal I of S is said to be *homogeneous* if there is a generating set for I consisting of homogeneous polynomials. Furthermore, the ideal I is said to be *monomial* if there is a generating set for I consisting of monomials. Characteristic for monomial ideals is that they each have a *unique* set of generators (up to scalar multiplication) that is minimal in terms of cardinality (see e.g. [39, Lemma 1.2]).

It is elementary that an ideal of S is monomial if and only if it is an $\mathbb{N}_0^n$-graded submodule of S (with the standard grading). Similarly, an ideal is homogeneous if and only if it is an $\mathbb{N}_0$-graded submodule. Thus, since a monomial ideal is obviously homogeneous, it is clear that monomial ideals are also $\mathbb{N}_0$-graded. (This is just a particular case of the fact that every $\mathbb{Z}^n$-graded S-module is also $\mathbb{Z}$-graded.)

If $I \subset S$ is a monomial ideal, then the factor ring S/I is an $\mathbb{N}_0^n$-graded S-module with graded parts $(S/I)_{\mathbf{a}} = S_{\mathbf{a}}/I_{\mathbf{a}} = S_{\mathbf{a}}/(S_{\mathbf{a}} \cap I)$. Likewise for homogeneous ideals, in which case S/I is $\mathbb{N}_0$-graded with $(S/I)_t = S_t/I_t = S_t/(S_t \cap I)$. The graded S-modules of interest to us in this thesis are the monomial ideals of S, and their factor rings.

1.1.2 Graded Betti numbers

Now, if M is a finitely generated, $\mathbb{N}_0$-graded S-module we may find a minimal generating set for M consisting of homogeneous elements. If $\{g_1, g_2, \ldots, g_r\}$ is such a generating set and g_i has degree d_i, then the S-linear map

$$\phi_0 : S(-d_1) \oplus S(-d_2) \oplus \cdots \oplus S(-d_r) \longrightarrow M$$

that sends the ith basis vector to g_i is a homogeneous, surjective homomorphism. In other words, there is a short exact sequence

$$0 \longleftarrow M \xleftarrow{\phi_0} F_0 \longleftarrow \ker(\phi_0) \longleftarrow 0,$$

where $F_0 = S(-d_1) \oplus S(-d_2) \oplus \cdots \oplus S(-d_r)$.

As $\ker(\phi_0)$ inherits the $\mathbb{N}_0$-grading from F_0, it too is a finitely generated, $\mathbb{N}_0$-graded S-module. We may therefore find a minimal generating set $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ for $\ker(\phi_0)$ where the jth component of the vector $\mathbf{g}_i$ is a homogeneous polynomial of degree $d_i' - d_j$, for some d_i'. Letting $F_1 = S(-d_1') \oplus S(-d_2') \oplus \cdots \oplus S(-d_s')$ we again have that the map $\phi_1 : F_1 \longrightarrow \ker(\phi_0)$ sending the ith basis vector to $\mathbf{g}_i$ is a homogeneous, surjective homomorphism. Continuing like this, we obtain what is known as an $\mathbb{N}_0$-*graded minimal free resolution* of M:

$$0 \longleftarrow M \xleftarrow{\phi_0} F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \longleftarrow \cdots$$

Note that this sequence is exact everywhere. Furthermore, that each ϕ_i maps the basis vectors of F_i to a minimal set of generators of F_{i-1} is equivalent to $\ker(\phi_i) \subseteq \mathbf{m}F_i$ for each i, where $\mathbf{m}$ is the maximal ideal $\langle x_1, \ldots, x_n \rangle \subseteq S$. (See e.g. [23, Lemma A.2.1] for the proof of this equivalence.)

Now let us consider the case when M is not only $\mathbb{N}_0$-graded, but $\mathbb{N}_0^n$-graded as well. Similar to before, we let $\{g_1, g_2, \ldots, g_r\}$ be a minimal generating set - with $g_i \in M_{\mathbf{a}_i}$. If

$$F_0 = S(-\mathbf{a}_1) \oplus S(-\mathbf{a}_2) \oplus \cdots \oplus S(-\mathbf{a}_r),$$

the S-linear map $\phi_0 : F_0 \longrightarrow M$ that sends the ith basis-vector to $\mathbf{g}_i$ is an $\mathbb{N}_0^n$-graded, surjective S-module homomorphism. In other words, the sequence

$$0 \longleftarrow M \xleftarrow{\phi_0} F_0 \longleftarrow \ker(\phi_0) \longleftarrow 0$$

is exact. Since $\ker(\phi_0)$ is an $\mathbb{N}_0^n$-graded submodule of F_0, it has a minimal set of generators $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s\}$ where the jth component of $\mathbf{g}_i$ is in $S_{\mathbf{b}_i - \mathbf{a}_j}$, for some $\mathbf{b}_i \in \mathbb{N}_0^n$. Consequently, letting

$$F_1 = S(-\mathbf{b}_1) \oplus S(-\mathbf{b}_2) \oplus \cdots \oplus S(-\mathbf{b}_s)$$

the S-linear map $\phi_1 : F_1 \longrightarrow \ker(\phi_0)$ sending the ith basis vector to $\mathbf{g}_i$ is a homogeneous S-module homomorphism onto $\ker(\phi_0)$. Continuing like this, finding minimal homogeneous generating sets for – and creating grade preserving homomorphisms onto – $\ker(\phi_i)$, we obtain an $\mathbb{N}_0^n$-*graded minimal free resolution* of M.

In the $\mathbb{N}_0^n$-graded case, let $\beta_{i,\mathbf{a}}(M;\Bbbk)$ be the number of times $S(-\mathbf{a})$ occurs in F_i, and, likewise, in the $\mathbb{N}_0$-graded case, let $\beta_{i,j}(M;\Bbbk)$ be the number of times $S(-j)$ occurs in F_i. That is, let

$$F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}_0^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}(M;\Bbbk)}$$

in the $\mathbb{N}_0^n$-graded case, and

$$F_i = \bigoplus_{j \in \mathbb{N}_0} S(-j)^{\beta_{i,j}(M;\Bbbk)}$$

in the $\mathbb{N}_0$-graded case – with $\beta_{i,\mathbf{a}}(M;\Bbbk)$ and $\beta_{i,j}(M;\Bbbk)$ nonnegative integers. Since any two ($\mathbb{N}_0$- or $\mathbb{N}_0^n$-graded) minimal free resolutions are isomorphic [15, Theorem 3.13], the numbers $\beta_{i,\mathbf{a}}(M;\Bbbk)$ and $\beta_{i,j}(M;\Bbbk)$ are unique; they do not depend upon the particular generators chosen at each step in the construction of the minimal free resolution – only on the field $\Bbbk$.

The *(graded) Hilbert syzygy theorem* (see e.g. [15, Theorem 3.8]) states that $F_i = 0$ for all $i > n$. The largest p such that $F_p \neq 0$ (but $F_{p+1} = 0$) is the *length* of the minimal free resolution. By the famous Quillen-Suslin Theorem [44, Theorem 4], every finitely generated, projective module over a principal domain is free. This implies that the length p is equal to the *projective dimension* of M. In fact, we shall take this length as our definition of projective dimension, and denote it $\operatorname{p.d.} M$.

By considering the $\mathbb{N}_0$- and $\mathbb{N}_0^n$-graded minimal generating sets for $\ker(\phi_i)$, it is easily seen that

$$\beta_{i,j}(M;\Bbbk) = \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(M;\Bbbk),$$

where $|\mathbf{a}| = a_1 + a_2 + \cdots + a_n$. In particular, we have $\beta_{i,\mathbf{a}}(M;\Bbbk) = 0$ for all $i > n$. Collectively, we refer to $\{\beta_{i,\mathbf{a}}(M;\Bbbk)\}$ and $\{\beta_{i,j}(M;\Bbbk)\}$, respectively, as the $\mathbb{N}_0^n$- and $\mathbb{N}_0$-graded *Betti numbers* of M. The ith *ungraded* Betti number $\beta_i(M;\Bbbk)$ is simply the rank of F_i. In other words, we have

$$\beta_i(M;\Bbbk) = \sum_{j\in\mathbb{N}_0} \beta_{i,j}(M;\Bbbk) = \sum_{\mathbf{a}\in\mathbb{N}_0^n} \beta_{i,\mathbf{a}}(M;\Bbbk).$$

If, for a fixed i, there is a d such that $\beta_{i,j}(M;\Bbbk) = 0$ for all $j \le d$, then $\beta_{i+1,j+1}(M;\Bbbk) = 0$ for all $j \le d$ [17, Proposition 1.9]. We therefore have the following compact way of presenting the $\mathbb{N}_0$-graded Betti numbers:

	0	1	$\cdots$	p
j	$\beta_{0,j}(M;\Bbbk)$	$\beta_{1,j+1}(M;\Bbbk)$	$\cdots$	$\beta_{p,j+p}(M;\Bbbk)$
$j+1$	$\beta_{0,j+1}(M;\Bbbk)$	$\beta_{1,j+2}(M;\Bbbk)$	$\cdots$	$\beta_{p,j+p+1}(M;\Bbbk)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$\beta_{0,k}(M;\Bbbk)$	$\beta_{1,k+1}(M;\Bbbk)$	$\cdots$	$\beta_{p,k+p}(M;\Bbbk)$

We refer to this as the *Betti table* of M (over $\Bbbk$), and denote it $\beta[M;\Bbbk]$.

An $\mathbb{N}_0$-graded minimal free resolution is said to be *pure* if each F_i has only one summand, that is, if each F_i is of the form $S(-j)^n$ for some $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$. A *linear* resolution is a (pure) $\mathbb{N}_0$-graded minimal free resolution of the form

$$0 \xleftarrow{\phi_0} S\big(-r\big)^{n_0} \xleftarrow{\phi_1} S\big(-(r+1)\big)^{n_1} \longleftarrow \cdots \xleftarrow{\phi_p} S\big(-(r+p)\big)^{n_p} \longleftarrow 0.$$

Remark. A very important observation is that for a monomial ideal I we have $\beta_{i-1,\sigma}(I;\Bbbk) = \beta_{i,\sigma}(S/I;\Bbbk)$ for all $i \ge 1$, while

$$\beta_{0,\sigma}(S/I;\Bbbk) = \begin{cases} 1, \sigma = \emptyset \\ 0, \text{otherwise.} \end{cases}$$

Consequently, for a homogeneous ideal $I \subset S$ we have $\beta_{i-1,j}(I;\Bbbk) = \beta_{i,j}(S/I;\Bbbk)$ for all $i \ge 1$, while $\beta_{0,j}(S/I;\Bbbk) = \begin{cases} 1, j = 0 \\ 0, j \ne 0. \end{cases}$

1.2 Some general concepts and terminology from commutative algebra

In Paper 3 we consider the concept of dimension both for simplicial complexes and their corresponding algebraic objects. In particular, we use properties of the Hilbert series to arrive at the main result of that paper. The following is a brief summary of the concepts touched upon concerning the dimension of algebraic objects such as rings, modules, and vector spaces; it presents the (standard) notation employed throughout. To begin with, for a $\Bbbk$-module V, we let $\dim_{\Bbbk} V$ denote the dimension of V as a vector space over $\Bbbk$.

If M is a finitely generated $\mathbb{N}_0$-graded S-module, the function $H(M,-) : \mathbb{Z} \to \mathbb{N}_0$ defined by $H(M,i) = \dim_{\Bbbk} M_i$ is called the *Hilbert function* of M. The *Hilbert series* $H_M(t)$ of M is the formal Laurent series $H_M(t) = \sum_{i \in \mathbb{Z}} H(M,i)t^i$.

The *Krull dimension* $\dim R$ of a commutative ring R is the supremum of the lengths of all chains of prime ideals. We thus have $\dim R \in \mathbb{N}_0 \cup \infty$. By [23, Theorem 6.1.3], there is a Laurent polynomial $Q_M(t) = \sum_{i=r}^{s} h_i t^i$ such that $H_M(t) = \frac{Q_M(t)}{(1-t)^d}$, where $d = \dim(R/\operatorname{Ann}(M))$. The coefficients of $Q_M(t)$ form the so-called *h-vector* $(h_r, h_{r+1}, \dots, h_s)$ of M.

Also by [23, Theorem 6.1.3], we have that $\min\{i : h_i \neq 0\} = \min\{i : M_i \neq 0\}$. In particular, if $I \subset S$ is a homogeneous ideal then the h-vector of S/I is of the form $(h_0, h_1, \dots)$.

A sequence $g_1, \dots, g_r \in \langle x_1, x_2, \dots, x_n \rangle$ is said to be a *regular M-sequence* if g_{i+1} is not a zero-divisor on $M/(g_1 M + \dots + g_i M)$. The *depth* of M is the common length of a longest regular M-sequence. By the *Auslander-Buchsbaum Theorem* we have

$$\operatorname{p.d.} M + \operatorname{depth} M = n.$$

Naturally, depth R denotes the depth of R seen as an R-module. In general we have $\operatorname{depth} R \leq \dim R$. In the case of equality, when $\operatorname{depth} R = \dim R$, the ring R is said to be *Cohen-Macaulay*.

1.3 Simplicial complexes and matroids

There are two complementing ways of seeing and defining a simplicial complex; one is as a topological space, the other, which we shall adhere to, is as a set construction. Our reason for choosing the latter of these approaches is our particular interest in those simplicial complexes that are matroids, together with the fact that we do not in particular concern ourselves with the geometrical or topological aspects.

A *simplicial complex* on E is a family Δ of subsets of E with the property that if $\sigma_1 \in \Delta$ and $\sigma_2 \subseteq \sigma_1$, then $\sigma_2 \in \Delta$. The elements of Δ are referred to as *faces*, while a subset of X which is not in Δ is referred to as a *nonface*. A *facet* is a face which is not properly contained in another face, that is, an inclusion maximal face. If all facets have the same cardinality, we say that the simplicial complex is *pure*.

If Δ and Δ' are two simplicial complexes, on $E(\Delta)$ and $E(\Delta')$, respectively, we say that Δ and Δ' are *isomorphic*, and write $\Delta \cong \Delta'$, if there is a bijection $\phi : E(\Delta) \to E(\Delta')$ with the property that $\phi[X] \in \Delta' \Leftrightarrow X \in \Delta$.

We shall have occasion to consider two dual constructions. The *dual* complex $\overline{\Delta}$ of Δ is the simplicial complex whose facets are the complements of facets of Δ. We shall see this dual a lot – but mostly in a matroid context. The *Alexander dual* Δ^* is the simplicial complex defined by

$$\Delta^* = \{\overline{\tau} \subseteq E : \tau \notin \Delta\}.$$

The *dimension* $\dim(\sigma)$ of $\sigma \in \Delta$ is 1 less than the cardinality of σ. In particular, we have (for a nonempty simplicial complex) that $\dim(\emptyset) = -1$. The dimension of the simplicial complex Δ itself is $\dim(\Delta) = \max\{\dim(\sigma) : \sigma \text{ is a facet of } \Delta\}$.

Central to Paper 3 is the subcomplex construction referred to as an i-skeleton. For $0 \leq i \leq \dim(\Delta)$, the *$i$-skeleton* $\Delta_{(i)}$ of Δ is defined by

$$\Delta_{(i)} = \{\sigma \in \Delta | \dim(\sigma) \leq i\}.$$

Observe that $\Delta_{(\dim(\Delta))} = \Delta$. We shall see that the $\mathbb{N}_0$-graded Betti numbers of $S/I_{\Delta_{(\dim(\Delta)-1)}}$ can be expressed as a $\mathbb{Z}$-linear combination of those of S/I_Δ.

A *shelling* of a pure simplicial complex Δ is a linear ordering $\lambda_1, \lambda_2, \ldots, \lambda_t$ of its facets such that for each pair of facets λ_i, λ_j with $1 \leq i < j \leq t$ there is a facet λ_k with $1 \leq k < j$ and an element $x \in \lambda_k$ such that

$$\lambda_i \cap \lambda_j \subseteq \lambda_k \cap \lambda_j = \lambda_j \smallsetminus \{x\}.$$

The geometric interpretation of this is that each of the facets λ_i, with $2 \leq i \leq t$, intersects with the complex generated by their predecessors in a non-void union of maximal proper faces. A (pure) simplicial complex which permits a shelling is said to be *shellable*.

The simplicial complex Δ is said to be *Cohen-Macaulay* over $\Bbbk$ if the Stanley-Reisner ring S/I_Δ is Cohen-Macaulay. The complex is said to be Cohen-Macaulay if it is Cohen-Macaulay over some field. By [48, Theorem 2.5], a shellable simplicial complex is Cohen-Macaulay over every field. A partial converse to this is [23, Lemma 8.1.5], which says that every Cohen-Macaulay complex is pure. The Cohen-Macaulay property is relevant to all three papers, although it is most directly discussed in Paper 3.

1.3.1 The homology spaces

Let $\mathscr{F}_i(\Delta)$ denote the set of i-dimensional faces of Δ, and let $\Bbbk^{\mathscr{F}_i(\Delta)}$ be the free $\Bbbk$-vector space on $\mathscr{F}_i(\Delta)$. The *(reduced) chain complex* of Δ over $\Bbbk$ is the complex

$$0 \longleftarrow \Bbbk^{\mathscr{F}_{-1}(\Delta)} \xleftarrow{\delta_0} \cdots \longleftarrow \Bbbk^{\mathscr{F}_{i-1}(\Delta)} \xleftarrow{\delta_i} \Bbbk^{\mathscr{F}_i(\Delta)} \longleftarrow \cdots \xleftarrow{\delta_{\dim(\Delta)}} \Bbbk^{\mathscr{F}_{\dim(\Delta)}(\Delta)} \longleftarrow 0,$$

where the *boundary maps* δ_i are defined as follows: Given a total ordering on E, set $\operatorname{sign}(j,\sigma) = (-1)^{r-1}$ if j is the rth element of $\sigma \subseteq E$, and let

$$\delta_i(\sigma) = \sum_{j\in\sigma} \operatorname{sign}(j,\sigma)\,\big(\sigma \setminus \{j\}\big).$$

Extending δ_i $\Bbbk$-linearly, we obtain a $\Bbbk$-linear map from $\Bbbk^{\mathscr{F}_i(\Delta)}$ to $\Bbbk^{\mathscr{F}_{i-1}(\Delta)}$.

So, for example, if $\{1,3,5\}$ and $\{1,3,4\}$ are faces of a simplical complex, and $\alpha_1,\alpha_2 \in \Bbbk$, then $\delta_3(\alpha_1\{1,3,5\}+\alpha_2\{1,3,4\}) = \alpha_1(\{3,5\}-\{1,5\})+\alpha_2(\{3,4\}-\{1,4\})+(\alpha_1+\alpha_2)\{1,3\}$.

The ith *(reduced) homology* of Δ over $\Bbbk$ is the vector space

$$\tilde{H}_i(\Delta;\Bbbk) = \ker(\delta_i)/\operatorname{im}(\delta_{i+1}).$$

It is explained in [23, p. 81] how

$$\dim_\Bbbk \tilde{H}_i(\Delta;\Bbbk) = \dim_\Bbbk \tilde{H}^i(\Delta;\Bbbk),$$

where $\dim_\Bbbk \tilde{H}^i(\Delta;\Bbbk)$ denotes the ith *(reduced) cohomology* of Δ over $\Bbbk$. Although we shall not use nor mention cohomology again, it is in general useful to be aware of the above identity, as some results may only be formulated using one of the two homologies – while in fact being valid for both.

For $X \subseteq E$, the set $\{\sigma \cap X : \sigma \in \Delta\}$ is a simplicial complex on X. We denote this complex $\Delta_{|X}$ and refer to it as the restriction of Δ to X. The identity (1.1), below, first appeared in [27], and is normally referred to as *Hochster's formula*.

$$\beta_{i,\sigma}(S/I_\Delta;\Bbbk) = \dim_\Bbbk \tilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma};\Bbbk). \tag{1.1}$$

It is one of the most celebrated and important results in the intersection between algebra and combinatorics. One could perhaps describe Hochster's formula as *the* prototypical result of combinatorial commutative algebra, in that it establishes a connection between an algebraic entity (the Betti numbers of S/I_Δ) and a combinatorial/topological one (the homology spaces of Δ). We shall use it repeatedly throughout the thesis.

The *face numbers* $f_i(\Delta)$ of a matroid are defined by $f_i(\Delta) = |\mathscr{F}_i(\Delta)|$. Not only are they important to us in their own right, in that we use them to derive our main result of Paper 2, but their connection (in fact, equivalence) to the h-vector of S/I_Δ was one of the main reasons research on what is now referred to as Stanley-Reisner theory begun in the first place. Recall that since $\min\{i : h_i(M) \neq 0\} = \min\{i : M_i \neq 0\}$ for a finitely generated $\mathbb{N}_0$-graded S-module M, we in particular have that the h-vector of S/I_Δ is of the form $(h_0, h_1, \ldots)$. By [5, Lemma 5.1.8], this vector is connected to the face numbers $f_i(\Delta)$ as follows:

$$\sum_i h_i t^i = \sum_{i=0}^{\dim(\Delta)+1} f_{i-1}(\Delta) t^i (1-t)^{\dim(\Delta)+1-i}.$$

This in turn implies that the h-vector of S/I_Δ has length at most $\dim(\Delta)+1$, and that for $0 \leq j \leq \dim(\Delta)+1$ we have

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{\dim(\Delta)+1-i}{j-i} f_{i-1}(\Delta)$$

and

$$f_{j-1}(\Delta) = \sum_{i=0}^{j} \binom{\dim(\Delta)+1-i}{j-i} h_i.$$

Since S/I_Δ being Cohen-Macauley puts some bounds on the h-vector, the above equations imply that Δ being Cohen-Macauley puts restrictions on $f_i(\Delta)$. This at least partially explains the significance of Cohen-Macaulayness to Stanley-Reisner theory. Concretely, we have by [5, Theorem 5.1.10] that if Δ is a Cohen-Macaulay simplicial complex on E, then

$$0 \leq h_i \leq \binom{n-\dim(\Delta)+i-2}{i},$$

for $0 \leq i \leq \dim(\Delta)+1$. Remarkably, it has been determined precisely which sequences $\{f_i\}$ that are the face numbers of some simplicial complex; for details, see [36] and [33].

The number

$$\tilde{\chi}(\Delta) = -f_{-1}(\Delta) + f_0(\Delta) - f_1(\Delta) + \cdots + (-1)^{\dim(\Delta)} f_{\dim(\Delta)}(\Delta)$$

is the *(reduced) Euler characteristic* of Δ. From [23, p. 102] we see that

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim(\Delta)} (-1)^i \dim_{\mathbb{k}} \tilde{H}_i(\Delta; \mathbb{k}).$$

This is a generalization of Euler's observation that for a convex polyhedron it is always the case that $f_0 - f_1 + f_2 = 2$.

1.3.2 Matroids

In Papers 1 and 2 we focus solely on those simplicial complexes that are matroids. These also receive special attention in Paper 3, although the topic of that paper is slightly more general in nature. Characteristic of matroid theory is that there are many different axiomatic systems that define a matroid. Thirteen such definitions are given in a table in the survey article [12], where it is also described how one can, with relative ease, move back and forth between any two such axiomatic systems. Although these definitions are all equivalent in the sense that the "matroid" stays the same, regardless of the definition chosen, they are often not *obviously* equivalent. Actually, these often seemingly unrelated sets of axioms, all in fact representing the same matroid, are one of the biggest strengths of matroid theory. Not only do they illustrate how a matroid captures some universal properties, to be found in structures stemming from widely different branches of mathematics, but it is also frequently the case that something which seems inexplicable using one axiomatic system, becomes obvious in another.

In this thesis we define a matroid in terms of its independent sets. This is convenient both when working with graphs and matrices, as well as with homology spaces and Betti numbers. One additional advantage of this, from our point of view, is that the underlying simplicial complex structure of a matroid becomes immediately apparent.

A *matroid* M on E is a family $I(M)$ of subsets of E with the property that

- $I(M)$ is a nonempty simplicial complex on E.
- If I_1 and I_2 are both in $I(M)$ and $|I_1| < |I_2|$, then there is an $x \in I_2$ such that $I_1 \cup \{x\} \in I(M)$.

The set $I(M)$ is referred to as the *set of independent sets* of M, and an element $I \in I(M)$ is said to be *independent*. In this vein, a *dependent* set is a subset of E that is not in $I(M)$. Two matroids M and M' are said to be *isomorphic* whenever their underlying simplicial complexes are, that is, when $I(M) \cong I(M')$

Matroid theory as a separate subject is a relatively recent one. It was initiated by Whitney's fundamental paper [54] of 1935, in which he isolates and studies the concept of independence as it is encountered in e.g. graph theory and linear algebra. He coined the term *matroid* as a set of linearly independent columns of a matrix, while pointing out that one could

> "... equally well consider points or vectors in an Euclidean space, polynomials, etc."([54, p. 509])

The word "matroid" itself carries an obvious allusion to the more familiar "matrices" of linear algebra. In fact, due to its motivation and origin from graph theory and linear algebra, matroid theory borrows much of its terminology, and many of its examples, from these subjects. That being said, there are a wide variety of constructions, from different mathematical disciplines, that in one way or another give rise to a matroid.

The discipline has seen a more or less steadily increasing research interest. For a short introduction, as well as an account of its early history, we recommend [56], while a modern and comprehensive introduction is e.g. [42].

There are, together with the already defined set of independent sets, two families of subsets of E that are repeatedly considered throughout this thesis. One is the set $C(M)$ of circuits of M, and the other is the set $B(M)$ of bases of M. These may both be defined directly from E by listing suitable axioms, similar to what we did for $I(M)$, thereby each providing their own definition of a matroid different from the definition given above. Alternatively, these families may be defined in terms of $I(M)$ as follows: A *circuit* is an inclusion minimal dependent set, while a *basis* is an inclusion maximal independent set. In other words, the independent sets of M are the faces of $I(M)$, the circuits are the minimal nonfaces, and the bases are the facets. It is a fundamental result in matroid theory that all bases have the same cardinality, making $I(M)$ a pure simplicial complex. That $I(M)$ is also shellable, was demonstrated in [4, Theorem 7.3.3]. As mentioned at the beginning of this section, this in turn implies that $I(M)$ is a *Cohen-Macaulay* complex.

As an example of how the different matroid definitions are all equivalent, note that if we had chosen to define a matroid in terms of its set of bases, then the independent sets would be defined as those subsets of E that are contained in a basis. Similarly, if we had chosen a definition in terms of circuits, then the independent sets would be defined as those subsets of E that do not contain a circuit.

For $X \subseteq E$, we denote the matroid we obtain by restricting $I(M)$ to X by $M_{|X}$. The *dual matroid* $\overline{M}$ is the matroid on E whose bases are the complements of the bases of M, that is,

$$B(\overline{M}) = \{E \smallsetminus B : B \in B(M)\}.$$

Or equivalently, $\overline{M}$ is the matroid whose underlying simplicial complex $I(M)$ is the dual complex $\overline{I(M)}$ of $I(M)$. It is worth mentioning that the Alexander dual of $I(M)$ does not necessarily form the set of independent sets of a matroid.

For $X \subseteq E$, the *rank function* r_M and *nullity function* n_M are defined by

$$r_M(X) = \max\{|I| : I \subseteq X \text{ and } I \in I(M)\}$$

and

$$n_M(X) = |X| - r(X).$$

The rank of M itself is $r(M) = r_M(E)$. Whenever the matroid M is clear from the context, we omit the subscript and write simply r and n.

We refer to [42] for the elementary proof that

1. $0 \leq r(X) \leq |X|$.
2. If $X \subseteq Y$, then $r(X) \leq r(Y)$.
3. $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

In fact, if $r' : 2^E \to \mathbb{N}_0$ is a map satisfying (1)-(3) above, then $\{X \subseteq E : r'(X) = |X|\}$ forms the set of independent sets of a matroid on E – and this matroid has rank function r'. Thus the rank function offers yet another possible way to define a matroid.

The rank and nullity function together "form" the *Tutte polynomial*

$$t_M(X,Y) = \sum_{X \subseteq E} (X-1)^{r(E)-r(X)}(Y-1)^{n(X)},$$

which carries information on several important invariants of M. For example, we have that $t_M(1,1)$ counts the number of bases of M, while $t_M(2,1)$ is the number of independent sets. See e.g. [30, Theorem 8.7] for the proof that $t_M(X,Y) = t_{\overline{M}}(Y,X)$.

1.3.3 The elongation and truncation of matroids

The *elongation of M to rank $r(M)+i$* is the matroid $M^{(i)}$ with

$$I(M^{(i)}) = \{X \subseteq E : n(X) \leq i\}.$$

We shall occasionally refer to $M_{(i)}$ as the ith elongation of M. We deal with these elongations in Paper 2, where, amongst other things, we demonstrate that the Betti numbers of $S/I_{M^{(1)}}$ are not quite determined by those of S/I_M.

The ith *truncation* $M_{(i)}$ of M is the matroid with

$$I(M_{(i)}) = \{I \in I(M) : r(X) \leq r(M) - i\}.$$

It is straightforward to verify that $\overline{M}_{(i)} = \overline{M^{(i)}}$ for $0 \leq i \leq n - r(M)$. Note also that the ith truncation of M is equal to the $(r(M)-1-i)$-skeleton of $I(M)$, when the latter is seen as a simplicial complex. Contrasting the above mentioned result of Paper 2, we demonstrate in Paper 3 that each of the Betti numbers of $S/I_{\Delta_{(d-1)}}$ is a $\mathbb{Z}$-linear combination of those of S/I_Δ.

1.3.4 The cycle matroid of a finite graph

We have already alluded to the connection between graphs and matroids, and we shall now describe this connection in more detail.

If G is a finite graph with n edges, one may, without loss of generality, assume that its set of edges is E. Let $M(G)$ be the matroid on E whose set of circuits is $C(M) = \{C \subseteq E : C \text{ is a cycle}\}$. (See e.g. [42] for a proof that $M(G)$ is indeed a matroid.) We refer to $M(G)$ as the *cycle matroid* of G. A matroid which is isomorphic to the cycle matroid of some graph, is called a *graphic* matroid. By Proposition [42, 1.2.9], every graphic matroid is isomorphic to the cycle matroid of a connected graph. Note that the independent sets of $M(G)$ are those subsets of E that are forests of G.

Although graphs are far from being a main topic of this thesis, they are, due to the above connection to matroids, always lurking in the background. In Paper 1, however, we deal explicitly with so-called *cactus graphs*, which are finite, connected graphs with the property that no two distinct cycles share an edge.

Cactus graphs are examples of outerplanar graphs. In general, a finite graph is said to be *planar* if it can be embedded in the plane. Naively, this means that it can be drawn without edges crossing (except in vertices). An *outerplanar* graph is a graph with the property that if you add a vertex that has an edge to each of the existing vertices, then the resulting graph is (still) planar. Clearly, every outerplanar graph is planar, while for example K_4, the complete graph on 4 vertices, is a planar graph that is not outerplanar. Outerplanar graphs were first defined and studied in [14], and by Theorem 1 of that article it follows that K_4 is in fact the smallest non-outerplanar graph (in terms of number of vertices).

In particular, cactus graphs are outerplanar, and in Paper 1 we have a closer look at their cycle matroids. Crucial to our arguments there is the concept of a 2-connected component – which translates to the blocks of a matroid (see the next section).

A *proper coloring* of the vertices of a graph is one that does not assign the same color to any pair of neighboring vertices. In an attempt to tackle the famous Four Color Conjecture (now Theorem), which says that every planar graph has a proper vertex coloring using at most 4 colors, it was demonstrated in [7] that the number of proper colorings of a planar graph G using λ colors is a polynomial in λ of degree m, where m is the number of vertices in G. The polynomial was successfully generalized to arbitrary finite graphs in [55]. It is commonly denoted $\chi_G(\lambda)$, and referred to as the *chromatic polynomial of* G. Its generalization to matroids, due to [16] and [46], is the *characteristic polynomial* $p(M,Z)$ of M defined by

$$p(M,Z) = \sum_{X \subseteq E} (-1)^{|X|} Z^{r(E)-r(X)}.$$

For a simple graph G we have (by e.g. [52, p. 262]) that $\chi_G(\lambda) = \lambda^{k(G)} p(M(G),\lambda)$, where $k(G)$ is the number of connected components of G.

There are several open problems related to the characteristic polynomial of a matroid. One example is the so-called *critical problem* of determining, or bounding, the number

$$\min\{j \in \mathbb{N}_0 : p(M,q^j) > 0\}$$

for a q-representable (see Section 1.5) matroid M.

Since the characteristic polynomial is equal to one of the polynomials we define for matroids in Paper 2, the main result of that paper applies to the characteristic polynomial as a special case. In particular, we shall see that the characteristic polynomial of M can be expressed in terms of certain Betti numbers associated to $\overline{M}$.

Finally, it is worth pointing out that the Tutte polynomial (of a graph) was originally conceived as a generalization of the chromatic polynomial (see [51] for an account of its genesis) – and it is easily verified that $p(M,Z) = (-1)^{r(E)} t_M(1-Z,0)$.

1.3.5 The blocks of a matroid

We say that two elements $e, f \in E$ belong to the same *block* of M if $e = f$ or if there is a circuit of M that contains both e and f. It is demonstrated in [42, Proposition 4.2.2] that the blocks of M constitute a partition of E. A matroid is said to be *connected* if this partition consists of a single block. This terminology should be seen in light of [42, Proposition 4.1.7], which states that if G is a loopless graph with at least three vertices, none of which are isolated, then $M(G)$ is connected if and only if G is 2-connected. Thus the inclusion maximal 2-connected subgraphs of a graph more or less correspond to the blocks of its cycle matroid.

In particular, the blocks of the cycle matroid of a cactus graph stem from either a cycle or a single edge. We use this observation in Paper 1 to demonstrate that the set of higher weights determine the Betti numbers of the so-called facet ideal associated to the cycle matroid of a loop-free cactus graph.

1.4 Stanley-Reisner and facet ideals

A monomial $\mathbf{x}^{\mathbf{a}}$ is said to be *squarefree* if $\mathbf{a} \in \{0,1\}^n$, and a *squarefree monomial ideal* is one which is generated by squarefree monomials. Note that $\mathbf{x}^{\mathbf{a}} \mapsto \mathrm{Supp}(\mathbf{a})$ gives a $1-1$ correspondence between the set of squarefree monomials in S and subsets of E.

There are in particular two squarefree monomial ideals we shall consider in this thesis, both of which are derived from a simplicial complex. The *Stanley-Reisner ideal* I_Δ is the ideal generated by monomials corresponding to (minimal) nonfaces of Δ. That is,

$$I_\Delta = \langle \mathbf{x}^\tau : \tau \notin \Delta \rangle.$$

Equally of interest to us is the *Stanley-Reisner ring* S/I_Δ. We point out, however, that as long as the object of study is (graded or ungraded) Betti numbers, it is, in light of Remark 1.1.2, inconsequential whether one considers the ideal I_Δ or the ring S/I_Δ. In other words, this is a matter of convention – and what is most convenient in a particular instance. Note that by [5, Theorem 5.1.4] we have $\dim S/I_\Delta = \dim(\Delta) + 1$.

Although simplicial complexes have been studied in topology ever since Poincaré used them in [43] to calculate homology groups by way of triangulation, methodically studying them in connection to squarefree monomial ideals is a quite recent idea. The face ideal (now Stanley-Reisner ideal) made its first appearances in the 70's, in the work of Stanley [49] and Reisner [45]. As mentioned at the beginning of this introduction, one could say it all began with the upper bound conjecture (UBC) – first proposed for polytopes in [41].

Upper Bound Conjecture. *Let Δ be a d-dimensional simplicial complex on E, and assume that the geometric realization (see e.g. [48]) of Δ is homeomorphic to the d-dimensional sphere $\mathbb{S}^d$. Then $f_i(\Delta) \le f_i(C(n, d+1))$ for $1 \le i \le d$, where $C(n, d+1)$ is the convex hull of any n distinct points on the curve $\{(x, x^2, \ldots, x^{d+1}) \in \mathbb{R}^{d+1} : x \in \mathbb{R}\}$.*

In the above mentioned [49], the connection between the h-vector of S/I_Δ and the face numbers $\{f_i(\Delta)\}$ of Δ is exploited to prove that if the geometric realization of Δ is homeomorphic to $\mathbb{S}^d$ and Δ is Cohen-Macaulay, then UBC holds for Δ. Since it followed from results in [45] that every simplicial complex Δ whose geometric realization is homeomorphic to the $\dim(\Delta)$-dimensional sphere is in fact Cohen-Macaulay, the UBC was thereby solved.

The *facet ideal* $\mathscr{F}(\Delta)$ is defined by

$$\mathscr{F}(\Delta) = \langle \mathbf{x}^\sigma : \sigma \text{ is a facet of } \Delta \rangle.$$

From the outset, it is important to be aware that the facet ideal is also a Stanley-Reisner ideal; an elementary proof that

$$\mathscr{F}(\Delta) = I_{(\overline{\Delta})^*} \tag{1.2}$$

can be found in e.g. [23, Lemma 1.5.3] or in Paper 1.

It is clear that each of these ideal constructions establish a $1-1$ correspondence between squarefree monomial ideals of S and simplicial complexes on E. We also point out that both constructions reverse inclusions, in that

$$\Delta_1 \subseteq \Delta_2 \Leftrightarrow \mathscr{F}(\Delta_2) \subseteq \mathscr{F}(\Delta_1)$$

and

$$\Delta_1 \subseteq \Delta_2 \Leftrightarrow I_{\Delta_2} \subseteq I_{\Delta_1}.$$

Both the set of facets and the set of minimal nonfaces are examples of so-called *clutters* on E (see e.g. [42]), the definition of which is a set of subsets of E with the property that no subset is properly contained in another. Ideals generated by (the squarefree) monomials corresponding to a clutter – which is the case for both the facet and the Stanley-Reisner ideal – are extensively studied in the survey paper [40] (which contains several new results as well). Their focus lies in particular on these ideals' algebraic properties such as the

> "...Cohen-Macaulayness property, the stability of associated primes,
> and the connection between torsion-freeness and combinatorial problems"([40]).

They also investigate numerical invariant such as projective dimension, depth, and Krull dimension. Several of these topics are highly relevant to this thesis, although they do not explicitly study Betti numbers the way we do.

Regarding our S-modules $\mathscr{F}(\Delta)$, I_Δ, and S/I_Δ, we shall in particular be interested in their graded minimal free resolutions. Something that immediately sets these apart from those of a general S-module, is that their nonzero Betti numbers all lie in squarefree degrees, that is, by [39, Corollary 1.40], we have $\beta_{i,\sigma}(S/I_\Delta;\Bbbk) = 0$ for all $\sigma \in \mathbb{N}_0^n \setminus \{0,1\}^n$. Furthermore, it is clear that for a Stanley-Reisner ideal (and thus also for a facet ideal) the $\mathbb{N}_0^n$-graded (and thus also the $\mathbb{N}_0$-graded) Betti numbers in homological degree 0 are independent of the base field $\Bbbk$ – as these simply correspond to the minimal nonfaces.

By a slight abuse of notation, we denote the Stanley-Reisner and facets ideals of a matroid M by I_M and $\mathscr{F}(M)$, instead of $I_{I(M)}$ and $\mathscr{F}(I(M))$. In fact, we shall frequently treat M as a simplicial complex, both in our notation and in our reasoning, although it is, strictly speaking, the set $I(M)$ of independent sets which is a simplicial complex. Observe that the (minimal) generators of I_M correspond to the circuits of M, while the generators of $\mathscr{F}(M)$ correspond to the bases.

By the famous Eagon-Reiner Theorem [18, Theorem 3], the ideal I_Δ has linear resolution if and only if the Alexander dual Δ^* is Cohen-Macaulay over $\Bbbk$. But if M is a matroid then $(\overline{M}^*)^* = \overline{M}$ is a matroid, and hence Cohen-Macaulay. We conclude from (1.2) that the facet ideal of a matroid has linear resolution. By the same reasoning, it follows from [18, Corollary 5] that the Betti numbers of this linear resolution is independent of the base field $\Bbbk$. Thus, since

$$\beta_{0,j}(\mathscr{F}(M);\Bbbk) = \begin{cases} |B(M)|, & j = r(M) \\ 0, & \text{otherwise} \end{cases},$$

the $\mathbb{N}_0$-graded and ungraded Betti numbers of $\mathscr{F}(M)$ contain precisely the same information. This explains the notation of Paper 1, where we write simply β_i – omitting any reference to either the base field $\Bbbk$ or the shift j.

Since their proofs go through otherwise unchanged if considering the chain complex over some field in place of over $\mathbb{Z}$, it follows from [4, Proposition 7.4.7 and Theorem 7.8.1] that

$$\dim_\Bbbk \tilde{H}_i(M;\Bbbk) = \begin{cases} (-1)^{i-1}\tilde{\chi}(M), & i = r(M) \\ 0, & i \neq r(M). \end{cases}$$

By Hochster's formula, we conclude that also the Betti numbers associated to the Stanley-Reisner ideal of a matroid are independent of $\Bbbk$.

Summarizing, we see that when it comes to matroids all Betti numbers are independent of base field, and, in addition, their facet ideals have linear resolutions.

1.4.1 The *i*th skeleton ideal

It was demonstrated in [26, Corollary 2.6] that

$$\operatorname{depth} S/I_\Delta = \max\{j : \Delta_{(j)} \text{ is Cohen-Macaulay}\}. \tag{1.3}$$

Wishing to obtain a similar result for arbitrary monomial ideal, the Stanley-Reisner ideal of a skeleton is generalized elegantly in [25] and [24]. We shall briefly describe (a slightly simplified version of) the construction as it is found in those papers, of the so-called jth generalized skeleton ideal. In Paper 3 we give a counterexample showing that our main result of that paper – concerning Betti numbers of Stanley-Reisner rings of skeletons – does not necessarily hold for the generalized skeleton ideals. Throughout this subsection we let $\mathbf{a}(i)$ denote the ith coordinate of $\mathbf{a} \in \mathbb{N}_0^n$.

For $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^n$ we say that $\mathbf{a} \leq \mathbf{b}$ if $\mathbf{a}(i) \leq \mathbf{b}(i)$ for $1 \leq i \leq n$. Clearly, this constitutes a partial order on $\mathbb{N}_0^n$. Let $I \subset S$ be a monomial ideal with (unique) minimal generating set $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}\}$, and let $\mathbf{g} \in \mathbb{N}_0^n$ be such that $\mathbf{a}_i \leq \mathbf{g}$ for all $1 \leq i \leq r$. Define the *characteristic poset* $P_{S/I}^{\mathbf{g}}$ of S/I with respect to $\mathbf{g}$ to be

$$P_{S/I}^{\mathbf{g}} = \{\mathbf{b} \in \mathbb{N}_0^n : \mathbf{b} \leq \mathbf{g}, \mathbf{b} \ngeq \mathbf{a}_i \text{ for all } i\}.$$

For $\mathbf{b} \in \mathbb{N}_0^n$, let $\rho(\mathbf{b}) = |i : \mathbf{b}(i) = \mathbf{g}(i)|$. It is demonstrated in [25, Corollary 2.6] that $\dim S/I = \max\{\rho(\mathbf{b}) : \mathbf{b} \in P_{S/I}^{\mathbf{g}}\}$.

The jth *generalized skeleton ideal* I_j is the ideal generated by $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}\} \cup \{\mathbf{x}^{\mathbf{b}} : \mathbf{b} \in \mathbb{N}_0^n, \rho(\mathbf{b}) > j\}$. By [24, Corollary 2.5] these ideals form a chain

$$I = I_d \subseteq I_{d-1} \subseteq \cdots \subseteq I_0 \subseteq S$$

with the property that S/I_j is Cohen-Macauley for all $j \leq \operatorname{depth} S/I$, and that $\operatorname{depth} S/I = \max\{j : S/I_j \text{ is Cohen-Macauley}\}$. In other words, these ideals successfully generalize (1.3). Furthermore, in the special case $I = I_\Delta$ and $\mathbf{g} = (1, 1, \ldots, 1)$, we have $I_j = I_{\Delta_{(j)}}$.

1.5 Linear codes

We have already mentioned how matroids capture the notion of independence familiar from linear algebra. A natural implication would seem to be that to each set of vectors, over some field, there corresponds a matroid. This is true, and, in particular, it is true for a set of vectors forming the columns of a matrix. The following is perhaps the most typical and intuitive of matroid constructions.

Let A be an $m \times n$ matrix over $\Bbbk$, and label the columns of A by elements of E. If we take as independent sets those subsets of E that correspond to a set of $\Bbbk$-linearly independent columns, this constitutes a matroid on E. We call this the *vector matroid* of A, and denote it $M(A)$. A matroid is said to be *representable over* $\Bbbk$ if it is isomorphic to a vector matroid of some matrix with coefficients in $\Bbbk$. One of the most active areas of research in matroid theory is that of determining which (isomorphism) classes of matroids are representable over which fields. In this thesis, the aim is different; we shall only consider those vector matroids that stem from a linear code.

By definition, a *q-ary linear* $[n,k]$*-code* C is a k-dimensional subspace of $\mathbb{F}_q^n$. The elements of C are referred to as *words*. Each set of k linearly independent words form the rows of a *generator matrix* G of C. Such a matrix by definition has the property that $C = \{\mathbf{x}G : \mathbf{x} \in \mathbb{F}_q^k\}$. A *parity check matrix* H for C is an $(n-k) \times n$ matrix with the property that $H\mathbf{w}^t = 0$ if and only if $\mathbf{w} \in C$. The *dual code* of C is the orthogonal complement $C^\perp$. It is easily verified that H is a parity check matrix for C if and only if it is a generator matrix for $C^\perp$. Since we shall only consider linear codes, over finite fields, we shall by "code" always be referring to a linear q-ary code – for some prime power q.

The support $\mathrm{Supp}(\mathbf{w})$ of a word $\mathbf{w}$ is a subset of $\{1,\dots,n\}$ with the property that $i \in \mathrm{Supp}(\mathbf{w})$ if and only if the ith coordinate of $\mathbf{w}$ is nonzero. The *weight* $\mathrm{wt}(\mathbf{w})$ of $\mathbf{w}$ is defined by $\mathrm{wt}(\mathbf{w}) = |\mathrm{Supp}(\mathbf{w})|$.

The *minimum distance* of C is

$$d = \min\{\mathrm{wt}(\mathbf{w}) : \mathbf{w} \in C \setminus \{0\}\}.$$

This number is very important when it comes to the actual encoding of messages, as a large minimum distance implies that the code is efficient at detecting errors in transmission.

If D is a subcode of C, we define

$$\mathrm{Supp}(D) = \bigcup_{\mathbf{w} \in D} \mathrm{Supp}(\mathbf{w}).$$

For $1 \leq i \leq k$, the ith *generalized Hamming weight* of C is

$$d_i(C) = \min\{|\operatorname{Supp}(D)| : D \subseteq C \text{ and } \dim_{\mathbb{F}_q}(D) = i\}.$$

These were first studied in [22] and [34]. For example, it was demonstrated in [22] (and, independently, in [53]) that $d_i < d_{i+1}$. In [53] it is explained how the set of generalized Hamming weights of a code completely determines that code's performance on a wire-tap channel of type II, and Ashikmin, Barg and Litsyn states in [2] that

> "Knowing generalized weights is of importance in the analysis of cryptographic resistance of codes in the wire-tap channel, estimation of trellis complexity, design of codes for the switching multiple-access channel, etc"(p. 1258).

In the above mentioned article [53], by Victor Wei, a close relation is found between the generalized Hamming weights of C and those of $C^{\perp}$. The following identity is now commonly referred to as *Wei-duality*:

$$\{d_i(C) : 1 \leq i \leq k\} = \{1, \dots, n\} \setminus \{n + 1 - d_i(C^{\perp}) : 1 \leq i \leq n\}.$$

Since the generalized Hamming weights are strictly increasing, Wei-duality implies that the weights of a code are determined by those of its dual code. Note also that $d = d_1(C)$. A recent effort to find upper bounds for $d_i(C)$ for different types of linear codes is [2].

Inspired by the generalized Hamming weights of a linear code, an analogous invariant for matroids was first introduced in [37]: The *higher weight* hierarchy $\{d_i(M) : 1 \leq i \leq n - r(M)\}$ of M is defined by

$$d_i(M) = \min\{|X| : X \subseteq E \text{ and } n(X) = i\}.$$

Also in [37], it is demonstrated that these matroidal higher weights satisfy the same Wei-duality as their code predecessors, that is, we have

$$\{d_i(M) : 1 \leq i \leq n - r(M)\} = \{1, \dots, n\} \setminus \{n + 1 - d_i(\overline{M}) : 1 \leq i \leq r(M)\}.$$

The matroidal higher weights are a topic of all three papers. For example, in regard to the next section (on elongation and truncation) we demonstrate that the higher weights of M determine both those of its elongations (Paper 2) and those of its truncations (Paper 3).

If G and G' are two generator matrices for the linear code C, then $M(G) = M(G')$. The same goes for parity-check matrices, of course. We may therefore speak of *the* matroid corresponding to a generator (or parity-check) matrix of C, and write $M(G)$ and $M(H)$ without giving specific generator or parity-check matrices G and H.

Thus, to a linear code C there naturally correspond two matroids: $M(G)$ and $M(H)$. We shall mostly consider $M(H)$, but this is insignificant since duality results abound and $M(H) = \overline{M(G)}$. It is readily checked that $r(M(G)) = \dim(C)$, and $r(M(H)) = \dim(C^{\perp})$. Note also that the above mentioned generalization in [37] is a good one, in the sense that $d_i(M(H)) = d_i(C)$.

For $0 \leq j \leq n$, let $A_{C,j} = |\{\mathbf{w} \in C : \mathrm{wt}(\mathbf{w}) = j\}|$. The *weight enumerator* of C is the bivariate, homogeneous polynomial

$$W_C(X,Y) = \sum_{j=0}^{n} A_{C,j} X^{n-j} Y^{j}.$$

The weight enumerator has practical applications. For example, it is shown in [31, Proposition 1.14] that the probability of undetected error on a q-ary symmetric channel with cross-over probability p is equal to $W_C(1-p, \frac{p}{q-1}) - (1-p)^n$. More immediately clear, however, is that $W_C(1,0) = A_{C,0} = 1$ and $W_C(1,1) = |C| = q^k$.

The weight enumerator has been generalized in several interesting ways. For example, the *generalized weight enumerator* $W_C^{(r)}(X,Y)$ of C is defined by

$$W_C^{(r)}(X,Y) = \sum_{i=0}^{k} A_C^{(r)} X^{n-i} Y^{i},$$

where $A_C^{(r)} = |\{D \subseteq C : \dim_{\mathbb{F}_q}(D) = i\}|$. It is easily verified that $W_C(X,Y) = W_C^{(0)}(X,Y) + (q-1)W_C^{(1)}(X,Y)$. Note also that $d_i(C) = \min\{i : A_C^{(i)} \neq 0\}$.

Most relevant to Paper 2 of this thesis, however, is the generalization to a q^r-ary extension of C.

1.5.1 Extension codes

Let $Q = q^r$ for some $r \geq 1$. Then $C \otimes_{\mathbb{F}_q} \mathbb{F}_Q$ is an $[n,k]$-code over $\mathbb{F}_Q$. Since a generator matrix for C is also a generator matrix for $C \otimes_{\mathbb{F}_q} \mathbb{F}_Q$, the code $C \otimes_{\mathbb{F}_q} \mathbb{F}_Q$ is isomorphic to the code whose words are all the $\mathbb{F}_Q$-linear combinations of words of C. Let $A_{C,j}(Q)$ be defined by $A_{C,0}(Q) = 1$ and

$$A_{C,j}(Q) = |\{\mathbf{w} \in C \otimes_{\mathbb{F}_q} \mathbb{F}_Q : \mathrm{wt}(\mathbf{w}) = j\}|$$

for $1 \leq j \leq n$. It is clear from e.g. [31] (or Paper 2) that $A_{C,j}(Q)$ is a polynomial in Q, and it is thus referred to as the jth *weight polynomial* of C. These polynomials were extensively studied in [22] and [34]. The *extended weight enumerator*, first introduced in [31], is

$$W_C(X,Y,Q) = \sum_{j=0}^{n} A_{C,j}(Q) X^{n-j} Y^j.$$

By definition then, we have $W_C(X,Y,Q) = W_{C \otimes_{\mathbb{F}_q} \mathbb{F}_Q}(X,Y)$. Although the extended weight enumerator had not been introduced yet, it follows from [22, Theorem 3.2] that

$$W_C(1,Y,q^r) = \sum_{i=0}^{n} \sum_{j=0}^{k} A_i^{(j)} \Big(\prod_{l=0}^{j-1} (q^r - q^l) \Big) Y^i.$$

As a side note, we point out that a set of vectors in $\mathbb{F}_q^n$ is linearly independent over $\mathbb{F}_q$ if and only if they are linearly independent over $\mathbb{F}_{q^r}$. This implies that the matroid associated to a parity check matrix of C is the same as the one associated to a parity check matrix of $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$. (This is equivalent to our above remark that you can choose a generator matrix for $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$ with entries in C.)

In Paper 2 we (further) generalize the polynomials $A_{C,j}(Q)$ and $W_C(X,Y,Q)$ to matroids, and, as our main result, demonstrate that they are determined by the $\mathbb{N}_0$-graded Betti numbers associated to elongations.

1.6 About the papers

What follows is a discussion of the contents of each of the three papers. We first give a brief summary of our main results, and then provide some context in terms of earlier, related results. In all three papers the examples and counterexamples were usually found using MAGMA (see [1]).

1.6.1 Paper 1

Our main result is that the Betti numbers associated to an $\mathbb{N}_0$-graded minimal free graded resolution of $\mathscr{F}(M)$ are determined by those associated to $\mathbb{N}_0$-graded minimal free resolutions of the facet ideals $\mathscr{F}(B_i) \subset \Bbbk[x_j : j \in B_i] \subseteq S$ of blocks of M.

Note first that if B is a block in M then $\mathscr{F}(B)$ may be considered as an S-ideal through its embedding $S\mathscr{F}(B)$. Furthermore, since S is flat over $\Bbbk[x_j : j \in B]$, we have that $\beta_i(\mathscr{F}(B)) = \beta_i(S\mathscr{F}(B))$. In a slightly more general setting, we write $S = \Bbbk[X;Y]$, let M_1 be a $\Bbbk[X]$ module and M_2 a $\Bbbk[Y]$ module. We explicitly construct an $\mathbb{N}_0$-graded minimal free resolution of

$$(S \otimes_{\Bbbk[X]} M_1) \otimes_S (S \otimes_{\Bbbk[Y]} M_2).$$

By induction, this clearly extends to any finite number of modules $M_i \subset \Bbbk[X_i]$. Our main result then follows by:

1. Observing that
$$(S \otimes_{\Bbbk[X]} I) \otimes_S (S \otimes_{\Bbbk[Y]} J) \cong (SI)(SJ)$$
for ideals $I \subset \Bbbk[X]$ and $J \subset \Bbbk[Y]$. And

2. Establishing that
$$\mathscr{F}(M) = \prod_B S\mathscr{F}(B).$$

Our original intention was to investigate the connection between Betti numbers associated to $\mathscr{F}(M)$ and the set of higher weights of M – analogous to what was done for the Stanley-Reisner ideal in [28]. A first indication that the Betti numbers associated to the facet ideal carry less information than those of the Stanley-Reisner ideal is the classic [18, Proposition 7], which implies that $\mathscr{F}(M)$ has linear resolution. This is in itself a fact we use repeatedly.

In [28] it is demonstrated that

$$d_i(M) = \min\{j : \beta_{i,j}(S/I_M;\Bbbk) \neq 0\}.$$

A similar result can not be hoped for when it comes to the facet ideal. There are plenty of examples of pairs of matroids with equal Betti numbers but different generalized Hamming weight hierarchy. One such example is given at the end of the paper. In our specific example, the matroids are cycle matroids of outerplanar graphs.

For a subclass of outerplanar graphs called cactus graphs it *does* however hold true that the Betti numbers determine the higher weight. As we worked to demonstrate this, it became apparent that it would be useful and interesting to isolate and generalize certain parts of our proof. Observing that circuits and single edges of a cactus graph constitute the blocks of its cycle matroid, we thus discovered our more general main result in an attempt to describe the Betti numbers of such a cycle matroid. The result concerning cactus graphs we now consider as an application of our main result.

Complementing our main result, we also demonstrate that knowing the higher weight hierarchy of each of the blocks enables one to calculate the higher weights of M itself easily by way of

$$d_i(M) = \min\left\{\sum_{j=1}^{t} d_{k_j}(B_j) : \sum_{j=1}^{t} k_j = i\right\}.$$

Previous work on the algebraic properties of facet ideals (of simplicial complexes) include Faridi's [19], where the graph theoretical concept of a tree is generalized to simplicial complexes. Extending a result from [47] on the edge ideal of a graph, it is demonstrated that the facet ideal of a simplicial tree has so-called sliding depth. As a result, the Rees ring (see e.g. [5, p. 182]) of such a facet ideal is seen to be both normal and Cohen-Macaulay ([19, Corollaries 3.9 and 3.12]). In [20] the theory of simplicial trees are developed further. The concept of a *grafted* simplicial complex is introduced, and it is demonstrated that such complexes are Cohen-Macaulay. Also in [20] can be found a generalization of König's Theorem (equivalent to the perhaps better known Hall's Marriage Theorem) of graph theory to simplicial trees.

The thread is picked up by Zheng in [57], where both the term "facet ideal" and "tree" has been adopted and these concepts are further investigated. The contents of this paper is closer to that of ours, in that it deals more directly with $\mathbb{N}_0$-graded minimal free resolutions and their Betti numbers – although mostly those associated to the facet ideal of a simplicial tree. It is demonstrated that if the facet ideal of a simplicial tree T has linear resolution (such trees are referred to as *linear*), then $\mathscr{F}(T)$ is pure and connected in codimension 1 (see e.g. [23, p. 161] for a definition). If T is a d-dimensional linear simplicial tree with m facets, then, by [57, Corollary 3.10], its associated Betti numbers are

$$\beta_i(S/\mathscr{F}(T)) = \begin{cases} m, & i = 1 \\ \sum_{\dim(\sigma)=d-1} \binom{m(\sigma)}{i}, & i \geq 2, \end{cases}$$

where $m(\sigma) = |\{\tau \in \mathscr{F}_{\dim(\Delta)}(T) : \sigma \subseteq \tau\}|$. Also, it is demonstrated in [57, Proposition 3.6] that if T is pure and connected in codimension 1 (although not necessarily linear), then

$$1 + \sum_{i>0} \beta_{i,i+d}(S/\mathscr{F}(T); \Bbbk) = 0.$$

Lastly, one could say that Paper 1 contributes to the work done by Morey and Villarreal in [40] for edge ideals of clutters. For although the facet ideal is but one type of "clutter ideal", we consider that looking at the connection between Betti numbers of edge ideals and those of substructures of the clutter (in our case, the blocks) would be interesting in general.

1.6.2 Paper 2

For a matroid M on E, we introduce the univariate polynomials $P_{M,0}(Z) = 1$ and

$$P_{M,j}(Z) = (-1)^j \sum_{|\sigma|=j} \sum_{\gamma\subseteq\sigma} (-1)^{|\gamma|} Z^{n_M(\gamma)} \text{ for } 1 \leq j \leq n.$$

This is a good generalization of $A_{C,j}$ in the sense that if C is a linear code with parity check matrix H, then $A_{C,j}(Q) = P_{M(H),j}(Q)$. Another property of $P_{M,j}$, immediate from the definition, is that $d_i(M) = \min\{j : \deg P_{M,j} = i\}$.

Analogous to how $W_C(X,Y,Q)$ is defined in terms of the polynomials $A_{C,j}$, we use our *generalized weight polynomials* $P_{M,j}$ to define an entirely matroidal *enumerator* of M by

$$W_M(X,Y,Z) = \sum_{i=0}^{n} P_{M,i}(Z) X^{n-i} Y^i.$$

This way, we have the desired

$$W_C(X,Y,Q) = W_{M(H)}(X,Y,Q).$$

In [31] it is demonstrated that the the polynomial $W_C(X,Y,Q)$ is equivalent to the Tutte polynomial of $M(H)$. We extend this result to arbitrary matroids and their matroid enumerator, establishing

$$W_M(X,Y,Z) = (X-Y)^{n-r(M)} Y^{r(M)} t_M\left(\frac{X}{Y}, \frac{X+(Z-1)Y}{X-Y}\right),$$

and

$$t_M(X,Y) = (X-1)^{-(n-r(M))} X^n W_M(1, X^{-1}, (X-1)(Y-1)).$$

As in Greene's [21], this immediately yields a MacWilliams identity for matroid enumerators.

Our main result is that the coefficients of $P_{M,j}$ can be expressed as a $\mathbb{Z}$-linear sum of the Betti numbers associated to the Stanley-Reisner ideals of elongations of M. We find that the coefficient of Z^l in $P_{M,j}$ is equal to

$$\sum_{i=0}^{n}(-1)^i\Big(\beta_{i,j}(I_{M^{(l-1)}})-\beta_{i,j}(I_{M^{(l)}})\Big). \tag{1.4}$$

The first observation en route to this result is that the constant term of $P_{M,n}$ is equal to $(-1)^{n+1}\tilde{\chi}(M)$. By Hochster's formula and two results by Björner that combined imply

$$H_i(M;\Bbbk)\cong\begin{cases}\Bbbk^{(-1)^{i-1}\tilde{\chi}(M)}, & i=r(M)\\ 0, & i\neq r(M),\end{cases}$$

we have

$$(-1)^{n+1}\tilde{\chi}(M)=(-1)^{n(M)+1}\beta_{n(M)-1,E}(I_M). \tag{1.5}$$

Equation (1.5), expressing the constant term of $P_{M,n}$ as a Betti number, constitutes the base case of our argument. In order to find the remaining coefficients of $P_{M,n}$ we reduce to the above base case using elongations of M. This can be done for $P_{M,j}$ with $1\leq j<n$ as well, by considering restrictions $M_{|\sigma}$ to subsets $\sigma\subseteq E$ with $|\sigma|=j$. The end result is the above stated (1.4).

A somewhat similar argument tells us that the generalized weight polynomial of M determines that of $M^{(1)}$ very directly through

$$P_{M^{(1)}}(Z)=\frac{P_M(Z)+P_M(0)(Z-1)}{Z}.$$

This in turn implies $d_i(M^{(1)})=d_{i+1}(M)$, for $1\leq i\leq n(M)-1$.

Using the identity $r_{\overline{M}}(X)=r_M(E\smallsetminus X)+|X|-r_M(E)$ it is readily verified that $p(M,Z)=P_{\overline{M},n}(Z)$. Thus an immediate consequence of our main result is that the characteristic polynomial of M can be expressed in terms of the Betti numbers of $\overline{M}$ and its elongations, the precise expression being

$$p(M,Z)=\sum_{l=0}^{n}\Big(\sum_{i=0}^{n}(-1)^i\big(\beta_{i,n}(I_{\overline{M}^{(l-1)}})-\beta_{i,n}(I_{\overline{M}^{(l)}})\big)\Big)Z^l.$$

Towards the end of the paper we give counterexamples demonstrating that a) the Betti numbers of elongations are needed in our expression (2) - the Betti numbers of M do not suffice by themselves. And b) the generalized weight polynomials do not determine the Betti table of M, not to mention the Betti table of all its elongations.

There has been quite some research on the weight enumerator of a q-ary linear code. To begin with only the univariate version $\sum_{\mathbf{w}\in C} Z^{\mathrm{wt}(\mathbf{w})} = W_C(1,Z)$ was considered, but this is insignificant since $W_C(X,Y) = X^n W_C(1,YX^{-1})$. There are in particular two very influential early results, both of which have strongly influenced later research efforts. One is the *MacWilliams identity*, named after the British mathematician Jessie MacWilliams. She demonstrated in 1963 [38, Theorem 2.8] that $W_{C^\perp}(1,Z) = q^k\big(1+(q-1)Z\big)^n W_C\big(1, \frac{1-Z}{1+(q-1)Z}\big)$, or, equivalently, that

$$W_{C^\perp}(X,Y) = q^{-k} W_C(X+(q-1)Y, X-Y). \tag{1.6}$$

We shall refer to similar results – linking versions of enumerators of dual structures – as *MacWilliams-type* identities.

The other is *Greene's Theorem*. In 1976, Curtis Greene demonstrated ([21, Corollary 4.5] that

$$W_C(X,Y) = (X-Y)^{n-k} Y^k t_{M(H)}\left(\frac{X}{Y}, \frac{X+(q-1)Y}{X-Y}\right), \tag{1.7}$$

thus establishing an elegant connection between the weight enumerator of C and the Tutte polynomial of $M(H)$, where H is a parity check matrix for C. We shall refer to results giving enumerators as an evaluation of some Tutte polynomial as *Greene-type* identities. Greene's first application of (1.7) was to give a new and much shorter proof of the MacWilliams identity; a method which is emulated in most later generalizations of (1.6).

In [22], Helleseth, Kløve, and Mykkeltveit explicitly find the weight enumerator of various cyclic codes, of different block lengths. The paper also contains several more general results on the weight enumerator. For example, they demonstrate that

$$W_{C\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}}(1,Z) = \sum_{i=0}^{n}\sum_{j=0}^{k}\prod_{l=0}^{j-1}(q^r-q^l)A_{i,C}^{(j)}Z^i, \tag{1.8}$$

which is equivalent to Kløve's

$$W_{C\otimes_{\mathbb{F}_q}\mathbb{F}_{q^r}}(1,Z) = \sum_{j=0}^{k}\prod_{l=0}^{j-1}(q^r-q^l)A_{C}^{(j)}(Z),$$

of [35].

By applying MacWilliams identity on (1.8), Kløve demonstrates that

$$\sum_{j=0}^{r}[r]_j W_{C^\perp}^{(j)}(1,Z) = q^{-rk}\big(1+(q^r-1)Z\big)^n\Big(\sum_{j=0}^{r}[r]_j W_C^{(j)}\big(1,\frac{1-Z}{1+(q^r-1)Z}\big)\Big),$$

where $[r]_j = \prod_{i=0}^{j-1}(q^r-q^i)$. From this equation he manages, in [34, Theorem 2.6], through lengthy but direct computations, to find an expression for $W_{C^\perp}^{(r)}(1,Z)$ in terms of $W_C^{(r)}(1,Z)$ – thus establishing a MacWilliams type duality for each r. A considerably shorter proof (of a homogeneous version) can be found in [30, Theorem 8.8].

In [3], Barg studies what he refers to as the mth support weight enumerator:

$$D^m(x) = \sum_{i=0}^{n} D_i^m x^i,$$

where $D^m(x) = \sum_{u=0}^{m}\prod_{i=0}^{m-1}(q^m-q^i)A_{i,C}^{(u)}$. Although he does not explicitly state that the mth support weight enumerator of C is equal to the weight enumerator of $C\otimes_{\mathbb{F}_q}\mathbb{F}_{q^m}$, it follows from [35, Lemma 4] that $D^m(Y)$ is equal to what in the terminology of [30], [31] and Paper 2 is denoted $W_C(1,Y,q^m)$. Barg argues that the function $(1-u)^{-k}u^{k-n}D^m(u)$ is a so-called Tutte-Grothendieck invariant of $M(H)$ – where H is a parity check matrix for C. Since (by e.g. [13, Theorem 6.2.2], or the original [11]) every Tutte-Grothendieck invariant is an evaluation of the Tutte polynomial, this enables Barg to establish that

$$D^m(u) = (1-u)^k u^{n-k} t_M\Big(\frac{1+(q^m-1)u}{1-u},\frac{1}{u}\Big).$$

As pointed out in [30, p. 59], however, this result is actually equivalent to Greene's original theorem, since $C\otimes_{\mathbb{F}_q}\mathbb{F}_{q^m}$ has a generator matrix with coefficients in $\mathbb{F}_q$. A proof of the converse result, that the extended weight enumerator determines the Tutte polynomial, is found in [30, Theorem 8.5]. As an immediate consequence of the above Greene's theorem for extended weight enumerators, Barg also establishes a MacWilliams-type identity between $D_{C^\perp}^m(u)$ and $D_C^m(u)$ – analogous to how Corollary 4.1 follows immediately from Theorem 4.1 in our paper. In [3, Theorem 3] Barg gives bounds on $|C|$ in terms of the numbers $A_i^u(C)$ and so-called Krawtchouk numbers $P_j^m(i) = \sum_{l=0}^{n}\binom{i}{l}\binom{n-i}{j-l}(q^m-1)^{j-l}$.

In [8], Britz introduces a "finer" generalized weight enumerator A, in n variables, defined by

$$A_C^{(r)}(z_1,z_2,\cdots,z_n) = \sum_{E'\subseteq E} A_{E'}^{(r)}\prod_{i\in E'} z_i,$$

where $A_{E'}^{(r)} = |\{D \subseteq C : \dim_k(D) = r, \cup_{\mathbf{w} \in D} \operatorname{Supp}(\mathbf{w}) = E'\}|$. Britz's [10, Theorem 7] is a MacWilliams identity relating $A_C^{(r)}(z_1, z_2, \cdots, z_n)$ and $A_{C^\perp}^{(r)}(z_1, z_2, \cdots, z_n)$; this result generalizes several earlier results, in that setting $z_i = Z$ for all i reduces the identity in [10, Theorem 7] to the MacWilliams identity for the generalized weight enumerators $W_C^{(r)}(1, Z)$ and $W_{C^\perp}^{(r)}(1, Z)$, while in addition setting $r = 1$ yields MacWilliams's original identity for the (regular) weight enumerator.

Britz's efforts at generalizing weight enumerators and Greene- and MacWilliams-type identities for codes culminate in [10, Theorem 24], which, although a bit too technical to restate here, is

> "...the most general MacWilliams-type identity for linear codes over fields and which generalizes almost all previous such generalizations of the MacWilliams identity"([10, p. 4350]).

Generalizing weight enumerators to matroids, instead of the above described generalizations to finer enumerators of (occasionally larger) codes, was first done in [9]. The *generalized coboundary polynomial* $W_M^*(\lambda, \mathbf{x}, \mathbf{y})$ of M, in $2n+1$ variables, is defined by

$$W_M^*(\lambda, \mathbf{x}, \mathbf{y}) = \sum_{T \subseteq E} p(M.T; \lambda) \mathbf{x}^{E \setminus T} \mathbf{y}^T,$$

where p is the characteristic polynomial and $M.T$ is the contraction $M/(E \setminus T)$ (see e.g. [42]). This polynomial generalizes the standard weight enumerator of a code C in the sense that if G is a generator matrix, each $y_i = Z$ and each $x_i = 1$, then

$$W_{M(G)}^*(q, \mathbf{x}, \mathbf{y}) = W_C(1, Z).$$

Recently, several new results have been found by Jurrius. As mentioned in Section 1.5.1 it was first demonstrated in [22, Theorem 3.2] that, for codes, the extended weight enumerator can be expressed as a sum of the generalized weight enumerators. A converse to this was demonstrated by Jurrius in [30, Theorem 2.25]. Jurrius also demonstrates that the Tutte polynomial of $M(H)$ is an evaluation of the extended weight enumerator – thereby strengthening Greene's classic [21, Corollary 4.5] to a complete equivalence. These two equivalence are in turn used to prove a third one, namely that the set of generalized weight enumerators and the Tutte polynomial completely determine each other (see [30, Theorem 8.6]). In [29] Jurrius determines the generalized and extended weight enumerator of the q-ary simplex code and the q-ary first order Reed Muller code.

1.6.3 Paper 3

This paper is in a sense a continuation of the results of Paper 2; for matroids, the operations of truncation and elongation should be regarded as opposites. Indeed, there is the identity $\overline{M}^{(i)} = \overline{M_{(i)}}$.

As we have seen, truncation is but the more general concept of i-skeleton – applied to those simplicial complexes that are matroids. Where we in Paper 2 were investigating the connection between the matroidal weight enumerator and the Betti numbers of Stanley-Reisner ideals of elongations of M, we here consider the relationship between Betti numbers of a Stanley-Reisner ring of a d-dimensional simplicial complex Δ and those of its skeletons.

Our main result is that the Betti numbers of $S/I_{\Delta_{(d-1)}}$ can be expressed as a $\mathbb{Z}$-linear sum of the Betti numbers of S/I_Δ.

$$\beta_{i,j}(S/I_{\Delta_{(d-1)}};\Bbbk) = \begin{cases} \beta_{i,j}(S/I_\Delta;\Bbbk), & j \le d+i-1 \\ \beta_{i,d+i}(S/I_\Delta;\Bbbk) - \beta_{i-1,d+i}(S/I_\Delta;\Bbbk) + \binom{n-d-1}{i-1}\delta, & j = d+i, \\ 0, & j \ge d+i-1 \end{cases}$$

where

$$\delta = \begin{cases} \sum_{k=0}^{n}(-1)^{n+d+k+1}\sum_{l\ge n-d-1}\binom{l}{n-d-1}\beta_{k,l}(S/I_\Delta;\Bbbk), & 1 \le i \le n-d \\ 0, & i > n-d. \end{cases} \tag{1.9}$$

It follows from [26, Corollary 2.6] that

$$\operatorname{p.d.} S/I_\Delta \le \operatorname{p.d.} S/I_{\Delta_{(d-1)}} \le \operatorname{p.d.} S/I_\Delta + 1.$$

(The rightmost inequality follows from our main result as well.) Also, applying Hochster's formula to the definition of a skeleton, we find that $\beta_{i,j}(S/I_\Delta;\Bbbk) = \beta_{i,j}(S/I_{\Delta_{(d-1)}};\Bbbk)$ for all i and $j \le d+i-1$. Since $\beta_{i,j}(S/I_\Delta;\Bbbk) = 0$ for all i and $j \ge d+i+2$, we thus conclude that if

	0	1	$\cdots$	p
0	1	$\beta_{1,1}(S/I_\Delta;\Bbbk)$	$\cdots$	$\beta_{p,p}(S/I_\Delta;\Bbbk)$
1	0	$\beta_{1,2}(S/I_\Delta;\Bbbk)$	$\cdots$	$\beta_{p,p+1}(S/I_\Delta;\Bbbk)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d+1$	0	$\beta_{1,d+2}(S/I_\Delta;\Bbbk)$	$\cdots$	$\beta_{p,p+d+1}(S/I_\Delta;\Bbbk)$

is the Betti table of S/I_Δ, then the Betti table of $S/I_{\Delta_{(d-1)}}$ is

$$\begin{array}{r|cccccc}
 & 0 & 1 & \cdots & p & p+1 \\
\hline
0 & 1 & \beta_{1,1}(S/I_\Delta;\Bbbk) & \cdots & \beta_{p,p}(S/I_\Delta;\Bbbk) & 0 \\
1 & 0 & \beta_{1,2}(S/I_\Delta;\Bbbk) & \cdots & \beta_{p,p+1}(S/I_\Delta;\Bbbk) & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d-1 & 0 & \beta_{1,d}(S/I_\Delta;\Bbbk) & \cdots & \beta_{p,p+d-1}(S/I_\Delta;\Bbbk) & 0 \\
d & 0 & * & \cdots & * & *
\end{array}$$

– where the $*$'s remain to be found.

From properties (in [23, Chapter 6]) of the Hilbert series we derive

$$\sum_{i=0}^{d+1} f_{i-1}(\Delta)t^i(1-t)^{n-i} = \sum_{i=0}^{n}(-1)^i \sum_j \beta_{i,j}(S/I_\Delta;\Bbbk)t^j,$$

and

$$\sum_{i=0}^{d} f_{i-1}(\Delta_{(d-1)})t^i(1-t)^{n-i} = \sum_{i=0}^{n}(-1)^i \sum_j \beta_{i,j}(S/I_{\Delta_{(d-1)}};\Bbbk)t^j.$$

Applying techniques from calculus on the above equations we demonstrate that

$$\beta_{i,d+i}(S/I_{\Delta_{(d-1)}};\Bbbk) = \beta_{i,d+i}(S/I_\Delta;\Bbbk) - \beta_{i-1,d+i}(S/I_\Delta;\Bbbk) + \binom{n-d-1}{i-1}\delta \quad (1.10)$$

for all $0 \le i \le n$, where δ is as in (1.9) above.

From equation (1.10) it follows that $\mathrm{p.d.}\, S/I_{\Delta_{(d-1)}} \le \mathrm{p.d.}\, S/I_\Delta + 1$. This in turn is sufficient to show, by the Auslander-Buchsbaum Theorem, that $\Delta_{(d-1)}$ is Cohen-Macaulay whenever Δ is.

Since a matroid's ith truncation corresponds to the $(d-i)$-skeleton of its underlying simplicial complex, our main result extends immediately to truncations of matroids. In contrast, the concept of matroid elongation does not have a canonical generalization to simplicial complexes. We do however have a result, specifically for matroids, analogous to our main result applied to truncations: In [28] it is demonstrated that

$$\beta_{i,\sigma}(S/I_M) \neq 0 \iff \sigma \text{ is inclusion-minimal with } n_M(\sigma) = i,$$

from which it follows that for $i \ge 1$ we have

$$\beta_{i,j}(S/I_M) \neq 0 \iff \beta_{i-1,j}(S/I_{M^{(1)}}) \neq 0.$$

In other words, the Betti table of $S/I_{M^{(1)}}$ looks like the Betti table of S/I_M with its second column removed – but only in terms of zeros and nonzeros.

As explained at the beginning of Section 1.4, studying the Betti numbers of Stanley-Reisner rings is equivalent to studying Betti numbers of squarefree monomial ideals. Good sources of known results on this topic is [39] or the slightly more specialized [23] and [48]. A modern classic is the previously mentioned theorem by Eagon and Reiner from '96, which says that I_{Δ^*} has linear resolution if and only if S/I_Δ is Cohen-Macauley (for a proof, see e.g. [39, Theorem 5.56], or the original one in [18]).

A question of continuous interest is under which conditions the (graded or ungraded) Betti numbers of S/I_Δ are independent of the base field $\Bbbk$. As we mentioned in Section 1.4, this is always the case for the (graded and ungraded) Betti numbers in homological degree one – as these simply correspond to the minimal nonfaces of Δ. Furthermore, Bruns and Herzog demonstrated in [6, Corollary 5.4] that $\dim_\Bbbk \tilde{H}_{|\Delta|-3}(\Delta;\Bbbk)$ is in fact independent of $\Bbbk$. By Hochster's formula this implies (indeed, is equivalent to) that the $\mathbb{N}_0^n$-graded Betti numbers in homological degree two are also independent of $\Bbbk$.

In [50], Terai and Hibi show that when I_Δ is generated by squarefree monomials of degree two, the ungraded Betti numbers $\beta_3(S/I_\Delta;\Bbbk)$ and $\beta_4(S/I_\Delta;\Bbbk)$ are also independent of the base field. They also point out that there exists a minimal free resolution of $\mathbb{Z}/I_\Delta$ over the polynomial ring $\mathbb{Z}[x_1,\ldots,x_n]$ if and only if all the ungraded Betti numbers β_i are independent of the base field. Also, recall that in Section 1.4 we saw that the $\mathbb{N}_0^n$-graded Betti numbers of a matroid are independent of $\Bbbk$

By [26, Corollary 2.4], the skeleton of a Cohen-Macaulay complex is Cohen-Macaulay. And [26, Corollary 2.6] is Hibi's classic result

$$\operatorname{depth} S/I_\Delta = \max\{j : \Delta_{(j)} \text{ is Cohen-Macaulay}\}. \tag{1.11}$$

In [24], Herzog, Zheng, and Jahan generalize (1.11) using the jth skeleton ideal construction described in Section 1.4.1 – demonstrating that for a (not necessarily squarefree) monomial ideal I, we have

$$\operatorname{depth} S/I = \max\{j : S/I_j \text{ is Cohen-Macaulay}\}.$$

A natural question is whether a result similar to our main result holds for the $\mathbb{N}_0$-graded Betti numbers of generalized skeleton ideals of [24]. In Paper 3, however, we give a counterexample showing two ideals with identical Betti tables but whose first generalized skeletons have nonidentical Betti tables. It would be interesting, however, to see for which ideals and which values of $\mathbf{g} \in \mathbb{N}_0^n$ we do get a positive result.

Truncations of matroids are studied in [32] (in particular in connection to matroidal polynomials) where, amongst other things, results are given on their representability.

Bibliography

[1] Bosma, W., Cannon, W., Playoust, C.: *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24**, 235-265 (1997)

[2] Ashikmin, A., Barg, A., Litsyn, S.: *New upper bounds on generalized weights*, IEEE Trans. Inform. Theory **45**, 1258-1263 (1999)

[3] Barg, A.: *Matroid of supports of a linear code*, AAECC **8**, 165-172 (1997)

[4] Björner, A.: *Homology and shellability*, In: Matroid Applications, pp. 226-283, Cambridge University Press (1992)

[5] Bruns, W., Herzog, J.: *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics **39**, Cambridge University Press (1993)

[6] Bruns, W., Herzog, J.: *On multigraded resolutions*, Math. Proc. Camb. Phil. Soc. **118**, 245-257 (1995)

[7] Birkhoff, G.: *A determinant formula for the number of ways of coloring a map*, Annals of Mathematics, **14**, 42-46 (1912)

[8] Britz, T.: *MacWilliams identities and matroid polynomials*, The Electronic Journal of Combinatorics **9** (2002)

[9] Britz, T.: *A MacWilliams type identity for matroids*, Discrete Mathematics **308**, 4551-4559 (2008)

[10] Britz, T.: *Code enumerators and Tutte polynomials*, IEEE Trans. Inform. Theory **56**, 4350-4358 (2010)

[11] Brylawski, T.: *A decomposition for combinatorial geometries*, Transactions of the American Mathematical Society **171**, 235-282 (1972)

[12] Brylawski, T.: *Constructions*, In: Theory of Matroids, pp. 127-223, Cambridge University Press (1986)

[13] Brylawski, T., Oxley, J.: *The Tutte polynomial and its applications*, In: Matroid Applications, pp. 123-215, Cambridge University Press (1992)

[14] Chartrand, G, Hartrand, F.:.: *Planar permutation graphs*, Annales de lI. H. P., section B, **3**, 433-438 (1967)

[15] Cox, D., Little, J., O'Shea, D.: *Using Algebraic Geometry*, Graduate Texts in Mathematics **185**, Springer (2000)

[16] Crapo, H.: *The Tutte polynomial*, Aequationes Math. **3**, 211-229 (1969)

[17] Eisenbud, D.: *The Geometry of Syzygies*, Graduate Texts in Mathematics **229**, Springer (2005)

[18] Eagon, E., Reiner, V.: *Resolutions of Stanley-Reisner rings and Alexander duality*, Journal of Pure and Applied Algebra **130**, 265-275 (1998)

[19] Faridi, S.: *The facet ideal of a simplicial complex*, Manuscripta Math.**109**, 159-174 (2002)

[20] Faridi, S.: *Cohen-Macaulay properties of squarefree monomial ideals*, Journal of Combinatorial Theory, Series A. **109**, 299-329 (2005)

[21] Greene, C: *Weight enumeration and the geometry of linear codes*, Studies in Appl. Math **55**, 119-128 (1976)

[22] Helleseth, T., Kløve, T., Mykkeltveit, J.: *The weight distribution of irreducible cyclic codes with block lengths $n_1((q^l-1)/N)$*, IEEE Trans. Inform. Theory **18**, 179-211 (1977)

[23] Herzog, J., Hibi, T.: *Monomial Ideals*, Graduate Texts in Mathematics **260**, Springer (2011)

[24] Herzog, J., Jahan, A., Zheng, X.: *Skeletons of monomial ideals*, Math. Nachr. **283** no. 10, 1403-1408 (2010)

[25] Herzog, J., Vladoiu, M., Zheng, X.: *How to compute the Stanley depth of a monomial ideal*, J. Algebra **88**, 2020-224 (1983)

[26] Hibi, T.: *Quotient algebras of Stanley-Reisner rings and local cohomology*, Journal of Algebra **140**, 336-343 (1991)

[27] Hochster, M.: *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, In: Ring theory II. Lect. Notes in Pure and Appl. Math. **26**, pp. 171-223, M. Dekker (1977)

[28] Johnsen, J., Verdure, H.: *Hamming weights and Betti numbers of Stanley-Reisner rings associated to matroids*, AAECC **24** no. 1, 73-93 (2013)

[29] Jurrius, R.: *Weight enumeration of codes from finite spaces*, Des. Codes Cryptogr. **63**, 321-330 (2012)

[30] Jurrius, R.: *Codes, arrangements, matroids, and their polynomial links*, PhD thesis. Eindhoven University of Technology (2012)

[31] Jurrius, R., Pellikaan, R.: *Codes, arrangements and matroids*. In: Algebraic geometry modeling in information theory, pp. 219-325, London: World Scientific (2013)

[32] Jurrius, R., Pellikaan, R.: *Truncation formulas for invariant polynomials of matroids and geometric lattices*. Mathematics in Computer Science **6**, 121-133 (2012)

[33] Katona, G.: *A theorem for finite sets*, In: Theory of graphs, pp. 187-207, Academic Press (1968)

[34] Kløve, T.: *The weight distribution of linear codes over $GF(q^l)$ having generator matrices over $GF(q)$*, Discr. Math. **23**, 159-168 (1978)

[35] Kløve, T.: *Support weight distribution of linear codes*, Discr. Math. **106/107**, 311-316 (1992)

[36] Kruskal, J.: *The number of simplices in a complex*, In: Mathematica optimization techniques, pp. 251-278, University of California Press (1963)

[37] Larsen, A.: *Matroider og lineære koder*, Master's thesis, University of Bergen (2005)

[38] MacWilliams, J.: *A theorem on the distribution of weights in a systematic code*, Bell Sustem Tech. J. **42**, 79-94 (1963)

[39] Miller, E., Sturmfels, B.: *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics **227**, Springer (2005).

[40] Morey, S., Villarreal, R.: *Edge Ideals: Algebraic and Combinatorial Properties*, In: Progress in Commutative Algebra 1, pp. 85-126, De Gruyter (2012)

[41] Motzkin, T.: *Comontone curves and polyhedra, Abstract 11*, Bull. Amer. Math. Soc. **63**, 35 (1957)

[42] Oxley, J.: *Matroid Theory*, 2nd Edition, Oxford University Press Inc., New York (1992)

[43] Poincaré, H.: *Analysis situs*, J. de l'Ecole Polytechnique, 1-121 (1895)

[44] Quillen, D.: *Projective modules over polynomial rings*, Inventiones Mathematicae **36**, 167-171 (1976)

[45] Reisner, G.: *Cohen-Macaulay quotients of polynomial rings*, Adv. in Math. **21**, 30-49 (1976)

[46] Rota, G.: *On the foundations of combinatorial theory I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 340-368 (1964)

[47] Simis, A., Vasconcelos, W, Villareal, R.: *On the ideal theory of graphs*, J. Algebra **167** (2), 389-416 (1994)

[48] Stanley, R.: *Combinatorics and Commutative Algebra (2nd edition)*, Progress in Mathematics **41**, Birkhäuser (1996).

[49] Stanley, R.: *The upper bound conjecture and Cohen-Macaulay rings*, Stud. Appl. Math., **54**, 135-142 (1975).

[50] Terai, N., Hibi, T.: *Some results on Betti numbers of Stanley-Reisner rings*, Discrete Mathematics, **157**, 311-320 (1996).

[51] Tutte, W.: *Graph-polynomials*, Adv. in Appl. Math., **32**, 5-9 (2004).

[52] Welsh, D.: *Matroid theory*, Academic Press Inc., London (1976)

[53] Wei, V.: *Generalized Hamming Weights for Linear Codes*, IEEE Transactions on Information Theory **37**, 1412-1418 (1991)

[54] Whitney, H.: *On the abstract properties of linear dependence*, American Journal of Mathematics **57**, 509-533 (1935)

[55] Whitney, H.: *On the coloring of graphs*, Annals of Mathematics **33**, 688-718 (1932)

[56] Wilson, R.: *An introduction to matroid theory*, The American Mathematical Monthly **80**, 50-525 (1973)

[57] Zheng, X.: *Resolutions of facet ideals*, Communications in Algebra **32**, No. 6, 2301-2324 (2004)