Part VII

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D. Lukkassen, R. Høibakk and A. Meidell, Appl. Math. Lett. **25** (2012), 937-940.

Applied Mathematics Letters 25 (2012) 937–940

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Nonlinear laminates where the effective conductivity is integer valued

Dag Lukkassen [a,b,*], Ralph Høibakk [a], Annette Meidell [a,b]

[a] *Narvik University College, P.O.B. 385 N-8505 Narvik, Norway*

[b] *Norut Narvik, P.O. Box 250, N-8504 Narvik, Norway*

ARTICLE INFO

Article history:
Received 21 May 2010
Received in revised form 22 October 2011
Accepted 31 October 2011

Keywords:
Nonlinear laminates
Homogenization theory
Integer values

ABSTRACT

We consider laminates with a power-law relation between the temperature gradient and the heat flux characterized by some constant $\tau > 1$. In particular, we discuss the problem of determining what positive integer combinations of the local conductivities and the power $-r = 1/(\tau - 1)$ which make the effective conductivity integer valued. The problem is settled for the case when the number of layers, k, equals 2. For $k > 2$ the problem is settled for the case $r = -1$, but for lower values, we can only identify certain classes of solutions.

1. Introduction

We consider nonlinear heat conduction through a periodic laminate which consists of k layers of equal thickness in each period. The temperature u is only dependent of the direction x orthogonal to the layers and the local heat flux $Q(x)$ is assumed to be given by the following power law:

$$Q(x) = \lambda(x)\left|\frac{du}{dx}\right|^{\tau-2}\frac{du}{dx},$$

for some constant $\tau > 1$, where $\lambda(x)$ is called the local conductivity. The conductivity of the layers in one period of the laminate is denoted $\lambda_1, \lambda_2, \ldots, \lambda_k$ (see Fig. 1). Assuming no heat-source inside the wall, the conservation of energy gives that $dQ/dx = 0$. Hence, $Q(x) = K$ for some constant K, and we obtain that

$$K^{\frac{1}{1-\tau}}\lambda^{\frac{1}{1-\tau}}(x) = \frac{du}{dx}.$$

By integration we then obtain "average" relation

$$\langle Q(x)\rangle = q^*\left|\left\langle\frac{du}{dx}\right\rangle\right|^{\tau-2}\left\langle\frac{du}{dx}\right\rangle,$$

where $\langle\cdot\rangle$ denotes the average taken over one period, and q^*, called the *effective conductivity*, is given by

$$q^* = \langle\lambda^{\frac{1}{1-\tau}}(x)\rangle^{1-\tau} = \left(\frac{\lambda_1^{\frac{1}{1-\tau}} + \cdots + \lambda_k^{\frac{1}{1-\tau}}}{k}\right)^{1-\tau}.$$

The average process mentioned above represents the simplest possible case of homogenization of monotone operators. For more detailed information we refer to the literature, see e.g. the papers [1–3]. Roughly speaking, the main idea of

* Corresponding author at: Narvik University College, P.O.B. 385 N-8505 Narvik, Norway.
E-mail address: dl@hin.no (D. Lukkassen).

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doi:10.1016/j.aml.2011.10.039

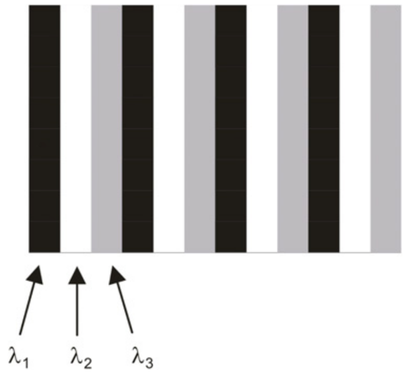

Fig. 1. Laminate consisting of $k = 3$ layers in each period.

the homogenization method is to estimate the parameters of a homogeneous material or media whose effective behavior corresponds to the composite structure we want to study. For an elementary introduction to the subject, see e.g. the book by Persson et al. [4] (see also [5]).

For all practical purposes the conductivities of the layers, and hence, also the effective conductivity, are always positive values. In this paper we assume that the quantity $-r = 1/(\tau - 1)$ and the conductivities $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all positive integers and discuss the problem of determining what combinations of these quantities which make the effective conductivity q^* integer valued.

2. A useful lemma

We recall that the r-th power mean $P_r = P_r(a_1, a_2, \ldots, a_k)$ (with equal weights) of the positive numbers $a_1, a_2, \ldots, a_k$ is defined by

$$P_r = \begin{cases} \left(\dfrac{a_1^r + a_2^r + \cdots + a_k^r}{k}\right)^{1/r} & \text{if } r \neq 0, \\ \sqrt[k]{a_1 a_2 \cdots a_k} & \text{if } r = 0. \end{cases}$$

Lemma 1. *Let $a_1, a_2, \ldots, a_k$ and z be positive integers such that the greatest common divisor $\gcd(a_1, a_2, \ldots, a_k, z) = 1$. Then $z = P_r(a_1, a_2, \ldots, a_k)$ if and only if there exist positive integers $a_1', a_2', \ldots, a_k'$ and z' such that $\gcd(a_1', a_2', \ldots, a_k', z') = 1$, $a_i = d/a_i'$, $z = d/z'$ and $z' = P_{-r}(a_1', a_2', \ldots, a_k')$, where d is the least common multiple $d = \mathrm{lcm}(a_1', a_2', \ldots, a_k', z')$.*

Proof. If $z = P_r(a_1, a_2, \ldots, a_k)$ we have that $a_1^r + a_2^r + \cdots + a_k^r = kz^r$. Dividing with d^r, we obtain that $(a_1')^{-r} + (a_2')^{-r} + \cdots + (a_k')^{-r} = k(z')^{-r}$. Hence,

$$z' = \left(\frac{(a_1')^{-r} + (a_2')^{-r} + \cdots + (a_k')^{-r}}{k}\right)^{-1/r},$$

i.e. $z' = P_{-r}(a_1', a_2', \ldots, a_k')$. If $a_1', a_2', \ldots, a_k'$ and z' are integers it is clear that $a_i = d/a_i'$ and $z = d/z'$ are integer valued. If, in addition, $\gcd(a_1', a_2', \ldots, a_k', z') = 1$, we obtain that $\gcd(a_1, a_2, \ldots, a_k, z) = 1$ (which is a consequence of the fact that $d = \mathrm{lcm}(a_1', a_2', \ldots, a_k', z')$ and $\gcd(a_1', a_2', \ldots, a_k', z') = 1$). Conversely, if $a_1, a_2, \ldots, a_k$ and z are integers and $d = \mathrm{lcm}(a_1, a_2, \ldots, a_k, z)$, it is clear that $a_i' = d/a_i$ and $z' = d/z$ are integer valued. If, in addition, $\gcd(a_1, a_2, \ldots, a_k, z) = 1$ we find by similar arguments as above that $\gcd(a_1', a_2', \ldots, a_k', z') = 1$ and $d = \mathrm{lcm}(a_1', a_2', \ldots, a_k', z')$. □

3. The case $k = 2$

For the case $k = 2$ and $r = -1$ we obtain the following result.

Corollary 2. *The integers a_1 and a_2 making $z = P_{-1}(a_1, a_2)$ integer valued (and, hence, q^* integer valued if $\tau = 2$) are precisely those on the form*

$$a_1 = tp(p + q), \qquad a_2 = tq(p + q) \tag{1}$$

(in this case $P_{-1} = 2tqp$) and the form

$$a_1 = t(2p + 1)(p + q + 1), \qquad a_2 = t(2q + 1)(p + q + 1) \tag{2}$$

(in this case $P_{-1} = t(2q + 1)(2p + 1)$), where p, q and t are integers.

A proof of this result can be found in [6]. Here, we give an independent proof based on Lemma 1.

Proof. Let us assume that

$$\gcd(a_1', a_2', z') = 1. \tag{3}$$

It is easily seen that the only integers a_1' and a_2' which make $z' = P_1(a_1', a_2')$ integer valued are precisely those of the form

$$(a_1', a_2') = (2q + 1, 2p + 1) \tag{4}$$

or

$$(a_1', a_2') = 2(q, p), \tag{5}$$

for some integers p and q. Hence, $z' = p + q + 1$ or $z' = p + q$, respectively. In the case (4), it follows by (3) and the fact that odd numbers are not divisible by 2 that

$$\gcd(2p + 1, 2q + 1) = \gcd(2p + 1, p + q + 1) = \gcd(2q + 1, p + q + 1) = 1.$$

Hence,

$$d = \mathrm{lcm}(a_1', a_2', z') = (2p + 1)(2q + 1)(p + q + 1).$$

In the case (5), it follows by (3) that

$$\gcd(p, q) = \gcd(p, p + q) = \gcd(q, p + q) = 1,$$

which gives that

$$d = \mathrm{lcm}(a_1', a_2', z') = 2pq(p + q).$$

Hence, (1) and (2) follow by Lemma 1 for the case $t = 1$. The general case $\gcd(a_1, a_2, z) = t$ follows by multiplying this parametrization with t.

In 1952 Denés [7] conjectured that there exist no positive integers x, y and z, where $x \neq y$ such that $x^r + y^r = 2z^r$ for any integer $r \geq 3$. He even proved this result for $2 < r < 31$. Motivated by the famous breakthrough which led to the proof of Fermat's Last Theorem, Ribet [8] proved this result when r is divisible by a prime which is congruent to 1 mod 4. Darmon and Merel [9] were able to give the final proof of Denés Conjecture in 1997 by retracing the steps in Ribet's argument. □

Remark 3. Thanks to this result we know that there exist no integers $a_1 \neq a_2$ such that $P_r(a_1, a_2)$ becomes integer valued for any integer $r \geq 3$. According to Lemma 1 we therefore obtain that q^* can never be integer valued when $k = 2$ if $1/(1-\tau)$ is an integer less than -2.

The only integers a_1 and a_2 making $P_2(a_1, a_2)$ integer valued are precisely those on the form

$$a_1 = t|p^2 - 2pq - q^2|, \qquad a_2 = t|p^2 + 2pq - q^2| \tag{6}$$

(giving the value $P_2(a_1, a_2) = t(p^2 + q^2)$), where p, q and t are integers (see e.g. [10, pp. 437–438]).

Remark 4. Using this result it is possible to show (see [6]) that the integers a_1 and a_2 making $P_{-2}(a_1, a_2)$ (hence also q^* if $\tau = 3/2$) integer valued are precisely those on the form

$$a_1 = t(p^2 + q^2)|p^2 - 2pq - q^2|, \qquad a_2 = t(p^2 + q^2)|p^2 + 2pq - q^2| \tag{7}$$

(giving the value $P_{-2}(a_1, a_2) = t(p^2 + q^2)|p^2 - 2pq - q^2|\,|p^2 + 2pq - q^2|$).

4. The case $k \geq 3$

Due to the fact that there exist no integers $a_1 \neq a_2$ such that $P_r(a_1, a_2)$ becomes integer valued for any integer $r \geq 3$, one might expect that this result could be generalized, namely that $P_r(a_1, a_2, \ldots, a_k)$ cannot be an integer for any integer $r \geq 3$ and any combination of integers $a_1, a_2, \ldots, a_k$ except for the trivial case $a_1 = a_2 = \cdots = a_k$. However, this is not true. For example, $P_3(10, 17, 108) = 75$. Utilizing this fact we easily obtain the following more general result.

Proposition 5. *For $k \geq 3$ there exist nontrivial combinations of integers $a_1, a_2, \ldots, a_k$ such that $P_3(a_1, a_2, \ldots, a_k)$ is integer valued.*

Proof. By inspection we see that

$$(10)^3 + (17)^3 + (108)^3 = 3 \cdot (75)^3,$$

or equivalently,

$$(10)^3 + (17)^3 + (108)^3 + (k - 3)(75)^3 = k(75)^3.$$

Hence,

$$\left(\frac{(10)^3+(17)^3+(108)^3+(k-3)(75)^3}{k}\right)^{1/3}=75.$$

Setting $a_1 = 10, a_2 = 17, a_3 = 108$ and $a_i = 75$ for $4 \leq i \leq k$, we then clearly obtain that $P_3(a_1, a_2, \ldots, a_k) = 75$. □

Remark 6. In addition to above result we have found we have found several other nontrivial examples. Here are two examples: $P_3(1, 2, 7, 8) = 6$ and $P_3(1, 1, 1, 2, 2, 2, 4, 5) = 3$ (in addition to all multiples of such combinations). It turns out that the smallest integer value that $P_3(a_1, a_2, \ldots, a_k)$ can attain is 2 (except the trivial case $P_3(1, 1, \ldots, 1) = 1$). One such example is $P_3(1, 1, 1, 1, 1, 1, 1, 1, 4) = 2$. More generally, we have that

$$P_r(\underbrace{1,\ldots 1}_{2^r\text{-times}}, 4, 2, \ldots, 2) = 2. \tag{8}$$

Remark 7. Using the fact that $P_3(1, 2, 7, 8, 6, 6, \ldots, 6) = 6$ we obtain from Lemma 1 that $P_{-3}(168, 84, 24, 21, 28, 28, \ldots, 28) = 28$. Similarly, by (8) and Lemma 1 we obtain that

$$P_{-r}(\underbrace{4,\ldots,4}_{2^r\text{-times}}, 1, 2, \ldots, 2) = 2.$$

This shows that q^* can be integer valued if $k \geq 3$ even if $1/(1-\tau) \leq -3$, in contrast to the case $k = 2$ (see Remark 3).

Remark 8. The fact that $P_3(10, 17, 108) = 75$ was kindly pointed out by an anonymous referee and is added to the revised version of this paper. By utilizing results of [11] the referee also pointed out that 75 is the smallest possible integer value of $P_3(a_1, a_2, a_3)$ when a_1, a_2 and a_3 are (nonequal) integers.

Using the same arguments as we did in obtaining Corollary 2 we easily obtain the following result.

Proposition 9. *The integers $a_1, a_2, \ldots, a_k$ making $z = P_{-1}(a_1, a_2, \ldots, a_k)$ integer valued (and, hence, q^* integer valued if $\tau = 2$) are precisely those of the form*

$$a_i = \frac{\text{lcm}(q_1, \ldots, q_k, q_1 + \cdots + q_k)}{2q_i},$$

where $q_1, \ldots, q_k$ are integers which sum is even (in this case $P_{-1}(a_1, a_2, \ldots, a_k) = \text{lcm}(q_1, \ldots, q_k, q_1 + \cdots + q_k)/(q_1 + \cdots + q_k)$).

It is easy to check that if $a'_1, a'_2, \ldots, a'_k$ are given by

$$a'_i = (q_1^2 + \cdots + q_k^2) - 2q_i(q_1 + \cdots + q_k), \tag{9}$$

where $q_1, q_2, \ldots, q_k$ are arbitrary integers, then $P_2(a'_1, a'_2, \ldots, a'_k) = q_1^2 + \cdots + q_k^2$, i.e. P_2 becomes integer valued.

Remark 10. From the class of integers (9) we can by Lemma 1 obtain the corresponding class of integers $a_1, a_2, \ldots, a_k$ making $P_{-2}(a_1, a_2, \ldots, a_k)$ integer valued (and, hence, q^* integer valued if $\tau = 3/2$).

Acknowledgments

We are grateful to two anonymous reviewers for helping us to improve the presentation of this paper, especially for pointing out the valuable information stated in Remark 8.

References

[1] J.-L. Lions, D. Lukkassen, L.-E. Persson, P. Wall, Reiterated homogenization of monotone operators, C. R. Acad. Sci., Paris, Ser. I, Math. 330 (8) (2000) 675–680.
[2] J.-L. Lions, D. Lukkassen, L.-E. Persson, P. Wall, Reiterated homogenization of nonlinear monotone operators, Chin. Ann. Math., Ser. B 22 (1) (2001) 1–14.
[3] D. Lukkassen, A. Meidell, P. Wall, Multiscale homogenization of monotone operators, Discrete Contin. Dyn. Syst. - Ser. A. 22 (3) (2008) 711–727.
[4] L.E. Persson, L. Persson, N. Svanstedt, J. Wyller, The Homogenization Method: An Introduction, Studentlitteratur, Lund, 1993.
[5] A. Meidell, R. Høybakk, D. Lukkassen, G. Beeri, Two-component composites whose effective conductivities are power means of the local conductivities, European J. Appl. Math. 19 (2008) 507–517.
[6] R. Høibakk, D. Lukkassen, Power means with integer values, Elem. Math. 64 (2009) 122–128.
[7] P. Denés, Über de Diophantische Gleichung $x^l + y^l = cz^l$, Acta Math. 88 (1952) 241–251.
[8] K. Ribet, On the equation $a^p + 2^\alpha b^p + c^p = 0$, Acta Arith. 79 (1) (1997) 7–16.
[9] H. Darmon, L. Merel, Winding quotients and some variants of Fermat's last theorem, Journal für die reine und angewandte Mathematik 490 (1997) 81–100.
[10] L.E. Dickson, History of the Theory of Numbers, in: Diophantine Analysis, vol. 2, Dover, New York, 2005.
[11] D.R. Heath-Brown, The density of rational points on nonsingular hyper-surfaces, Proc. Indian Acad. Sci. Math. Sci. 104 (1) (1994) 13–29. K.G. Ramanathan memorial issue.