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Solutions to some problems related to Diophantine equation, power means and homogenization theory

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Abstract

This Ph.D. thesis consists of an introduction and 7 papers where we investigate the requirements for finding integer or rational solutions to a selection of Diophantine equations leading to problems connected to power means and homogenization.

In Paper 1 we present a modern view of classic number theory in a historic context. We have also included some new interpretations of importance for this thesis. Some of the basic discoveries and tools developed by Euclid and Diophantus in the classic period 300 BC - 300 AD are discussed. We also focus on some aspects about how Diophantus found a remarkable way of solving a third degree equation in rational numbers, which, as far as we know, have not been offered much attention in the literature. For the period 1650 - 1850 we investigate some important works by Fermat, Euler and Gauss in a new light. Special attention has been given to Euler's work on rational or integer solutions to fourth degree equations. Especially, we suggest some generalizations which are of importance for the new research result of this thesis. Concerning the discoveries over the last 150 years, we have focused on the theory of elliptic curves. In particular, by using this theory, we are able to solve a set of four simultaneous second degree equations. This represents new results corresponding to a generalized Crossed Ladders Problem. Finally, we present a summary of the discoveries that led up to Wiles' proof of Fermat's Last Theorem.

In Paper 2 we introduce the Crossed Ladders Problem and present a proof of an infinite and complete parametric representation of integer valued solutions to a set of corresponding Diophantine equations. Moreover, we point out a connection between certain classes of the solutions and the Pell numbers series.

In Paper 3 we investigate a particular form of the Crossed Ladders Problem, finding many parametrized solutions, some polynomial, and some involving Fibonacci and Lucas sequences. We establish a connection between this particular form and a quartic equation studied by Euler, giving corresponding solutions to the latter.

In Paper 4 we study the connection between the crossed ladders problem and certain power means. We prove that we geometrically can construct a number of power means of two variables of different lengths using the crossed ladders geometric structure.

In Paper 5 we consider the problem of determining integers a and b such that the corresponding power mean of order k becomes integer valued. By using a variant of Fermat's Last Theorem we show that the problem has no solutions for the case $|k| \geq 3$. All solutions for the cases $k = 0$, ± 1, ± 2, and combinations of these, are found.

In Paper 6 we study a scale of two-component composite structures of equal proportions with infinitely many microlevels. The structures are obtained recursively and we find that their effective conductivities are power means of the local conductivities.

In Paper 7 we consider laminates with a power-law relation between the temperature gradient and the heat flux characterized by some constant $\tau > 1$. In particular, we discuss the problem of determining what positive integer combinations of the local conductivities and the power $-r = 1/(\tau - 1)$ which make the effective conductivity integer valued.

The papers in this thesis

This Ph.D. thesis consists of seven papers (Papers 1, 2, 3, 4, 5, 6 and 7) and an Introduction, which puts these papers into a more general frame.

1. R. Høibakk, A modern view of classic number theory, Research report (2016), to be submitted.

2. R. Høibakk, T. Jorstad, D. Lukkassen and L.-P. Lystad, Integer Crossed Ladders; parametric representation and minimal integer values, Normat **56** (2008), no. 2, 68-79.

3. A. Bremner, R. Høibakk and D. Lukkassen, Crossed ladders and Euler's quartic, Ann. Math. Inform. **36** (2009), 29-41.

4. R. Høibakk and D. Lukkassen, Crossed ladders and power means, Elem. Math. **63** (2008), 137-140.

5. R. Høibakk and D. Lukkassen, Power means with integer values, Elem. Math. **64** (2009), 122-128.

6. A. Meidell, R. Høibakk, D. Lukkassen and G. Beeri, Two-component composites whose effective conductivities are power means of the local conductivities, European J. Appl. Math. **19** (2008), 507-517.

7. D. Lukkassen, R. Høibakk and A. Meidell, Nonlinear laminates where the effective conductivity is integer valued, Appl. Math. Lett. **25** (2012), 937-940.

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Introduction - Short description of the publications in this Ph.D.-thesis

1 A modern view of classic number theory, Paper 1

In Paper 1 we present a modern view of classic number theory in a historic context. We have also included some new interpretations of importance for this thesis.

Some of the basic discoveries and tools developed by Euclid and Diophantus in the classic period 300 BC - 300 AD are discussed. We also focus on some aspects about how Diophantus found a remarkable way of solving a third degree equation in rational numbers, which, as far as we know, have not been offered much attention in the literature. There is an important line of thought between this particular discovery and recent analyses of cubic equations called elliptic curves. Therefore, in this introduction we include a condensed version of Diophantus solution of

$$Q^2 = K^3 + 2,$$

in addition to some words about our development of a modern extension of Diophantus' method. Diophantus introduced a parameter m and defined

$$K = m - 1,$$

$$Q = \frac{3}{2}m + 1.$$

Inserting these values for K and Q in the cubic equation, he obtained

$$(\frac{3}{2}m + 1)^2 = (m - 1)^3 + 2$$

giving

$$m_1 = \frac{21}{4}, \text{and } (K_1, Q_1) = (\frac{17}{4}, \frac{71}{8}).$$

A modern extension of Diophantus method can be used to determine an infinite number of rational solutions to the above cubic equation. This is done by defining a recursive relation between an assumed rational solution K_1, Q_1 and a possible next order solution K_2, Q_2. We set

$$K_2 = K_1 + m_2,$$

$$Q_2 = Q_1 + \frac{3}{2}\frac{K_1^2}{Q_1}m_2.$$

By inserting these values in the cubic equation we find

$$m_2 = \frac{1}{4Q_1^2}\left(9K_1^4 - 12K_1Q_1^2\right).$$

This lead to the next order solutions to the cubic equation

$$K_2 = \frac{1}{4}K_1\frac{9K_1^3 - 8Q_1^2}{Q_1^2},$$

$$Q_2 = \frac{1}{8}\frac{27K_1^6 + 8Q_1^4 - 36K_1^3Q_1^2}{Q_1^3}.$$

Inserting the values determined by Diophantus $(K_1, Q_1) = (\frac{17}{4}, \frac{71}{8})$ into these formulas, we get the following new set of rational solutions to the cubic equations:

$$(K_2, Q_2) = (\frac{66\,113}{80\,656}, \frac{36\,583\,777}{22\,906\,304}).$$

By the above formulas we can determine infinitely many rational solutions. We also show that if we insert $(K_0, Q_0) = (-1, 1)$ (seen by inspection) into the formulas for K_2 and Q_2, we regenerate Diophantus' original solution $(K_1, Q_1) = (\frac{17}{4}, \frac{71}{8})$.

The use of parameters in determining a rational solution to a polynomial equation, and of a recursive procedure to find more, possibly infinitely many, is a recurring theme in Paper 1. We use these methods extensively both in the description of the works of Euler and in some new developments presented in the last part of Paper 1.

For the period 1650 - 1850 we investigate some important works by Fermat, Euler and Gauss in a new light. Special attention has been given to Euler's work on rational or integer solutions to fourth degree equations. Especially, we suggest some generalizations by applying the above described use of parameters and of a recursive search for more solutions. These results are of importance for some of the discoveries presented in this thesis.

Concerning the discoveries over the last 150 years, we have focused on the theory of elliptic curves. In particular, by using this theory, we are able to solve a set of four simultaneous second degree equations. This represents new results corresponding to a generalized Crossed Ladders Problem. Finally, we present a summary of the discoveries that led up to Wiles' proof of Fermat's Last Theorem.

Since the solution of the generalized Crossed Ladders Problem represent some new discovery and applies recently developed methods, we have included a short presentation here. We first develop a set of parametric identities that give an infinite number of integer solutions to the variables x, y, z, u, v, r in the following set of simultaneous equations

$$x^2(1, 1, 1) - 3(y^2, z^2, (y + z)^2) = (u^2, v^2, r^2),$$

namely

$$(2(m^2 - mp + p^2))^2 - 3(m(m - 2p))^2 = \left(2mp + m^2 - 2p^2\right)^2,$$

$$(2(m^2 - mp + p^2))^2 - 3(p(2m - p))^2 = \left(-2mp + 2m^2 - p^2\right)^2,$$

$$(2(m^2 - mp + p^2))^2 - 3(m^2 - p^2)^2 = \left(-4mp + m^2 + p^2\right)^2,$$

where m, p are integers.

This set of identities was used by J. Leech [4] to solve the problem of finding integer solutions to the variables x, y, z, u, v, r, s in a more complex set of equations

$$x^2(1, 1, 1, 1) - 3(y^2, z^2, (y + z)^2, (y - z)^2) = (u^2, v^2, r^2, s^2). \quad (1)$$

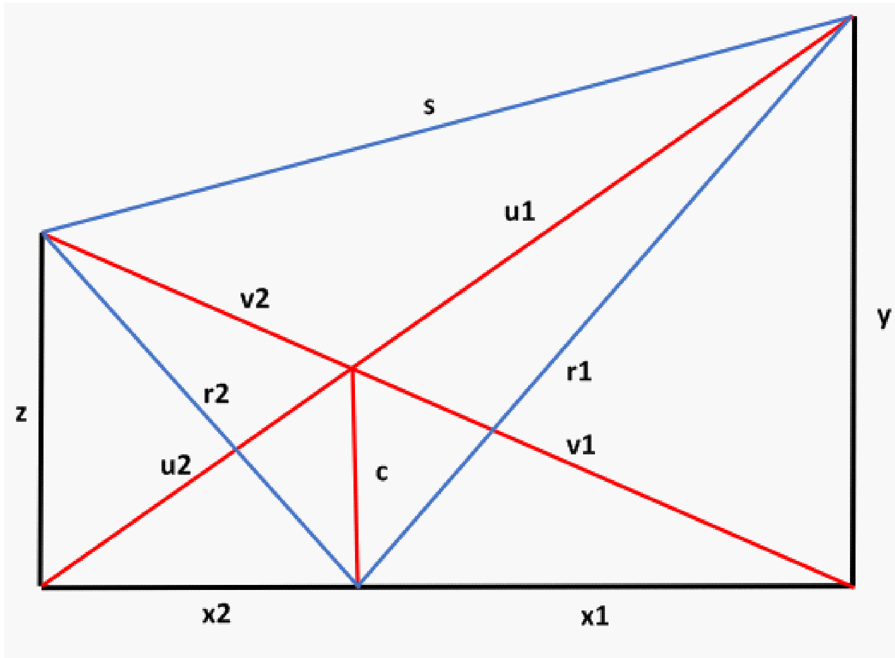

Figure 1: The generalized Crossed Ladders Problem.

A solution to this set of equation requires integer solution to m, p, s in the quartic equation

$$s^2 = m^4 + 16m^3p - 42m^2p^2 + 16mp^3 + p^4. \tag{2}$$

By transformation of this equation to the elliptic curve

$$j^2 = (k + 108)(k + 96)(k + 72),$$

we can determine infinitely many rational solutions to this elliptic and transform these solutions back to give integer solutions for $m, p, s.$ in (2) Thereby we obtain infinitely many integer solutions to the variables in (1).

We further in Paper 1 describe a generalized version of the Crossed Ladders Problem where the challenge consists of finding integer solutions to all lines and line segments in Figure 1

The requirements for a solution to this problem lead to finding integer solutions to the following set of equations

$$x^2(1, 1, 1, 1) + (y^2, z^2, (y + z)^2, (y - z)^2) = (u^2, v^2, r^2, s^2) \tag{3}$$

A method discovered by J. Leech in 1981 [4] that give infinitely many integer solutions to the variables in (3) is described, whereby we determine solutions also to the generalized version of the Crossed Ladders Problem. To our knowledge this discovery has not been published before.

2 Crossed Ladders Problem, Papers 2, 3 and 4.

The Crossed Ladders Problem (CLP) requires integer solutions to the set of simultaneous DEs illustrated in Figure 2:

$$x^2(1, 1) + (y^2, z^2) = (a^2, b^2)$$

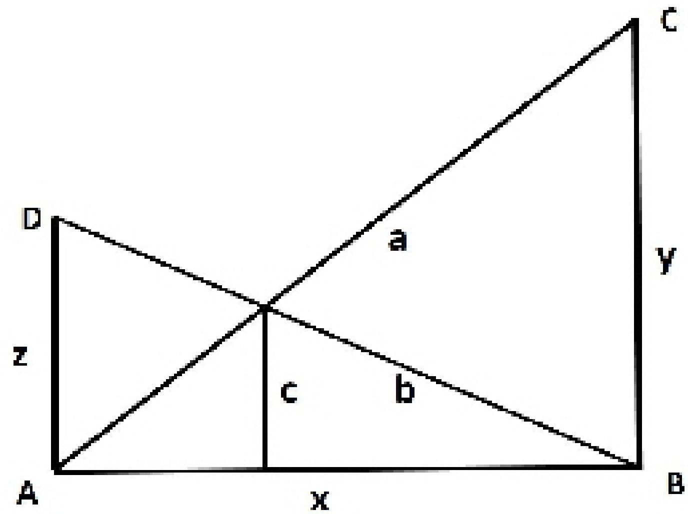

Figure 2: The Crossed Ladders Problem.

and

$$c = \frac{yz}{y+z}.$$

In Paper 2 we prove that the following formulas give a complete integer parametric representation to the variables x,y,z,a,b,c:

$$x = S2m_1n_1m_2n_2,$$

$$y = Sm_2n_2(m_1^2 - n_1^2),$$
$$z = Sm_1n_1(m_2^2 - n_2^2),$$
$$a = Sm_2n_2(m_1^2 + n_1^2),$$
$$b = Sm_1n_1(m_2^2 + n_2^2)$$

and

$$c = \frac{m_1n_1m_2n_2(m_1^2 - n_1^2)(m_2^2 - n_2^2)}{\gcd F},$$

where

$$F = (m_1n_1m_2n_2(m_1^2 - n_1^2)(m_2^2 - n_2^2),\ m_2n_2(m_1^2 - n_1^2) + m_1n_1(m_2^2 - n_2^2))$$

and

$$S = \frac{m_2n_2(m_1^2 - n_1^2) + m_1n_1(m_2^2 - n_2^2)}{\gcd F}.$$

Here, m_1,n_1,m_2 and n_2 are positive integers $m_1 > n_1$ and $m_2 > n_2$.

We further demonstrate the following infinite integer representation of the CLP using the Pell numbers series $\{P_i\}$ for $i \geq 1$:

$$(x, y, z, a, b, c) = (4P_{2i}P_{2i+1}, 4P_{2i}^2P_{2i+1}^2 - 1, 2(P_{2i+1}^2 - P_{2i}^2), 4P_{2i}^2P_{2i+1}^2 + 1, 2(P_{2i+1}^2 + P_{2i}^2), 2(2P_{2i}P_{2i+1} - 1)).$$

Recall that Pell number series is defined by

$$P_i = 2p_{i-1} + P_{i-2}, \text{ with } P_0 = 0 \text{ and } P_1 = 1, \; i = 0, 1, 2, ...$$

From Figure 2 it can be seen that the length of the line c is half of the harmonic mean of y and z. Moreover, the length of the line DC (not drawn in the figure) is twice the square mean of y and z in the case when $x = y + z$. This led us to investigate if it is possible geometrically to construct other power means of two variables. This is the object of Paper 4, where we demonstrate that the power means

$$P_{-2}^2, P_{-1}^2, P_{-\frac{1}{2}}^2, P_0^2, P_{\frac{1}{2}}^2, P_1^2 \text{ and } P_2^2$$

all can be constructed geometrically in the same compact figure. For alternative considerations of some of the cases, see e.g. [7] and [3].

Another question concerns the relation between integer valued y and z given in Figure 2. In Paper 3 we consider the cases when the value of M in

$$M = \frac{y}{z} = \frac{m_2n_2(m_1^2 - n_1^2)}{m_1n_1(m_2^2 - n_2^2)} \tag{4}$$

is required to be an integer. We show that this problem is closely connected to finding integer solutions to the so called Euler‘s quartic

$$r^4 + dr^2s^2 + s^4 = w^2. \tag{5}$$

We find that the representation (4) can be transformed to the problem of finding integer valued M that give non-trivial integer solutions to the equation

$$X^4 + (4M^2 - 2)X^2Y^2 + Y^4 = Z^2. \tag{6}$$

The literature on Euler‘s quartic is substantial. In particular, A. Bremner and J. W. Jones [1] have determined all values for $d < 3000$ that give non-trivial solutions to (5). From (6) and (5) we see that we are searching for solutions to d such that $d = 4M^2 - 2$ also gives that M is integer valued. The smallest such value is $d = 194$, giving $M = 7$. This result is obtained for two sets of solutions in the Crossed Ladders Problem, namely

$$(x, y, z, a, b, c) = (96, 280, 40, 296, 104, 35) \text{ and } (70, 168, 24, 182, 74, 21).$$

Considering (4) we obtain several sets of parametric representations for m_1, n_1, m_2, n_2 that give integer valued M. In addition, we demonstrate a connection between Euler‘s quartic, integer M, and recurring series like Fibonacci, Lucas and Pell number series, leading to infinite new integer solutions for (4). Of some interest is the following representation:

$$(x, y, z, a, b) = (2F_{k+1}F_k, \; F_{k+1}^2F_k^2 - 1, \; F_{k+1}^2 - F_k^2, \; F_{k+1}^2F_k^2 + 1, \; F_{k+1}^2 + F_k^2),$$

where F_k is the k-th Fibonacci number given by $F_k = F_{k-1} + F_{k-2}$, with $F_0 = 0$ and $F_1 = 1$, $i = 0, 1, 2...$. Hence,

$$M = \frac{y}{z} = \frac{F_{k+1}^2 F_k^2 - 1}{F_{k+1}^2 - F_k^2} = F_{k+1}F_k - (-1)^k.$$

For $k \geq 3$ this gives an infinite number of integer values for M :

$$M = 7, 14, 41, 103, 274, ... \tag{7}$$

The equation (5) will then have an infinite number of non-trivial solutions that gives integer valued r, s, w, d :

$$(r, s, w, d) = (F_{k+2},\ F_{k-1},\ 2(F_{k+1}^2 F_k^2 + 1),\ 4(F_{k+1}F_k - (-1)^k))^2 - 2.$$

In particular, for $k = 3$ we find the following solution to (5)

$$(r, s, w, d) = (5, 1, 74, 194),$$

since

$$5^4 + 194 \times 5^2 \times 1^2 + 1^4 = 74^2.$$

3 Power means, Papers 5, 6 and 7

Averages and means of a number of variables have fascinated mathematicians since antiquity. Mathematically there are many different types of means. Some of the most used ones are the power means (of some positive numbers $a_1, a_2, ...a_n$), defined as follows:

$$P_k^n = \left[\frac{a_1^k + a_2^k + ... + a_n^k}{n}\right]^{\frac{1}{k}},\ \text{if } k \neq 0$$

and

$$P_0^n = [a_1 a_2 .... a_n]^{\frac{1}{n}},\ \text{if } k = 0.$$

There is a substantial literature on the subject of power means. It has been shown that $P_k^n > P_l^n$ if $k > l$ if all the a_i are not identical, that P_k^n converge towards $\max a_i$ $1 \leq i \leq n$ when $k \to \infty$ and towards $\min a_i$ $1 \leq i \leq n$ when $k \to -\infty$, see e.g. [2], [5], [6] and [8]. Moreover it is easy to see that $P_k^n(Ka_1, Ka_2, ..., Ka_n) = KP_k^n(a_1, a_2, ..., a_n)$.

The most commonly used power means are the arithmetic mean ($A = P_1^n$), the geometric mean ($G = P_0^n$), the harmonic mean ($H = P_{-1}^n$) and the quadratic mean, also called the root mean square ($Q = P_2^n$).

Physical problems often lead to interconnected sets of variables where we search for the effective property ("average value") of the variables, for instance the effective heat or current conductivity. In many instances the effective property takes the form of the k-th power mean of the variables. In Papers 5 and 6 we consider cases where the effective conductivity corresponds to identified power means.

In Paper 1 one of the requirements for a solution is that c in

$$c = \frac{yz}{y+z}$$

must be an integer (where y and z also are integers). Recognizing the similarities with the harmonic mean of two variables, we continue in Paper 5 and study sets of two variables that lead to integer valued power means. We there present complete parametric representation to the variables that give integer valued power means for $P^2_{-2}, P^2_{-1}, P^2_0, P^2_1$, and P^2_2, and also to some combinations of these means simultaneously.

For the harmonic mean

$$P^2_{-1}(a,b) = \frac{2ab}{a+b}$$

to be an integer, we found that a and b must be precisely of the form

$$a = tp(p+q),$$

$$b = tq(p+q),$$

or of the form

$$a = t(2p+1)(p+q+1),$$
$$b = t(2q+1)(p+q+1),$$

where t,p,q are positive integers. This leads to the form

$$P^2_{-1}(a,b) = 2tpq$$

or the form

$$P^2_{-1}(a,b) = t(2p+1)(2q+1),$$

respectively.
For the harmonic mean, the arithmetic mean and the geometric mean, that is P^2_{-1}, P^2_0, and P^2_1, to be integer valued simultaneously, a and b must be precisely of the form

$$a = 2tp^2(p^2+q^2),$$

$$b = 2tq^2(p^2+q^2),$$

or of the form

$$a = t(2p+1)^2(2p^2+2p+2q^2+2q+1),$$
$$b = t(2q+1)^2(2p^2+2p+2q^2+2q+1).$$

The integers a and b making $P^2_2(a,b)$ integer valued must be precisely of the form

$$a = t\left|p^2-2pq-q^2\right|,$$

$$b = t\left|p^2+2pq-q^2\right|,$$

giving

$$P_2^2(a,b) = t(p^2+q^2).$$

For $P_{-2}^2(a,b)$ to be integer valued a and b must be precisely of the form

$$a = t(p^2+q^2)\left|p^2-2pq-q^2\right|,$$

$$b = t(p^2+q^2)\left|p^2+2pq-q^2\right|,$$

leading to

$$P_{-2}^2(a,b) = t\left|p^2-2pq-q^2\right|\left|p^2+2pq-q^2\right|.$$

A further development of the methods developed in Paper 4 to study the requirements that lead to integer valued power means when the number of variables $n>2$, is an area for future research.

In Paper 6 we study a two-component composite of laminate, chess-board and combined structures whose effective conductivities are power means of the individual conductivities. We identify two-component structures, where the effective conductivity matrix, σ^*, satisfies

$$\sigma^* = \begin{bmatrix} P_k^2(\lambda_b,\lambda_w) & 0 \\ 0 & P_{-k}^2(\lambda_b,\lambda_w) \end{bmatrix}.$$

Here P_k^2 and P_{-k}^2 are the power means of the individual conductivities, λ_w and λ_b, of the two components. We further describe structures having this effective conductivity matrix for $k=\frac{1}{2^n}$, where n is any integer and k is the power of the k-th power mean. These structures are obtained recursively by using a self-similar structure combined with a laminate structure. For every fixed n we have identified two-component composite structures with local conductivity matrices

$$C_b = \begin{bmatrix} \mu_{b,1} & 0 \\ 0 & \mu_{b,2} \end{bmatrix} \text{ and } C_w = \begin{bmatrix} \mu_{w,1} & 0 \\ 0 & \mu_{w,2} \end{bmatrix},$$

of equal proportions, such that the effective conductivity matrix is

$$\sigma^* = \begin{bmatrix} P_{\frac{1}{2^n}}(\mu_{b,1},\mu_{w,1}) & 0 \\ 0 & P_{-\frac{1}{2^n}}(\mu_{b,2},\mu_{w,2}) \end{bmatrix},$$

for all $\mu_{b,i}$ and $\mu_{w,i}$.
This result is obtained by using the fact that

$$P_k(a,b)P_{-k}(a,b) = ab,$$

and by proving that

$$P_{\frac{1}{2^{n+1}}}(a,b) = P_{\frac{1}{2^n}}(P_{\frac{1}{2^n}}(a,b), P_0(a,b))$$

and that

$$P_{-\frac{1}{2^{n+1}}}(a,b) = P_{-\frac{1}{2^n}}(P_{-\frac{1}{2^n}}(a,b), P_0(a,b)).$$

In Paper 7 we consider laminates with a power-law relation between the temperature gradient and the heat flux. The effective conductivity, λ^*, orthogonal to the layers is given by

$$\lambda^* = P^k_{\frac{1}{1-\tau}}(\lambda_1, ..., \lambda_k) = \left(\frac{\lambda_1^{\frac{1}{1-\tau}} + ... + \lambda_k^{\frac{1}{1-\tau}}}{k} \right)^{1-\tau},$$

where $\tau > 1$ is the heat flux constant, λ_i is the conductivity of layer i, and k is the number of layers. In particular we study the case where $k = 2$ and $\frac{1}{1-\tau} = -1$, *i.e.*, $\tau = 2$. For the effective conductivity in this case to be integer valued, we prove that the individual conductivities must be precisely of the form

$$\lambda_1 = tp(p+q),\ \lambda_2 = tq(p+q)$$

or of the form

$$\lambda_1 = t(2p+1)(p+q+1),\ \lambda_2 = t(2q+1)(p+q+1),$$

where p,q and t are positive integers. We then obtain that the effective conductivities are of the form

$$\lambda^* = 2tpq,$$

or of the form

$$\lambda^* = t(2p+1)(2q+1).$$

We also verify that the individual conductivities for $k = 2$, and $\frac{1}{1-\tau} = -2$, *i.e.*, $\tau = \frac{3}{2}$, that give integer effective conductivity, must be precisely of the form

$$\lambda_1 = t(p^2+q^2)\left|p^2 - 2pq - q^2\right| \text{ and } \lambda_2 = t(p^2+q^2)\left|p^2 + 2pq - q^2\right|.$$

This leads to

$$\lambda^* = t\left|p^2 - 2pq - q^2\right|\left|p^2 + 2pq - q^2\right|.$$

We further prove that for $k = 2$ and $\frac{1}{1-\tau} \leq -3$, *i.e.*, $\tau = \frac{n+1}{n}$ where $n \geq 3$, there are no values of $\lambda_1 \neq \lambda_2$ that will result in an integer effective conductivity, λ^*.

For the number of layers $k \geq 3$ we show that for certain values of $\frac{1}{1-\tau} \leq -3$, and for particular individual conductivities, the effective conductivity λ^* can still be integer valued.

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Part I
A modern view of classic number theory

R. Høibakk, Research report (2016), to be submitted.

Part II

Integer Crossed Ladders; parametric representation and minimal integer values

R. Høibakk, T. Jorstad, D. Lukkassen and L.-P. Lystad, Normat **56** (2008), no. 2, 68-79.

Part III

Crossed ladders and Euler‘s quartic

A. Bremner, R. Høibakk and D. Lukkassen, , Ann. Math. Inform. **36** (2009), 29-41.

Part IV
Crossed ladders and power means

R. Høibakk and D. Lukkassen, Elem. Math. **63** (2008), 137-140.

Part V
Power means with integer values

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Part VI

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Part VII

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