CONFORMAL DIFFERENTIAL INVARIANTS

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ABSTRACT. We compute the Hilbert polynomial and the Poincaré function counting the number of fixed jet-order differential invariants of conformal metric structures modulo local diffeomorphisms, and we describe the field of rational differential invariants separating generic orbits of the diffeomorphism pseudogroup action. This resolves the local recognition problem for conformal structures.

INTRODUCTION

Differential invariants play a central role in the classification problems of geometric structures. Often the fundamental invariants have tensorial character, but for resolution of the equivalence problem scalar invariants are required to be derived from those.

For instance, the fundamental invariant of a Riemannian metric g on a manifold M is the Riemann tensor $R_g \in \Gamma(\Lambda^2 T^*M \otimes \mathfrak{so}(TM))$. Scalar differential invariants are Weyl curvature invariants [15], separating generic orbits of the diffeomorphism pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M)$ acting on the space of jets of metrics $J^\infty(S^2_{\mathrm{ndg}} T^*M)$, where $S^2_{\mathrm{ndg}} T^*M$ is the complement in $S^2 T^*M$ to the cone of degenerate quadrics, and they are obtained by contractions of the tensor products of the covariant derivatives of the curvature tensor R_g. Their number depending on the jet-order was computed by Zorawski [16] and Haskins [5], see also [14].

In this paper we do the same for conformal metric structures $(M, [g])$ of arbitrary signature in dimensions $n = \dim M > 2$. Notice that for $n = 2$ the conformal group is too large and, due to Gauß theorem on existence of isothermal coordinates, there are no local invariants of conformal structures, and hence no differential invariants in 2D.

The fundamental invariants C of the conformal structure are the Cotton tensor for $n = 3$ and the Weyl tensor for $n > 3$. Similarly to Weyl scalar invariants for Riemannian metrics, one could expect scalar invariants to be derived from the fundamental tensor invariants, and this was done in [3, 1, 4], and will be discussed in the next section. These scalar invariants are however defined on the (proper jet-lift of

the) ambient space $\hat{M}$ to our M, dim $\hat{M} = n+2$, so that the constructed scalars are covariants rather than invariants.

There is however an easy approach to construct differential invariants for generic conformal structures. It is based on the folklore result that in the domain $U \subset M$, where $\|C\|_g^2 \neq 0$ for some (and hence any) representative $g \in [g]$, one can uniquely fix (actually up to $\pm$ if the signature is split) a metric g_0 in the conformal class $[g]$ by the normalization $\|C\|_{g_0}^2 = \pm 1$ (the sign is always $+$ in the Riemannian case, but can be any in the indefinite case). Then the conformal invariants are derived from the (pseudo-)Riemannian metric ones (Weyl curvature invariants or those from [7, 10]).

This however does not yield the number[1] of scalar differential invariants $H_n(k)$ depending on the jet-order k (we count so-called "pure order", see below). The classical approach to computing these numbers is the Lie method of elimination of group parameters (or algebra parameters), see [5, 16, 12]. This involves calculation of ranks of large matrices. Instead we rely on some simple algebraic ideas and compute the Hilbert polynomial $H_n(k)$, the first values of which are given below:

$n \setminus k$	1	2	3	4	$\dots$	k
3	0	0	1	9	$\dots$	$k^2 - 4$
4	0	3	36	91	$\dots$	$\frac{1}{6}(k+2)(k+3)(5k-7)$
5	0	24	135	350	$\dots$	$\frac{1}{24}(k+2)(k+3)(k+4)(9k-11)$

Then we derive the Poincaré function encoding these numbers. We also indicate a different set of conformal differential invariants, now rational, and describe the field they generate.

1. The algebras and fields of differential invariants

The scalar conformal invariants mentioned in the introduction are constructed via the ambient metric construction of Fefferman and Graham [3] roughly as follows. Consider the bundle $\bar{M} = M \times \mathbb{R}_+$ over M consisting of all representatives g of $[g]$ with its natural horizontal metric $\bar{g}$ (tautological structure: $\bar{g}_g = g \circ d_g\pi$, where $\pi : \bar{M} \to M$), and let $\hat{M} = \bar{M} \times (-1, 1)$. The ambient metric $\hat{g}$ is $\mathbb{R}_+$-scaling weight 2 homogeneous Ricci flat Lorentzian metric on $\hat{M}$ restricting to $\bar{g}$ on $\bar{M} \times \{0\}$. This exists on the infinite jet of $\bar{M} \times \{0\} \subset \hat{M}$ for odd n,

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[1]We have to fix the signature (p, q) of $[g]$, $p+q = n$. The formulas for invariants vary a bit with this (p, q), but the number of invariants depends only on n.

and up to order $n/2$ for even n. Taking the Weyl metric curvature invariants of $\hat{g}$ yields scalar invariants of $[g]$, which give a complete set of polynomial invariants[2] for odd n and the same to a finite order for even n, see [1, 4]. The definite advantage of these invariants is that they are defined for all conformal structures.

There are however two basic problems with these ambient Weyl conformal invariants, similar to the classical Weyl metric curvature invariants. First of all, the algebra generated by these polynomial invariants is not finitely generated. Secondly, it is not apriori clear which of these differential invariants are separating for the orbits of the diffeomorphism pseudogroup action (on infinite or any finite jet-level).

The second problem is solved by passing to rational differential invariants: since the action is algebraic, its prolongations are algebraic too [9], and in any finite jet-order there exists a rational quotient by the action due to the Rosenlicht theorem [13]. From this viewpoint the field $\mathfrak{F}$ of rational differential invariants is useful and simpler. The invariants obtained in this way will be presented below.

The first problem is a bit more complicated, as it is clear that the transcendence degree $\operatorname{trdeg}(\mathfrak{F}) = \infty$, so just passing to rational invariants does not resolve infinite generation. In the early days of differential invariants theory it was suggested and motivated by Sophus Lie and Arthur Tresse that the algebra of differential invariants is generated by a finite number of differential invariants $I_1, \ldots, I_t$ and a finite number of invariant derivations $\nabla_1, \ldots, \nabla_s$. This was later proved in several versions, see [9] and the references therein.

In more details, consider the algebra $\mathfrak{A}_l$ of differential invariants that are rational by the jets of order $\leq l$ and polynomial by the jets of higher order (l is determined by the structure in question, we will see that in the case of conformal structures $l = 4$ for $n = 3$ and $l = 3$ for $n > 3$). This $\mathfrak{A}_l$ is called the algebra of rational-polynomial invariants.

The main result of [9] states that $\mathfrak{A}_l$ is finitely generated by I_i, ∇_j, i.e. any differential invariant from $\mathfrak{A}_l$ is a polynomial of $\nabla_J I_i$ for ordered multi-indices $J = (j_1, \ldots, j_r)$ with rational coefficients of I_k.

Now the field of rational differential invariants $\mathfrak{F}$ is generated by $\mathfrak{A}_l$ for some l, and so is also finitely generated in the Lie-Tresse sense as above. The algebra $\mathfrak{A}_l$ separates the orbits of the G-action on the space of jets of conformal structures $J^\infty(\mathcal{C}_M)$, where

$$\mathcal{C}_M = S^2_{\mathrm{ndg}} T^* M / \mathbb{R}_+,$$

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[2]When we write "polynomial" here and beyond we mean only with respect to jets of order > 0, allowing division by the determinant of g everywhere.

and we get a finite separating set of invariants $\nabla_J I_i$, $|J| \leq k - \deg(I_i)$ for the restriction of the action on J^k. Thus we obtain a set of generators for the field $\mathfrak{F}_k$ of rational invariants of order k that filter the field $\mathfrak{F}$.

2. Scalar invariants of conformal metric structures

Let us generate conformal differential invariants of generic conformal structures $[g]$. This will in turn generate rational differential invariants on the space of jets of all conformal structures $J^\infty(\mathcal{C}_M)$.

We begin with the case $n \geq 4$. Since there are no metric invariants of order < 2, there are no conformal invariants of lower order too. The lowest order conformal invariants live in 2-jets. It is well-known that the complete invariant there is the conformal Weyl tensor C, considered as a $(3, 1)$ tensor. Indeed, the only invariant of the 2-jet of a Riemannian metric g is the Riemann curvature tensor R_g (due to existence of normal geodesic coordinates), and conformal re-scalings of g leave invariant only the Weyl part of it [2].

The space of conformal Weyl tensors $\mathcal{W}$ (at one point) has dimension $\dim \mathcal{W} = \frac{1}{12}(n-3)n(n+1)(n+2)$, the conformal linear group $CO(g)$ acts effectively on $\mathcal{W}$, and so the codimension of a generic orbits equals

$$H_n(2) = \dim \mathcal{W} - \dim CO(n) = \tfrac{1}{12}(n^4 - 13n^2 - 12).$$

Since $CO(g) = SO(g) \times \mathbb{R}_+$ is a reductive Lie group acting in algebraic manner on $\mathcal{W}$, the Hilbert invariant theorem [6] implies existence of invariants separating generic orbits.

Moreover, $SO(g)$-invariants can be taken to be polynomial, but presence of $\mathbb{R}_+$-factor compels to extend to rational invariants (in this case, however dividing by the determinant of metric is enough). The geometric invariant theory [11] provides a method to construct these, and we can choose among them $H_n(2)$ functionally independent invariants separating orbits on a Zariski open set in $\mathcal{W}$.

Let us indicate first how to construct non-algebraic invariants, that are obtained by a finite algebraic extension. Consider a generic element $C \in \mathcal{W}$, at this point meaning only $\|C\|_g^2 \neq 0$. Fix a metric $g_0 \in [g]$ by the condition $\|C\|_{g_0}^2 = \pm 1$. Then we can convert C to a $(2, 2)$ tensor, interpreted as a linear map $C : \Lambda^2 T \to \Lambda^2 T$, where $T = T_a M$ is the tangent space to M at the considered point a. This map is g-symmetric, traceless and has unit norm. Therefore its spectrum gives $d = \binom{n}{2} - 2$ real scalar invariants $\lambda_1, \ldots \lambda_d$ (alternatively, pass to algebraic invariants $\mathrm{Tr}(C^i)$, $1 < i \leq d + 1$). Notice that the spectrum $\mathrm{Sp}(C)$ is simple for C from a Zariski open set in $\mathcal{W}$.

Let $\sigma_i \in \Lambda^2 T$ be the eigenvectors corresponding to λ_i, normalized by the condition[3] $\|\sigma_i\|_{g_0}^2 = \pm 1$ (the sign again can be arbitrary in the case of indefinite signature of $[g]$; our genericity assumption implicitly implies that $\|\sigma_i\|_{g_0}^2 \neq 0$, so σ_i can be rescaled to get the desired normalization). Then the operators $A_i = g_0^{-1} \sigma_i$ carry a lot of invariants, for instance $\operatorname{Tr}(A^\sigma)$ for $A^\sigma = A_1^{k_1} \cdots A_d^{k_d}$ when $\sigma = (k_1, \ldots, k_d)$, and we can extract $H_n(2)$ independent among them.

Let us also notice that the normalized eigenbases of one these operators A^σ give us a canonical frame $e_1, \ldots, e_n$ (provided that one of them has simple spectrum - this is yet another requirement for C being generic). This frame depends on the 2-jet of a conformal structure.

Remark. *The skew-symmetric operators A_i have purely imaginary spectrum, so one has to consider their products, or work in complexification and take the real parts, to derive non-trivial real invariants. The case $n = 4$ is however an exception ($n = 3$ is an exception too, to be considered later).*

In this case, due to exceptional isomorphism $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, the algebra of operators is the sum of these two algebras, interpreted as the action of unit purely imaginary quaternions $\mathbb{S}^2 \subset \operatorname{Im} \mathbb{H}$ from left and right on $\mathbb{H}$. Denote these operators by J_i^{left} and J_i^{right}, $i = 1, .., 3$. They are in quaternionic relations and the left ones commute with the right ones. The operators $B_i = J_i^{left} \cdot J_i^{right}$ have $\operatorname{Sp}(B_i) = \{\pm 1, \pm 1\}$, and the corresponding eigenspace decomposition is $T = \Pi_i^- \oplus \Pi_i^+$, $\dim \Pi_i^\pm = 2$. The intersections $\Pi_i^\pm \cap \Pi_j^\pm$ yield the splitting of T into direct sum of 4 lines, whence the frame $e_1, \ldots, e_4$ obtained by g_0-normalization (a residual finite symmetry related to numeration of e_i and change of sign remains here, but can be eliminated on further steps).

Writing the canonical representative $g_0 \in [g]$ in this frame we obtain all other differential invariants. This can be formulated in the framework of Lie-Tresse theorem, since $\nabla_j = D_{e_j}$ (horizontal lift to J^∞) form the basis of invariant derivations, and we can choose I_i among the second order invariants already constructed.

Now we achieve algebraicity as follows. The algebraic extension is given by variables $y_1, \ldots, y_p$ that are in algebraic relations with vertical coordinates on the space $J^2(\mathcal{C}_M)$. These enter both I_i and ∇_j, but the derived differential invariants $\nabla_J I_i$ are algebraic by higher order jets. Considering the algebra of invariants generated by these on both second and higher jets, we can eliminate (for instance, via a Gröbner basis)

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[3]Here for σ_i as well as before for g_0 a possible freedom of the sign choice is hidden. This can be locked, but we prefer to ignore it for simplicity of the exposition.

the y-variables and get a system of separating algebraic invariants that generate the fields $\mathfrak{F}_k$ for $k = 2$ and $k > 2$ by the Rosenlicht theorem. We can also eliminate y-parameters in the coefficients of $\nabla_i = \kappa_i^j D_j$ (by taking linear combinations with invariant coefficients) to have rational invariant derivations.

It is not difficult to see (it also follows from considerations in the next section) that the denominators in the rational differential invariants can be chosen supported in the 3rd jets for $n > 3$, and in the 4th jets for $n = 3$. Hence we can take $l = 3$ in the algebra $\mathfrak{A}_l$ of rational-polynomial invariants discussed in the previous section for $n > 3$, and $l = 4$ for $n = 3$. This establishes the following statement for $n \geq 4$.

Theorem 1. *The algebra $\mathfrak{A}_l$ of rational-polynomial invariants for $l = 3 + \delta_{n3}$ as well the field $\mathfrak{F}$ of rational differential invariants of conformal metric structures are both generated by a finite number of (the indicated) differential invariants I_i and invariant derivations ∇_j, and the invariants from this algebra/field separate generic orbits in $J^\infty(\mathcal{C}_M)$.*

Let us consider now the exceptional case $n = 3$ and justify the above theorem in this case. There are no conformal invariants in 3D of order 2, and all differential invariants of order 3 are derived from the Cotton tensor C considered as $(3, 0)$ tensor. The space of Cotton tensors has dimension 5, is acted upon effectively by $CO(g)$ of dimension 4, so the generic orbit has codimension $H_3(3) = 1$ (this fact was also checked independently by a straightforward computation in Maple).

For generic C we have $\|C\|_g^2 \neq 0$, and so we can fix the metric representative $g_0 \in [g]$ by $\|C\|_{g_0}^2 = \pm 1$. Then we convert C, using the Hodge $*$-operator of g_0, to the $(1, 1)$ Cotton-York tensor $C : T \to T$. Again by genericity the spectrum $\mathrm{Sp}(C) = \{\lambda_1, \lambda_2, \lambda_3\}$ is simple, and the relations $\sum \lambda_i = 0$, $\max |\lambda_i| = 1$ yield precisely one scalar invariant of order 3; we can take, for instance, the polynomial invariant $\mathrm{Tr}(C^2)$.

In addition, we have the (g-normalized) eigenbasis e_1, e_2, e_3 (that depends on the 3-jet of generic conformal structures). This produces invariant derivations $\nabla_1, \nabla_2, \nabla_3$, as before, and writing g_0 in this frame we get all 4th and higher order differential invariants I_i sufficient for Lie-Tresse generating property. These invariants will be indeed separating, and eliminating non-algebraicity as before, we derive the fields of rational invariants $\mathfrak{F}_k$ with $\cup_k \mathfrak{F}_k = \mathfrak{F}$.

3. Stabilizers of generic jets

Our method to compute the number of independent differential invariants of order k follows the approach of [10]. We will use the jet-language from the formal theory of PDE, and refer the reader to [8].

Fix a point $a \in M$. Denote by $\mathbb{D}_k$ the Lie group of k-jets of diffeomorphisms preserving the point a. This group is obtained from $\mathbb{D}_1 = \mathrm{GL}(T)$ by successive extensions according to the exact 3-sequence

$$0 \to \Delta_k \longrightarrow \mathbb{D}_k \longrightarrow \mathbb{D}_{k-1} \to \{e\},$$

where $\Delta_k = \{[\varphi]_x^k : [\varphi]_x^{k-1} = [\mathrm{id}]_x^{k-1}\} \simeq S^k T^* \otimes T$ is Abelian $(k > 1)$.

Denote $V_M = T_{[g]}^{\mathrm{vert}}(\mathcal{C}_M)$ the tangent to the fiber of $\mathcal{C}_M$.

Lemma 2. *The following is a natural isomorphism:*

$$V_M = \mathrm{End}_0^{\mathrm{sym}}(T) = \{A : T \to T \mid g(Au, v) = g(u, Av), \mathrm{Tr}(A) = 0\}.$$

Proof. For a curve $[g + \epsilon\,\sigma] = [g(\mathbf{1} + \epsilon\, g^{-1}\sigma)]$ in $\mathcal{C}_M$ let us associate to its tangent vector the endomorphism $A = g^{-1}\sigma - \frac{1}{n}\mathrm{Tr}(g^{-1}\sigma)\mathbf{1}$. Since

$$\mathbf{1} + \epsilon g^{-1}\sigma = \left(1 + \tfrac{\epsilon}{n}\mathrm{Tr}(g^{-1}\sigma)\right)\mathbf{1} + \epsilon A = \left(1 + \tfrac{\epsilon}{n}\mathrm{Tr}(g^{-1}\sigma)\right)\cdot\left(\mathbf{1} + \tfrac{\epsilon n}{n + \epsilon\,\mathrm{Tr}(g^{-1}\sigma)}A\right)$$

removal of the trace part of σ (equivalent to conformal rescaling of the representative) is in the kernel of this map. Since A is obviously g-symmetric (for any representative g), this map is the required isomorphism. $\square$

Thus the symbol of the bundle $J^k(\mathcal{C}_M)$ is

$$\mathfrak{g}_k = \mathrm{Ker}[d\pi_{k,k-1} : TJ^k(\mathcal{C}_M) \to TJ^{k-1}(\mathcal{C}_M)] = S^k T^* \otimes V_M.$$

The differential group $\mathbb{D}_{k+1}$ acts on $J_a^k(\mathcal{C}_M)$, and hence Δ_{k+1} acts on $\mathfrak{g}_k$. Let $a_k \in J_a^k(\mathcal{C}_M)$ be a generic point. The next statement is obtained by a direct computation of the symbol of Lie derivative.

Lemma 3. *The space $\Delta_{k+1} \cdot a_k \subset \mathfrak{g}_k$ is the image $\mathrm{Im}(\zeta_k)$ of the map ζ_k that is equal to the following composition*

$$S^{k+1}T^* \otimes T \xrightarrow{\delta} S^k T^* \otimes (T^* \otimes T) \xrightarrow{\mathbf{1} \otimes \Pi} S^k T^* \otimes V_M.$$

Here δ is the Spencer operator and $\Pi : T^* \otimes T \to V_M \subset T^* \otimes T$ is the projection given by

$$\langle p, \Pi(B)u \rangle = \tfrac{1}{2}\langle p, Bu \rangle + \tfrac{1}{2}\langle u_\flat, Bp^\sharp \rangle - \tfrac{1}{n}\mathrm{Tr}(B)\langle p, u \rangle,$$

where $u \in T, p \in T^*, B \in T^* \otimes T$ are arbitrary, $\langle \cdot, \cdot \rangle$ denotes the pairing between T^* and T, and we use the musical isomorphisms $\flat$ (flat) and $\sharp$ (sharp) that depend on the choice of representative $g \in [g]$, but the right-hand side is independent of it. In the index notations:

$$\Pi(B)_j^i = \tfrac{1}{2}(B_j^i + g^{ik}B_k^l g_{lj}) - \tfrac{1}{n}B_k^k \delta_j^i.$$

One should, of course, check that the image of Π belongs to V_M, but this is straightforward. Recall that i-th prolongation of a Lie algebra $\mathfrak{h} \subset \mathrm{End}(T)$ is defined by the formula $\mathfrak{h}^{(i)} = S^{i+1}T^* \otimes T \cap S^i T^* \otimes \mathfrak{h}$. As is well-known, for the conformal algebra of $[g]$ of any signature (p, q), $n = p + q > 2$, it holds: $\mathfrak{co}(g)^{(1)} = T^*$ and $\mathfrak{co}(g)^{(i)} = 0$ for $i > 1$.

Lemma 4. *We have:* $\mathrm{Ker}(\zeta_k) = 0$ *for* $k > 1$.

Proof. If $\zeta_k(\Psi) = 0$, then $\delta(\Psi) \in S^k T^* \otimes \mathfrak{co}(g)$, where $\mathfrak{co}(g) \subset \mathrm{End}(T)$ is the conformal algebra. This means that $\Psi \in \mathfrak{co}(g)^{(k+1)} = 0$, if $k > 1$. Thus we conclude injectivity of ζ_k. $\qquad\qquad\square$

Denote by $\mathrm{St}_k \subset \mathbb{D}_{k+1}$ the stabilizer of a generic point $a_k \in J_a^k(\mathcal{C}_M)$, and by St_k^0 its connected component of unity. Then Lemma 4 implies that $\Delta_{k+1} \cap \mathrm{St}_k = \{e\}$ for $k > 1$, so the projectors $\rho_{k+1,k} : \mathbb{D}_{k+1} \to \mathbb{D}_k$ induce the injective homomorphisms $\mathrm{St}_k \to \mathrm{St}_{k-1}$ and $\mathrm{St}_k^0 \to \mathrm{St}_{k-1}^0$.

The stabilizers of low order (for any $n \geq 3$) are the following. For any $a_0 \in \mathcal{C}_M$ its stabilizer is $\mathrm{St}_0 = CO(g)$. Next, the stabilizer $\mathrm{St}_1 \subset \mathbb{D}_2$ of $a_1 \in J^1(\mathcal{C}_M)$ is the extension (by derivations) of St_0 by $\mathfrak{co}(g)^{(1)} = T^* \overset{\iota}{\hookrightarrow} \Delta_2$, where $\iota : T^* \to S^2 T^* \otimes T$ is given by

$$\iota(p)(u, v) = \langle p, u \rangle v + \langle p, v \rangle u - \langle u_\flat, v \rangle p^\sharp,$$

for $p \in T^*$, $u, v \in T$, or by using indices: $\iota(p)_{kl}^j = p_k \delta_l^j + p_l \delta_k^j - g^{ij} p_i g_{kl}$. In other words, we have $\mathrm{St}_1 = CO(g) \ltimes T$.

Notice also that for $n = 3$ due to absence of second order differential invariants and equality $\dim \Delta_3 = \dim \mathfrak{g}_2$ we have $\mathrm{St}_2^0 = \mathrm{St}_1^0$. Then by dimensional reasons St_2 for $n \geq 4$ and St_3 for $n = 3$ are nontrivial $(\dim \mathbb{D}_3 > \dim J_a^2(\mathcal{C}_M) - H_n(2)$ for $n > 3$, resp. $\dim \mathbb{D}_4 > \dim J_a^3(\mathcal{C}_M) - H_3(3)$ for $n = 3$).

Lemma 5. *If* $k \geq 3$, $n \geq 4$ *of if* $k \geq 4$, $n = 3$, *then* $\mathrm{St}_k^0 = \{e\}$.

Proof. In Section 2 we constructed a canonical frame $e_1, \dots, e_n$ on T depending on (generic) jet $a_i \in J^i(\mathcal{C}_M)$, where $i = 2$ for $n > 3$ and $i = 3$ for $n = 3$. In other words, we constructed a frame on the bundle $\pi_i^* TM$ over a Zariski open set in $J^i(\mathcal{C}_M)$.

The elements from St_i^0 shall preserve this frame. Since $\mathrm{St}_i^0 \subset \mathrm{St}_1 = CO(g) \ltimes T$ this eliminates the liner conformal freedom (first factor).

Choosing a point $a_{i+1} \in \pi_{i+1,i}^{-1}(a_i) \in J^{i+1}(\mathcal{C}_M)$ the elements of $\mathrm{St}_{i+1}^0 \subset \mathrm{St}_1$ should also stabilize 1-jet of this frame (realized via the canonical lift $L(a_{i+1}) \subset T_{a_i} J^i(\mathcal{C}_M)$ of $T = T_a M$), and this eliminates the remaining freedom $\mathfrak{co}(g)^{(1)} = T$, yielding $\mathrm{St}_{i+1}^0 = 0$ (we take the connected component because of the undetermined signs $\pm$ in the normalizations above). Hence the stabilizers St_k^0 for $k \geq i + 1$ are trivial as well. $\qquad\square$

4. Hilbert polynomial and Poincaré function

Now we can compute the number of independent differential invariants. Since G acts transitively on M, the codimension of the orbit of G in $J^k(\mathcal{C}_M)$ is equal to the codimension of the orbit of $\mathbb{D}_{k+1}$ in $J^k_a(\mathcal{C}_M)$ (where $a \in M$ is a fixed point). Denoting the orbit of G through a generic point $a_k \in J^k_a(\mathcal{C}_M)$ by $\mathcal{O}_k \subset J^k_a(\mathcal{C}_M)$ we have:

$$\dim(\mathcal{O}_k) = \dim \mathbb{D}_{k+1} - \dim \mathrm{St}_k .$$

Notice that

$$\mathrm{codim}(\mathcal{O}_k) = \dim J^k_a(\mathcal{C}_M) - \dim(\mathcal{O}_k) = \mathrm{trdeg}\, \mathfrak{F}_k$$

is the number of functionally (or, in our context, algebraically) independent scalar differential invariants of order k.

The Hilbert function is the number of "pure order" k differential invariants $H_n(k) = \mathrm{trdeg}\, \mathfrak{F}_k - \mathrm{trdeg}\, \mathfrak{F}_{k-1}$. It is known to be a polynomial (we refer to [9] for the proof in our context), so we will refer to it as the Hilbert polynomial.

The results of Section 3 and the formulae above allow to compute the values $H_n(k)$, giving the table in the introduction.

Theorem 6. *For $n > 3$ the number of ord=2 differential invariants is*

$$H_n(2) = \tfrac{1}{12}(n^4 - 13n^2 - 12),$$

the number of "pure" order 3 differential invariants is

$$H_n(3) = \tfrac{1}{24}n(n^4 + 2n^3 - 5n^2 - 14n - 32),$$

and the number of "pure" order $k > 3$ differential invariants is

$$H_n(k) = \frac{n(k-1)}{2}\binom{n+k-1}{k+1} - \binom{n+k-1}{k}.$$

For the exceptional case $n = 3$ we have:

$$H_3(3) = 1, \quad H_3(4) = 9, \quad and \quad H_3(k) = k^2 - 4 \text{ for } k \geq 5.$$

Notice that $H_n(k) \sim \frac{n^2-n-2}{2}\frac{1}{(n-1)!}k^{n-1}$, which confirms the (obvious) fact that moduli of conformal structures are parametrized by $\binom{n}{2} - 1$ functions of n arguments.

Proof. From Lemma 5 we have: $H_n(3) = \dim J^3_a(\mathcal{C}_M) - \dim \mathbb{D}_4 - H_n(2)$ for $n \geq 4$ and $H_3(4) = \dim J^4_a(\mathcal{C}_M) - \dim \mathbb{D}_5 - H_3(3)$ for $n = 3$. These numbers are positive, and we have by Lemma 5: $H_n(k) = \dim \mathfrak{g}_k - \dim \Delta_{k+1}$ for $k > 3$, $n \geq 4$ and for $k > 4$, $n = 3$. $\square$

The Poincaré function is the generating function for the Hilbert polynomial, defined by $P_n(z) = \sum_{k=0}^{\infty} H_n(k)z^k$. This is a rational function with the only pole $z = 1$ of order equal to the minimal number of invariant derivations in the Lie-Tresse generating set [9].

Depending on dimension $n > 2$ the Poincaré functions are:

$$P_3(z) = \frac{z^3(1 + 6z - 3z^2 - 5z^3 + 3z^4)}{(1-z)^3},$$

$$P_4(z) = \frac{z^2(3 + 24z - 35z^2 + 8z^3 + 9z^4 - 4z^5)}{(1-z)^4},$$

$$P_5(z) = \frac{z^2(24 + 15z - 85z^2 + 74z^3 - 10z^4 - 14z^5 + 5z^6)}{(1-z)^5}.$$

$$P_n(z) = \frac{(n+1)nz - 2(n+z)}{2z(1-z)^n} + n\Big(\frac{1}{z} + z - z^3\Big) + \Big(\binom{n}{2} + 1\Big)(1 - z^2).$$

5. CONCLUSION

We have described the lowest degree differential invariants: 1 invariant I_1 of order $k = 3$ for $n = 3$, 3 invariants I_1, I_2, I_3 of order $k = 2$ for $n = 4$, etc. How to see the next invariants of order $(k + 1)$?

For $n = 3$ there are 9 differential invariants of "pure" order 4. They can be extracted from $\nabla_j(I_1)$ ($\nabla_j = \hat{e}_j$) and the structure constants c_{ij}^k given by $[e_i, e_j] = c_{ij}^k e_k$. This gives $3 + 9 > 9$ invariants of order 4.

For $n = 4$ we can take $\nabla_j(I_i)$, structure constants c_{ij}^k, but we can also add the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection of the normalized metric g_0 in the basis $\{e_i\}$ as invariants of order 3. This gives $12 + 24 + 40 > 36$ invariants of order 3.

However the amount of invariants obtained in this way exceeds the number of the independent invariants we have found. This is because there exist algebraic relations between them (like differential Bianchi's identity). In terms of Lie-Tresse representation of invariants, these relations are called differential syzygies. The important problem of understanding these syzygies remains open.

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