MASTER'S THESIS IN MATHEMTAICS

Differential invariants of the 2D conformal Lie algebra action

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Chapter 1

Introduction

The space

$$CO(2) = \left\{ \begin{bmatrix} \lambda\cos(t) & -\lambda\sin(t) \\ \lambda\sin(t) & \lambda\cos(t) \end{bmatrix} \mid t \in S^1 = \mathbb{R} \mod 2\pi,\ \lambda \in \mathbb{R}^+ \right\}$$

is the linear conformal Lie group. The Lie algebra of $CO(2)$ is

$$\mathrm{co}(2) = \langle -y\partial_x + x\partial_y,\ x\partial_x + y\partial_y \rangle .$$

Consider the $4-$dimensional Lie group

$$CO(2) \ltimes \mathbb{R}^2 = \left\{\varphi \in \mathrm{Aff}\left(\mathbb{R}^2, \mathbb{R}^2\right) :\ \varphi(x) = Ax + b \mid A \in CO(2),\ b \in \mathbb{R}^2\right\}.$$

The Lie algebra of $CO(2) \ltimes \mathbb{R}^2$ is

$$\mathrm{co}(2) \ltimes \mathbb{R}^2 = \langle -y\partial_x + x\partial_y,\ x\partial_x + y\partial_y,\ \partial_x,\ \partial_y \rangle .$$

It is known [S, KL2] that the conformal Lie algebra

$$\mathfrak{g} = \{V_g = g_1(x,y)\partial_x + g_2(x,y)\partial_y \mid g_{1x} = g_{2y},\ g_{1y} = -g_{2x}\} \subset \mathcal{D}(\mathbb{R}^2)$$

is the completion of the $\infty-$prolongation of $\mathrm{co}(2) \ltimes \mathbb{R}^2$. Hence $\mathfrak{g}$ is the Lie algebra that corresponds to the Lie pseudogroup of all conformal transformations of $\mathbb{R}^2$

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y)),$$

$$\begin{bmatrix} \dfrac{\partial \varphi_1}{\partial x} & \dfrac{\partial \varphi_1}{\partial y} \\ \dfrac{\partial \varphi_2}{\partial x} & \dfrac{\partial \varphi_2}{\partial y} \end{bmatrix} \in CO(2).$$

The conformal Lie algebra is canonically represented as the Lie algebra of vector fields in $\mathbb{R}^2$. In Chapter 4 we find all possible representations of $\mathfrak{g}$ via vector fields in

$$J^0\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3(x,y,u)$$

which project to the canonical representation. Namely, for any function $F(u) \in C^\infty\left(J^0\mathbb{R}^2\right)$ and constant $b = b_1 + ib_2 \in \mathbb{C}$ the inclusion map

$$I_{Fb} : \mathfrak{g} \longrightarrow \mathcal{D}(J^0\mathbb{R}^2),$$

$$I_{Fb}(V_g) = V_g + F(u)(b_1 g_1 - b_2 g_2)\partial_u,$$

is an injective Lie algebra homomorphism and these are all representations of the form $V_g \mapsto V_g + \lambda\partial_u$. Let

$$\mathfrak{g}_{Fb} = \mathrm{Im}(I_{Fb}) = \{V_g + F(u)(b_1 g_1 - b_2 g_2)\partial_u \mid g_{1x} = g_{2y},\ g_{1y} = -g_{2x}\}$$

denote the image of the map.

In this thesis we will find the algebra of $\mathfrak{g}_{Fb}-$differential invariants.

Theorem 1 *The algebra* $\mathcal{G}^{Fb}$ *of* $\mathfrak{g}_{Fb}-$*differential invariants is generated by* I_0, I_2, ∇_1 *and*

∇_2, *where for* $F=0$

$$\begin{aligned} I_0 &= u, \\ I_2 &= \frac{u_{20}+u_{02}}{u_{10}^2+u_{01}^2}, \\ \nabla_1 &= \frac{1}{u_{10}^2+u_{01}^2}\left(u_{10}\mathcal{D}_x+u_{01}\mathcal{D}_y\right), \\ \nabla_2 &= \frac{1}{u_{10}^2+u_{01}^2}\left(u_{01}\mathcal{D}_x-u_{10}\mathcal{D}_y\right), \end{aligned}$$

and for $F\neq 0$

$$\begin{aligned} I_0 &= \int\frac{du}{F(u)}-b_1x+b_2y, \\ I_2 &= \frac{(-u_{01}^2-u_{10}^2)F_u(u)+F(u)(u_{02}+u_{20})}{\left(b_1F(u)-u_{10}\right)^2+\left(b_2F(u)+u_{01}\right)^2}, \\ \nabla_1 &= \left(\frac{F(u)^2}{\left(u_{10}-b_1F(u)\right)^2+\left(u_{01}+b_2F(u)\right)^2}\right)\left(\left(\frac{u_{10}}{F(u)}-b_1\right)\mathcal{D}_x+\left(\frac{u_{01}}{F(u)}+b_2\right)\mathcal{D}_y\right), \\ \nabla_2 &= \left(\frac{F(u)^2}{\left(u_{10}-b_1F(u)\right)^2+\left(u_{01}+b_2F(u)\right)^2}\right)\left(\left(\frac{u_{01}}{F(u)}+b_2\right)\mathcal{D}_x+\left(-\frac{u_{10}}{F(u)}+b_1\right)\mathcal{D}_y\right). \end{aligned}$$

Hence, any function $f\in\mathcal{G}^{Fb}$ *of order* m *has the form*

$$f=f\left(I_0,I_2,I_{3,1},I_{3,2},...,I_{m,1},...,I_{m,m-1}\right),$$

where

$$I_{k,j}=\nabla_1^{k-2-j}\nabla_2^{j}\left(I_2\right),\quad j,k\in\mathbb{Z}_{\geq 2},\ k>j.$$

The invariants $\{I_{k,j}\}$ *are functionally independent.*

We will also show that if f is a $\mathfrak{g}_{Fb}$–differential invariant and $h(x,y)\in C^\infty(\mathbb{R}^2)$ is a solution of the PDE $\mathcal{E}=\{f=0\}$, then the function

$$u(x,y) = h(g_1(x,y),g_2(x,y)),\ F=0, \tag{1.1}$$

$$u(x,y) = G^{-1}\left(b_1(x-g_1(x,y))-b_2(y-g_2(x,y))+G(h(g_1(x,y),g_2(x,y)))\right), \tag{1.2}$$

$$F \neq 0,\ G(u)=\int\frac{du}{F(u)},$$

is a solution of $\mathcal{E}$ for any analytic function $g(z) = g_1(x,y) + ig_2(x,y)$ on domains where $g_z \neq 0$. Thus we get a collection of $PDEs$ $\mathcal{E}$ with $\mathrm{sym}(\mathcal{E}) \supseteq \mathfrak{g}$. This provides a large family of solutions for any differential equation from this collection.

Structure of the thesis.

In Chapter 2 we collect some basic concepts from complex analysis and describe our main object, the Lie algebra $\mathfrak{g}$.

In Chapter 3 we describe the the space of jets, the Cartan distribution, invariant differentiations and the Lie-Tresse theorem. In the last part of this chapter we will use three different methods to find the differential invariants of the canonical representation of $\mathfrak{g}$. The three descriptions of the algebra turns out to be equivalent.

n Chapter 4 we will find the differential invariants of the deformed representations of $\mathfrak{g}$. We use the best method from Chapter 3 to generate the invariants.

In Chapter 5 we justify the above claim that Formulas (1.1) and (1.2) represent solutions of the $\mathfrak{g}-$invariant equations. In the last part of this chapter we will represent $\mathfrak{g}$ as a Lie algebra of vector fields in $\mathbb{R}^2 = J^0\mathbb{R}$, and find differential invariants of some finite dimensional Lie subalgebras of $\mathfrak{g}$.

Conventions.

Most of the results in this thesis are defined locally, restricted to regular domains where the $\mathfrak{g}-$differential invariants are well defined. We will not specify locations in the text.

In this thesis we will extensively use complexification, which work nicely with real-analytic objects. Thus we adopt the following convention: depending on the context C^∞ can mean smooth or analytic functor. The coordinate function

$$z = x + iy,\ \bar{z} = x - iy,$$

are used when we assume analyticity. The convention is helpful because the main results concerning $\mathfrak{g}-$differential invariants hold in smooth category. Thus we will be using the freedom of extending and shrinking the space of functions, vector fields etc.

Chapter 2

The Lie Algebra $\mathfrak{g}$

2.1 Vector Bundles over a Complex Manifold

In this section we will describe some basic concepts that will be important in the rest of the text. Most of the material is well known, see [KN].

2.1.1 Algebras of Functions on a Complex Manifold

Let M be a complex smooth manifold of dimension n. Consider the spaces of functions

$$C^\infty(M) = \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}\,,$$

$$\mathcal{O}(M) = \{f : M \to \mathbb{C} \mid f \text{ is complex analytic}\}\,,$$

$$C^\infty(M, \mathbb{C}) = \{f : M \to \mathbb{C} \mid f \text{ is smooth}\}\,.$$

The spaces of functions $C^\infty(M,\mathbb{C})$ and $\mathcal{O}(M)$ are $\mathbb{C}-$algebras, and the space of functions $C^\infty(M)$ is an $\mathbb{R}-$algebra. Moreover, $C^\infty(M,\mathbb{C})$ is equal to the tensor product

$$C^\infty(M,\mathbb{C}) = C^\infty(M)\otimes\mathbb{C}.$$

Let $U\subset M$ be a chart with local coordinates

$$(z_1 = x_1 + iy_1, ..., z_n = x_n + iy_n).$$

There exist projections (restrictions)

$$C^\infty(M) \longrightarrow C^\infty(U),$$

$$\mathcal{O}(M) \longrightarrow \mathcal{O}(U),$$

$$C^\infty(M)\otimes\mathbb{C} \longrightarrow C^\infty(U)\otimes\mathbb{C}.$$

The functions $f_1 \in C^\infty(U)$, $f_2 \in \mathcal{O}(U)$ and $f_3 \in C^\infty(U)\otimes\mathbb{C}$ have the forms

$$f_1 = f_1(x_1, y_1, ..., x_n, y_n),$$

$$f_2 = f_2(z_1, ..., z_n),$$

$$f_3 = F_1(x_1, y_1, ..., x_n, y_n) + iF_2(x_1, y_1, ..., x_n, y_n).$$

The inclusion map

$$\mathcal{O}(M) \hookrightarrow C^\infty(M)\otimes\mathbb{C}$$

is an injective $\mathbb{C}-$algebra homomorphism. Hence $\mathcal{O}(M)$ is a $\mathbb{C}-$subalgebra of $C^\infty(M)\otimes\mathbb{C}$.

The inclusion $\mathbb{R}\hookrightarrow\mathbb{C}$ induces the inclusion

$$I : C^\infty(M) \hookrightarrow C^\infty(M)\otimes\mathbb{C},$$

$$I(f) = f(x_1, y_1, ..., x_n, y_n) + i0.$$

The projection maps $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \to \mathbb{R}$ induce the projections

$$\mathrm{Re} : C^\infty(M) \otimes \mathbb{C} \longrightarrow C^\infty(M),$$

$$\mathrm{Re}(f_1(x_1, y_1, ..., x_n, y_n) + if_2(x_1, y_1, ..., x_n, y_n)) = f_1(x_1, y_1, ..., x_n, y_n),$$

$$\mathrm{Im} : C^\infty(M) \otimes \mathbb{C} \longrightarrow C^\infty(M),$$

$$\mathrm{Im}(f_1(x_1, y_1, ..., x_n, y_n) + if_2(x_1, y_1, ..., x_n, y_n)) = f_2(x_1, y_1, ..., x_n, y_n),$$

with

$$\mathrm{Re}\, I = \mathrm{Im}(iI) = Id_{C^\infty(M)}.$$

The inclusion I is an injective $\mathbb{R}-$algebra homomorphism. Hence $C^\infty(M)$ is an $\mathbb{R}-$subalgebra of $C^\infty(M) \otimes \mathbb{C}$.

2.1.2 Vector Spaces and Vector Bundles

Let X_1 be an $\mathbb{R}-$linear map and X_2 and X_3 be $\mathbb{C}-$linear maps

$$X_1 : C^\infty(M) \longrightarrow \mathbb{R},$$

$$X_2 : \mathcal{O}(M) \longrightarrow \mathbb{C},$$

$$X_3 : C^\infty(M) \otimes \mathbb{C} \longrightarrow \mathbb{C}.$$

The linear map X_j, for $j \in \{1, 2, 3\}$, is a derivation if it satisfies the equation

$$X_j(f_j g_j) = f_j X_j g_j + g_j X_j f_j \tag{2.1}$$

for all functions $f_1, g_1 \in C^\infty(M)$, $f_2, g_2 \in \mathcal{O}(M)$ and $f_3, g_3 \in C^\infty(M) \otimes \mathbb{C}$. The linear map X_j is a derivation at the point $p \in M$ if it satisfies Equation (2.1) at p.

For any point $p \in M$ the spaces of all derivations at p of the algebras $\mathcal{O}(M)$ and $C^\infty(M) \otimes \mathbb{C}$ are complex vector spaces, and the space of all derivations at p of the algebra $C^\infty(M)$ is a real vector space.

Let us use the following notation for the spaces of all derivations at p of the algebras $C^\infty(M)$ and $\mathcal{O}(M)$:

$$T_pM = \mathrm{Der}_{\mathbb{R}}(C^\infty(M))_p,$$

$$T_p^{1,0}M = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(M))_p.$$

Lemma 2 *Let p be a point of the manifold M. Then the space of all derivations at p of the algebra $C^\infty(M) \otimes \mathbb{C}$ is equal to the tensor product*

$$\mathrm{Der}_{\mathbb{C}}(C^\infty(M) \otimes \mathbb{C})_p = \mathrm{Der}_{\mathbb{R}}(C^\infty(M))_p \otimes \mathbb{C}.$$

Proof. For all functions $f \in C^\infty(M) \otimes \mathbb{C}$ there exist functions $f_1, f_2 \in C^\infty(M)$ such that

$$f = f_1 + if_2.$$

Hence we have the $\mathbb{C}-$linear inclusion map

$$I : \mathrm{Der}_{\mathbb{R}}(C^\infty(M))_p \otimes \mathbb{C} \hookrightarrow \mathrm{Der}_{\mathbb{C}}(C^\infty(M) \otimes \mathbb{C})_p,$$

$$(IY)(f) = Y(f_1) + iY(f_2).$$

The algebra $C^\infty(M)$ is an $\mathbb{R}-$subalgebra of $C^\infty(M) \otimes \mathbb{C}$. Hence if we restrict $\tilde{Y} \in \mathrm{Der}_{\mathbb{C}}(C^\infty(M) \otimes \mathbb{C})_p$ to $C^\infty(M)$, then $\tilde{Y}$ is an $\mathbb{R}-$linear map

$$\tilde{Y}|_{C^\infty(M)} : C^\infty(M) \longrightarrow \mathbb{C}$$

such that for all functions $f, g \in C^\infty(M)$

$$\tilde{Y}(fg)(p) = f(p)\tilde{Y}g(p) + g(p)\tilde{Y}f(p).$$

Hence we have the $\mathbb{C}-$linear map

$$R : \mathrm{Der}_{\mathbb{C}}(C^\infty(M) \otimes \mathbb{C})_p \longrightarrow \mathrm{Der}_{\mathbb{R}}(C^\infty(M))_p \otimes \mathbb{C},$$

$$R(\tilde{Y}) = \tilde{Y}|_{C^\infty(M)}.$$

The map R is surjective since

$$RI = \mathrm{Id}_{\mathrm{Der}_{\mathbb{R}}(C^\infty(M))_p \otimes \mathbb{C}}\,.$$

Suppose that for an element $\tilde{Y} \in \mathrm{Der}_{\mathbb{C}}(C^\infty(M) \otimes \mathbb{C})_p$ $\tilde{Y}(f_j) = 0$ for all functions $f_j \in C^\infty(M)$. Then

$$\tilde{Y}(f) = \tilde{Y}(f_1) + i\tilde{Y}(f_2) = 0, \ \forall f = f_1 + if_2 \in C^\infty(M) \otimes \mathbb{C}.$$

Hence $\mathrm{Ker}(R) = \{0\}$. It follows that the map R is bijective. ■

The inclusion map

$$T_p^{1,0}M \hookrightarrow T_pM \otimes \mathbb{C} = \mathrm{Der}_{\mathbb{C}}(C^\infty(M) \otimes \mathbb{C})_p$$

is an injective $\mathbb{C}-$linear map. Hence $T_p^{1,0}M$ is a $\mathbb{C}-$subspace of $T_pM \otimes \mathbb{C}$.

The inclusion map

$$I : T_pM \hookrightarrow T_pM \otimes \mathbb{C}$$

where

$$\mathrm{Re}\, I = \mathrm{Im}(iI) = \mathrm{Id}_{T_pM}$$

is an injective $\mathbb{R}-$linear map. Hence T_pM is an $\mathbb{R}-$subspace of $T_pM \otimes \mathbb{C}$.

Consider the $\mathbb{C}-$subspace of $T_pM \otimes \mathbb{C}$

$$T_p^{0,1}M = \overline{T_p^{1,0}M}.$$

Lemma 3 *[KN] $T_pM \otimes \mathbb{C}$ is equal to the direct sum*

$$T_pM \otimes \mathbb{C} = T_p^{1,0}M \oplus T_p^{0,1}M.$$

Let

$$(U, (z_1 = x_1 + iy_1, ...z_n = x_n + iy_n))$$

be any smooth chart containing p. T_pM is a real vector space of dimension $2n$

$$T_pM = \langle \partial_{x_1}|_p, \partial_{y_1}|_p....\partial_{x_n}|_p, \partial_{y_n}|_p\rangle_{\mathbb{R}}\,.$$

$T_p^{1,0}M$ and $T_p^{0,1}M$ are complex vector spaces of dimension n

$$T_p^{1,0}M = \langle \partial_{z_1}|_p, ..., \partial_{z_n}|_p\rangle_{\mathbb{C}} = \left\langle \tfrac{1}{2}(\partial_{x_1} - i\partial_{y_1})|_p, ..., \tfrac{1}{2}(\partial_{x_n} - i\partial_{y_n})|_p\right\rangle_{\mathbb{C}},$$

$$T_p^{0,1}M = \langle \partial_{\bar{z}_1}|_p, ..., \partial_{\bar{z}_n}|_p\rangle_{\mathbb{C}} = \left\langle \tfrac{1}{2}(\partial_{x_1} + i\partial_{y_1})|_p, ..., \tfrac{1}{2}(\partial_{x_n} + i\partial_{y_n})|_p\right\rangle_{\mathbb{C}}.$$

$T_pM \otimes \mathbb{C}$ is a complex vector space of dimension $2n$

$$\begin{aligned} T_pM \otimes \mathbb{C} &= T_p^{1,0}M \oplus T_p^{0,1}M = \langle \partial_{x_1}|_p, \partial_{y_1}|_p....\partial_{x_n}|_p, \partial_{y_n}|_p\rangle_{\mathbb{C}} \\ &= \langle \partial_{z_1}|_p, .., \partial_{z_n}|_p, \partial_{\bar{z}_1}|_p, .., \partial_{\bar{z}_n}|_p\rangle_{\mathbb{C}}\,. \end{aligned}$$

Consider the spaces

$$TM = \bigcup_{p\in M} T_pM,$$

$$T^{1,0}M = \bigcup_{p\in M} T_p^{1,0}M,$$

$$T^{0,1}M = \underset{p \in M}{\cup} T_p^{0,1}M,$$

$$TM \otimes \mathbb{C} = \underset{p \in M}{\cup} T_pM \otimes \mathbb{C}.$$

By standard topological arguments TM, $T^{1,0}M$, $T^{0,1}M$ and $TM \otimes \mathbb{C}$ are vector bundles over M. The bundle TM is a real subbundle of $TM \otimes \mathbb{C}$, and $T^{1,0}M$ and $T^{0,1}M$ are complex subbundles of $TM \otimes \mathbb{C}$.

Remark 4 [*KN*] *The above constructions work as well for the case when M is an almost complex manifold, i.e. M is a real manifold with a tensor field J which is, at every point p of M, an endomorphism of the tangent space T_pM such that $J^2 = -1$, where 1 denotes the identity transformation of T_pM.*

2.1.3 Vector Fields on a Complex Manifold

Consider the spaces of smooth sections of the vector bundles TM, $T^{1,0}M$, $T^{0,1}M$ and $TM \otimes \mathbb{C}$

$$\mathcal{D}(M) = C^\infty(TM) = \mathrm{Der}_{\mathbb{R}}(C^\infty(M)),$$

$$\mathcal{D}^{1,0}(M) = C^\infty(T^{1,0}M),$$

$$\mathcal{D}^{0,1}(M) = C^\infty(T^{0,1}M),$$

$$\mathcal{D}(M) \otimes \mathbb{C} = \mathrm{Der}_{\mathbb{R}}(C^\infty(M)) \otimes \mathbb{C}.$$

The space $\mathcal{D}(M)$ is a module over the algebra $C^\infty(M)$, and the spaces $\mathcal{D}^{1,0}(M)$, $\mathcal{D}^{0,1}(M)$ and $\mathcal{D}(M) \otimes \mathbb{C}$ are modules over the algebra $C^\infty(M) \otimes \mathbb{C}$.

Consider the space of $\mathbb{C}-$analytic sections

$$C^\omega(T^{1,0}M) = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(M)).$$

The space $C^\omega(T^{1,0}M)$ is a module over $\mathcal{O}(M)$.

Let us write these vector fields in local coordinates. There exist projections (restrictions) for $\varphi \in \{(), (1,0), (0,1)\}$

$$\mathcal{D}^\varphi(M) \longrightarrow \mathcal{D}^\varphi(U),$$

$$\mathcal{D}(M) \otimes \mathbb{C} \longrightarrow \mathcal{D}(U) \otimes \mathbb{C},$$

$$C^\omega(T^{1,0}M) \longrightarrow C^\omega(T^{1,0}U).$$

For the vector fields $X_1 \in \mathcal{D}(U)$, $X_2 \in \mathcal{D}(U) \otimes \mathbb{C}$, $X_3 \in \mathcal{D}^{1,0}(U)$, $X_4 \in \mathcal{D}^{0,1}(U)$ and $X_5 \in C^\omega(T^{1,0}U)$ there exist functions $f_{1j},\ f_{2j} \in C^\infty(U)$, $h_{1j},\ h_{2j},\ q_{1j},\ q_{2j} \in C^\infty(U) \otimes \mathbb{C}$ and $g_j \in \mathcal{O}(U)$ such that

$$X_1 = \sum_{j=1}^{n} f_{1j}\partial_{x_j} + f_{2j}\partial_{y_j},$$

$$X_2 = \sum_{j=1}^{n} h_{1j}\partial_{x_j} + h_{2j}\partial_{y_j},$$

$$X_3 = \sum_{j=1}^{n} q_{1j}\partial_{z_j},$$

$$X_4 = \sum_{j=1}^{n} q_{2j}\partial_{\bar{z}_j},$$

$$X_5 = \sum_{j=1}^{n} g_j\partial_{z_j}.$$

The spaces $C^\omega(T^{1,0}M)$, $\mathcal{D}^\varphi(M)$ and $\mathcal{D}(M)\otimes\mathbb{C}$ are infinite dimensional Lie algebras with the Lie bracket being the commutator.

For $\beta \in \{(1,0), (0,1)\}$ the inclusion maps

$$C^\omega(T^{1,0}M) \hookrightarrow \mathcal{D}^{1,0}(M),$$

$$\mathcal{D}^\beta(M) \hookrightarrow \mathcal{D}(M) \otimes \mathbb{C},$$

are Lie algebra homomorphisms. Hence $C^\omega(T^{1,0}M)$ is an infinite dimensional Lie subalgebra of $\mathcal{D}^{1,0}(M)$, and $\mathcal{D}^\beta(M)$ is an infinite dimensional Lie subalgebra of $\mathcal{D}(M)\otimes\mathbb{C}$.

The inclusion map

$$I : \mathcal{D}(M) \hookrightarrow \mathcal{D}(M)\otimes\mathbb{C},$$

where

$$\operatorname{Re} I = \operatorname{Im}(iI) = \operatorname{Id}_{\mathcal{D}(M)},$$

is a Lie algebra homomorphism. Hence $\mathcal{D}(M)$ is an infinite dimensional Lie subalgebra of $\mathcal{D}(M)\otimes\mathbb{C}$.

2.2 The Lie Algebra $\mathfrak{g}$

Consider the subspace $\mathfrak{g}\subset\mathcal{D}(\mathbb{R}^2)$

$$\mathfrak{g} = \{g_1\partial_x + g_2\partial_y | g_{1x} = g_{2y},\ g_{1y} = -g_{2x}\}.$$

Any element of $\mathfrak{g}$ has the form

$$V_g = g_1\partial_x + g_2\partial_y,$$

where $g = g_1 + ig_2 \in \mathcal{O}$.

Proposition 5 *The space $\mathfrak{g}$ is a Lie algebra.*

Proof. For any numbers $a, b\in\mathbb{R}$ and any functions $v, w\in\mathcal{O}$

$$aV_v + bV_w = (av_1 + bw_1)\partial_x + (av_2 + bw_2)\partial_y = V_{av+bw}\in\mathfrak{g}.$$

Hence $\mathfrak{g}$ is a linear subspace of $\mathcal{D}(\mathbb{R}^2)$.

$$[V_v, V_w] = \tilde{u}_1\partial_x + \tilde{u}_2\partial_y$$

$$= (v_1w_{1x} - v_2w_{2x} - w_1v_{1x} + w_2v_{2x})\partial_x + (v_1w_{2x} + v_2w_{1x} - w_1v_{2x} - w_2v_{1x})\partial_y.$$

It is left to show that the Cauchy-Riemann equations hold for the function $\tilde{u}_1 + i\tilde{u}_2$.

$$\frac{\partial}{\partial x}\tilde{u}_1 = v_1w_{1xx} - v_2w_{2xx} - w_1v_{1xx} + w_2v_{2xx} + v_{1x}w_{1x} - v_{2x}w_{2x} - w_{1x}v_{1x} + w_{2x}v_{2x},$$

$$\frac{\partial}{\partial x}\tilde{u}_2 = v_1w_{2xx} + v_2w_{1xx} - w_1v_{2xx} - w_2v_{1xx} + v_{1x}w_{2x} + v_{2x}w_{1x} - w_{1x}v_{2x} - w_{2x}v_{1x},$$

$$\frac{\partial}{\partial y}\tilde{u}_1 = -v_1w_{2xx} - v_2w_{1xx} + w_1v_{2xx} + w_2v_{1xx} - v_{2x}w_{1x} - v_{1x}w_{2x} + w_{2x}v_{1x} + w_{1x}v_{2x},$$

$$\frac{\partial}{\partial y}\tilde{u}_2 = v_1w_{1xx} - v_2w_{2xx} - w_1v_{1xx} + w_2v_{2xx} - v_{2x}w_{2x} + v_{1x}w_{1x} + w_{2x}v_{2x} - w_{1x}v_{1x}.$$

Thus we see that

$$\tilde{u}_{1x} = \tilde{u}_{2y},\ \ \tilde{u}_{1y} = -\tilde{u}_{2x},$$

and $[V_v, V_w] \in \mathfrak{g}$. Hence $\mathfrak{g}$ is closed under the commutator bracket. ∎

2.2.1 Lie Algebra Structure on $\mathcal{O}$

Consider the map

$$L : \mathfrak{g} \longrightarrow \mathcal{O},$$

$$L(V_g) = V_g(z) = V_g(x) + iV_g(y) = g.$$

L is an isomorphism of vector spaces over $\mathbb{R}$. Since $\mathfrak{g}$ is a Lie algebra we are able to introduce a Lie algebra structure on the space of analytic functions. Namely, define the bracket on $\mathcal{O}$ by the following rule

$$[V_v, V_w] \stackrel{def}{=} V_{[v,w]}.$$

In coordinates

$$\begin{aligned}[V_v, V_w] &= V_v(V_w) - V_w(V_v) \\ &= (v_1 w_{1x} - v_2 w_{2x} - w_1 v_{1x} + w_2 v_{2x})\partial_x + (v_1 w_{2x} + v_2 w_{1x} - w_1 v_{2x} - w_2 v_{1x})\partial_y.\end{aligned}$$

Hence the bracket on $\mathcal{O}$ is

$$\begin{aligned}[v, w] &= [V_v, V_w](z) = V_v(w) - V_w(v) \\ &= (v_1 w_{1x} - v_2 w_{2x} - w_1 v_{1x} + w_2 v_{2x}) + i(v_1 w_{2x} + v_2 w_{1x} - w_1 v_{2x} - w_2 v_{1x}).\end{aligned}$$

Note that the formula for the bracket on $\mathcal{O}$ in complex coordinates is

$$\{f(z), g(z)\} = f(z)g'(z) - f'(z)g(z). \tag{2.2}$$

This leads to an isomorphism of the space $\mathcal{O}$ equipped with the bracket defined in Equation (2.2) with the space of linear in momenta holomorphic functions on $T^*\mathbb{C}$ equipped with the standard Poisson structure.

The Lie algebra $(\mathcal{O}, \{\})$ is simple, i.e. it contains no ideals, but it does contain subalgebras. For instance, consider the subspace

$$\mathrm{sl}_2 = \left\langle 1, z, z^2 \right\rangle \subset \mathcal{O}.$$

The space sl_2 is a linear subspace of $\mathcal{O}$. Moreover, for $j, k \in \{0, 1, 2\}$

$$\left\{ z^j, z^k \right\} = (k - j) z^{k+j-1} \in \mathrm{sl}_2\,.$$

Hence sl_2 is a Lie subalgebra of $\mathcal{O}$ isomorphic to $\mathrm{sl}_2(\mathbb{C})$. It follows that the subspace

$$\langle V_1, V_i, V_z, V_{iz}, V_{z^2}, V_{iz^2} \rangle \subset \mathfrak{g}$$

is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathrm{sl}_2(\mathbb{C})_{\mathbb{R}}$.

2.2.2 The Manifold $\mathbb{C}$

In this subsection we will use the theory of Section 2.1 for the manifold $\mathbb{C}$. Note that $\mathbb{C}$ is a complex manifold of dimension 1, and $\mathbb{R}^2 \simeq \mathbb{C}$ is a real manifold of dimension 2.

The space $C^\infty(\mathbb{R}^2) \otimes \mathbb{C}$ is an algebra with subalgebras $C^\infty(\mathbb{R}^2)$ and $\mathcal{O}$.

For any point $z_0 \in \mathbb{C}$ we have the following vector spaces

$$T_{z_0}\mathbb{R}^2 = \langle \partial_x|_{z_0}, \partial_y|_{z_0} \rangle_{\mathbb{R}} ,$$

$$T^{1,0}_{z_0}\mathbb{C} = \langle \partial_z|_{z_0} \rangle_{\mathbb{C}} , \quad T^{0,1}_{z_0}\mathbb{C} = \langle \partial_{\bar{z}}|_{z_0} \rangle_{\mathbb{C}} ,$$

$$T_{z_0}\mathbb{R}^2 \otimes \mathbb{C} = \langle \partial_x|_{z_0}, \partial_y|_{z_0} \rangle_{\mathbb{C}} = \langle \partial_z|_{z_0}, \partial_{\bar{z}}|_{z_0} \rangle_{\mathbb{C}} .$$

The spaces of smooth sections of the vector bundles $T\mathbb{R}^2$, $T^{1,0}\mathbb{C}$, $T^{0,1}\mathbb{C}$ and $T\mathbb{R}^2 \otimes \mathbb{C}$ are

$$\mathcal{D}\left(\mathbb{R}^2\right) = C^\infty(T\mathbb{R}^2) = \left\{ f_1\partial_x + f_2\partial_y \mid f_1, f_2 \in C^\infty\left(\mathbb{R}^2\right) \right\} ,$$

$$\mathcal{D}^{1,0}\left(\mathbb{C}\right) = C^\infty(T^{1,0}\mathbb{C}) = \left\{ f\partial_z \mid f \in C^\infty(\mathbb{R}^2) \otimes \mathbb{C} \right\} ,$$

$$\mathcal{D}^{0,1}\left(\mathbb{C}\right) = C^\infty(T^{0,1}\mathbb{C}) = \left\{ f\partial_{\bar{z}} \mid f \in C^\infty(\mathbb{R}^2) \otimes \mathbb{C} \right\} ,$$

$$\mathcal{D}\left(\mathbb{R}^2\right) \otimes \mathbb{C} = C^\infty(T\mathbb{R}^2) \otimes \mathbb{C} = \left\{ f_1\partial_x + f_2\partial_y \mid f_1, f_2 \in C^\infty(\mathbb{R}^2) \otimes \mathbb{C} \right\} .$$

Let $\mathfrak{h}$ denote the space of all derivations of the space $\mathcal{O}$

$$\mathfrak{h} = \{ g\partial_z \mid g \in \mathcal{O} \} .$$

The space $\mathfrak{h}$ is a free 1-dimensional module over $\mathcal{O}$.

The spaces $\mathcal{D}\left(\mathbb{R}^2\right)$, $\mathcal{D}^{1,0}\left(\mathbb{C}\right)$, $\mathcal{D}^{0,1}\left(\mathbb{C}\right)$ and $\mathfrak{h}$ are infinite dimensional Lie subalgebras of $\mathcal{D}\left(\mathbb{R}^2\right) \otimes \mathbb{C}$.

Consider the space of anti-holomorphic functions

$$\bar{\mathcal{O}}=\left\{h = h_1 + ih_2 \in C^{\infty}\left(\mathbb{R}^2\right)\otimes\mathbb{C} \mid h_{1x} = -h_{2y},\ h_{2x} = h_{1y}\right\} = \{\bar{g} \mid g \in \mathcal{O}\}.$$

The space $\bar{\mathcal{O}}$ is a subalgebra of $C^{\infty}(\mathbb{R}^2)\otimes\mathbb{C}$.

Let $\bar{\mathfrak{h}}$ denote the space of all derivations of $\bar{\mathcal{O}}$

$$\bar{\mathfrak{h}} = \{\bar{g}\partial_{\bar{z}} \mid g \in \mathcal{O}\}.$$

The space $\bar{\mathfrak{h}}$ is a free 1-dimensional module over $\bar{\mathcal{O}}$ and an infinite dimensional Lie subalgebra of $\mathcal{D}\left(\mathbb{R}^2\right)\otimes\mathbb{C}$.

2.2.3 Almost Complex Structure on $\mathbb{TR}^2$

The tensor

$$J = \partial_y \otimes dx - \partial_x \otimes dy$$

is an almost complex structure on $\mathbb{TR}^2$

$$J(\partial_x) = \partial_y, \quad J(\partial_y) = -\partial_x.$$

If the vector field

$$V = g_1\partial_x + g_2\partial_y \in \mathcal{D}(\mathbb{R}^2)$$

is a symmetry of the tensor J, then

$$\begin{aligned} L_V(J) &= -[V,\partial_x]\otimes dy - \partial_x\otimes d(g_2) + [V,\partial_y]\otimes dx + \partial_y \otimes d(g_1) \\ &= -(g_{2x}+g_{1y})\,\partial_x\otimes dx + (-g_{2y}+g_{1x})\,\partial_y\otimes dx \\ &\quad + (g_{1x}-g_{2y})\,\partial_x\otimes dy + (g_{2x}+g_{1y})\,\partial_y\otimes dy = 0. \end{aligned}$$

Hence $\mathfrak{g}$ is the Lie algebra of symmetries of the tensor J. This shows that there must exist a Lie algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{h}$. In the next subsection we will find it.

2.2.4 A Relation Between the Lie Algebras $\mathfrak{g}$ and $\mathfrak{h}$

Consider the $\mathbb{R}-$linear map

$$2\,\mathrm{Re}:\mathcal{D}\left(\mathbb{R}^2\right)\otimes\mathbb{C}\longrightarrow\mathcal{D}\left(\mathbb{R}^2\right).$$

We have that

$$2\,\mathrm{Re}\left[ix\partial_x, i\partial_x\right]=2\partial_x\neq\left[2\,\mathrm{Re}\left(ix\partial_x\right), 2\,\mathrm{Re}\left(i\partial_x\right)\right]=0.$$

Hence $2\,\mathrm{Re}$ is not a Lie algebra homomorphism between the Lie algebras $\mathcal{D}\left(\mathbb{R}^2\right)\otimes\mathbb{C}$ and $\mathcal{D}\left(\mathbb{R}^2\right)$. We have that

$$g\partial_z=\tfrac{1}{2}(g_1\partial_x+g_2\partial_y+i(-g_1\partial_y+g_2\partial_x)).$$

Thus the map $2\,\mathrm{Re}$ restricted to $\mathfrak{h}$ is

$$2\,\mathrm{Re}\left(g\partial_z\right)=V_g.$$

Proposition 6 *The $\mathbb{R}-$linear map*

$$2\,\mathrm{Re}:\mathfrak{h}\longrightarrow\mathfrak{g}$$

is a Lie algebra isomorphism.

Proof. Using the Poisson bracket defined on $\mathcal{O}$ in Subsection 2.2.1 we get

$$[2\,\mathrm{Re}(g\partial_z), 2\,\mathrm{Re}(f\partial_z)]=[V_g, V_f]=V_{\{g,f\}}=2\,\mathrm{Re}((g_zf-f_zg)\partial_z)=2\,\mathrm{Re}\left[g\partial_z, f\partial_z\right].$$

Hence $2\,\mathrm{Re}$ preserves the bracket.

By definition $V_g\in\mathfrak{g}$ if and only if $g\in\mathcal{O}$, i.e. $g\partial_z\in\mathfrak{h}$. Hence $2\,\mathrm{Re}$ is bijective. ■

Example 7 *It was shown in Subsection 2.2.1 that*

$$\langle V_1, V_i, V_z, V_{iz}, V_{z^2}, V_{iz^2} \rangle \subset \mathfrak{g}$$

is a Lie algebra. Hence

$$\mathfrak{s} = \left\langle \partial_z, z\partial_z, z^2\partial_z \right\rangle \subset \mathfrak{h}$$

is a Lie algebra. Moreover, $\mathfrak{s}$ *is isomorphic to* $\mathrm{sl}_2(\mathbb{C})$.

The $\mathbb{R}-$linear map

$$2\,\mathrm{Im} : \mathfrak{h} \longrightarrow \mathfrak{g},$$

$$2\,\mathrm{Im}\,(g\partial_z) = -V_{ig},$$

is an isomorphism of vector spaces over $\mathbb{R}$. It follows from Proposition 6 that for any functions $g, h \in \mathcal{O}$

$$[2\,\mathrm{Im}(g\partial_z), 2\,\mathrm{Im}(h\partial_z)] = [2\,\mathrm{Re}(ig\partial_z), 2\,\mathrm{Re}(ih\partial_z)] = -2\,\mathrm{Re}\,[g\partial_z, h\partial_z] = -2\,\mathrm{Im}\,(i\,[g\partial_z, h\partial_z])\,.$$

Hence the map is not a Lie algebra isomorphism.

The complexification of the Lie algebra $\mathfrak{g}$ and the direct sum of the Lie algebras $\mathfrak{h}$ and $\bar{\mathfrak{h}}$

$$\begin{aligned} \mathfrak{g} \otimes \mathbb{C} &= \{V_g + iV_h \mid g, h \in \mathcal{O}\}\,, \\ \mathfrak{h} \oplus \bar{\mathfrak{h}} &= \left\{g\partial_z + \bar{h}\partial_{\bar{z}} \mid g, h \in \mathcal{O}\right\}, \end{aligned}$$

are Lie subalgebras of $\mathcal{D}(\mathbb{R}^2)\otimes\mathbb{C}$.

Theorem 8 *The map*

$$\psi : \mathfrak{h} \oplus \bar{\mathfrak{h}} \longrightarrow \mathfrak{g} \otimes \mathbb{C},$$

$$\psi(g\partial_z + \bar{h}\partial_{\bar{z}}) = \mathrm{Re}(g\partial_z + \bar{h}\partial_{\bar{z}}) + i\,\mathrm{Im}(g\partial_z + \bar{h}\partial_{\bar{z}}) = \tfrac{1}{2}\left(V_{g+h} + iV_{i(h-g)}\right),$$

is a $\mathbb{C}-$*linear Lie algebra isomorphism.*

Proof. For any functions $g, h \in \mathcal{O}$

$$\begin{aligned} \psi(i\left(g\partial_z + \bar{h}\partial_{\bar{z}}\right)) &= \psi(ig\partial_z - \overline{ih}\partial_{\bar{z}}) = \tfrac{1}{2}\left(V_{ig-ih} + iV_{i(-ig-ih)}\right) \\ &= \tfrac{i}{2}\left(V_{g+h} + iV_{i(h-g)}\right) = i\psi(g\partial_z + \bar{h}\partial_{\bar{z}}). \end{aligned}$$

Hence the map ψ is $\mathbb{C}-$linear.

For any function $g \in \mathcal{O}$

$$\psi(g\partial_z + \bar{g}\partial_{\bar{z}}) = V_g.$$

Hence ψ is bijective.

We see that

$$\begin{aligned} \psi(\left[g\partial_z + \bar{h}\partial_{\bar{z}}, f\partial_z + \bar{q}\partial_{\bar{z}}\right]) &= \psi([g\partial_z, f\partial_z] + \left[\bar{h}\partial_{\bar{z}}, \bar{q}\partial_{\bar{z}}\right]) \\ &= \tfrac{1}{2}\left(V_{[g,f]+[h,q]} + iV_{i(-[g,f]+[h,q])}\right), \\ \left[\psi\left(g\partial_z + \bar{h}\partial_{\bar{z}}\right), \psi\left(f\partial_z + \bar{q}\partial_{\bar{z}}\right)\right] &= \left[\tfrac{1}{2}\left(V_{g+h} + iV_{i(h-g)}\right), \tfrac{1}{2}\left(V_{f+q} + iV_{i(q-f)}\right)\right] \\ &= \tfrac{1}{4}\left(V_{[g+h,f+q]+[h-g,q-f]} + iV_{i([g+h,q-f]+[h-g,q+f])}\right) \\ &= \tfrac{1}{2}\left(V_{[g,f]+[h,q]} + iV_{i(-[g,f]+[h,q])}\right). \end{aligned}$$

Hence ψ preserves the bracket. ∎

Chapter 3

Invariant Functions of the Lie Algebra $\mathfrak{g}^k$

3.1 The Space of Jets

3.1.1 Quotient Algebras

For any point $z_0 = x_0 + iy_0 \in \mathbb{C}$ the space

$$\mu_{z_0} = \left\{ f \in C^\infty\left(\mathbb{R}^2\right) \mid f(x_0, y_0) = 0 \right\}$$

is a maximal ideal of the algebra $C^\infty\left(\mathbb{R}^2\right)$.

The space

$$\left(\mu_{z_0}\right)^{k+1} = \left\{ f \in C^\infty\left(\mathbb{R}^2\right) \mid f = \sum f_1 ... f_{k+1},\ f_j \in \mu_{z_0} \right\} \tag{3.1}$$

is an ideal of $C^\infty\left(\mathbb{R}^2\right)$ for any integer $k \in \mathbb{Z}_{\geq 0}$. It follows from Equation (3.1) that

$$\left(\mu_{z_0}\right)^{k+1} \subset \left(\mu_{z_0}\right)^{k} ... \subset \left(\mu_{z_0}\right)^{2} \subset \mu_{z_0}.$$

Hence for $k > 0$ the ideal $\left(\mu_{z_0}\right)^{k+1}$ is not maximal.

The quotient space

$$C^{\infty}\left(\mathbb{R}^2\right) / \left(\mu_{z_0}\right)^{k+1}$$

is an $\mathbb{R}-$algebra.

For any smooth function $f(x,y) \in C^{\infty}\left(\mathbb{R}^2\right)$ the corresponding equivalence class $[f(x,y)]^k_{z_0} \in C^{\infty}\left(\mathbb{R}^2\right) / \left(\mu_{z_0}\right)^{k+1}$ has the following representative

$$[f]^k_{z_0} \equiv f(x_0, y_0) + \sum_{m+n\leq k} \frac{m!n!}{(m+n)!} \frac{\partial^{m+n} f}{\partial x^m \partial y^n}(x_0, y_0)(x - x_0)^m (y - y_0)^n.$$

3.1.2 Algebra of Functions on the Space of Jets

For any pair of integers $m, n \in \mathbb{Z}_{\geq 0}$ such that $m + n \leq k$ there exists an $\mathbb{R}-$linear map

$$u_{mn} : C^{\infty}\left(\mathbb{R}^2\right) / \mu^k_{z_0} \longrightarrow \mathbb{R},$$

$$u_{mn}([f]^k_{z_0}) = \frac{\partial^{m+n} f}{\partial x^m \partial y^n}(x_0, y_0).$$

The space

$$J^k_{z_0}\mathbb{R}^2 = C^{\infty}\left(\mathbb{R}^2\right) / \left(\mu_{z_0}\right)^{k+1}$$

is a real vector space of dimension $(k+1)(k+2)/2$

$$J^k_{z_0}\mathbb{R}^2 \simeq \langle u_{mn} \mid m + n \leq k \rangle_{\mathbb{R}}.$$

Consider the space

$$J^k\mathbb{R}^2 = \bigcup_{z_0 \in \mathbb{C}} J^k_{z_0}\mathbb{R}^2.$$

By standard topological arguments $J^k\mathbb{R}^2$ is a vector bundle over $\mathbb{C}$. Its total space is diffeomorphic to

$$J^k\mathbb{R}^2 \simeq \mathbb{R}^{(k+1)(k+2)/2+2}(x, y, u_{mn}|m+n \leq k).$$

Consider the following spaces

$$C^\infty\left(J^k\mathbb{R}^2\right) = \left\{f : J^k\mathbb{R}^2 \to \mathbb{R} \mid f \text{ is smooth}\right\},$$

$$C^\infty\left(J^k\mathbb{R}^2, \mathbb{C}\right) = \left\{f : J^k\mathbb{R}^2 \to \mathbb{C} \mid f \text{ is smooth}\right\}.$$

The space $C^\infty\left(J^k\mathbb{R}^2\right)$ is an algebra over $\mathbb{R}$ and $C^\infty\left(J^k\mathbb{R}^2, \mathbb{C}\right)$ is an algebra over $\mathbb{C}$.

The algebra $C^\infty\left(J^k\mathbb{R}^2, \mathbb{C}\right)$ is equal to the tensor product

$$C^\infty\left(J^k\mathbb{R}^2, \mathbb{C}\right) = C^\infty\left(J^k\mathbb{R}^2\right) \otimes \mathbb{C}.$$

The inclusion map

$$I : C^\infty\left(J^k\mathbb{R}^2\right) \hookrightarrow C^\infty\left(J^k\mathbb{R}^2\right) \otimes \mathbb{C},$$

where

$$\operatorname{Re} I = \operatorname{Im}(iI) = \operatorname{Id}_{C^\infty\left(J^k\mathbb{R}^2\right)},$$

is an injective $\mathbb{R}-$algebra homomorphism. Hence $C^\infty\left(J^k\mathbb{R}^2\right)$ is an $\mathbb{R}-$subalgebra of $C^\infty\left(J^k\mathbb{R}^2\right) \otimes \mathbb{C}$.

3.1.3 The Tangent- and the Complexified Tangent Bundle of $J^k\mathbb{R}^2$

For any point $p \in J^k\mathbb{R}^2$ the space

$$T_p\left(J^k\mathbb{R}^2\right) = \operatorname{Der}_{\mathbb{R}}\left(C^\infty\left(J^k\mathbb{R}^2\right)\right)_p$$

is a real vector space. It follows from Lemma 2 that

$$T_p\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}=\mathrm{Der}_{\mathbb{C}}\left(C^\infty\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}\right)_p.$$

The real dimension of $T_p\left(J^k\mathbb{R}^2\right)$ is $(k+1)(k+2)/2+2$

$$T_p\left(J^k\mathbb{R}^2\right)=\langle\partial_x|_p,\partial_y|_p,\partial_{u_{nm}}|_p \mid m+n\leq k\rangle_{\mathbb{R}}.$$

The complex dimension of $T_p\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$ is $(k+1)(k+2)/2+2$

$$T_p\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}=\langle\partial_x|_p,\partial_y|_p,\partial_{u_{nm}}|_p \mid m+n\leq k\rangle_{\mathbb{C}}.$$

The inclusion map

$$I:T_p\left(J^k\mathbb{R}^2\right)\hookrightarrow T_p\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C},$$

where

$$\mathrm{Re}\,I=\mathrm{Im}(iI)=\mathrm{Id}_{T_p\left(J^k\mathbb{R}^2\right)},$$

is an injective $\mathbb{R}-$linear map. Hence $T_p\left(J^k\mathbb{R}^2\right)$ is an $\mathbb{R}-$linear subspace of $T_p\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$.

Consider the spaces

$$T\left(J^k\mathbb{R}^2\right)=\bigcup_{p\in J^k\mathbb{R}^2}T_p\left(J^k\mathbb{R}^2\right),$$

$$T\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}=\bigcup_{p\in J^k\mathbb{R}^2}T_p\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}.$$

The space $T\left(J^k\mathbb{R}^2\right)$ is an $\mathbb{R}-$vector bundle and $T\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$ is a $\mathbb{C}-$vector bundle over $J^k\mathbb{R}^2$. The bundle $T\left(J^k\mathbb{R}^2\right)$ is an $\mathbb{R}-$subbundle of $T\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$.

3.1.4 Vector Fields on $J^k\mathbb{R}^2$

Consider the spaces of all smooth sections of the vector bundles $T\left(J^k\mathbb{R}^2\right)$ and $T\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$

$$\mathcal{D}\left(J^k\mathbb{R}^2\right) = C^\infty(T\left(J^k\mathbb{R}^2\right)) = \mathrm{Der}_{\mathbb{R}}\left(C^\infty\left(J^k\mathbb{R}^2\right)\right),$$

$$\mathcal{D}\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C} = C^\infty(T\left(J^k\mathbb{R}^2\right),\mathbb{C}) = \mathrm{Der}_{\mathbb{R}}\left(C^\infty\left(J^k\mathbb{R}^2\right)\right)\otimes\mathbb{C}.$$

The space $\mathcal{D}\left(J^k\mathbb{R}^2\right)$ is a module over the algebra $C^\infty\left(J^k\mathbb{R}^2\right)$, and the space $\mathcal{D}\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$ is a module over the algebra $C^\infty\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$

$$\mathcal{D}(J^k\mathbb{R}^2) = \left\{\tilde{f}_1\partial_x + \tilde{f}_2\partial_y + \sum_{m+n\leq k} f_{mn}\partial_{u_{mn}} \mid \tilde{f}_1,\tilde{f}_2,f_{mn}\in C^\infty\left(J^k\mathbb{R}^2\right)\right\},$$

$$\mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C} = \left\{\tilde{f}_1\partial_x + \tilde{f}_2\partial_y + \sum_{m+n\leq k} f_{mn}\partial_{u_{mn}} \mid \tilde{f}_1,\tilde{f}_2,f_{mn}\in C^\infty(J^k\mathbb{R}^2)\otimes\mathbb{C}\right\}.$$

The spaces $\mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C}$ and $\mathcal{D}(J^k\mathbb{R}^2)$ are infinite dimensional Lie algebras with the Lie bracket being the commutator. The inclusion map

$$I : \mathcal{D}(J^k\mathbb{R}^2) \hookrightarrow \mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C},$$

where

$$\mathrm{Re}\, I = \mathrm{Im}(iI) = \mathrm{Id}_{\mathcal{D}(J^k\mathbb{R}^2)},$$

is an injective Lie algebra homomorphism. Hence $\mathcal{D}(J^k\mathbb{R}^2)$ is an infinite dimensional Lie subalgebra of $\mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C}$.

3.2 The Contact Distribution and the Cartan Distribution

3.2.1 The Contact Distribution on $J^1\mathbb{R}^2$

The 4−dimensional distribution on $J^1\mathbb{R}^2$

$$C_0 = \mathrm{Ker}(\omega_0),\ \omega_0 = du - u_{10}dx - u_{01}dy$$

is called the contact distribution. The distribution is spanned by the four vector fields

$$C_0 = \langle X_1 = \partial_x + u_{10}\partial_u, X_2 = \partial_y + u_{01}\partial_u, Y_1 = \partial_{u_{10}}, Y_2 = \partial_{u_{01}} \rangle\,. \tag{3.2}$$

There exists no integral manifold of dimension four, since

$$[Y_j, X_j] = \partial_u \notin C_0,\ j \in \{1,2\}.$$

Every smooth function $f \in C^\infty(\mathbb{R}^2)$ determines a 2−dimensional submanifold of $J^1\mathbb{R}^2$

$$L_f = \left\{ u = f(x,y),\ u_{10} = \frac{\partial f}{\partial x}(x,y),\ u_{01} = \frac{\partial f}{\partial y}(x,y) \right\}, \tag{3.3}$$

which is an integral manifold of the contact distribution since

$$\omega_0|_{L_f} = 0.$$

3.2.2 Contact Transformations and Contact Vector Fields

A diffeomorphism

$$F : J^1\mathbb{R}^2 \longrightarrow J^1\mathbb{R}^2$$

is called a contact transformation if it preserves the contact distribution, i.e.

$$F^*(\omega_0) = \lambda_F \omega_0,\ \lambda_F \in C^\infty(J^1\mathbb{R}^2).$$

A vector field $X \in \mathcal{D}(J^1\mathbb{R}^2)$ is called a contact vector field if its flow consists of contact transformations. If X is a contact vector field, then

$$L_X(\omega_0) = \lambda_X \omega_0,\ \ \lambda_X \in C^\infty(J^1\mathbb{R}^2).$$

It is known that all contact vector fields on $J^1\mathbb{R}^2$ have the form

$$X_f = f\partial_u + X_1(f)Y_1 + X_2(f)Y_2 - Y_1(f)X_1 - Y_2(f)X_2,$$

where X_j and Y_j are given in Equation (3.2) and the function $f \in C^\infty(J^1\mathbb{R}^2)$ is equal to

$$f = \omega_0(X_f).$$

The space of all contact vector fields is an infinite dimensional Lie algebra denoted $\mathrm{Cont}(J^1\mathbb{R}^2)$.

Consider the subspace of $\mathcal{D}(J^1\mathbb{R}^2)\otimes\mathbb{C}$ that consists of all the complexified vector fields that preserve the contact distribution

$$\left\{\ Y \in \mathcal{D}(J^1\mathbb{R}^2)\otimes\mathbb{C} \mid L_Y(\omega_0) = \lambda_Y \omega_0,\ \ \lambda_Y \in C^\infty(J^1\mathbb{R}^2)\otimes\mathbb{C}\right\} \subset \mathcal{D}(J^1\mathbb{R}^2)\otimes\mathbb{C}. \tag{3.4}$$

For any vector field $Y \in \mathcal{D}(J^1\mathbb{R}^2)\otimes\mathbb{C}$ there exist vector fields $Y_1, Y_2 \in \mathcal{D}(J^1\mathbb{R}^2)$ such that $Y = Y_1 + iY_2$. Hence if Y preserve the contact distribution, then

$$L_Y(\omega_0) = L_{Y_1}(\omega_0) + iL_{Y_2}(\omega_0) = \lambda_Y \omega_0.$$

It follows that Y_1 and Y_2 are contact vector fields. Hence there exist functions $f_1, f_2 \in C^\infty(J^1\mathbb{R}^2)$ and $f = f_1 + if_2 \in C^\infty(J^1\mathbb{R}^2)\otimes\mathbb{C}$, such that

$$Y = Y_1 + iY_2 = X_{f_1} + iX_{f_2} = X_f.$$

So the subspace described in Equation (3.4) is the complexification of the Lie algebra of contact vector fields

$$\mathrm{Cont}(J^1\mathbb{R}^2)\otimes\mathbb{C} = \left\{\ Y \in \mathcal{D}(J^1\mathbb{R}^2)\otimes\mathbb{C} \mid Y = X_f,\ \ f \in C^\infty(J^1\mathbb{R}^2)\otimes\mathbb{C}\right\}.$$

The inclusion map

$$I : \mathrm{Cont}(J^1\mathbb{R}^2) \hookrightarrow \mathrm{Cont}(J^1\mathbb{R}^2)\otimes\mathbb{C},$$

where

$$\mathrm{Re}\, I = \mathrm{Im}(iI) = \mathrm{Id}_{\mathrm{Cont}(J^1\mathbb{R}^2)},$$

is an injective Lie algebra homomorphism. Hence $\mathrm{Cont}(J^1\mathbb{R}^2)$ is a Lie subalgebra of $\mathrm{Cont}(J^1\mathbb{R}^2)\otimes\mathbb{C}$.

3.2.3 Prolongation of $\mathcal{D}(J^0\mathbb{R}^2)$ and $\mathrm{Cont}(J^1\mathbb{R}^2)$

Consider the vector field $W_1 = a_1\partial_x + b_1\partial_y + c_1\partial_u \in \mathcal{D}(J^0\mathbb{R}^2)$ and the complex vector field $W_2 = a_2\partial_x + b_2\partial_y + c_2\partial_u \in \mathcal{D}(J^0\mathbb{R}^2)\otimes\mathbb{C}$. The first prolongation of W_1 is $X_{f_1} \in \mathrm{Cont}(J^1\mathbb{R}^2)$ and the first prolongation of W_2 is $X_{f_2} \in \mathrm{Cont}(J^1\mathbb{R}^2)\otimes\mathbb{C}$, where

$$f_j = c_j - a_j u_{10} - b_j u_{01},\ j \in \{1, 2\}.$$

It is known [KL1] that the k^{th} prolongation of the vector fields $X_{f_1} \in \mathrm{Cont}(J^1\mathbb{R}^2)$ and $X_{f_2} \in \mathrm{Cont}(J^1\mathbb{R}^2)\otimes\mathbb{C}$ is

$$X^{(k)}_{f_j} = \sum_{m=0}^{k}\sum_{n=0}^{k-m} \mathcal{D}_x^m \mathcal{D}_y^n (f_j)\partial_{u_{mn}} - \partial_{u_1}(f_j)\mathcal{D}_x|_{J^k} - \partial_{u_2}(f_j)\mathcal{D}_y|_{J^k},\ j \in \{1, 2\},$$

where

$$\begin{aligned}
\mathcal{D}_x &= \partial_x + \sum_{m,n\geq 0} u_{(m+1)n}\partial_{u_{mn}}, \quad \mathcal{D}_y = \partial_y + \sum_{m,n\geq 0} u_{m(n+1)}\partial_{u_{mn}}, \\
\mathcal{D}_x|_{J^k} &= \partial_x + \sum_{m=0}^{k}\sum_{n=0}^{k-m} u_{(m+1)n}\partial_{u_{mn}}, \quad \mathcal{D}_x|_{J^k} = \partial_y + \sum_{m=0}^{k}\sum_{n=0}^{k-m} u_{m(n+1)}\partial_{u_{mn}}.
\end{aligned}$$

3.2.4 The Cartan Distribution on $J^k\mathbb{R}^2$

The distribution on $J^k\mathbb{R}^2$

$$C_k = \mathrm{Ker}(\omega_{mn} \mid m+n<k),\ \omega_{mn} = du_{mn} - u_{(m+1)n}dx - u_{m(n+1)}dy$$

is called the Cartan distribution. Note that when $k=1$ the Cartan distribution is the contact distribution.

It is known [KLV] that if $L \subset J^k\mathbb{R}^2$ is an integral manifold of the Cartan distribution such that the map

$$\pi_k : L \longrightarrow \mathbb{R}^2$$

is a diffeomorphism, then there exists a unique function $h \in C^\infty(\mathbb{R}^2)$ such that L is equal to the k^{th} prolongation of the integral manifold L_h defined in Equation (3.3)

$$L = L_h^{(k)}.$$

3.2.5 Lie Transformations and Lie Vector Fields

A diffeomorphism

$$F : J^k\mathbb{R}^2 \longrightarrow J^k\mathbb{R}^2$$

is called a Lie transformation of $J^k\mathbb{R}^2$ if for any pair of integers $i,j \in \mathbb{Z}_{\geq 0}$ with $i+j<k$

$$F^*(\omega_{ij}) \equiv 0 \,(\mathrm{mod}\,\langle \omega_{nm} \mid m+n<k\rangle)\,.$$

A vector field $X \in \mathcal{D}(J^k\mathbb{R}^2)$ is called a Lie vector field on $J^k\mathbb{R}^2$ if its flow consists of Lie transformations. Let $\mathrm{Lie}(J^k\mathbb{R}^2)$ denote the space of all Lie vector fields on $J^k\mathbb{R}^2$. If $Y \in \mathrm{Lie}(J^k\mathbb{R}^2)$, then

$$L_Y(\omega_{ij}) = \sum_{m+n<k} \lambda_{Y_{mn}}\omega_{mn},\ \lambda_{Y_{mn}} \in C^\infty(J^k\mathbb{R}^2).$$

It follows from the Lie-Bäcklund theorem that all Lie transformations are prolongations of contact transformations, see [KLV]. Hence the space of Lie vector fields on $J^k\mathbb{R}^2$ is the k^{th} prolongation of the space of contact vector fields on $J^1\mathbb{R}^2$

$$\mathrm{Lie}(J^k\mathbb{R}^2) = \mathrm{Cont}(J^1\mathbb{R}^2)^k = \left\{ X_f^{(k)} \mid f \in C^\infty(J^1\mathbb{R}^2) \right\}.$$

Consider the subspace of $\mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C}$ that consists of all vector fields that preserve the Cartan distribution

$$\left\{ Y \in \mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C} \mid L_Y(\omega_{ij}) = \sum_{m+n<k} \lambda_{Y_{mn}}\omega_{mn}, \lambda_{Y_{mn}} \in C^\infty(J^k\mathbb{R}^2)\otimes\mathbb{C} \right\} \subset \mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C}. \tag{3.5}$$

For any vector field $Y \in \mathcal{D}(J^k\mathbb{R}^2)\otimes\mathbb{C}$ there exist vector fields $Y_1, Y_2 \in \mathcal{D}(J^k\mathbb{R}^2)$ such that $Y = Y_1 + iY_2$. Hence if Y preserve the Cartan distribution, then

$$L_Y(\omega_{ij}) = L_{Y_1}(\omega_{ij}) + iL_{Y_2}(\omega_{ij}) = \sum_{m+n<k} \lambda_{Y_{mn}}\omega_{mn}.$$

It follows that $Y_1, Y_2 \in \mathrm{Lie}(J^k\mathbb{R}^2)$. Hence the subspace described in Equation (3.5) is the complexification of $\mathrm{Lie}(J^k\mathbb{R}^2)$

$$\mathrm{Lie}(J^k\mathbb{R}^2)\otimes\mathbb{C} = \mathrm{Cont}(J^1\mathbb{R}^2)^k\otimes\mathbb{C} = \left(\mathrm{Cont}(J^1\mathbb{R}^2)\otimes\mathbb{C}\right)^k = \left\{ X_f^{(k)} \mid f \in C^\infty(J^1\mathbb{R}^2)\otimes\mathbb{C} \right\}.$$

The inclusion map

$$I : \mathrm{Lie}(J^k\mathbb{R}^2) \hookrightarrow \mathrm{Lie}(J^k\mathbb{R}^2)\otimes\mathbb{C},$$

where

$$\mathrm{Re}\, I = \mathrm{Im}(iI) = \mathrm{Id}_{\mathrm{Lie}(J^k\mathbb{R}^2)},$$

is an injective Lie algebra homomorphism. Hence $\mathrm{Lie}(J^k\mathbb{R}^2)$ is a Lie subalgebra of $\mathrm{Lie}(J^k\mathbb{R}^2)\otimes\mathbb{C}$.

3.2.6 Invariant Functions and Differential Invariants

Let $\mathfrak{f}$ be a Lie subalgebra of Cont $\left(J^1\mathbb{R}^2\right)$. The space of functions

$$\mathcal{F}_k = \left\{h \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid X_f^{(k)}(h) = 0,\ \forall X_f \in \mathfrak{f}\right\} \tag{3.6}$$

is the algebra of invariant functions under the action of $\mathfrak{f}$ on $C^\infty(J^k\mathbb{R}^2)$.

Let $\mathfrak{j}$ be a Lie subalgebra of Cont $\left(J^1\mathbb{R}^2\right) \otimes \mathbb{C}$. The space of functions

$$\mathcal{J}_k = \left\{h \in C^\infty_{loc}(J^k\mathbb{R}^2) \otimes \mathbb{C} \mid X_f^{(k)}(h) = 0,\ \forall X_f \in \mathfrak{j}\right\} \tag{3.7}$$

is the algebra of invariant functions under the action of $\mathfrak{j}$ on $C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C}$.

Proposition 9 *Let $\mathfrak{q}$ be any Lie subalgebra of* Cont $\left(J^1\mathbb{R}^2\right)$, *and let $\mathcal{Q}_k$ be the algebra of invariant functions under the action of $\mathfrak{q}$ on $C^\infty(J^k\mathbb{R}^2)$. Then the algebra of invariant functions under the action of $\mathfrak{q} \otimes \mathbb{C}$ on $C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C}$ is $\mathcal{Q}_k \otimes \mathbb{C}$.*

Proof. $\Longrightarrow$For all functions $h \in \mathcal{Q}_k \otimes \mathbb{C}$ there exist functions $h_1, h_2 \in \mathcal{Q}_k$ such that

$$h = h_1 + ih_2.$$

Hence for all contact vector fields $X_f \in \mathfrak{q}$

$$X_f^{(k)}(h) = X_f^{(k)}(h_1) + iX_f^{(k)}(h_2) = 0.$$

$\Longleftarrow$Suppose that the function $h \in C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C}$ is a $\mathfrak{q}-$differential invariant

$$X_f^{(k)}(h) = 0,\ \forall X_f \in \mathfrak{q}.$$

There exist functions $h_1, h_2 \in C^\infty(J^k\mathbb{R}^2)$ such that

$$h = h_1 + ih_2.$$

Hence

$$X_f^{(k)}(h) = X_f^{(k)}(h_1) + iX_f^{(k)}(h_2) = 0,\ \forall X_f \in \mathfrak{q}.$$

It follows that $h_1, h_2 \in \mathcal{Q}_k$ and

$$h \in \mathcal{Q}_k \otimes \mathbb{C}.$$

■

The projection map for any integer $k \in \mathbb{Z}_+$

$$\pi_{k,k-1} : J^k\mathbb{R}^2 \longrightarrow J^{k-1}\mathbb{R}^2$$

induces the following exact maps for any point $p \in J^k\mathbb{R}^2$

$$0 \longrightarrow C^\infty(J^{k-1}\mathbb{R}^2) \overset{\pi^*_{k,k-1}}{\longrightarrow} C^\infty(J^k\mathbb{R}^2),$$

$$T_p(J^k\mathbb{R}^2) \overset{(\pi_{k,k-1})_*}{\longrightarrow} T_p(J^{k-1}\mathbb{R}^2) \longrightarrow 0,$$

$$0 \longrightarrow C^\infty(J^{k-1}\mathbb{R}^2) \otimes \mathbb{C} \overset{\pi^*_{k,k-1}}{\longrightarrow} C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C},$$

$$T_p(J^k\mathbb{R}^2) \otimes \mathbb{C} \overset{(\pi_{k,k-1})_*}{\longrightarrow} T_p(J^{k-1}\mathbb{R}^2) \otimes \mathbb{C} \longrightarrow 0.$$

For any vector fields $X_1 \in \mathcal{D}(J^0\mathbb{R}^2)$ and $X_2 \in \mathcal{D}(J^0\mathbb{R}^2) \otimes \mathbb{C}$, the k^{th} prolongation of X_1 is a Lie vector field $X_1^{(k)} \in \text{Lie}(J^k\mathbb{R}^2)$ and the k^{th} prolongation of X_2 is the complexification of a Lie vector field $X_2^{(k)} \in \text{Lie}(J^k\mathbb{R}^2) \otimes \mathbb{C}$. Hence the vector fields X_1 and X_2 are $\left(\pi_{k,k-1}\right)_*$ −projectable. So for $j \in \{1, 2\}$

$$\left(\pi_{k,k-1}\right)_* X_j^{(k)} = X_j^{(k-1)}.$$

It follows that for any smooth functions $f_1 \in C^\infty(J^{k-1}\mathbb{R}^2)$ and $f_2 \in C^\infty(J^{k-1}\mathbb{R}^2) \otimes \mathbb{C}$

$$X_j^{(k-1)}(f_j) = \left(\pi_{k,k-1}\right)_* X_j^{(k)}(f_j) = X_j^{(k)}(\pi^*_{k,k-1} f_j),$$

for $j \in \{1,2\}$. Hence the map $\pi_{k,k-1}$ induces the canonical inclusions

$$(\pi_{k,k-1})_* \quad : \quad \mathcal{F}_{k-1} \hookrightarrow \mathcal{F}_k,$$

$$(\pi_{k,k-1})_* \quad : \quad \mathcal{J}_{k-1} \hookrightarrow \mathcal{J}_k,$$

where $\mathcal{F}_k$ and $\mathcal{J}_k$ are the algebras defined in Equation (3.6) and (3.7).

Definition 10 *The algebra of $\mathfrak{f}-$differential invariants is the following injective limit*

$$\mathcal{F} = \lim_{k\to\infty} \mathcal{F}_k = \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathcal{F}_k.$$

Definition 11 *The algebra of $\mathfrak{j}-$differential invariants is the following injective limit*

$$\mathcal{J} = \lim_{k\to\infty} \mathcal{J}_k = \bigcup_{k\in\mathbb{Z}_{\geq 0}} \mathcal{J}_k.$$

3.3 The Lie Algebra $\mathfrak{g}^k$

The inclusion maps

$$I_1 \quad : \quad \mathcal{D}(\mathbb{R}^2) \hookrightarrow \mathcal{D}(J^0\mathbb{R}^2),$$

$$I_2 \quad : \quad \mathcal{D}(\mathbb{R}^2) \otimes \mathbb{C} \hookrightarrow \mathcal{D}(J^0\mathbb{R}^2) \otimes \mathbb{C},$$

$$I_j(f_1\partial_x + f_2\partial_y) = f_1\partial_x + f_2\partial_y + 0\partial_u, \ j \in \{1,2\},$$

are injective Lie algebra homomorphisms. Hence any Lie subalgebra of $\mathcal{D}(\mathbb{R}^2)$ is a Lie subalgebra of $\mathcal{D}(J^0\mathbb{R}^2)$ and any Lie subalgebra of $\mathcal{D}(\mathbb{R}^2) \otimes \mathbb{C}$ is a Lie subalgebra of $\mathcal{D}(J^0\mathbb{R}^2) \otimes \mathbb{C}$.

Consider the following spaces for $k \in \mathbb{Z}_+$

$$\begin{aligned} \mathfrak{h}^k &= \left\{ (g\partial_z)^{(k)} \mid g \in \mathcal{O} \right\}, \\ \bar{\mathfrak{h}}^k &= \left\{ (\bar{g}\partial_{\bar{z}})^{(k)} \mid g \in \mathcal{O} \right\}, \\ \mathfrak{g}^k &= \left\{ V_g^{(k)} \mid g \in \mathcal{O} \right\}. \end{aligned}$$

The spaces $\mathfrak{h}^k$, $\bar{\mathfrak{h}}^k$ and $\mathfrak{g}^k \otimes \mathbb{C}$ are infinite dimensional Lie subalgebras of $\mathrm{Lie}(J^k\mathbb{R}^2) \otimes \mathbb{C}$, and $\mathfrak{g}^k$ is an infinite dimensional Lie subalgebra of $\mathrm{Lie}(J^k\mathbb{R}^2)$. If we consider $\mathfrak{g}^k \subset \mathrm{Lie}(J^k\mathbb{R}^2) \otimes \mathbb{C}$, then

$$(g\partial_z)^{(k)} = g\partial_z - \sum_{l=1}^{k} \frac{\partial^l g}{\partial z^l} \left(\sum_{m=l}^{k} \sum_{n=0}^{k-m} \binom{m}{l} u_{(m+1-l)\bar{n}} \partial_{u_{m\bar{n}}} \right), \tag{3.8}$$

$$V_g^{(k)} = (g\partial_z)^{(k)} + (\bar{g}\partial_{\bar{z}})^{(k)}. \tag{3.9}$$

Definition 12 *Let $\mathcal{H}_k$ denote the algebra of invariant functions under the action of $\mathfrak{h}$ on $C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C}$*

$$\mathcal{H}_k = \left\{ h \in C^\infty_{loc}(J^k\mathbb{R}^2) \otimes \mathbb{C} \mid (g\partial_z)^{(k)}(h) = 0, \ \forall g \in \mathcal{O} \right\}.$$

Definition 13 *Let $\mathcal{G}_k$ denote the algebra of invariant functions under the action of $\mathfrak{g}$ on $C^\infty(J^k\mathbb{R}^2)$*

$$\mathcal{G}_k = \left\{ h \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid V_g^{(k)}(h) = 0, \ \forall g \in \mathcal{O} \right\}.$$

Proposition 14 *The algebra of invariant functions under the action of $\mathfrak{g} \otimes \mathbb{C}$ on $C^\infty_{loc}(J^k\mathbb{R}^2) \otimes \mathbb{C}$ is*

$$\mathcal{G}_k \otimes \mathbb{C} = \bar{\mathcal{H}}_k \cap \mathcal{H}_k.$$

Proof. $\Longrightarrow$For any function $f \in \bar{\mathcal{H}}_k \cap \mathcal{H}_k$

$$V_g^{(k)}(f) = (g\partial_z)^{(k)}(f) + (\bar{g}\partial_{\bar{z}})^{(k)}(f) = 0, \ \forall g \in \mathcal{O}.$$

Hence

$$\mathcal{G}_k \otimes \mathbb{C} \supseteq \bar{\mathcal{H}}_k \cap \mathcal{H}_k.$$

$\Longleftarrow$For any function $f \in \mathcal{G}_k \otimes \mathbb{C}$

$$\begin{aligned} (V_g - iV_{ig})^{(k)}(f) &= 2(g\partial_z)^{(k)}(f) = 0, \ \forall g \in \mathcal{O}, \\ (V_g + iV_{ig})^{(k)}(f) &= 2(\bar{g}\partial_{\bar{z}})^{(k)}(f) = 0, \ \forall g \in \mathcal{O}. \end{aligned}$$

Hence

$$\mathcal{G}_k \otimes \mathbb{C} \subseteq \bar{\mathcal{H}}_k \cap \mathcal{H}_k.$$

■

3.3.1 The Distribution Π^k

It is known [KLR] that if M is a real $(n+m)-$dimensional smooth manifold and

$$\pi : TM \otimes \mathbb{C} \longrightarrow M$$

is the complexification of the tangent bundle, then a complex distribution P on M is a smooth field

$$P : a \in M \mapsto P_a = P(a) \subset T_aM \otimes \mathbb{C}$$

of complex subspaces of $\dim_{\mathbb{C}} P_a = m$.

A complex distribution P of rank m on a $(n+m)-$dimensional real manifold M is called completely integrable if it has locally n functionally independent first integrals, i.e. complex-valued functions $I_j \in C^{\infty}(M) \otimes \mathbb{C}$ such that

$$Ann(P) = \langle dI_1, .., dI_n \rangle_{\mathbb{C}}.$$

A complex distribution P is involutive if $[X, Y] \in \mathcal{D}(P)$ for any $X, Y \in \mathcal{D}(P)$.

Theorem 15 *[KLR] Let P be a complex involutive distribution such that $P+\bar{P}$ is an involutive distribution and $\dim_{\mathbb{C}}(P\cap\bar{P}) = const$. Then P is a completely integrable distribution.*

Prolongations of the holomorphic vector fields define the following complex distribution on $J^k\mathbb{R}^2$

$$\Pi^k = \left\langle (g\partial_z)^{(k)} \ \mid g \in \mathcal{O} \right\rangle_{\mathbb{C}}. \tag{3.10}$$

It follows from Equation (3.8) that

$$\Pi^k = \left\langle \partial_z, \sum_{m=l}^{k}\sum_{n=0}^{k-m} \binom{m}{l} u_{(m+1-l)\bar{n}} \partial_{u_{m\bar{n}}}, \mid l \in \{1, ..., k\} \right\rangle_{\mathbb{C}}. \tag{3.11}$$

Hence Π^k has complex dimension $k+1$.

The conjugate of the complex distribution Π^k is

$$\bar{\Pi}^k = \left\langle (\bar{g}\partial_{\bar{z}})^{(k)} \mid g \in \mathcal{O} \right\rangle_{\mathbb{C}} = \left\langle \partial_{\bar{z}}, \sum_{m=l}^{k}\sum_{n=0}^{k-m} \binom{m}{l} u_{n\overline{(m+1-l)}} \partial_{u_{n\bar{m}}}, \mid l \in \{1, ..., k\} \right\rangle_{\mathbb{C}}. \tag{3.12}$$

Since

$$\Pi^k \cap \bar{\Pi}^k = 0,$$

it follows that the complex distribution

$$\Pi^k \oplus \bar{\Pi}^k = \left\langle ((h\partial_z)^{(k)} + (\bar{g}\partial_{\bar{z}})^{(k)} \mid g, h \in \mathcal{O} \right\rangle_{\mathbb{C}} = \left\langle V_g^{(k)} \mid g \in \mathcal{O} \right\rangle_{\mathbb{R}} \otimes \mathbb{C} \tag{3.13}$$

has complex dimension $2\,(k+1)$.

Corollary 16 *The distribution Π^k is completely integrable.*

Proof. For any functions $g, h \in \mathcal{O}$

$$\left[(g\partial_z)^{(k)}, (h\partial_z)^{(k)}\right] = ((gh_z - hg_z)\partial_z)^{(k)} \in \Pi^k,$$

$$\left[V_g{}^{(k)}, V_h^{(k)}\right] = V_{[g,h]}^{(k)} \in \Pi^k \oplus \bar{\Pi}^k.$$

Hence the distributions Π^k and $\Pi^k \oplus \bar{\Pi}^k$ are involutive. So by Theorem 15 the distribution Π^k is completely integrable. ■

It follows from Equation (3.13) that the first integrals of the complex distribution $\Pi^k \oplus \bar{\Pi}^k$ are invariant functions under the action of $\mathfrak{g} \otimes \mathbb{C}$ on $C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C}$. By Proposition 9 the algebra of invariant functions under the action of $\mathfrak{g} \otimes \mathbb{C}$ on $C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C}$ is $\mathcal{G}_k \otimes \mathbb{C}$, where $\mathcal{G}_k$ is the algebra of invariant functions under the action of $\mathfrak{g}$ on $C^\infty(J^k\mathbb{R}^2)$. Therefore for $K = (k+1)(k+2)/2 + 2$, the complex distribution $\Pi^k \oplus \bar{\Pi}^k$ has locally $K - 2(k+1)$ functionally independent real first integrals

$$\{J_j\}_{j=1}^{K-2(k+1)} \in C^\infty_{loc}(J^k\mathbb{R}^2).$$

Consider the inclusion defined in Subsection 3.2.6

$$\mathcal{G}_k \hookrightarrow \mathcal{G}_{k+1}.$$

By the argument above the distribution $\Pi^k \oplus \bar{\Pi}^k$ has one first integral of order 0

$$I_0 \in C^\infty_{loc}(J^k\mathbb{R}^2)$$

and $l-1$ first integrals of order l

$$\{I_{l,j}\}_{j=2}^{l-1} \in C^\infty_{loc}(J^k\mathbb{R}^2)$$

for $1 \le l \le k$, such that locally the $K - 2(k+1)$ functions

$$\{I_0\} \bigcup_{2 \le l \le k} \{I_{l,j}\}_{j=1}^{l-1}$$

are functionally independent.

It follows from Equation (3.10) that the first integrals of the distribution Π^k are invariant functions under the action of $\mathfrak{h}$ on $C^\infty(J^k\mathbb{R}^2)\otimes\mathbb{C}$. By Equation (3.11) and (3.12)

$$\bar{z}, u_{0\bar{j}} \in C^\infty(J^k\mathbb{R}^2)\otimes\mathbb{C},\ 1 \le j \le k$$

are first integrals of the distribution Π^k and not first integrals of $\bar{\Pi}^k$. Hence locally the $K-(k+1)$ functionally independent functions

$$\{\bar{z}\} \cup \{u_{0\bar{j}}\}_{j=1}^{k} \cup \{I_0\} \bigcup_{2\le l\le k} \{I_{l,j}\}_{j=1}^{l-1}$$

are first integrals of the distribution Π^k.

The following table shows the number of locally functionally independent first integrals of pure order from 0 to k for the distributions Π^k, $\bar{\Pi}^k$ and $\Pi^k \oplus \bar{\Pi}^k$.

Order	Π^k	$\bar{\Pi}^k$	$\Pi^k \oplus \bar{\Pi}^k$
k	k	k	$k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
l	l	l	$l-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	0
0	2	2	1

The algebras of invariant functions under the action of $\mathfrak{h}$ and $\bar{\mathfrak{h}}$ on $C^\infty(J^k\mathbb{R}^2)\otimes\mathbb{C}$ and $\mathfrak{g}$ on $C^\infty(J^k\mathbb{R}^2)$ are

$$\begin{aligned}
\mathcal{H}_k &= \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2)\otimes\mathbb{C} \mid f = f(\bar{z}, u_{0\bar{1}}, ..., u_{0\bar{k}}, I_0, ..., I_{k,k-1}) \right\}, \\
\bar{\mathcal{H}}_k &= \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2)\otimes\mathbb{C} \mid f = f(z, u_{1\bar{0}}, ..., u_{1\bar{0}}, I_0, ..., I_{k,k-1}) \right\}, \\
\mathcal{G}_k &= \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid f = f(I_0, I_2, I_{3,1}, I_{3,2}, ..., I_{k,k-1}) \right\}.
\end{aligned}$$

3.3.2 Invariant Functions of Order 0, 1, 2 and 3

For any function $g \in \mathcal{O}$ the first, second and third prolongations of the vector fields $V_g \in \mathfrak{g}$ and $g\partial_z \in \mathfrak{h}$ are

$$V_g^{(1)} = g_1\partial_x + g_2\partial_y - (g_{1x}u_{10} - g_{1y}u_{01})\,\partial_{u_{10}} - (g_{1x}u_{01} + g_{1y}u_{10})\,\partial_{u_{01}},$$

$$(g\partial_z)^{(1)} = g\partial_z - g_z u_{1\bar{0}}\partial_{u_{1\bar{0}}},$$

$$\begin{aligned} V_g^{(2)} \;\; = \;\; & g_1\partial_x + g_2\partial_y - (g_{1x}u_{10} - g_{1y}u_{01})\,\partial_{u_{10}} - (g_{1x}u_{01} + g_{1y}u_{10})\,\partial_{u_{01}} \\ & +(-2u_{20}g_{1x} + 2u_{11}g_{1y} - u_{10}g_{1xx} + u_{01}g_{1xy})\partial_{u_{20}} \\ & +(-u_{20}g_{1y} - 2g_{1x}u_{11} + g_{1y}u_{02} - u_{10}g_{1xy} - u_{01}g_{1xx})\partial_{u_{11}} \\ & +(-2u_{11}g_{1y} - 2u_{02}g_{1x} + u_{10}g_{1xx} - u_{01}g_{1xy})\partial_{u_{02}}, \end{aligned}$$

$$(g\partial_z)^{(2)} = g\partial_z - g_z u_{1\bar{0}}\partial_{u_{1\bar{0}}} + (-g_{zz}u_{1\bar{0}} - 2g_z u_{2\bar{0}})\partial_{u_{2\bar{0}}} - g_z u_{1\bar{1}}\partial_{u_{1\bar{1}}},$$

$$\begin{aligned} V_g^{(3)} \;\; = \;\; & V_g^{(2)} + (-3g_{1xx}u_{20} + 3g_{1xy}u_{11} - 3g_{1x}u_{30} + 3g_{1y}u_{21} - g_{1xxx}u_{10} + g_{1xxy}u_{01})\partial_{u_{30}} \\ & +(-2g_{1xy}u_{20} - g_{1xx}3u_{11} - g_{1y}u_{30} + 2g_{1y}u_{12} - 3g_{1x}u_{21} + g_{1xy}u_{02} - g_{1xxy}u_{10} \\ & -g_{1xxx}u_{01})\partial_{u_{21}} \\ & +(g_{1xx}u_{20} - g_{1xy}3u_{11} - 3g_{1x}u_{12} - 2g_{1y}u_{21} + g_{1y}u_{03} - 2g_{1xx}u_{02} \\ & +g_{1xxx}u_{10} - g_{1xxy}u_{01})\partial_{u_{12}} \\ & +(3g_{1xx}u_{11} - 3g_{1xy}u_{02} - 3g_{1x}u_{03} - 3g_{1y}u_{12} + g_{1xxy}u_{10} + g_{1xxx}u_{01})\partial_{u_{03}}, \end{aligned}$$

$$(g\partial_z)^{(3)} = (g\partial_z)^{(2)} - (g_{zzz}u_{1\bar{0}} + 3g_{zz}u_{2\bar{0}} + 3g_z u_{3\bar{0}})\partial_{u_{3\bar{0}}} - (g_{zz}u_{1\bar{1}} + 2g_z u_{2\bar{1}})\partial_{u_{2\bar{1}}} - g_z u_{1\bar{2}}\partial_{u_{1\bar{2}}},$$

The algebras of invariant functions under the action of $\mathfrak{h}$ and $\bar{\mathfrak{h}}$ on $C^\infty(J^3\mathbb{R}^2)\otimes\mathbb{C}$

and $\mathfrak{g}$ on $C^\infty(J^3\mathbb{R}^2)$ are

$$
\begin{aligned}
\mathcal{H}_3 &= \left\{ f \in C^\infty_{loc}(J^3\mathbb{R}^2)\otimes\mathbb{C} \mid f = f\left(\bar{z}, u, u_{0\bar{1}}, \frac{u_{1\bar{1}}}{u_{0\bar{1}}u_{1\bar{0}}}, u_{0\bar{2}}, u_{0\bar{3}}, I_{3,1}, I_{3,2}\right)\right\},\\
\bar{\mathcal{H}}_3 &= \left\{ f \in C^\infty_{loc}(J^3\mathbb{R}^2)\otimes\mathbb{C} \mid f = f\left(z, u, u_{1\bar{0}}, \frac{u_{1\bar{1}}}{u_{0\bar{1}}u_{1\bar{0}}}, u_{2\bar{0}}, u_{3\bar{0}}, I_{3,1}.I_{3,2}\right)\right\},\\
\mathcal{G}_3 &= \left\{ f \in C^\infty_{loc}(J^3\mathbb{R}^2) \mid \tilde{f}\left(u, \frac{u_{20}+u_{02}}{u_{10}^2+u_{01}^2}, I_{3,1}, I_{3,2}\right)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{3,1} &= \frac{-(u_{0\bar{1}}^2u_{2\bar{1}}u_{1\bar{0}} - u_{0\bar{1}}^2u_{1\bar{1}}u_{0\bar{1}} - u_{0\bar{2}}u_{1\bar{0}}^2u_{1\bar{1}} + u_{1\bar{2}}u_{1\bar{0}}^2u_{0\bar{1}})}{u_{1\bar{1}}^3}\\
&= \frac{-2}{(u_{20}+u_{02})^3}(u_{10}^3u_{30} + u_{10}^3u_{12} - u_{10}^2u_{20}^2 + u_{10}^2u_{21}u_{01} + u_{10}^2u_{02}^2 + u_{01}^2u_{20}^2\\
&\quad +u_{10}u_{01}^2u_{30} + u_{10}^2u_{03}u_{01} + u_{10}u_{01}^2u_{12} + u_{01}^3u_{03} + u_{01}^3u_{21} - u_{01}^2u_{02}^2\\
&\quad -4u_{10}u_{01}u_{20}u_{11} - 4u_{10}u_{01}u_{02}u_{11}),
\end{aligned}
$$

$$
\begin{aligned}
I_{3,2} &= \frac{i(-u_{0\bar{1}}^2u_{2\bar{1}}u_{1\bar{0}} + u_{0\bar{1}}^2u_{1\bar{1}}u_{2\bar{0}} - u_{0\bar{2}}u_{1\bar{0}}^2u_{1\bar{1}} + u_{1\bar{2}}u_{1\bar{0}}^2u_{0\bar{1}})}{u_{1\bar{1}}^3}\\
&= \frac{2}{(u_{20}+u_{02})^3}(u_{10}^2u_{30}u_{01} - u_{10}u_{01}^2u_{21} + u_{10}^2u_{12}u_{01} - u_{10}u_{01}^2u_{03} + 2u_{10}^2u_{20}u_{11}\\
&\quad +2u_{10}u_{01}u_{02}^2 - 2u_{10}u_{01}u_{20}^2 + 2u_{10}^2u_{02}u_{11} - 2u_{01}^2u_{02}u_{11} - 2u_{01}^2u_{20}u_{11}\\
&\quad -u_{10}^3u_{03} + u_{01}^3u_{12} + u_{01}^3u_{30} - u_{10}^3u_{21}).
\end{aligned}
$$

Remark 17 *It is not possible to find $\mathfrak{g}-$differential invariants of pure order* 3 *by standard methods with* Maple 11. *However, it is possible to find $\mathfrak{h}-$differential invariants of pure order* 3. *The function* $h = \dfrac{-u_{0\bar{2}}u_{1\bar{0}}^2u_{1\bar{1}} + u_{1\bar{2}}u_{1\bar{0}}^2u_{0\bar{1}}}{u_{1\bar{1}}^3}$ *and it conjugate $\bar{h}$ are $\mathfrak{h}-$differential invariants. Hence $I_{3,1} = h + \bar{h}$ and $I_{3,2} = i(h - \bar{h})$ are $\mathfrak{g}-$differential invariants. The $\mathfrak{h}-$differential invariants of pure order* 3 *are computed in Maple Worksheet "h_ diff_ inv_ 3", see Appendix 6.*

3.4 Invariant Differentiations and Differential Invariants

Let $\mathfrak{q}$ be a Lie subalgebra of $\mathrm{Cont}(J^1\mathbb{R}^2)$, and let $\mathcal{Q}_k$ be the algebra of invariant functions under the action of $\mathfrak{q}$ on $C^\infty(J^k\mathbb{R}^2)$. Consider the derivation operator

$$\nabla = \lambda_1 \mathcal{D}_x + \lambda_2 \mathcal{D}_y,$$

where $\lambda_1, \lambda_2 \in C^\infty(J^p\mathbb{R}^2)$ and p is the maximum order of the functions λ_1 and λ_2. The derivation operator ∇ is an invariant derivative of $\mathfrak{q}$ if the following diagram commutes for all contact vector fields $X_f \in \mathfrak{q}$ and all integers $k \geq \max\{p-1, 1\}$

$$\begin{array}{ccc} C^\infty(J^k\mathbb{R}^2) & \stackrel{X_f^{(k)}}{\longrightarrow} & C^\infty(J^k\mathbb{R}^2) \\ \downarrow \nabla & & \downarrow \nabla \\ C^\infty(J^{k+1}\mathbb{R}^2) & \stackrel{X_f^{(k+1)}}{\longrightarrow} & C^\infty(J^{k+1}\mathbb{R}^2) \end{array}.$$

If ∇ is an invariant derivative of $\mathfrak{q}$, then

$$\left[X_f^{(\infty)}, \nabla\right] = 0, \ \forall X_f \in \mathfrak{q}, \tag{3.14}$$

where

$$X_f{}^{(\infty)} = \sum_{m+n\geq 0} (\mathcal{D}_x)^m (\mathcal{D}_y)^n (f)\partial_{u_{mn}} - \partial_{u_1}(f)\mathcal{D}_x - \partial_{u_2}(f)\mathcal{D}_y.$$

For any function $q \in \mathcal{Q}_k$

$$X_f^{(k+1)}(\nabla(q)) = \nabla\left(X_f^{(k)}(q)\right) = 0, \ \forall X_f \in \mathfrak{q}$$

Hence

$$\nabla : \mathcal{Q}_k \longrightarrow \mathcal{Q}_{k+1}.$$

Let $\mathfrak{f}$ be a Lie subalgebra of $\mathrm{Cont}\left(J^1\mathbb{R}^2\right)\otimes\mathbb{C}$, and let $\mathcal{F}_k$ be the algebra of invariant functions under the action of $\mathfrak{f}$ on $C^\infty(J^k\mathbb{R}^2)\otimes\mathbb{C}$. Consider the complex derivation operator

$$\nabla = \lambda_1 \mathcal{D}_x + \lambda_2 \mathcal{D}_y$$

where $\lambda_1, \lambda_2 \in C^\infty(J^p\mathbb{R}^2) \otimes \mathbb{C}$ and p is the maximum order of the functions λ_1 and λ_2. The derivation operator ∇ is a complex invariant derivative of $\mathfrak{f}$ if the following diagram commutes for all vector fields $X_f \in \mathfrak{f}$ and all integers $k \geq \max\{p-1, 1\}$

$$\begin{array}{ccc} C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C} & \overset{X_f^{(k)}}{\longrightarrow} & C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C} \\ \downarrow \nabla & & \downarrow \nabla \\ C^\infty(J^{k+1}\mathbb{R}^2) \otimes \mathbb{C} & \overset{X_f^{(k+1)}}{\longrightarrow} & C^\infty(J^{k+1}\mathbb{R}^2) \otimes \mathbb{C} \end{array}.$$

Moreover,

$$\left[X_f^{(\infty)}, \nabla\right] = 0, \ \forall X_f \in \mathfrak{f}, \tag{3.15}$$

$$\nabla : \mathcal{F}_k \longrightarrow \mathcal{F}_{k+1}.$$

It follows from Subsection 3.2.2 that if $\mathfrak{q}$ is a Lie subalgebra of $\mathrm{Cont}\left(J^1\mathbb{R}^2\right)$, then $\mathfrak{q}$ is a Lie subalgebra of $\mathrm{Cont}\left(J^1\mathbb{R}^2\right) \otimes \mathbb{C}$.

Proposition 18 *Let $\mathfrak{q}$ be a Lie subalgebra of* $\mathrm{Cont}\left(J^1\mathbb{R}^2\right)$. *If*

$$\nabla = (\lambda_{11} + i\lambda_{12})\,\mathcal{D}_x + (\lambda_{21} + i\lambda_{22})\,\mathcal{D}_y$$

is a complex invariant derivative of $\mathfrak{q}$, then

$$\begin{aligned} \mathrm{Re}(\nabla) &= \lambda_{11}\mathcal{D}_x + \lambda_{21}\mathcal{D}_y, \\ \mathrm{Im}(\nabla) &= \lambda_{21}\mathcal{D}_x + \lambda_{22}\mathcal{D}_y, \end{aligned}$$

are real invariant derivatives of $\mathfrak{q}$.

Proof. It follows from Equation(3.15) that

$$\begin{aligned} & \left[X_f^{(\infty)}, (\lambda_{11} + i\lambda_{12})\,\mathcal{D}_x + (\lambda_{21} + i\lambda_{22})\,\mathcal{D}_y\right] \\ = & \left[X_f^{(\infty)}, \lambda_{11}\mathcal{D}_x + \lambda_{21}\mathcal{D}_y\right] + i\left[X_f^{(\infty)}, \lambda_{12}\mathcal{D}_x + \lambda_{22}\mathcal{D}_y\right] = 0 \end{aligned}$$

for all contact vector fields $X_f \in \mathfrak{q}$. Hence

$$\left[X_f^{(\infty)}, \lambda_{11}\mathcal{D}_x + \lambda_{21}\mathcal{D}_y\right] = \left[X_f^{(\infty)}, \lambda_{12}\mathcal{D}_x + \lambda_{22}\mathcal{D}_y\right] = 0.$$

So by Equation (3.14) $\mathrm{Re}(\nabla)$ and $\mathrm{Im}(\nabla)$ are invariant derivatives of $\mathfrak{q}$. ■

3.4.1 Tresse Derivation

The total differential of a function $h \in C^\infty(J^k\mathbb{R}^2)$ is

$$\hat{d}h = \mathcal{D}_x(h)dx + \mathcal{D}_y(h)dy.$$

It is known [KL1] that if the total differentials of two functions $h_1, h_2 \in C^\infty(J^k\mathbb{R}^2) \otimes \mathbb{C}$ are independent, i.e.

$$\hat{d}h_1 \wedge \hat{d}h_2 \neq 0$$

on a domain $U \in J^k\mathbb{R}^2$, then $\left\langle \hat{d}h_1, \hat{d}h_2 \right\rangle_{\mathbb{C}}$ is a cobasis of $(\pi_k)^*T\mathbb{R}^2|_U$. Hence for any function $h \in C^\infty(J^l\mathbb{R}^2)$ the total differential of h is

$$\hat{d}h = \left(\frac{\mathcal{D}h}{\mathcal{D}h_1}\right)\hat{d}h_1 + \left(\frac{\mathcal{D}h}{\mathcal{D}h_2}\right)\hat{d}h_2,$$

where

$$\begin{bmatrix} \dfrac{\mathcal{D}h}{\mathcal{D}h_1} \\ \dfrac{\mathcal{D}h}{\mathcal{D}h_2} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x(h_1) & \mathcal{D}_x(h_2) \\ \mathcal{D}_y(h_1) & \mathcal{D}_y(h_2) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{D}_x(h) \\ \mathcal{D}_y(h) \end{bmatrix}$$

are the Tresse derivations of the function h.

Let $\mathfrak{q}$ be a Lie subalgebra of $\mathrm{Cont}\left(J^1\mathbb{R}^2\right)$, and let $\mathcal{Q}$ be the algebra of the $\mathfrak{q}-$differential invariants. If the total differentials of two functions q_1, $q_2 \in \mathcal{Q}$ are independent, it is known [KL1] that for any function $q \in \mathcal{Q}$ the Tresse derivations of q are $\mathfrak{q}-$differential invariants

$$\frac{\mathcal{D}q}{\mathcal{D}q_2}, \frac{\mathcal{D}q}{\mathcal{D}q_1} \in \mathcal{Q}.$$

Hence the two derivation operators

$$\begin{bmatrix} \dfrac{\mathcal{D}}{\mathcal{D}q_1} \\ \dfrac{\mathcal{D}}{\mathcal{D}q_2} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x(q_1) & \mathcal{D}_x(q_2) \\ \\ \mathcal{D}_y(q_1) & \mathcal{D}_y(q_2) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{D}_x \\ \\ \mathcal{D}_y \end{bmatrix}$$

are invariant derivatives of the Lie algebra $\mathfrak{q}$. Moreover, these invariant derivatives commute

$$\left[\frac{\mathcal{D}}{\mathcal{D}q_1}, \frac{\mathcal{D}}{\mathcal{D}q_2} \right] = 0.$$

Let $\mathfrak{f}$ be a Lie subalgebra of $\mathrm{Cont}\left(J^1\mathbb{R}^2\right) \otimes \mathbb{C}$, and let $\mathcal{F}$ be the algebra of the complex valued $\mathfrak{f}-$differential invariants. If the total differentials of two functions f_1, $f_2 \in \mathcal{F}$ are independent, then the derivation operators

$$\begin{bmatrix} \dfrac{\mathcal{D}}{\mathcal{D}f_1} \\ \dfrac{\mathcal{D}}{\mathcal{D}f_2} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x(f_1) & \mathcal{D}_x(f_2) \\ \\ \mathcal{D}_y(f_1) & \mathcal{D}_y(f_2) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{D}_x \\ \\ \mathcal{D}_y \end{bmatrix}$$

are complex invariant derivatives of $\mathfrak{f}$. Moreover,

$$\left[\frac{\mathcal{D}}{\mathcal{D}f_1}, \frac{\mathcal{D}}{\mathcal{D}f_2} \right] = 0.$$

3.4.2 Lie-Tresse Theorem

Let $\mathfrak{q}$ be a Lie subalgebra of $\mathrm{Cont}\left(J^1\mathbb{R}^2\right)$. It is known [L, T, KL1] that there exist $\mathfrak{q}-$differential invariants, $I_{g_1},\ I_{g_2}, J_{k_1}, J_{k_2}, ..., J_{k_s}$ such that if J is an $\mathfrak{q}-$differential invariant, then

$$J = J\left(I_{g_1}, I_{g_2}, \left(\frac{\mathcal{D}}{\mathcal{D}I_{g_1}}\right)^{m_1}\left(\frac{\mathcal{D}}{\mathcal{D}I_{g_2}}\right)^{n_1}(J_{k_1}), ..., \left(\frac{\mathcal{D}}{\mathcal{D}I_{g_1}}\right)^{m_s}\left(\frac{\mathcal{D}}{\mathcal{D}I_{g_2}}\right)^{n_s}(J_{k_s})\right).$$

3.5 Invariant Derivatives of the Lie Algebra $\mathfrak{g}$

In this section we will find invariant derivatives of the Lie algebra $\mathfrak{g}$ by using three different methods. The first two methods require $\mathfrak{g}-$differential invariants of order three, while in the third method we only need two $\mathfrak{h}-$differential invariants of order zero.

3.5.1 Invariant Derivatives of $\mathfrak{g}$, Method 1

In this subsection we will use the theory of Subsection 3.4.1 to find two invariant derivatives of $\mathfrak{g}$.

So far, we have found four invariant functions $I_0, I_2, I_{3,1}, I_{3,2} \in \mathcal{G}$ that are independent on some regular domains in $J^3\mathbb{R}^2$

$$\begin{aligned}
I_0 &= u,\\
I_2 &= \frac{u_{20}+u_{02}}{u_{10}^2+u_{01}^2},\\
I_{3,1} &= \frac{-2}{(u_{20}+u_{02})^3}(u_{10}^3u_{30}+u_{10}^3u_{12}-u_{10}^2u_{20}^2+u_{10}^2u_{21}u_{01}+u_{10}^2u_{02}^2+u_{01}^2u_{20}^2\\
&\quad +u_{10}u_{01}^2u_{30}+u_{10}^2u_{03}u_{01}+u_{10}u_{01}^2u_{12}+u_{01}^3u_{03}+u_{01}^3u_{21}-u_{01}^2u_{02}^2\\
&\quad -4u_{10}u_{01}u_{20}u_{11}-4u_{10}u_{01}u_{02}u_{11}),\\
I_{3,2} &= \frac{2}{(u_{20}+u_{02})^3}(u_{10}^2u_{30}u_{01}-u_{10}u_{01}^2u_{21}+u_{10}^2u_{12}u_{01}-u_{10}u_{01}^2u_{03}+2u_{10}^2u_{20}u_{11}\\
&\quad +2u_{10}u_{01}u_{02}^2-2u_{10}u_{01}u_{20}^2+2u_{10}^2u_{02}u_{11}-2u_{01}^2u_{02}u_{11}-2u_{01}^2u_{20}u_{11}\\
&\quad -u_{10}^3u_{03}+u_{01}^3u_{12}+u_{01}^3u_{30}-u_{10}^3u_{21}).
\end{aligned}$$

The functions $I_{3,1}$ and $I_{3,2}$ have independent symbols

$$(\partial_{u_{30}},\partial_{u_{21}},\partial_{u_{12}},\partial_{u_{03}})(I_{3,1}) = 2\left(I_2(u_{20}+u_{02})^2\right)^{-1}(u_{10},u_{01},u_{10},u_{01}), \tag{3.16}$$

$$(\partial_{u_{30}},\partial_{u_{21}},\partial_{u_{12}},\partial_{u_{03}})(I_{3,2}) = -2\left(I_2(u_{20}+u_{02})^2\right)^{-1}(u_{01},-u_{10},u_{01},-u_{10}). \tag{3.17}$$

It follows from Subsection 3.4.1 that the two derivation operators

$$\begin{bmatrix} \frac{\mathcal{D}}{\mathcal{D}I_0} \\ \frac{\mathcal{D}}{\mathcal{D}I_2} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x(I_0) & \mathcal{D}_x(I_2) \\ \mathcal{D}_y(I_0) & \mathcal{D}_y(I_2) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \begin{bmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{bmatrix}$$

are invariant derivatives of $\mathfrak{g}$. The maximum order of λ_{ij}, for $i, j \in \{1, 2\}$, is 3. These invariant derivatives are computed in Maple Worksheet "tresse_inv_der", see Appendix 6.

It follows from Equation (3.16) and (3.17) that

$$\begin{aligned}
\frac{\mathcal{D}}{\mathcal{D}I_0}(I_{3,1}) &= f_{01} + 2\left(I_2(u_{20}+u_{02})^2\right)^{-1} \\
&\quad (\lambda_{11}\left(u_{10}u_{40} + u_{01}u_{31} + u_{10}u_{22} + u_{01}u_{13}\right) \\
&\quad +\lambda_{12}\left(u_{10}u_{31} + u_{01}u_{22} + u_{10}u_{13} + u_{01}u_{04}\right)), \\
\frac{\mathcal{D}}{\mathcal{D}I_0}(I_{3,2}) &= f_{02} - 2\left(I_2(u_{20}+u_{02})^2\right)^{-1} \\
&\quad (\lambda_{11}\left(u_{01}u_{40} - u_{10}u_{31} + u_{01}u_{22} - u_{10}u_{13}\right) \\
&\quad +\lambda_{12}\left(u_{01}u_{31} - u_{10}u_{22} + u_{01}u_{13} - u_{10}u_{04}\right)), \\
\frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,1}) &= f_{21} + 2\left(I_2(u_{20}+u_{02})^2\right)^{-1} \\
&\quad (\lambda_{21}\left(u_{10}u_{40} + u_{01}u_{31} + u_{10}u_{22} + u_{01}u_{13}\right) \\
&\quad +\lambda_{22}\left(u_{10}u_{31} + u_{01}u_{22} + u_{10}u_{13} + u_{01}u_{04}\right)), \\
\frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,2}) &= f_{22} - \left(I_2(u_{20}+u_{02})^2\right)^{-1} \\
&\quad (\lambda_{21}\left(u_{01}u_{40} - u_{10}u_{31} + u_{01}u_{22} - u_{10}u_{13}\right) \\
&\quad +\lambda_{22}\left(u_{01}u_{31} - u_{10}u_{22} + u_{01}u_{13} - u_{10}u_{04}\right)),
\end{aligned}$$

where f_{ij} are smooth functions of order less than 4, for $i \in \{0, 2\}$, $j \in \{1, 2\}$. Hence

$$g_1(I_0, I_2, I_{31}, I_{32})\frac{\mathcal{D}}{\mathcal{D}I_{m_1}}(I_{3,j_1}) + g_2(I_0, I_2, I_{31}, I_{32})\frac{\mathcal{D}}{\mathcal{D}I_{m_2}}(I_{3,j_2}) = 0$$

if and only if $g_1 = g_2 = 0$, for $m_1, m_2 \in \{0, 2\}$ and $j_1, j_2 \in \{1, 2\}$.

It follows from computations in Maple Worksheet "dep_inv" that

$$\frac{\mathcal{D}}{\mathcal{D}I_0}(I_{3,2}) = \tfrac{1}{2}I_2^2((I_{3,1}I_2+2)\frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,2}) - I_2 I_{3,2}\frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,1})),$$

$$g_1(I_0, I_2, I_{31}, I_{32})\frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,1}) + g_2(I_0, I_2, I_{31}, I_{32})\frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,2}) + g_2(I_0, I_2, I_{31}, I_{32})\frac{\mathcal{D}}{\mathcal{D}I_0}(I_{3,2}) = 0,$$

if and only if $g_j = 0$, for $j \in \{1, 2, 3\}$. Hence

$$\mathcal{G}_4 = \left\{ f \in C^\infty_{loc}\left(\mathbb{R}^2\right) | f = \ f\left(I_0, I_2, I_{3,1}, I_{3,2}, \frac{\mathcal{D}}{\mathcal{D}I_0}(I_{3,1}), \frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,1}), \frac{\mathcal{D}}{\mathcal{D}I_2}(I_{3,2})\right)\right\}.$$

Theorem 19 *For any integer $m \in \mathbb{Z}_+$ the $m+2$ functions*

$$I_{m+3,m+2} = \left(\frac{\mathcal{D}}{\mathcal{D}I_2}\right)^m (I_{3,2}), \quad I_{m+3,j+1} = \left(\frac{\mathcal{D}}{\mathcal{D}I_0}\right)^{m-n}\left(\frac{\mathcal{D}}{\mathcal{D}I_2}\right)^n (I_{3,1}), \ n \in \{0, 1, ..., m\},$$

are $\mathfrak{g}-$differential invariants of order $m+3$. Moreover, these $\mathfrak{g}-$differential invariants are independent, i.e.

$$\sum_{j=1}^{m+2} g_j I_{m+3,j} = 0, \ g_j \in \mathcal{G}_{m+2} \implies g_j = 0, \ j \in \{1, ..., m+2\}.$$

For any integer $k \in \mathbb{Z}_+$ the algebra of invariant functions under the action of $\mathfrak{g}$ on $J^k(\mathbb{R}^2)$ is

$$\mathcal{G}_k = \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid f = f(I_0, I_2, I_{3,1}, I_{3,2}, ..., I_{k,k-1})\right\}.$$

Theorem 19 will be proved in Subsection 3.5.3.

3.5.2 Invariant Derivatives of $\mathfrak{g}$, Method 2

In this subsection we are seeking invariant derivatives of order less than 3. The first step is to find derivation operators ∇_1 and ∇_2 such that

$$\begin{aligned} \nabla_1(I_2) &= h_1(I_0, I_2, I_{3,1}), \\ \nabla_2(I_2) &= h_2(I_0, I_2, I_{3,2}), \end{aligned}$$

where $\dfrac{\partial h_i}{\partial I_{3,i}} \neq 0$, for $i \in \{1,2\}$, and the $\mathfrak{g}-$differential invariants $I_{3,1}$ and $I_{3,2}$ are as defined in Subsection 3.5.1.

The second step is to compute the commutator for $j \in \{1,2\}$

$$\left[V_g{}^{(\infty)}, \nabla_j\right].$$

If the commutator is zero for any function $g \in \mathcal{O}$, then ∇_1 and ∇_2 are invariant derivatives.

Let us start at the first step and find a derivation operator ∇_1 such that $\nabla_1(I_2) = h_1(I_0, I_2, I_{3,1})$.

$$\begin{aligned}
\nabla_1(I_2) &= \left(A\mathcal{D}_x + B\mathcal{D}_y\right)(I_2) \\
&= A\left(\frac{\partial I_2}{\partial x} + u_{10}\frac{\partial I_2}{\partial u} + u_{20}\frac{\partial I_2}{\partial u_{10}} + u_{11}\frac{\partial I_2}{\partial u_{01}}\right) + \\
&\quad B\left(\frac{\partial I_2}{\partial y} + u_{01}\frac{\partial I_2}{\partial u} + u_{11}\frac{\partial I_2}{\partial u_{10}} + u_{02}\frac{\partial I_2}{\partial u_{01}}\right) + \\
&\quad u_{30}\left(A\frac{\partial I_2}{\partial u_{20}}\right) + u_{21}\left(A\frac{\partial I_2}{\partial u_{20}} + B\frac{\partial I_2}{\partial u_{20}}\right) + \\
&\quad u_{12}\left(A\frac{\partial I_2}{\partial u_{02}} + B\frac{\partial I_2}{\partial u_{11}}\right) + u_{03}\left(B\frac{\partial I_2}{\partial u_{02}}\right).
\end{aligned}$$

The function $I_{3,1}$ is linear in the coordinate functions of third order. Hence the functions A and B must satisfy the four equations

$$\begin{aligned}
A\frac{\partial I_2}{\partial u_{20}} &= \frac{\partial I_{3,1}}{\partial u_{30}} f(I_0, I_2) = 2\left(I_2(u_{20}+u_{02})^2\right)^{-1} u_{10} f(I_0, I_2), \\
A\frac{\partial I_2}{\partial u_{11}} + B\frac{\partial I_2}{\partial u_{20}} &= \frac{\partial I_{3,1}}{\partial u_{21}} f(I_0, I_2) = 2\left(I_2(u_{20}+u_{02})^2\right)^{-1} u_{01} f(I_0, I_2), \\
A\frac{\partial I_2}{\partial u_{02}} + B\frac{\partial I_2}{\partial u_{11}} &= \frac{\partial I_{3,1}}{\partial u_{12}} f(I_0, I_2) = 2\left(I_2(u_{20}+u_{02})^2\right)^{-1} u_{10} f(I_0, I_2), \\
B\frac{\partial I_2}{\partial u_{02}} &= \frac{\partial I_{3,1}}{\partial u_{03}} f(I_0, I_2) = 2\left(I_2(u_{20}+u_{02})^2\right)^{-1} u_{01} f(I_0, I_2),
\end{aligned}$$

for some smooth function f.

The four equations hold for the functions

$$B = \frac{u_{01}}{u_{10}^2 + u_{01}^2}, \; A = \frac{u_{10}}{u_{10}^2 + u_{01}^2}, \; f(I_0, I_2) = -I_2^3/2.$$

Hence we get the derivation operator

$$\nabla_1 = \frac{u_{10}}{u_{10}^2 + u_{01}^2}\mathcal{D}_x + \frac{u_{01}}{u_{10}^2 + u_{01}^2}\mathcal{D}_y.$$

Now, let us find ∇_2 such that $\nabla_2(I_2) = h_2(I_0, I_2, I_{3,2})$. By following the procedure above, we get the derivation operator

$$\nabla_2 = \frac{u_{01}}{u_{10}^2 + u_{01}^2}\mathcal{D}_x - \frac{u_{10}}{u_{10}^2 + u_{01}^2}\mathcal{D}_y.$$

Note that if ∇_1 and ∇_2 are invariant derivatives of $\mathfrak{g}$, then $\nabla_j(I_0) \in \mathcal{G}$ for $j \in \{0, 1\}$. Let us check that this is true before we do the last step

$$\nabla_1(I_0) = \left(\frac{-u_{01}}{u_{10}^2 + u_{01}^2}\mathcal{D}_x + \frac{u_{10}}{u_{10}^2 + u_{01}^2}\mathcal{D}_y\right)(u) = 0 \in \mathcal{G},$$

$$\nabla_2(I_0) = \left(\frac{u_{10}}{u_{10}^2 + u_{01}^2}\mathcal{D}_x + \frac{u_{01}}{u_{10}^2 + u_{01}^2}\mathcal{D}_y\right)(u) = 1 \in \mathcal{G}.$$

Let us compute the commutator

$$\begin{aligned}
&[(V_g)^\infty, \nabla_j] = (g_1\mathcal{D}_x(\lambda_{1j}) + g_2\mathcal{D}_y(\lambda_{1j}) - \lambda_{1j}g_{1x} - \lambda_{2j}g_{1y} \\
&\left. - \sum_{1 \geq m+n \geq 0} (\mathcal{D}_x)^m (\mathcal{D}_y)^n (u_{10}g_1 + u_{01}g_2)\partial_{u_{mn}}(\lambda_{1j})\right)\mathcal{D}_x \\
&+ (g_1\mathcal{D}_x(\lambda_{2j}) + g_2\mathcal{D}_y(\lambda_{2j}) + \lambda_{1j}g_{1y} - \lambda_{2j}g_{1x} \\
&\left. - \sum_{1 \geq m+n \geq 0} (\mathcal{D}_x)^m (\mathcal{D}_y)^n (u_{10}g_1 + u_{01}g_2)\partial_{u_{mn}}(\lambda_{2j})\right)\mathcal{D}_y
\end{aligned}$$

for $\nabla_j = \lambda_{1j}\mathcal{D}_x + \lambda_{2j}\mathcal{D}_y$. It follows from that Appendix **??** that

$$\left[{V_g}^{(\infty)}, \nabla_j\right] = 0, \; j \in \{1, 2\}, \; \forall g \in \mathcal{O}.$$

Hence the derivation operators

$$\nabla_1 = \frac{1}{u_{10}^2 + u_{01}^2} \left(u_{10} \mathcal{D}_x + u_{01} \mathcal{D}_y \right), \; \nabla_2 = \frac{1}{u_{10}^2 + u_{01}^2} \left(-u_{01} \mathcal{D}_x + u_{10} \mathcal{D}_y \right),$$

are invariant derivatives of $\mathfrak{g}$.

The commutator of ∇_1 and ∇_2 is

$$[\nabla_1, \nabla_2] = -I_2 \nabla_2.$$

Theorem 20 *For any integer $m \in \mathbb{Z}_+$ the $m+1$ functions*

$$I_{m+2,j+1} = (\nabla_1)^{m-j} (\nabla_2)^j (I_2), \; j \in \{0, 1, ..., k\}$$

are $\mathfrak{g}-$differential invariants of order $m+2$. Moreover, these $\mathfrak{g}-$differential invariants are independent, i.e.

$$\sum_{j=1}^{m+1} g_j I_{m+2,j} = 0, \; g_j \in \mathcal{G}_{m+1} \implies g_j = 0, \; j \in \{1, ..., m+1\}.$$

For any integer $k \in \mathbb{Z}_+$ the algebra of invariant functions under the action of $\mathfrak{g}$ on $C^\infty(J^k\mathbb{R}^2)$ is

$$\mathcal{G}_k = \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid f = f(I_0, I_2, I_{3,1}, I_{3,2}, ..., I_{k,k-1}) \right\}.$$

Theorem 20 will be proved in Subsection 3.5.3.

Remark 21 *For $k \geq 3$, the $\mathfrak{g}-$differential invariant $I_{k,j}$ defined in Theorem 20 is not equal to the $\mathfrak{g}-$differential invariant $I_{k,j}$ defined in Theorem 19. In the following sections $I_{k,,j}$ will denote the $\mathfrak{g}-$differential invariant defined in Theorem 20. The two $\mathfrak{g}-$differential invariants of pure order three used in Subsection 3.5.1 are*

$$I_{3,1}^{old} = -2 \frac{I_{3,1} + I_2^2}{I_2^3}, \; I_{3,2}^{old} = 2 \frac{I_{3,2}}{I_2^3}.$$

The computation is done in Maple Worksheet "dep_ inv_ n_ o".

3.5.3 Invariant Derivatives of $\mathfrak{g}$, Method 3

The methods used in Subsection 3.5.1 and 3.5.2 required $\mathfrak{g}-$differential invariants of order three to generate the algebra $\mathcal{G}$. In this subsection we will use two $\mathfrak{h}-$ differential invariants of order zero

$$u, \bar{z} \in \mathcal{H},$$

to generate the algebra $\mathcal{G}$.

The derivation operators

$$\begin{bmatrix} \frac{\mathcal{D}}{\mathcal{D}\bar{z}} \\ \frac{\mathcal{D}}{\mathcal{D}u} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_z(\bar{z}) & \mathcal{D}_z(u) \\ \mathcal{D}_{\bar{z}}(\bar{z}) & \mathcal{D}_{\bar{z}}(u) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{D}_z \\ \mathcal{D}_{\bar{z}} \end{bmatrix} = \begin{bmatrix} -\frac{u_{0\bar{1}}}{u_{1\bar{0}}} & 1 \\ \frac{1}{u_{1\bar{0}}} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{D}_z \\ \mathcal{D}_{\bar{z}} \end{bmatrix}$$

are invariant derivatives of $\mathfrak{h}$.

The derivation operator $\mathcal{D}_{\bar{z}}$ is an invariant derivative of $\mathfrak{h}$. Hence $\mathcal{D}_{\bar{z}}(u) = u_{0\bar{1}}$ is an $\mathfrak{h}-$differential invariant.

Note that for any integers $k, j \in \mathbb{Z}_{\geq 0}$ where $k \geq j$

$$\begin{aligned} &\left(\frac{\mathcal{D}}{\mathcal{D}\bar{z}}\right)^{k-j} \left(\frac{\mathcal{D}}{\mathcal{D}u}\right)^{j} (u_{0\bar{1}}) \\ = \quad & f + \left(\frac{1}{u_{1\bar{0}}}\right)^{j} \sum_{n=0}^{k-j} \binom{k-j}{n} \left(-\frac{u_{0\bar{1}}}{u_{1\bar{0}}}\right)^{k-j-n} (u_{(k-n)\overline{(1+n)}}), \end{aligned} \tag{3.18}$$

where f is a smooth functions of order less than $k+1$.

Theorem 22 *For any integer $m \in \mathbb{Z}_{\geq 0}$ the $m+1$ functions*

$$Q_{m+1,j+1} = \left(\frac{\mathcal{D}}{\mathcal{D}\bar{z}}\right)^{m-j} \left(\frac{\mathcal{D}}{\mathcal{D}u}\right)^{j} (u_{0\bar{1}}), \ j \in \{0, 1, ..., m\}$$

are $\mathfrak{h}-$differential invariants of of order $m+1$. Moreover, these $\mathfrak{h}-$differential invariants *are independent, i.e.*

$$\sum_{j=1}^{m+1} g_j Q_{m+1,j} = 0,\ g_j \in \mathcal{H}_m \implies g_j = 0,\ j \in \{1, .., m+1\}.$$

For any integer $k \in \mathbb{Z}_+$ the algebra of invariant functions under the action of $\mathfrak{h}$ on $C^\infty\left(J^k\mathbb{R}^2\right) \otimes \mathbb{C}$ is

$$\mathcal{H}_k = \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid f = f(\bar{z}, u, u_{0\bar{1}}, Q_{1,1}, ..., Q_{k,k}) \right\}.$$

Proof. The theorem follows from Subsection 3.3.1 and Equation (3.18). ■

Since the derivation operator $\mathcal{D}_{\bar{z}}$ is an invariant derivative of $\mathfrak{h}$ and $\mathcal{D}_z$ is an invariant derivative of $\bar{\mathfrak{h}}$, it follows that

$$\frac{\bar{\mathcal{D}}}{\mathcal{D}u} = \frac{1}{u_{1\bar{0}}}\mathcal{D}_z, \quad \frac{\mathcal{D}}{\mathcal{D}u} = -\frac{1}{u_{0\bar{1}}}\mathcal{D}_{\bar{z}}$$

are invariant derivatives of both $\mathfrak{h}$ and $\bar{\mathfrak{h}}$ and hence also invariant derivatives of the Lie algebra

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{h} \oplus \bar{\mathfrak{h}}.$$

Moreover,

$$\left[\frac{\bar{\mathcal{D}}}{\mathcal{D}u}, \frac{\mathcal{D}}{\mathcal{D}u}\right] = -I_2\left(\frac{\bar{\mathcal{D}}}{\mathcal{D}u} - \frac{\mathcal{D}}{\mathcal{D}u}\right).$$

It follows from Proposition 18 that the derivation operators

$$\begin{aligned} \nabla_1 &= \tfrac{1}{2}\left(\frac{\bar{\mathcal{D}}}{\mathcal{D}u} + \frac{\mathcal{D}}{\mathcal{D}u}\right) = \frac{1}{u_{10}^2 + u_{01}^2}\left(u_{10}\mathcal{D}_x + u_{01}\mathcal{D}_y\right), \\ \nabla_2 &= \tfrac{i}{2}\left(\frac{\mathcal{D}}{\mathcal{D}u} - \frac{\bar{\mathcal{D}}}{\mathcal{D}u}\right) = \frac{1}{u_{10}^2 + u_{01}^2}\left(u_{01}\mathcal{D}_x - u_{10}\mathcal{D}_y\right), \end{aligned}$$

are invariant derivatives of $\mathfrak{g}$.

Lemma 23 *For any integer* $m \in \mathbb{Z}_{\geq 0}$ *the* $m+1$ *functions*

$$I_{m+2,j+1} = (\nabla_1)^{m-j} (\nabla_2)^j (I_2), \ j \in \{0, 1, ..., m\}$$

are $\mathfrak{g}-$*differential invariants of order* $m+2$*. Moreover, these* $\mathfrak{g}-$*differential invariants are independent, i.e.*

$$\sum_{j=0}^{k} g_j I_{m+2,j+1} = 0, \ g_j \in \mathcal{G}_{m+1} \implies g_j = 0, \ j \in \{0, .., m\}.$$

For any integer $k \in \mathbb{Z}_+$ *the algebra of invariant functions under the action of* $\mathfrak{g}$ *on* $C^\infty(J^k\mathbb{R}^2)$ *is*

$$\mathcal{G}_k = \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid f = f(I_0, I_2, I_{3,1}, I_{3,2}, ..., I_{k,k-1}) \right\}.$$

Proof. The lemma follows from Theorem 22 . ■

The invariant derivatives of $\mathfrak{g}$ that we found in Subsection 3.5.2 are equal to ∇_1 and ∇_2. Hence Theorem 20 follows from Lemma 23.

Theorem 24 *Suppose that the invariant derivatives* $\hat{\nabla}_1$ *and* $\hat{\nabla}_2$ *are equal to*

$$\begin{bmatrix} \hat{\nabla}_1 \\ \hat{\nabla}_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} \nabla_1 \\ \nabla_2 \end{bmatrix}$$

where the maximum order of the functions $f_{ij} \in \mathcal{G}$ *is* $k+2$ *for* $k \in \mathbb{Z}_{\geq 0}$ *and*

$$f_{11}f_{22} - f_{12}f_{21} \neq 0, \ f_{22} \neq 0.$$

Then the $k+n+1$ *functions*

$$K_{n+k+2,i_1+1} = \hat{\nabla}_1^{n-i_1} \hat{\nabla}_2^{i_1} (I_{k+2,1}), \ i_1 \in \{0, .., n\},$$

$$K_{n+k+2,i_2+n+1} = \hat{\nabla}_2^{n} (I_{k+2,i_2+1}), \ i_2 \in \{1, ..., k\},$$

are $\mathfrak{g}-$differential invariants of order $n+k+2$. Moreover, these $\mathfrak{g}-$differential invariants are independent

$$\sum_{i_2=1}^{k} g_{i_1}\hat{\nabla}_2^n(I_{k+2,i_2+1}) + \sum_{i_1=0}^{n} g_{i_2}\hat{\nabla}_1^{n-i_1}\hat{\nabla}_2^{i_1}(I_{k+2,1}) = 0,\ g_{i_2}, g_{i_1} \in \mathcal{G}_{k+n+1} \implies g_{i_2} = g_{i_1} = 0.$$

For any integer $m \in \mathbb{Z}_+$ the algebra of invariant functions under the action of $\mathfrak{g}$ on $C^\infty(J^m\mathbb{R}^2)$ is

$$\mathcal{G}_m = \left\{ f \in C^\infty_{loc}(J^m\mathbb{R}^2) \mid f = f(I_0, ..., I_{k,k-1}, K_{k+1,1....}K_{m,m-1}) \right\}$$

Proof. For $i_2 \in \{1, ..., k\}$

$$\begin{aligned} \hat{\nabla}_2^n(I_{k+2,i_2+1}) &= (f_{21}\nabla_1 + f_{21}\nabla_2)^n(I_{k+2,i_2+1}) \\ &= h_{i_2} + (f_{22})^n(I_{k+n+2,i_2+1+n}) + \sum_{j=1}^{n} h_{ji_2}(f_{22})^{n-j}(f_{21})^j(I_{k+n+2,i_2+n-j+1}), \end{aligned}$$

where $h_{i_2}, h_{ji_2} \in \mathcal{G}_{k+n+1}$. Hence

$$\sum_{i_2=1}^{k} g_{i_1}\hat{\nabla}_2^n(I_{k+2,i_2+1}) = 0,\ g_{i_2} \in \mathcal{G}_{k+n+1} \implies g_{i_2} = 0.$$

We will prove that

$$\sum_{i_1=0}^{n} g_{i_2}\hat{\nabla}_1^{n-i_1}\hat{\nabla}_2^{i_1}(I_{k+2,1}) = 0,\ g_{i_1} \in \mathcal{G}_{k+n+1} \implies g_{i_1} = 0 \tag{3.19}$$

by induction.

For $n = 1$

$$\begin{aligned} \hat{\nabla}_1(I_{k+2,1}) &= f_{11}I_{k+3,1} + f_{12}I_{k+3,2}, \\ \hat{\nabla}_2(I_{k+2,1}) &= f_{12}I_{k+3,1} + f_{22}I_{k+3,2}. \end{aligned}$$

Since $f_{11}f_{22} - f_{12}f_{21} \neq 0$, it follows that Equation (3.19) holds for $n = 1$.

Suppose that Equation (3.19) holds for $n = m$. Then

$$\begin{aligned}\sum_{i_1=0}^{m} g_{i_2}\hat{\nabla}_1\hat{\nabla}_1^{m-i_1}\hat{\nabla}_2^{i_1}(I_{k+2,1}) &= 0,\ g_j \in \mathcal{G}_{k+m+2} \implies g_{i_1} = 0,\\ \sum_{i_1=0}^{m} g_{i_2}\hat{\nabla}_2\hat{\nabla}_1^{m-i_1}\hat{\nabla}_2^{i_1}(I_{k+2,1}) &= 0,\ g_j \in \mathcal{G}_{k+m+2} \implies g_{i_1} = 0.\end{aligned}$$

Since

$$g_1\hat{\nabla}_1^{m+1}(I_{k+2,1}) + g_2\hat{\nabla}_2^{m+1}(I_{k+2,1}) = 0,\ g_1, g_2 \in \mathcal{G}_{k+m+2} \implies g_1 = g_2 = 0,$$

it follows that Equation (3.19) holds for $n = m + 1$. Hence Equation (3.19) holds for any integer $n \in \mathbb{Z}_{\geq 0}$.

It follows that

$$\sum_{i_2=1}^{k} g_{i_1}\hat{\nabla}_2^{n}(I_{k+2,i_2+1}) + \sum_{i_1=0}^{n} g_{i_2}\hat{\nabla}_1^{n-i_1}\hat{\nabla}_2^{i_1}(I_{k+2,1}) = 0,\ g_j \in \mathcal{G}_{k+n+1} \implies g_{i_2} = g_{i_1} = 0.$$

■

We have that the invariant derivatives defined in Subsection 3.5.1 are equal to

$$\begin{bmatrix} \dfrac{\mathcal{D}}{\mathcal{D}I_0} \\ \dfrac{\mathcal{D}}{\mathcal{D}I_2} \end{bmatrix} = \frac{1}{I_{3,2}} \begin{bmatrix} I_{3,2} & -I_{3,1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nabla_1 \\ \nabla_2 \end{bmatrix}.$$

Hence Theorem 19 follows from Theorem 24.

3.5.4 Invariant Functions of the Lie Algebras $\mathrm{sl}_2(\mathbb{C})_{\mathbb{R}}$ and $\mathrm{co}(2)$

It follows from Subsection 2.2.1 and 2.2.4 that

$$\mathfrak{z} = \langle V_1, V_i, V_z, V_{iz}, V_{z^2}, V_{iz^2} \rangle \subset \mathfrak{g},$$

$$\mathfrak{s} = \left\langle \partial_z, z\partial_z, z^2\partial_z \right\rangle \subset \mathfrak{h},$$

are Lie algebras. Moreover,

$$\mathfrak{z} \otimes \mathbb{C} = \mathfrak{s} \oplus \bar{\mathfrak{s}}. \tag{3.20}$$

Let $\mathcal{S}$ denote the algebra of $\mathfrak{s}-$differential invariants and let $\mathcal{Z}$ denote the algebra of $\mathfrak{z}-$differential invariants. It follows from Equation (3.20) that

$$\mathcal{Z} \otimes \mathbb{C} = \mathcal{S} \cap \bar{\mathcal{S}}. \tag{3.21}$$

For an integer $k \in \{0, 1, 2\}$ the distribution defined in Subsection 3.3.1 is equal to

$$\Pi^k = \left\langle (\partial_z)^{(k)}, (z\partial_z)^{(k)}, (z^2\partial_z)^{(k)} \right\rangle_{\mathbb{C}}.$$

Hence

$$\mathcal{S}_k = \mathcal{H}_k,\ \ \mathcal{Z}_k = \mathcal{G}_k,\ \ k \in \{0, 1, 2\}.$$

For any integer $k \geq 3$ there exist locally $k+1$ functionally independent $\mathfrak{s}-$ and $\mathfrak{z}-$differential invariants of pure order k. Theorem 22 and Lemma 23 give us k $\mathfrak{s}-$ differential invariants and $k-1$ $\mathfrak{z}-$differential invariants of pure order k. Hence we are seeking two real functions $\hat{I}_{k,1}$ and $\hat{I}_{k,2}$ of pure order k such that

$$\hat{I}_{k,1}, \hat{I}_{k,2} \in \mathcal{Z}_k,\ \hat{I}_{k,1}, \hat{I}_{k,2} \notin \mathcal{G}_k,$$

$$g_1\hat{I}_{k,1} + g_2\hat{I}_{k,2} = 0,\ \ g_1, g_2 \in \mathcal{Z}_{k-1} \implies g_1 = g_2 = 0,$$

for all integers $k \geq 3$. It follows from Equation (3.21) that

$$\hat{I}_{k,1}, \hat{I}_{k,2} \in \mathcal{S} \cap \bar{\mathcal{S}}.$$

Since there exist locally k functionally independent $\mathfrak{h}-$ differential invariants of pure order

k, it follows that

$$g_1(\hat{I}_{k,1}, \hat{I}_{k,2}) = g_2(u_{1\bar{0}}, .., u_{k\bar{0}}) \in \bar{\mathcal{H}},$$
$$g_3(\hat{I}_{k,1}, \hat{I}_{k,2}) = g_4(u_{0\bar{1}}, .., u_{0\bar{k}}) \in \bar{\mathcal{H}},$$

for some nonzero functions $g_1, g_3 \in C^\infty_{loc}(\mathbb{R}^2)$, $g_2, g_4 \in C^\infty_{loc}(\mathbb{R}^k)$.

For $k = 3$ it is well known [KL2] that the Schwarz derivative is a differential invariant of the Lie algebra $\mathfrak{s}$

$$SD = \frac{2u_{1\bar{0}}u_{3\bar{0}} - 3u_{2\bar{0}}^2}{u_{1\bar{0}}^4}.$$

Note that

$$SD \in \bar{\mathcal{H}}, \overline{SD} \in \mathcal{H}, \;\; SD \notin \mathcal{H}, \overline{SD} \notin \bar{\mathcal{H}}.$$

Hence

$$\hat{I}_{3,1}, \hat{I}_{3,2} \in \mathcal{Z}_k, \; \hat{I}_{3,1}, \hat{I}_{3,2} \notin \mathcal{G}_k,$$

for

$$\begin{aligned}
\hat{I}_{3,1} &= -\tfrac{1}{2}\left(SD + \overline{SD}\right) \\
&= \frac{1}{(u_{10}^2 + u_{01}^2)^4}(3u_{10}^4u_{20}^2 + 6u_{01}^5u_{21} - 2u_{01}^5u_{03} - 12u_{10}^4u_{11}^2 - 12u_{01}^4u_{11}^2 + 48u_{10}^3u_{01}u_{20}u_{11} \\
&\quad -48u_{10}^3u_{01}u_{11}u_{02} + 72u_{10}^2u_{01}^2u_{11}^2 - 6u_{01}^4u_{20}u_{02} + 36u_{10}^2u_{01}^2u_{20}u_{02} - 12u_{10}^2u_{01}^3u_{21} \\
&\quad +4u_{10}^2u_{01}^3u_{03} + 6u_{10}u_{01}^4u_{30} - 18u_{10}u_{01}^4u_{12} - 48u_{10}u_{01}^3u_{20}u_{11} + 48u_{10}u_{01}^3u_{11}u_{02} \\
&\quad -6u_{10}^4u_{20}u_{02} - 18u_{10}^2u_{01}^2u_{02}^2 + 3u_{01}^4u_{20}^2 + 3u_{01}^4u_{02}^2 - 18u_{10}^4u_{01}u_{21} + 6u_{10}^4u_{01}u_{03} \\
&\quad +4u_{10}^3u_{01}^2u_{30} - 12u_{10}^3u_{01}^2u_{12} - 18u_{10}^2u_{01}^2u_{20}^2 - 2u_{10}^5u_{30} + 6u_{10}^5u_{12} + 3u_{10}^4u_{02}^2),
\end{aligned}$$

$$\begin{aligned}
\hat{I}_{3,2} &= -\tfrac{i}{2}\left(SD-\overline{SD}\right)\\
&= \frac{1}{(u_{10}^2+u_{01}^2)^4}(-2u_{10}^2u_{01}^3u_{30}+6u_{10}^2u_{01}^3u_{12}-6u_{10}u_{01}^3u_{02}^2-24u_{10}^3u_{01}u_{11}^2-6u_{10}u_{01}^3u_{20}^2\\
&\quad +u_{01}^5u_{30}-3u_{01}^5u_{12}+2u_{10}^3u_{01}^2u_{03}+3u_{10}u_{01}^4u_{03}-9u_{10}u_{01}^4u_{21}+24u_{10}u_{01}^3u_{11}^2\\
&\quad +6u_{01}^4u_{11}u_{02}+9u_{10}^4u_{01}u_{12}-3u_{10}^4u_{01}u_{30}-6u_{10}^4u_{20}u_{11}+6u_{10}^4u_{11}u_{02}-6u_{01}^4u_{20}u_{11}\\
&\quad +36u_{10}^2u_{01}^2u_{20}u_{11}-36u_{10}^2u_{01}^2u_{11}u_{02}+12u_{10}u_{01}^3u_{20}u_{02}+6u_{10}^3u_{01}u_{20}^2+6u_{10}^3u_{01}u_{02}^2\\
&\quad -12u_{10}^3u_{01}u_{20}u_{02}-6u_{10}^3u_{01}^2u_{21}-u_{10}^5u_{03}+3u_{10}^5u_{21}).
\end{aligned}$$

It follows from Subsection 3.5.3 that the derivation operators

$$\nabla=\frac{\bar{\mathcal{D}}}{\mathcal{D}u}=\frac{1}{u_{1\bar{0}}}\mathcal{D}_z,\ \bar{\nabla}=\frac{\mathcal{D}}{\mathcal{D}u}=\frac{1}{u_{0\bar{1}}}\mathcal{D}_{\bar{z}},$$

are invariant derivatives of the Lie algebras $\mathfrak{h}$, $\bar{\mathfrak{h}}$, $\mathfrak{s}$ and $\bar{\mathfrak{s}}$. Hence

$$\begin{aligned}
\nabla^k(SD) &\in \bar{\mathcal{H}},\ \mathcal{S}\cap\bar{\mathcal{S}},\\
\bar{\nabla}^k\left(\overline{SD}\right) &\in \mathcal{H},\ \mathcal{S}\cap\bar{\mathcal{S}},\\
\nabla^k(SD) &\notin \mathcal{H},\\
\bar{\nabla}^k\left(\overline{SD}\right) &\notin \bar{\mathcal{H}}.
\end{aligned}$$

Hence the two functions

$$\begin{aligned}
\hat{I}_{3+k,1} &= \nabla^k(SD)+\bar{\nabla}^k\left(\overline{SD}\right),\\
\hat{I}_{3+k,2} &= i\left(\nabla^k(SD)-\bar{\nabla}^k\left(\overline{SD}\right)\right),
\end{aligned}$$

are $\mathfrak{z}-$differential invariant of pure order $k+3$.

Therefore the algebra of invariant functions under the action of $\mathfrak{s}$ and $\bar{\mathfrak{s}}$ on $C^\infty(J^k\mathbb{R}^2)\otimes$

$\mathbb{C}$ and $\mathfrak{z}$ on $C^\infty(J^k\mathbb{R}^2)$ are

$$\begin{aligned}
\mathcal{S}_k &= \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2)\otimes\mathbb{C} \mid f = f(\bar{z}, u_{0\bar{1}}, .., u_{0\bar{k}}, u, , I_2, .., I_{k,k-1}, SD, .., \nabla^{k-3}\left(SD\right)) \right\}, \\
\bar{\mathcal{S}}_k &= \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2)\otimes\mathbb{C} \mid f = f(z, u_{1\bar{0}}, .., u_{1\bar{0}}, u, I_2, .., I_{k,k-1}, \overline{SD}, .., \bar{\nabla}^{k-3}\left(\overline{SD}\right)) \right\}, \\
\mathcal{Z}_k &= \left\{ f \in C^\infty_{loc}(J^k\mathbb{R}^2) \mid f = f(u, I_{2,}.., I_{k,k-1}, \hat{I}_{3,1}, \hat{I}_{3,2}, .., \hat{I}_{k,1}, \hat{I}_{k,2}) \right\},
\end{aligned}$$

where the functions $I_{j,i}$ are as defined in Lemma 23.

The subspaces

$$\mathfrak{c} = \langle V_1, V_i, V_z, V_{iz} \rangle \subset \mathfrak{z},$$

$$\mathfrak{w} = \langle \partial_z, z\partial_z \rangle \subset \mathfrak{s},$$

are Lie algebras. There exist locally three functionally independent $\mathfrak{c}-$ and $\mathfrak{w}-$differential invariants of pure order two. The function

$$\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2} \in \bar{\mathcal{H}}$$

is a $\mathfrak{w}-$differential invariant. Hence the two functions

$$\begin{aligned}
\tfrac{1}{2}\left(\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2} + \frac{u_{0\bar{2}}}{u_{01}^2}\right) &= \frac{(u_{20} - u_{02})\left(u_{10}^2 - u_{01}^2\right) + 4u_{11}u_{01}u_{10}}{\left(u_{10}^2 + u_{01}^2\right)^2}, \\
\tfrac{i}{2}\left(\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2} - \frac{u_{0\bar{2}}}{u_{01}^2}\right) &= \frac{2(u_{20} - u_{02})u_{01}u_{10} - 2u_{11}\left(u_{10}^2 - u_{01}^2\right)}{\left(u_{10}^2 + u_{01}^2\right)^2},
\end{aligned}$$

are $\mathfrak{c}-$differential invariants.

The function

$$\nabla\left(\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2}\right) = \frac{1}{u_{1\bar{0}}}\mathcal{D}_z\left(\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2}\right) = \frac{u_{3\bar{0}}}{u_{1\bar{0}}^3} - \left(\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2}\right)^2$$

is an $\mathfrak{c}-$differential invariant of order three. Note that

$$SD = 2\nabla\left(\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2}\right) - \left(\frac{u_{2\bar{0}}}{u_{1\bar{0}}^2}\right)^2.$$

Chapter 4

Differential Invariants of the Deformed Representations of $\mathfrak{g}$

This chapter is a generalization of Chapter 3.

4.1 The Lie Algebra $\mathfrak{g}_{Fb}$

4.1.1 The Lie Algebra Homomorphism $K_\lambda : \mathfrak{h} \rightarrow \mathcal{D}\left(J^0\mathbb{R}^2\right) \otimes \mathbb{C}$

In this subsection we are seeking a Lie algebra homomorphism

$$K_\lambda : \mathfrak{h} \longrightarrow \mathcal{D}\left(J^0\mathbb{R}^2\right) \otimes \mathbb{C},$$

$$K_\lambda(g\partial_z) = g\partial_z + \lambda(g,u)\partial_u,$$

where $\lambda(z,u) \in C^\infty(J^0\mathbb{R}^2) \otimes \mathbb{C}$. Hence the map K_λ must be linear and preserve the commutator bracket, i.e.

$$\lambda(\{g,h\},u) = gh_z\lambda_h(h,u) - hg_z\lambda_g(g,u) + \lambda(g,u)\lambda_u(h,u) - \lambda_u(g,u)\lambda(h,u), \tag{4.1}$$

where $\{g, h\}$ is the bracket defined on $\mathcal{O}$ in Subsection 2.2.1.

It follows from Equation (4.1) that for any constants $c_1, c_2 \in \mathbb{C}$

$$\lambda(\{c_1, c_2\}, u) = \lambda(c_1, u)\lambda_u(c_2, u) - \lambda_u(c_1, u)\lambda(c_2, u) = 0.$$

Hence the function $\lambda(z, u)$ is separable, i.e.

$$\lambda(z, u) = Z(z)U(u).$$

It follows from Equation (4.1) that

$$Z(\{1, z\}) = Z(1) = Z'(z).$$

Hence

$$Z(z) = zc,\ c \in \mathbb{C}.$$

Thus the map

$$K_\lambda : \mathfrak{h} \longrightarrow \mathcal{D}\left(J^0\mathbb{R}^2\right) \otimes \mathbb{C},$$

$$K_\lambda(g\partial_z) = g\partial_z + \lambda(g, u)\partial_u,$$

is a Lie algebra homomorphism if and only if

$$\lambda(z, u) = zU(u),\ U(u) \in C^\infty\left(J^0\mathbb{R}^2\right) \otimes \mathbb{C}.$$

Let $\mathfrak{h}_U = \mathrm{Im}(K_\lambda)$ denote the image of the Lie algebra homomorphism

$$\mathfrak{h}_U = \{\ g\left(\partial_z + U(u)\partial_u\right)\ \mid g \in \mathcal{O}\}.$$

4.1.2 A Lie Algebra Isomorphism

In this subsection we are seeking functions $G(u) = G_1(u) + iG_2(u) \in C^\infty\left(J^0\mathbb{R}^2\right) \otimes \mathbb{C}$ such that the $\mathbb{R}-$linear map

$$2\,\mathrm{Re} : \mathfrak{h}_G \longrightarrow \mathcal{D}\left(J^0\mathbb{R}^2\right)$$

is an injective Lie algebra homomorphism.

Since

$$\begin{aligned} & g\left(\partial_z + G(u)\partial_u\right) \\ = \ & \tfrac{1}{2}\ \left(g_1\partial_x + g_2\partial_y\right) + (g_1G_1(u) - g_2G_2(u))\partial_u + i\left(\tfrac{1}{2}\left(g_2\partial_x - g_1\partial_y\right) + (g_2G_1(u) + g_1G_2(u))\partial_u\right) \end{aligned}$$

we have

$$2\,\mathrm{Re}\left(g\left(\partial_z + G(u)\partial_u\right)\right) = V_g + 2(g_1G_1(u) - g_2G_2(u))\partial_u.$$

If the map $2\,\mathrm{Re}$ is a Lie algebra homomorphism when restricted to $\mathfrak{h}_F$, then

$$\begin{aligned} & \left[2\,\mathrm{Re}\left(g\left(\partial_z + G(u)\partial_u\right)\right), 2\,\mathrm{Re}\left(h\left(\partial_z + G(u)\partial_u\right)\right)\right] \\ = \ & \left[V_g + 2(g_1G_1(u) - g_2G_2(u))\partial_u, V_h + 2(h_1G_1(u) - h_2G_2(u))\partial_u\right] \\ = \ & V_{[g,h]} + 2\left(V_g(h_1G_1(u) - h_2G_2(u)) - V_h(g_1G_1(u) - g_2G_2(u))\right)\partial_u \\ & -4\left(g_1h_2 - h_1g_2\right)\left(G_1(u)G_2'(u) - G_1'(u)G_2(u)\right)\partial_u \end{aligned}$$

is equal to

$$\begin{aligned} & 2\,\mathrm{Re}\left[g\left(\partial_z + G(u)\partial_u\right), h\left(\partial_z + G(u)\partial_u\right)\right] \\ = \ & V_{[g,h]} + 2\left(V_g(h_1G_1(u) - h_2G_2(u)) - V_h(g_1G_1(u) - g_2G_2(u))\right)\partial_u. \end{aligned}$$

Hence

$$G_1(u)G_2'(u) - G_1'(u)G_2(u) = 0.$$

So it follows that the linear map $2\,\mathrm{Re} : \mathfrak{h}_G \longrightarrow \mathcal{D}\left(J^0\mathbb{R}^2\right)$ is an injective Lie algebra homomorphism if and only if

$$G(u) = F(u)b,$$

where $F(u) \in C^\infty\left(J^0\mathbb{R}^2\right)$ and $b = b_1 + ib_2 \in \mathbb{C}$.

Hence the three spaces

$$\begin{aligned}
\mathfrak{h}_{Fb} &= \left\{ gW_{Fb} = g(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right) \mid g \in \mathcal{O}\right\}, \\
\bar{\mathfrak{h}}_{Fb} &= \left\{ \bar{g}\bar{W}_{Fb} = \bar{g}(z)\left(\partial_{\bar{z}} + \tfrac{1}{2}F(u)\bar{b}\partial_u\right) \mid g \in \mathcal{O}\right\}, \\
\mathfrak{g}_{Fb} &= \left\{V_{Fbg} = V_g + F(u)(g_1b_1 - g_2b_2)\partial_u \mid g \in \mathcal{O}\right\},
\end{aligned}$$

are infinite dimensional Lie algebras.

If we consider $\mathfrak{g}_{Fb}$ as a Lie subalgebra of $\mathcal{D}\left(J^0\mathbb{R}^2\right) \otimes \mathbb{C}$, then

$$V_{Fbg} = g(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right) + \bar{g}\left(\partial_{\bar{z}} + \tfrac{1}{2}F(u)b\partial_u\right),\ \forall g \in \mathcal{O}.$$

Moreover, the complexification of $\mathfrak{g}_{Fb}$ is equal to the direct sum

$$\mathfrak{g}_{Fb} \otimes \mathbb{C} = \mathfrak{h}_{Fb} \oplus \bar{\mathfrak{h}}_{Fb}.$$

Consider the linear subspace of $\mathfrak{h}_{Fb}$

$$\mathfrak{s}_{Fb} = \left\langle z^j\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right) \mid j \in \{0,1,2\}\right\rangle_{\mathbb{C}} \subset \mathfrak{h}_{Fb}.$$

Since

$$\left[z^j\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right), z^l\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right)\right] = (l-j)z^{j-l}\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right) \in \mathfrak{s}_{Fb}$$

for $j, l \in \{0,1,2\}$, it follows that $\mathfrak{s}_{Fb}$ is a $3-$dimensional Lie subalgebra of $\mathfrak{s}_{Fb}$ isomorphic to $\mathrm{sl}_2(\mathbb{C})$. Hence the space

$$\mathfrak{z}_{Fb} = \langle V_{Fb_1}, V_{Fbi}, V_{Fbz}, V_{Fbiz}, V_{Fbz^2}, V_{Fbiz^2}\rangle \subset \mathfrak{g}_{Fb},$$

is a Lie algebra isomorphic to $\mathrm{sl}_2(\mathbb{C})_{\mathbb{R}}$.

4.2 The Lie Algebra $\mathfrak{g}^k_{Fb}$

In this section we will use the results from Chapter 3.

Consider the spaces

$$\mathfrak{g}^k_{Fb} = \left\{ V^{(k)}_{Fbg} = V^{(k)}_g + ((F(u)(b_1 g_1 - b_2 g_2))\partial_u)^{(k)} \ \mid g \in \mathcal{O} \right\},$$

$$\mathfrak{h}^k_{Fb} = \left\{ gW^{(k)}_{Fb} = \left(g(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right)\right)^{(k)} \ \mid g \in \mathcal{O} \right\}.$$

For $k \geq 1$ $\mathfrak{h}^k_{Fb}$ is a Lie subalgebra of Lie $\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$ and $\mathfrak{g}^k_{Fb}$ is a Lie subalgebra of Lie $\left(J^k\mathbb{R}^2\right)$. Let $\mathcal{H}^{Fb}$ denote the algebra of the $\mathfrak{h}_{Fb}-$differential invariants, and let $\mathcal{G}^{Fb}$ denote the algebra of the $\mathfrak{g}_{Fb}-$differential invariants.

Proposition 25 *We have that*

$$\mathcal{G}^{Fb} \otimes \mathbb{C} = \mathcal{H}^{Fb} \cap \bar{\mathcal{H}}^{Fb}$$

Proof. $\Longrightarrow$For any integer $k \in \mathbb{Z}_{\geq 0}$ and any function $f \in \mathcal{H}^{Fb}_k \cap \bar{\mathcal{H}}^{Fb}_k$

$$V^{(k)}_{Fbg}(f) = \left(g(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right)\right)^{(k)}(f) + \left(\bar{g}(z)\left(\partial_{\bar{z}} + \tfrac{1}{2}F(u)\bar{b}\partial_u\right)\right)^{(k)}(f) = 0, \ \forall g \in \mathcal{O}.$$

Hence

$$\mathcal{G}^{Fb} \otimes \mathbb{C} \supseteq \mathcal{H}^{Fb} \cap \bar{\mathcal{H}}^{Fb}$$

$\Longleftarrow$For any integer $k \in \mathbb{Z}_{\leq 0}$ and any function $f \in \mathcal{G}^{Fb}_k \otimes \mathbb{C}$

$$\begin{aligned}
\left(g(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right)\right)^{(k)}(f) &= V^{(k)}_{Fbg}(f) - iV^{(k)}_{Fbig}(f) = 0, \ \forall g \in \mathcal{O} \\
\left(\bar{g}(z)\left(\partial_{\bar{z}} + \tfrac{1}{2}F(u)\bar{b}\partial_u\right)\right)^{(k)}(f) &= V^{(k)}_{Fbg}(f) + iV^{(k)}_{Fbig}(f) = 0, \ \forall g \in \mathcal{O}.
\end{aligned}$$

■

4.2.1 The Distribution Ω^k_{Fb}

In this subsection we will use the results from Subsection 3.3.1.

For every function $F(u) \in C^\infty\left(J^0\mathbb{R}^2\right)$ and constant $b = b_1 + ib_2 \in \mathbb{C}$ the complex distribution

$$\Omega^k_{Fb} = \left\langle \left(g(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right)\right)^{(k)} \mid g \in \mathcal{O}\right\rangle_{\mathbb{C}}$$

has complex dimension $k+1$ and

$$\Omega^k_{Fb} \oplus \bar{\Omega}^k_{Fb} = \left\langle V^{(k)}_{Fbg} \mid g \in \mathcal{O}\right\rangle \otimes \mathbb{C}$$

has complex dimension $2(k+1)$.

For any functions $g, h \in \mathcal{O}$

$$\left[\left(g(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right)\right)^{(k)}, \left(h(z)\left(\partial_z + \tfrac{1}{2}F(u)b\partial_u\right)\right)^{(k)}\right] \in \Omega^k_{Fb},$$

$$\left[V^{(k)}_{Fbg}, V^{(k)}_{Fbh}\right] \in \Omega^k_{Fb} \oplus \bar{\Omega}^k_{Fb}.$$

Moreover

$$\Omega^k_{Fb} \cap \bar{\Omega}^k_{Fb} = 0.$$

Hence the distribution Ω^k_{Fb} is completely integrable.

The first integrals of the complex distribution $\Omega^k_{Fb} \oplus \bar{\Omega}^k_{Fb}$ are invariant functions under the action of $\mathfrak{g}_{Fb}$ on $C^\infty\left(J^k\mathbb{R}^2\right) \otimes \mathbb{C}$. Hence the distribution $\Omega^k_{Fb} \oplus \bar{\Omega}^k_{Fb}$ has one first integral of order zero

$$J_0 \in C^\infty_{loc}\left(J^k\mathbb{R}^2\right)$$

and $l-1$ first integrals of order l for $1 \le l \le k$

$$\{J_{l,j}\}_{j=1}^{l-1} \in C^\infty_{loc}\left(J^k\mathbb{R}^2\right),\ 2 \le l \le k$$

such that locally the $K-2(k+1)$ functions

$$J_0 \bigcup_{2\leq l\leq k} \{J_{l,j}\}_{j=1}^{l-1}$$

are functionally independent, where

$$K=(k+1)(k+2)/2+2.$$

The first integrals of the distribution Ω_{Fb}^k are invariant functions under the action of $\mathfrak{h}_{Fb}$ on $C^\infty\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$. Hence the distribution Ω_{Fb}^k has one first integral of pure order l

$$Q_l,\ \in C_{loc}^\infty\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C},\ 1\leq l\leq k,$$

such that locally the $K-(k+1)$ functions

$$J_0 \bigcup_{2\leq l\leq k} \{J_{l,j}\}_{j=1}^{l-1} \bigcup_{1\leq m\leq k} Q_m$$

are functionally independent.

The algebras of invariant functions under the action of $\mathfrak{h}_{Fb}$ and $\bar{\mathfrak{h}}_{Fb}$ on $C^\infty\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C}$ and $\mathfrak{g}_{Fb}$ on $C^\infty\left(J^k\mathbb{R}^2\right)$ are

$$\begin{aligned}
\mathcal{H}_k^{Fb} &= \left\{f\in C_{loc}^\infty\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C} \mid f=f(J_0,J_2,J_{3,1},J_{3,2},..,J_{k,k-1},Q_1,...,Q_k)\right\},\\
\bar{\mathcal{H}}_k^{Fb} &= \left\{f\in C_{loc}^\infty\left(J^k\mathbb{R}^2\right)\otimes\mathbb{C} \mid f=f(J_0,J_2,J_{3,1},J_{3,2},...,J_{k,k-1},\bar{Q}_1,..,\bar{Q}_k)\right\},\\
\mathcal{G}_k^{Fb} &= \left\{f\in C_{loc}^\infty\left(J^k\mathbb{R}^2\right) \mid f=f(J_0,J_2,J_{3,1},J_{3,2},...,J_{k,k-1})\right\}.
\end{aligned}$$

4.2.2 Invariant Derivatives of $\mathfrak{g}_{Fb}$

For any function $F(u)\neq 0\in C^\infty\left(J^k\mathbb{R}^2\right)$ and constant $b=b_1+ib_2\neq 0\in\mathbb{C}$, the following functions are $\mathfrak{h}_{Fb}-$differential invariants of order zero

$$\bar{z},J_0=\int\frac{1}{F(u)}du-\tfrac{1}{2}zb-\tfrac{1}{2}\bar{z}\bar{b}\in\mathcal{H}^{Fb}.$$

The derivation operators

$$\begin{bmatrix} \dfrac{\mathcal{D}}{\mathcal{D}\bar{z}} \\ \dfrac{\mathcal{D}}{\mathcal{D}J_0} \end{bmatrix} = \frac{-1}{\frac{u_{1\bar{0}}}{F(u)} - \frac{1}{2}b} \begin{bmatrix} \frac{u_{0\bar{1}}}{F(u)} - \frac{1}{2}\bar{b} & -\frac{u_{1\bar{0}}}{F(u)} + \frac{1}{2}b \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{D}_z \\ \mathcal{D}_{\bar{z}} \end{bmatrix}$$

are invariant derivatives of $\mathfrak{h}_{Fb}$.

The derivation operator $\mathcal{D}_{\bar{z}}$ is an invariant derivative of $\mathfrak{h}_{Fb}$. Hence the following function is a $\mathfrak{h}_{Fb}-$differential invariant of order one

$$\mathcal{D}_{\bar{z}}(J_0) + \tfrac{1}{2}\bar{b} = \frac{u_{0\bar{1}}}{F(u)}.$$

Note that for any integers $k, j \in \mathbb{Z}_{\geq 0}$ such that $k \geq j$

$$\begin{aligned} & \left(\frac{\mathcal{D}}{\mathcal{D}\bar{z}}\right)^{k-j} \left(\frac{\mathcal{D}}{\mathcal{D}J_0}\right)^{j} \left(\frac{u_{0\bar{1}}}{F(u)}\right) \\ = \quad & f + \frac{1}{F(u)} \left(\frac{-F(u)}{u_{1\bar{0}} - \frac{1}{2}F(u)b}\right)^{k} \sum_{n=0}^{k-j} \left(\frac{u_{0\bar{1}}}{F(u)} - \frac{1}{2}\bar{b}\right)^{k-j-n} \left(-\frac{u_{1\bar{0}}}{F(u)} + \frac{1}{2}b\right)^{n} u_{(k-n)\overline{(1+n)}}, \end{aligned} \tag{4.2}$$

where f is a smooth function of order less than $k+1$.

Theorem 26 *For any integer $k \in \mathbb{Z}_{\geq 0}$ the $k+1$ functions*

$$K_{k+1,j+1} \left(\frac{\mathcal{D}}{\mathcal{D}\bar{z}}\right)^{k-j} \left(\frac{\mathcal{D}}{\mathcal{D}J_0}\right)^{j} \left(\frac{u_{0\bar{1}}}{F(u)}\right), \ j \in \{0, .., k\}$$

are $\mathfrak{h}_{Fb}-$differential invariants of order $k+1$. Moreover,

$$\sum_{j=0}^{k} g_j \left(\frac{\mathcal{D}}{\mathcal{D}\bar{z}}\right)^{k-j} \left(\frac{\mathcal{D}}{\mathcal{D}J_0}\right)^{j} \left(\frac{u_{0\bar{1}}}{F(u)}\right) = 0, g_j \in \mathcal{H}_k^{Fb} \Longrightarrow g_j = 0, j \in \{0, .., k\}.$$

For any integer $m \in \mathbb{Z}_{+}$ the algebra of invariant functions under the action of $\mathfrak{h}_{Fb}$ on $C^{\infty}\left(J^m\mathbb{R}^2\right) \otimes \mathbb{C}$ is

$$\mathcal{H}_m^{Fb} = \left\{ f \in C_{loc}^{\infty}\left(J^m\mathbb{R}^2\right) \otimes \mathbb{C} \mid f = f(\bar{z}, J_0, K_{1,1}, K_{2,1}, K_{2,2}, ..., K_{m,m}) \right\}$$

Proof. The theorem follows from Equation (4.2) and Subsection 4.2.1. ■

Since $\mathcal{D}_{\bar{z}}$ is an invariant derivative of $\mathfrak{h}_{Fb}$ and $\mathcal{D}_z$ is an invariant derivative of $\bar{\mathfrak{h}}_{Fb}$ it follows that the derivation operators

$$\frac{\bar{\mathcal{D}}}{\mathcal{D}J_0} = \frac{1}{\frac{u_{1\bar{0}}}{F(u)} - \frac{1}{2}b}\mathcal{D}_z, \quad \frac{\mathcal{D}}{\mathcal{D}J_0} = \frac{1}{\frac{u_{0\bar{1}}}{F(u)} - \frac{1}{2}\bar{b}}\mathcal{D}_{\bar{z}},$$

are invariant derivatives for $\mathfrak{g}_{Fb}\otimes\mathbb{C}$. Since

$$\left[\frac{\bar{\mathcal{D}}}{\mathcal{D}J_0}, \frac{\mathcal{D}}{\mathcal{D}J_0}\right] = \frac{\mathcal{D}_z\mathcal{D}_{\bar{z}}(J_0)}{\mathcal{D}_z(J_0)\mathcal{D}_z(J_0)}\left(\frac{\bar{\mathcal{D}}}{\mathcal{D}J_0} - \frac{\mathcal{D}}{\mathcal{D}J_0}\right),$$

it follows that the function

$$J_2 = \frac{\mathcal{D}_z\mathcal{D}_{\bar{z}}(J_0)}{\mathcal{D}_z(J_0)\mathcal{D}_z(J_0)} = \frac{(-u_{01}^2 - u_{10}^2)F_u(u) + F(u)(u_{02} + u_{20})}{(b_1F(u) - u_{10})^2 + (b_2F(u) + u_{01})^2}$$

is an $\mathfrak{g}_{Fb}-$differential invariant of order two. Moreover,

$$\hat{\nabla}_1 = \left(\frac{F(u)^2}{(u_{10} - b_1F(u))^2 + (u_{01} + b_2F(u))^2}\right)\left(\left(\frac{u_{10}}{F(u)} - b_1\right)\mathcal{D}_x + \left(\frac{u_{01}}{F(u)} + b_2\right)\mathcal{D}_y\right),$$

$$\hat{\nabla}_2 = \left(\frac{F(u)^2}{(u_{10} - b_1F(u))^2 + (u_{01} + b_2F(u))^2}\right)\left(\left(\frac{u_{01}}{F(u)} + b_2\right)\mathcal{D}_x + \left(-\frac{u_{10}}{F(u)} + b_1\right)\mathcal{D}_y\right),$$

are invariant derivatives of $\mathfrak{g}_{Fb}$.

Theorem 27 *For any integer $k \in \mathbb{Z}_{\geq 0}$ the $k+1$ functions*

$$J_{k+2,j+1} = \left(\hat{\nabla}_1\right)^{k-j}\left(\hat{\nabla}_j\right)^j(J_2), \ j \in \{0,..,k\}$$

are $\mathfrak{g}_{Fb}-$differential invariants of order $k+2$. Moreover, these $\mathfrak{g}_{Fb}-$differential invariants are independent, i.e.

$$\sum_{j=0}^{k} g_j\left(\hat{\nabla}_1\right)^{k-j}\left(\hat{\nabla}_2\right)^j(J_2) = 0, \ g_j \in \mathcal{G}_{k+1}^{Fb} \implies g_j = 0, \ j \in \{0,..,k\}.$$

For any integer $m \in \mathbb{Z}_+$ the algebra of invariant functions under the action of $\mathfrak{g}_{Fb}$ on $C^\infty\left(J^m\mathbb{R}^2\right)$ is

$$\mathcal{G}_m^{Fb} = \left\{ f \in C^\infty_{loc}\left(J^m\mathbb{R}^2\right) \otimes \mathbb{C} \mid f = f(J_0, J_2, J_{3,1}, J_{3,2}, ..., J_{m,m-1}) \right\}.$$

Proof. The theorem follows from Theorem 26 . ■

Remark 28 *Note that for any nonzero function $F(u) \in C^\infty\left(J^0\mathbb{R}^2\right)$ and $b = b_1 + ib_2 \in \mathbb{C}$, we have that*

$$\lim_{b\to 0}(V_g + F(u)(b_1 g_1 + b_2 g_2)) = V_g,$$

$$\begin{aligned}
\lim_{b\to 0} J_0 &= \int \frac{du}{F(u)} = \int \frac{dI_0}{F(I_0)}, \\
\lim_{b\to 0} J_2 &= -F'(u) + F(u)\left(\frac{u_{20}+u_{02}}{u_{10}^2+u_{01}^2}\right) = -F'(I_0) + F(I_0)I_2, \\
\lim_{b\to 0} \hat{\nabla}_1 &= \frac{F(u)}{u_{10}^2+u_{01}^2}\left(u_{10}\mathcal{D}_x + u_{01}\mathcal{D}_y\right) = F(I_0)\nabla_1, \\
\lim_{b\to 0} \hat{\nabla}_2 &= \frac{F(u)}{u_{10}^2+u_{01}^2}\left(u_{01}\mathcal{D}_x - u_{10}\mathcal{D}_y\right) = F(I_0)\nabla_2.
\end{aligned}$$

Hence the results of Chapter 3 can be obtained from the results of this chapter.

4.2.3 Invariant Functions of the Lie Algebra $\mathfrak{z}_{Fb}$ and $\mathfrak{c}_{Fb}$

For any function $F(u) \neq 0 \in C^\infty\left(J^k\mathbb{R}^2\right)$ and constant $b = b_1 + ib_2 \neq 0 \in \mathbb{C}$ consider the Lie algebras

$$\mathfrak{c}_{Fb} = \langle V_{Fb1}, V_{Fbi}, V_{Fbz}, V_{Fbiz}\rangle \subset \mathfrak{z}_{Fb} = \langle V_{Fb1}, V_{Fbi}, V_{Fbz}, V_{Fbiz} V_{Fbz^2}, V_{Fbiz^2}\rangle \subset \mathfrak{g}_{Fb},$$

$$\mathfrak{w}_{Fb} = \left\langle z^j\left(\partial_z + \tfrac{b}{2}F(u)\partial_u\right) \mid j \in \{0,1\}\right\rangle \subset \mathfrak{s}_{Fb} = \left\langle z^j\left(\partial_z + \tfrac{b}{2}F(u)\partial_u\right) \mid j \in \{0,1,2\}\right\rangle \subset \mathfrak{h}_{Fb}.$$

For any integer $k \geq 2$ there exist locally $k+1$ functionally independent $\mathfrak{c}_{Fb}-$ and $\mathfrak{w}_{Fb}-$differential invariants of pure order k. The function

$$W_2 = \frac{u_{2\bar{0}}F(u) - F'(u)u_{1\bar{0}}^2}{(-2u_{1\bar{0}} + bF(u))^2} \in \bar{\mathcal{H}},\, W_2 \notin \mathcal{H},$$

is an $\mathfrak{w}_{Fb}$−differential invariant.

For any integer $k \geq 3$ there exist locally $k+1$ functionally independent $\mathfrak{z}_{Fb}$− and $\mathfrak{s}_{Fb}$−differential invariants of pure order k. The function

$$\begin{aligned} &\nabla(W_2) \\ = \; &\frac{2}{(-2u_{1\bar{0}} + bF(u))^4}\big(2F_u(u)u_{2\bar{0}}u_{1\bar{0}}^2 + 3u_{1\bar{0}}F_u(u)u_{2\bar{0}}bF(u) - 2F_{uu}(u)u_{1\bar{0}}^4 + \\ &bF_{uu}(u)F(u)u_{1\bar{0}}^3 - 2bF_u(u)^2u_{1\bar{0}}^3 - 4u_{2\bar{0}}^2F(u) + 2F(u)u_{3\bar{0}}u_{1\bar{0}} - F(u)^2u_{3\bar{0}}b\big) \end{aligned}$$

is an $\mathfrak{w}_{Fb}$−differential invariant, where $\nabla = \dfrac{\bar{\mathcal{D}}}{\mathcal{D}J_0}$ is the invariant derivative of $\mathfrak{h}$ from Subsection 4.2.2. Since $\mathfrak{w}_{Fb}$ is a Lie subalgebra of $\mathfrak{s}_{Fb}$, it follows that there must exist a function $W_3\left(W_2, \nabla(W_2)\right)$ that is an $\mathfrak{s}_{Fb}$−differential invariant. Indeed, the function

$$W_3 = 2W_2^2 + \nabla(W_2)$$

is an $\mathfrak{s}_{Fb}$−differential invariant.

Since

$$\mathfrak{z}_{Fb} \otimes \mathbb{C} = \mathfrak{s}_{Fb} \oplus \bar{\mathfrak{s}}_{Fb}, \quad \mathfrak{c}_{Fb} \otimes \mathbb{C} = \mathfrak{w}_{Fb} \oplus \bar{\mathfrak{w}}_{Fb},$$

it follows that the functions

$$\hat{J}_{2,1} = W_2 + \bar{W}_2, \quad \hat{J}_{2,2} = i\left(W_2 - \bar{W}_2\right),$$

are $\mathfrak{c}_{Fb}$−differential invariants and

$$\hat{J}_{3+k,1} = \nabla^k W_3 + \nabla^k \bar{W}_3, \quad \hat{J}_{3+k,2} = \nabla^k W_3 - \nabla^k \bar{W}_3, \; k \in \mathbb{Z}_+$$

are $\mathfrak{z}_{Fb}-$differential invariants.

The algebras of invariant functions under the action of $\mathfrak{c}_{Fb}$ and $\mathfrak{z}_{Fb}$ on $C^{\infty}\left(J^k\mathbb{R}^2\right)$ are

$$
\begin{aligned}
\mathcal{C}_k^{Fb} &= \left\{f \in C_{loc}^{\infty}\left(J^k\mathbb{R}^2\right) \mid f = f(J_0, .., J_{k,k-1}, \hat{J}_{2,1}, \hat{J}_{2,2}, ..., \hat{J}_{k,1}, \hat{J}_{k,2})\right\}, \\
\mathcal{Z}_k^{Fb} &= \left\{f \in C_{loc}^{\infty}\left(J^k\mathbb{R}^2\right) \mid f = f(J_0, .., J_{k,k-1}, \hat{J}_{3,1}, \hat{J}_{3,2}, ..., \hat{J}_{k,1}, \hat{J}_{k,2})\right\},
\end{aligned}
$$

where the functions $J_{i,j}$ are as defined in Subsection 4.2.2.

Chapter 5

Applications and Examples

5.1 Applications

For any function $Q \in C^{\infty}\left(J^k\mathbb{R}^2\right)$ the surface $\mathcal{E} = \{Q = 0\}$ is a PDE. A vector field $X_f \in \mathrm{Cont}\left(J^1\mathbb{R}^2\right)$ is a symmetry of $\mathcal{E}$ if

$$X_f^{(k)}(f) = \lambda_{X_f} f,\ \lambda_{X_f} \in C^{\infty}\left(J^k\mathbb{R}^2\right).$$

Let θ_t be the flow of a symmetry vector field. If $h \in C^{\infty}\left(\mathbb{R}^2\right)$ is a solution of $\mathcal{E}$, then

$$\theta_t(x, y, h(x, y)) = (x_t, y_t, h_t(x_t, y_t)),$$

where $\hat{h}_t \in C^{\infty}\left(\mathbb{R}^2\right)$ is a family of solutions of the PDE $\mathcal{E}$.

The Lie group corresponding to the Lie algebra $\mathfrak{g}$ consists of all conformal transformations of $\mathbb{R}^2$. Therefore we have the following theorem:

Theorem 29 *Let F be a $\mathfrak{g}-$differential invariant. If the function $h(x, y) \in C^{\infty}\left(\mathbb{R}^2\right)$ is a solution of the PDE $\mathcal{E} = \{F = 0\}$, then the function*

$$u(x, y) = h(g_1(x, y), g_2(x, y))$$

is a solution of $\mathcal{E}$ *for every function* $g(z) = g_1(x,y) + ig_2(x,y) \in \mathcal{O}$ *on domains where* $g_z \neq 0$.

Example 30 *The function* $I_2 = \dfrac{u_{20} + u_{02}}{u_{10}^2 + u_{01}^2} \in C^\infty\left(J^k\mathbb{R}^2\right)$ *is a* $\mathfrak{g}-$*differential invariant. Therefore the PDE*

$$\mathcal{E} = \{u_{20} + u_{02} = 0\}$$

is $\mathfrak{g}-$*invariant. Moreover,* $\mathfrak{g}$ *acts transitively on the space* $\mathrm{sol}(\mathcal{E})$ *of harmonic functions.*

Example 31 *The function* $\dfrac{u_{20} + u_{02}}{u_{10}^2 + u_{01}^2} - \dfrac{1}{u}$ *is a* $\mathfrak{g}-$*differential invariant.*

$$h(x,y) = x^2 + y^2.$$

is a solution of the PDE

$$\mathcal{E} = \left\{ \frac{u_{11} + u_{22}}{u_2^2 + u_1^2} - \frac{1}{u} = 0 \right\}.$$

We will verify that the function

$$F(x,y) = h(g_1(x,y), g_2(x,y)) = g_1(x,y)^2 + g_2(x,y)^2$$

is a solution of $\mathcal{E}$ *for any function* $g = g_1 + ig_2 \in \mathcal{O}$ *on domains where* $g_z \neq 0$:

$$\begin{aligned}
& \frac{F_{xx} + F_{yy}}{F_x^2 + F_y^2} - \frac{1}{F} \\
= & \frac{2(g_1 g_{1xx} + g_{1x}^2 - g_2 g_{1xy} + g_{1y}^2) + 2(g_1 g_{1yy} + g_{1x}^2 + g_2 g_{1xy} + g_{2y}^2)}{4(g_1 g_{1x} - g_2 g_{1y})^2 + 4(g_1 g_{1y} - g_2 g_{1x})^2} - \frac{1}{g_1^2 + g_2^2} \\
= & \frac{4(g_{1x}^2 + g_{1y}^2)}{4(g_1^2 + g_2^2)(g_{1x}^2 + g_{1y}^2)} - \frac{1}{g_1^2 + g_2^2} = 0.
\end{aligned}$$

Let as compute the flow of $V_{Fbg} \in \mathfrak{g}_{Fb}$

$$\begin{aligned} \frac{dx}{dt} &= g_1(x,y), \\ \frac{dy}{dt} &= g_2(x,y), \\ \frac{du}{dt} &= F(u)\left(b_1 g_1(x,y) - b_2 g_2(x,y)\right) = F(u)\left(b_1 \frac{dx}{dt} - b_2 \frac{dy}{dt}\right). \end{aligned}$$

For $F \neq 0$ we have

$$G(u) = \int \frac{du}{F(u)} = \left(b_1 x - b_2 y + c\right).$$

Hence

$$u = G^{-1}\left(b_1 x - b_2 y + G(u_0) - (b_1 x_0 - b_2 y_0)\right).$$

Substituting

$$x_0 = g_1(x,y),\ y_0 = g_2(x,y),\ u_0 = h(x_0,y_0) = h(g_1,g_2)$$

we get the following theorem:

Theorem 32 *Let J be a $\mathfrak{g}_{Fb}-$differential invariant for $F \neq 0$. If the function $h(x,y) \in C^\infty\left(\mathbb{R}^2\right)$ is a solution of the PDE $\mathcal{E} = \{J = 0\}$, then the function*

$$u(x,y) = G^{-1} b_1(x - g_1(x,y)) - b_2(y - g_2(x,y)) + G(h(g_1(x,y), g_2(x,y))).$$

is a solution of $\mathcal{E}$ for every function $g = g_1 + i g_2 \in \mathcal{O}$ on domains where $g_z \neq 0$.

Example 33 *Let $F = c \in \mathbb{R}\backslash 0$. The function $\dfrac{(u_{20} + u_{02})}{(cb_1 - u_{10})^2 + (cb_2 + u_{01})^2}$ is a $\mathfrak{g}_{Fb}-$differential invariant. The function*

$$h = \ln(xy) + c(b_1 x - b_2 y)$$

is a solution of the PDE

$$\mathcal{E} = \left\{ \frac{(u_{20} + u_{02})}{(cb_1 - u_{10})^2 + (cb_2 + u_{01})^2} + 1 = 0 \right\}.$$

We will verify that the function

$$H(x, y) = c(b_1 x - b_2 y) + \ln(g_1(x, y) g_2(x, y))$$

is a solution of $\mathcal{E}$ *for any function* $g(z) = g_1(x, y) + ig(x, y)_2 \in \mathcal{O}$ *on domains where* $g_z \neq 0$.

We have that

$$\begin{aligned} H_x &= \frac{g_{1x} g_2 - g_{1y} g_1}{g_1 g_2} + cb_1, \\ H_y &= \frac{g_{1y} g_2 + g_{1x} g_1}{g_1 g_2} - cb_2, \\ H_{xx} &= \frac{-g_{1x}^2 g_2^2 - g_{1y}^2 g_1^2 + g_{1xx} g_1 g_2^2 - g_{1xy} g_1^2 g_2}{g_1^2 g_2^2}, \\ H_{yy} &= \frac{-g_{1y}^2 g_2^2 - g_{1x}^2 g_1^2 - g_{1xx} g_1 g_2^2 + g_{1xy} g_1^2 g_2}{g_1^2 g_2^2}. \end{aligned}$$

So it follows that

$$\frac{(H_{xx} + H_{yy})}{(cb_1 - u_{10})^2 + (cb_2 + u_{01})^2} + 1 = \frac{(-2g_{1x}^2 g_2^2 - 2g_{1y}^2 g_1^2)}{(g_{1x} g_2 - g_{1y} g_1)^2 + (g_{1y} g_2 + g_{1x} g_1)} + 1 = 0.$$

5.2 The Action of $\mathfrak{g}$ on $J^k\mathbb{R}$

Consider the Lie algebra

$$\mathfrak{g} = \{ g_1 \partial_x + g_2 \partial_u \mid g_{1x} = g_{2u},\ g_{1u} = g_{2x} \} \subset \mathcal{D}(\mathbb{R}^2) = \mathcal{D}(J^0\mathbb{R}).$$

This canonical action on $J^0\mathbb{R}$ lifts to the action of $\mathfrak{g}$ on $J^k\mathbb{R}$.

All scalar $\mathfrak{g}-$differential invariants are constants. However there exist $\mathfrak{g}-$invariant differential equations $\mathcal{E} \subset J^k\mathbb{R}$, see [G].

Consider the 3−dimensional Lie algebras

$$\mathfrak{t} = \langle \partial_x, \partial_u, u\partial_x - x\partial_u \rangle\,,\ \ \mathfrak{a} = \langle \partial_x, \partial_u, x\partial_x + u\partial_u \rangle \subset \mathfrak{g}.$$

For $k \geq 2$ there exists $k-1$ functions that are invariant under the action of $\mathfrak{t}$ and $\mathfrak{a}$ on $J^k\mathbb{R}$.

The function u_1 is an $\mathfrak{a}-$differential invariant of order one and $\dfrac{u_2}{(1+u_1^2)^{3/2}}$ is a $\mathfrak{t}-$differential invariant of order two. The latter invariant is the well known curvature of a curve.

The spaces $\mathfrak{t}$ and $\mathfrak{a}$ are Lie subalgebras of the Lie algebra

$$\mathfrak{c} = \langle \partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u \rangle \subset \mathfrak{g}.$$

For $k \geq 3$ there exists one $\mathfrak{c}-$differential invariant of pure order k.

The functions

$$\begin{aligned} I_3 &= \frac{u_3u_1^2 - 3u_2^2u_1 + u_3}{u_2^2}, \\ I_4 &= \frac{u_4u_1^4 - 10u_1^3u_2u_3 + (2u_4 + 15u_2^3)u_1^2 - 10u_1u_2u_3 + u_4}{u_2^3}, \end{aligned}$$

are $\mathfrak{c}-$differential invariants. Hence for any integer $k \in \mathbb{Z}_+$ the following function

$$I_{k+4} = \left(\frac{1}{\mathcal{D}_x(I_3)}\mathcal{D}_x\right)^k (I_4)$$

is a $\mathfrak{c}-$differential invariant of pure order $4+k$.

Consider the Lie algebra

$$\mathfrak{f} = \left\langle \partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u, (x^2-y^2)\partial_x + 2xy\partial_y, (x^2-y^2)\partial_y + 2xy\partial_x \right\rangle \subset \mathfrak{g}.$$

For $k \geq 5$ there exists one $\mathfrak{f}-$differential invariant of pure order k, denoted J_k. Since $\mathfrak{c}$ is a Lie subalgebra of $\mathfrak{f}$ it follows that

$$J_k = f_k(I_3, ..., I_k).$$

Indeed, the functions

$$\begin{aligned}
J_5 &= \frac{1}{4I_3^3}(45 + 4I_3I_4I_5 - 12I_3^2I_4 + 40I_3^2 - 30I_4 + 5I_4^2 - 12I_3I_5 - 8I_5I_3^3), \\
J_6 &= \frac{1}{4I_3^{9/2}}(1/4(-405 - 108I_3I_4I_5 - 24I_6I_3^2I_4 + 15I_4^3 + 405I_4 + 18I_5I_4^2I_3 - 80I_3^4 + 24I_3^4I_4 \\
&\quad -42I_4^2I_3^2 + 24I_5I_3^5 + 16I_6I_3^6 - 48I_3^3I_4I_5 - 8I_5^2I_3^4 + 4I_5^2I_4I_3^2 - 16I_6I_3^4I_4 + 36I_6I_3^2 \\
&\quad +4I_6I_4^2I_3^2 - 135I_4^2 - 390I_3^2 + 48I_6I_3^4 + 144I_5I_3^3 - 12I_5^2I_3^2 + 256I_3^2I_4 + 162I_3I_5),
\end{aligned}$$

are $\mathfrak{f}-$differential invariants of pure order five and six. Hence for any integer $k \in \mathbb{Z}_+$ the following function

$$J_{k+6} = \left(\frac{1}{\mathcal{D}_x(I_5)}\mathcal{D}_x\right)^k (J_5)$$

is an $\mathfrak{f}-$differential invariant of pure order $6+k$.

The algebra of invariant functions under the action of $\mathfrak{t}$, $\mathfrak{a}$, $\mathfrak{c}$ and $\mathfrak{f}$ on $J^k\mathbb{R}$ are, respectively

$$\begin{aligned}
\mathcal{T}_k &= \left\{f \in C^\infty_{loc}(J^k\mathbb{R}) \mid f = f\left(\frac{u_2}{(1+u_1^2)^{3/2}}, I_3, I_4, J_5, J_6, ..., J_k\right)\right\}, \\
\mathcal{A}_k &= \left\{f \in C^\infty_{loc}(J^k\mathbb{R}) \mid f = f\left(u_1, I_3, I_4, J_5, J_6, ..., J_k\right)\right\}, \\
\mathcal{C}_k &= \left\{f \in C^\infty_{loc}(J^k\mathbb{R}) \mid f = f\left(I_3, I_4, J_5, J_6, ..., J_k\right)\right\}, \\
\mathcal{F}_k &= \left\{f \in C^\infty_{loc}(J^k\mathbb{R}) \mid f = f\left(J_5, J_6, ..., J_k\right)\right\}.
\end{aligned}$$

All the computations of this section are done in Maple Worksheet "Lie_sa_fin"

Example 34 *The ordinary differential equation*

$$y'' = K\left(1 + (y')^2\right)^{3/2}, \; K \in \mathbb{R}, \tag{5.1}$$

determines a $1-$*dimensional distribution on the manifold of* $1-$*jets* $J^1\mathbb{R}$*. The distribution is generated by the vector field*

$$\mathcal{D} = \partial_x + u_1\partial_u + K(1+u_1^2)^{3/2}\partial_{u_1},$$

or by the Cartan differential $1-$*forms*

$$\omega_1 = du - u_1 dx, \; \omega_2 = du_1 - K(1+u_1^2)^{3/2}dx.$$

The function $\dfrac{u_2}{(1+u_1^2)^{3/2}}$ *is a* $\mathfrak{t}-$*differential invariant, where*

$$\mathfrak{t} = \langle \partial_x, \partial_u, u\partial_x - x\partial_u \rangle .$$

Hence the following vector fields

$$\begin{aligned} S_1 &= \partial_u, \\ S_2 &= \partial_x - \mathcal{D} = -\left(u_1\partial_u + K(1+u_1^2)^{3/2}\partial_{u_1}\right), \end{aligned}$$

are commutative shuffling symmetries of the distribution.

Following the method of [KLR], we introduce the differential 1*-forms for* $K \neq 0$

$$\begin{bmatrix} \theta_1 \\ \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & -\dfrac{u_1}{K(1+u_1^2)^{3/2}} \\ 0 & -\dfrac{1}{K(1+u_1^2)^{3/2}} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \\ \omega_2 \end{bmatrix}.$$

The differential 1*-forms* $\langle dx, \theta_1, \theta_2 \rangle$ *are the dual basis for* $\langle \mathcal{D}, \, S_1, \, S_2 \rangle$*. In addition the differential* 1*-forms* θ_1 *and* θ_2 *are closed, i.e.*

$$\theta_1 = dH, \; \theta_2 = dG,$$

where

$$\begin{aligned} H &= \int du - \int \frac{u_1}{K(1+u_1^2)^{3/2}} du_1 = u + \frac{1}{K(1+u_1^2)^{1/2}} + c_1,\ c_1 \in \mathbb{R}, \\ G &= \int dx - \int \frac{1}{K(1+u_1^2)^{3/2}} du_1 = x - \frac{u_1}{K(1+u_1^2)^{1/2}} + c_1,\ c_1 \in \mathbb{R}. \end{aligned}$$

Since the functions H and G are first integrals of the distribution, we can express the solution as the curve

$$(x - c_1)^2 + (u - c_2)^2 = 1/K^2,\ c_1, c_2 \in \mathbb{R}.$$

This proves that the only curves of constant curvature $K \neq 0$ are circles of radius $1/K$.

Chapter 6

Appendix

In this thesis we used DifferentialGeometry Package of Maple 11 to compute differential invariants and invariant derivatives. The program used are:

-Maple Worksheet "h_diff_inv_3",

-Maple Worksheet "tresse_inv_der",

-Maple Worksheet "dep_inv",

-Maple Worksheet "dep_inv_n_o",

-Maple Worksheet "Lie_sa_fin".

They are available from the author on request.

Bibliography

[G] V. Gauhman, *The geometry of a second-order ordinary differential equation with respect to a conformal group of transformations in two variables* (Russian), Dokl. Akad. Nauk SSSR **140** (1961), 15–18.

[KL1] B.Kruglikov, V. Lychagin, *Invariants of Pseudogroup Actions: Homological Methods and Finiteness Theorem*, International Journal of Geometric Methods in Modern Physics Vol **3**, (2006), 1131-1165.

[KL2] B.Kruglikov, V. Lychagin, *Handbook of Global Analysis,* Ed. D.Krupka, D.Saunders, Elsevier, 725-771, 2007.

[KN] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Volume 2, Wiley Classics Libary, 1996.

[KLR] A. Kushner, V. Lychagin, V. Rubtsov, *Contact Geometry and Nonlinear Differential Equations,* Cambridge University Press, 2007.

[KLV] I. Krasilschik, V. Lychagin, A. Vinogradov, *Geometry of jet spaces and differential equations*, Gordon and Breach, 1986.

[L] S. Lie, *Gesammelte Abhandlungen* Zweiter Abschnitt (unter Mitwirkung von Prof. Dr. Friederich Engel), Teubner, Leipzig, 1890.

[S] S. Sternberg, *Lectures on Differential Geometry,* (Prentice-Hall, Englewood Cliffs, 1964.

[T] A. Tresse, *Sur les invariants differentiels des groupes continus de transformations*, Acta Math. **18** (1894) 1-88.