

Dual-quaternion backstepping control for a fully-actuated rigid-body*

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Abstract—In this paper dual-quaternions are used to model a fully actuated rigid-body. A backstepping-controller that solves the trajectory tracking problem is derived and proved to provide uniform asymptotical stabilization of the error dynamics. Numerical simulations are provided where the controller is compared to existing dual-quaternion tracking controllers and it is shown to have similar performance.

I. INTRODUCTION

The motion of rigid-bodies has been extensively researched within the framework of classical mechanics. The commonly known Newton-Euler equations completely describe the motion of a rigid-body having six degrees of freedom and have successfully been used for modelling of a wide range of dynamic systems, such as satellites, aircrafts and underwater vehicles. In this framework the rotational and translational movement are often considered separately, and control solutions usually deal with 3+3 DOF motion. In recent years a reformulation of the equations of motion has been considered using dual-quaternions. This formulation combines translation and rotation into a unified framework and allows for efficient and compact notation. By combining translation and rotational motion in a single framework the total motion of the rigid-body can be controlled with a single controller. Instead of 3+3 DOF motion we can consider the full 6DOF motion of the rigid-body in our controller design. This can be beneficial in systems where rotational and translational movement are highly-coupled; for instance fixed-wing aerial vehicles and quadrotors.

A considerable amount of research has been done on the use of dual-quaternions in theoretical kinematics. The original motivation for this is that the motion of a rigid-body in three-dimensional Euclidian space can be described by six parameters, which can be regarded as a point in a six-dimensional space. Work was carried out by Study in [1] to apply the work of Clifford [2] to the kinematics of rigid-bodies where the motion of a rigid-body can be seen as a point on a six-dimensional manifold in eight-dimensional space. This idea was further developed by [3] to represent Euclidian displacements using four coordinates in a

dual-space to study the kinematic motion of rigid-bodies. In [4] dual-numbers were used to express the six-dimensional motion of a rigid-body in a three-dimensional dual-space by the use of a dual inertia operator. This work was later used in [5] to study the control of satellite formations where a PD-like tracking controller based on the logarithm of dual-quaternions. The work of [3] was later used by [6] to solve the general dynamics problem using dual-quaternions. Additionally dual-quaternions has also been applied in the study of navigation in [7] where navigational equations of motion are derived using dual-quaternions and the resulting algorithms were shown to be suitable for high precision navigation systems. Although the study of dual-quaternions is prevalent in kinematics and the use of dual-numbers in rigid-body dynamics, less work has been done on the application of dual-quaternions on the dynamics of rigid-bodies. Most of the application of dual-quaternions has been centered around satellite pose control [5], [8], [9], [10]. In [9] the set-point regulation problem was studied and a velocity-free controller was derived. In [11] a velocity-free tracking controller was derived with a dual-quaternion filter that provides dampening injection. In [12] a discontinuous backstepping controller was derived that solved the maneuver problem of a rigid spacecraft while in [13] a PD-controller was developed for tracking taking into account unknown mass and inertia and unknown disturbances.

In this paper a continuous backstepping controller is proposed that solves the trajectory tracking problem for a general fully actuated rigid-body. This allows the controller to be easily adapted to a wide variety of systems such as satellites, fully actuated multirotors and composite systems such as quadrotors with camera gimbals. A simple methodology is introduced for deriving a backstepping controller by introducing an anti-diagonal matrix in the augmented Lyapunov function, which avoids the introduction of swap operators. A numerical simulation is presented and the result is compared to the controllers in [11] and [14].

The outline of this paper is as follows. In Section 2 presents the essential preliminaries associated with dual-quaternions. Section 3 outlines the problem statement, error kinematics, control design and stability analysis. Section 4 provides a numerical simulation results and a conclusion on the work is provided in Section 5.

II. PRELIMINARIES

A. Notation and Reference frames

Vectors are denoted as lower-case bold letters while scalars are non-bold for instance $\mathbf{x} \in \mathbb{R}^n$ is an n-dimensional vector while $a \in \mathbb{R}$ is a scalar. Matrices are upper-case bold letters

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where the transpose of an n by m matrix $M \in \mathbb{R}^{n \times m}$ is denoted M^T . The n by n identity matrix is denoted $I_{n \times n}$ while an n by m matrix with zero entries is denoted $\mathbf{0}_{n \times m}$. The derivative with respect to time is denoted as $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$. Reference frames are denoted $\mathcal{F}_{(\cdot)}$, and superscripts are used to denote a variables frame of reference, such that the vector \mathbf{x}^A is referenced in \mathcal{F}_A . The norm is denoted as $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$. The set of quaternions is defined as

$$\mathbb{H} := \{(q_0, \mathbf{q}_v) : q_0 \in \mathbb{R}, \mathbf{q}_v \in \mathbb{R}^3\}$$

where q_0 is the scalar part and \mathbf{q}_v is the vector part, while the set of unit-quaternions is defined as

$$\mathbb{H}_u := \{\mathbf{q} \in \mathbb{H} : \|\mathbf{q}\| = 1\}$$

The set of vector quaternions is defined as

$$\mathbb{H}_v := \{\mathbf{q} \in \mathbb{H} : q_0 = 0\}$$

The set of unit dual-quaternions is defined as

$$\hat{\mathbb{H}}_u := \{\mathbf{q}_p + \epsilon \mathbf{q}_d : \mathbf{q}_p \in \mathbb{H}_u, \mathbf{q}_p \otimes \mathbf{q}_d^* + \mathbf{q}_d \otimes \mathbf{q}_p^* = \mathbf{0}\}$$

where \mathbf{q}_p is called the primary part and \mathbf{q}_d is called the dual part. The set of dual-vectors is defined as

$$\hat{\mathbb{H}}_v := \{\mathbf{q}_p + \epsilon \mathbf{q}_d : \mathbf{q}_p \in \mathbb{H}_v, \mathbf{q}_d \in \mathbb{H}_v\}$$

where the dual unit ϵ satisfies $\epsilon \neq 0$ and $\epsilon^2 = 0$. The reference frames that are used in this work are given as follows:

Inertial frame This reference frame denoted \mathcal{F}_i has its origin at a fixed point in space and its axes are fixed.

Body frame This coordinate reference frame denoted \mathcal{F}_b is fixed at the rigid-body's centre of mass and the axes are fixed to the rigid-body.

Desired frame This coordinate reference frame denoted \mathcal{F}_d represents the rigid-body's desired pose.

B. Dual quaternions

In this section a brief introduction to dual-quaternions is given, for a more comprehensive treatment *c.f.* [3], [6], [13], [9]. A dual-quaternion is a quaternion where each element is a dual number instead of a real number. They were first introduced by Clifford in [2] and later on Study [15] applied them to the representation of rigid-body motion. The quaternion product between dual-quaternions can be written as

$$\hat{q}_1 \otimes \hat{q}_2 = \mathbf{q}_{1,p} \otimes \mathbf{q}_{2,p} + \epsilon (\mathbf{q}_{1,p} \otimes \mathbf{q}_{2,d} + \mathbf{q}_{1,d} \otimes \mathbf{q}_{2,p}) \quad (1)$$

while addition and subtraction of two dual-quaternions can be written as

$$\hat{q}_1 \pm \hat{q}_2 = \mathbf{q}_{1,p} \pm \mathbf{q}_{2,p} + \epsilon (\mathbf{q}_{1,d} \pm \mathbf{q}_{2,d}).$$

Two conjugates can be defined for dual-quaternions, the first denoted by $\bar{\hat{q}}$ defined as

$$\bar{\hat{q}} = \mathbf{q}_p - \epsilon \mathbf{q}_d.$$

and the second denoted \hat{q}^* defined as

$$\hat{q}^* = \mathbf{q}_p^* + \epsilon \mathbf{q}_d^*.$$

In this paper the inner product between two dual-quaternions is defined as

$$\langle \hat{q}_1, \hat{q}_2 \rangle = \langle \mathbf{q}_{1,p}, \mathbf{q}_{2,p} \rangle + \langle \mathbf{q}_{1,d}, \mathbf{q}_{2,d} \rangle$$

which defines the norm of a dual-quaternion as

$$\|\hat{q}\| = \langle \hat{q}, \hat{q} \rangle^{\frac{1}{2}} = \sqrt{(\mathbf{q}_p)^T \mathbf{q}_p + (\mathbf{q}_d)^T \mathbf{q}_d}$$

The identity unit dual-quaternion is defined as $\hat{q}_I = [1 \ 0 \ 0 \ 0] + \epsilon \mathbf{0}$ with the properties

$$\begin{aligned} \hat{q}_I \otimes \hat{q} &= \hat{q} \otimes \hat{q}_I = \hat{q} \\ \hat{q}^* \otimes \hat{q} &= \hat{q}_I \end{aligned}$$

and $\hat{0} = \mathbf{0} + \epsilon \mathbf{0}$ denotes the zero element for dual-vectors.

C. Kinematics

The position and orientation of a rigid-body relative to some inertial reference frame \mathcal{F}_i can be compactly expressed through a dual-quaternion

$$\hat{q} = \mathbf{q}_{i,b} + \epsilon \frac{1}{2} \mathbf{p}^i \otimes \mathbf{q}_{i,b} = \mathbf{q}_{i,b} + \epsilon \frac{1}{2} \mathbf{q}_{i,b} \otimes \mathbf{p}^b \quad (2)$$

where $\mathbf{p}^i \in \mathbb{H}_v$ is the rigid-body's inertial position while $\mathbf{q}_{i,b} \in \mathbb{H}_u$ represents the rigid-body's attitude. As the rigid-body moves and rotates the dual-quaternion will change over time which can be expressed as

$$\dot{\hat{q}} = \frac{1}{2} \hat{q} \otimes \hat{\omega}^b \quad (3)$$

where $\hat{\omega}^b = \omega_{i,b}^b + \epsilon \mathbf{v}^b$ is called the dual velocity. The quaternion product can be used to combine several dual-quaternions representing the combined rotation and translation. Given three reference frames \mathcal{F}_a , \mathcal{F}_b and \mathcal{F}_c related to each other by the dual-quaternions

$$\begin{aligned} \hat{q}_{a,b} &= \mathbf{q}_{a,b} + \epsilon \frac{1}{2} \mathbf{t}_{AB}^a \otimes \mathbf{q}_{a,b} \\ \hat{q}_{b,c} &= \mathbf{q}_{b,c} + \epsilon \frac{1}{2} \mathbf{t}_{BC}^b \otimes \mathbf{q}_{b,c} \end{aligned}$$

where we have introduced an intermediary notation $\hat{q}_{a,b}$ to help differentiate between the dual-quaternions. The combined dual-quaternion $\hat{q}_{a,c}$ can be defined as

$$\begin{aligned} \hat{q}_{a,c} &= \hat{q}_{a,b} \otimes \hat{q}_{b,c} \\ &= \hat{q}_{a,c} = \mathbf{q}_{a,c} + \epsilon \frac{1}{2} \mathbf{t}_{AC}^a \otimes \mathbf{q}_{a,c} \end{aligned}$$

where $\hat{q}_{a,c} = \hat{q}_{a,b} \otimes \hat{q}_{b,c}$ and $\mathbf{t}_{AC}^a = \mathbf{t}_{AB}^a + \mathbf{t}_{BC}^a$ which is the displacement from frame \mathcal{F}_a to \mathcal{F}_c . Similarly if we have two dual-quaternions $\hat{q}_{a,b}$ and $\hat{q}_{a,d}$ which relates from \mathcal{F}_a to \mathcal{F}_b and \mathcal{F}_a to \mathcal{F}_d respectively, the difference between \mathcal{F}_b and \mathcal{F}_d can be represented as

$$\hat{q}_{b,d} = \hat{q}_{a,b}^* \otimes \hat{q}_{a,d}.$$

D. Dynamics

It was shown in [16] that the dual velocity can be related to the dual momentum through a dual inertia operator

$$\hat{h}^b = \hat{M}\hat{\omega}^b \quad (4)$$

where $\hat{h} = \mathbf{h}_L + \epsilon \mathbf{h}_A$ with \mathbf{h}_L representing the linear momentum and \mathbf{h}_A representing the angular momentum. The dual inertia operator is a matrix with dual number elements, however it has also been shown that \hat{M} can be defined as in [17]

$$\hat{M} = \begin{bmatrix} 0 & \mathbf{0}_{1,3} & 1 & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3,1} & m\mathbf{I}_3 \\ 1 & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{J}^b & \mathbf{0}_{1 \times 3} & \mathbf{0}_{3,3} \end{bmatrix}.$$

This matrix always has an inverse and the product with its inverse yields the identity matrix as shown in [17]. The dual force is related to the derivative of the dual momentum which is expressed as

$$\hat{M}\dot{\hat{\omega}}^b = \hat{f}_u^b - \hat{f}_G^b - \hat{\omega}^b \times \hat{M}\hat{\omega}^b \quad (5)$$

where $\hat{f}_G^b = \mathbf{f}_G^b + \epsilon \mathbf{0}$ is the gravitational forces expressed in the body frame, while $\hat{f}_u^b = \mathbf{f}^b + \epsilon \boldsymbol{\tau}^b$ with $\mathbf{f}^b \in \mathbb{H}_v$ and $\boldsymbol{\tau}^b \in \mathbb{H}_v$ represents the combined applied forces and moments in the body frame and in this work it specifically it represents the control force to be designed.

III. CONTROLLER DESIGN

A. Problem statement

Consider a fully actuated rigid-body with kinematics and dynamics described by (3) and (5) respectively. Let a desired dual-quaternion be defined as $\hat{q}_d = \mathbf{q}_{i,d} + \epsilon \frac{1}{2} \mathbf{q}_{i,d} \otimes \mathbf{p}^d$ and let it be a two times continuously differentiable bounded time-varying trajectory. Design a feedback law that ensures that $\hat{q} \rightarrow \hat{q}_d$ and $\hat{\omega} \rightarrow \hat{\omega}_d$ as $t \rightarrow \infty$.

B. Error kinematics

The error dual-quaternion can be defined as

$$\begin{aligned} \hat{q}_e &= \hat{q}_d^* \otimes \hat{q} \\ &= \mathbf{q}_e + \epsilon \frac{1}{2} \mathbf{q}_e \otimes \mathbf{p}_e^b \end{aligned} \quad (6)$$

where $\mathbf{q}_e = \mathbf{q}_{i,d}^* \otimes \mathbf{q}_{i,b}$ is the orientation error and $\mathbf{p}_e^b = \mathbf{p}^b - \mathbf{p}_d^b$ is the position error expressed in the body frame. Taking the derivative of (6) gives us

$$\begin{aligned} \dot{\hat{q}}_e &= \frac{1}{2} \hat{q}_e \otimes \hat{\omega}_e^b \\ \hat{\omega}_e^b &= \hat{\omega}^b - \hat{\omega}_d^b = \boldsymbol{\omega}_e^b + \epsilon \mathbf{v}_e^b - \boldsymbol{\omega}_{i,d}^b \times \mathbf{p}_e^b \end{aligned} \quad (7)$$

where $\hat{\omega}_d^b = \boldsymbol{\omega}_{i,d}^b + \epsilon \mathbf{v}_d^b$ is the desired dual velocity, $\boldsymbol{\omega}_e^b = \boldsymbol{\omega}_{i,b}^b - \boldsymbol{\omega}_{i,d}^b$ is the angular velocity error expressed in the body frame and $\mathbf{v}_e^b = \mathbf{v}^b - \mathbf{v}_d^b$ is the velocity error expressed in the body frame.

C. Integrator backstepping

In this section an integrator backstepping control law is derived using control Lyapunov functions following the methodology of [18]. The following assumptions are assumed to hold:

a) *Assumption 1:* It is assumed that $q_0(t)q_0(t_0) \geq 0, \forall t > t_0$

b) *Assumption 2:* The mass and inertia matrix are assumed constant and there exists a set of principal axes such that \mathbf{J}^b is diagonal.

c) *Assumption 3:* \hat{q} , $\hat{\omega}_d^d$ and $\dot{\hat{\omega}}_d^d$ are continuous and bounded.

Step 1: Control of (6): Consider the Lyapunov function candidate

$$V(\hat{q}_e) = (\hat{q}_e - \hat{q}_I)^T (\hat{q}_e - \hat{q}_I) \quad (8)$$

where $\hat{q}^T \hat{q}$ means $\langle \hat{q}, \hat{q} \rangle$. The Lyapunov function candidate is clearly zero when $\hat{q}_e = \hat{q}_I$ and positive definite otherwise. The derivative of (8) is

$$\dot{V} = 2(\hat{q}_e - \hat{q}_I)^T \dot{\hat{q}}_e$$

which through inserting (7), expanding terms and simplifying can shown to be

$$\dot{V} = \hat{\zeta}^T (\hat{\omega}^b - \hat{q}_e^* \otimes \hat{\omega}_d^d \otimes \hat{q}_e)$$

where $\hat{\zeta} = \text{vec}(\mathbf{q}_e) + \epsilon \frac{1}{2} \mathbf{p}_e^b$. Let $\hat{\omega}^b$ be a virtual control input $\hat{\omega}_c^b$ and define it as

$$\hat{\omega}_c^b = \hat{q}_e^* \otimes \hat{\omega}_d^d \otimes \hat{q}_e - k_1 \hat{\zeta}. \quad (9)$$

Inserting (9) into (8) yields

$$\dot{V} = -k_1 \hat{\zeta}^T \hat{\zeta} \quad (10)$$

which is negative definite. Define \hat{z} as

$$\hat{z} = \hat{\omega}^b - \hat{\omega}_c^b \quad (11)$$

which we would like to drive to zero to ensure (10).

Step 2 backstepping for \hat{z} : The derivative of (11) is

$$\hat{M}\dot{\hat{z}} = \hat{M}\dot{\hat{\omega}}^b - \hat{M}\dot{\hat{\omega}}_c^b \quad (12)$$

and inserting (5) into (12) gives us

$$\hat{M}\dot{\hat{z}} = \hat{f}_u^b - \hat{f}_G^b - \hat{\omega}^b \times \hat{M}\hat{\omega}^b - \hat{M}\dot{\hat{\omega}}_c^b.$$

Consider now the augmented Lyapunov function

$$V(\hat{q}_e, \hat{z}) = (\hat{q}_e - \hat{q}_I)^T (\hat{q}_e - \hat{q}_I) + \frac{1}{2} \hat{z}^T \hat{K} \hat{M} \hat{z} \quad (13)$$

where \hat{K} is a square matrix such that $\hat{K}\hat{M}$ is diagonal, for instance

$$\hat{K} = \begin{bmatrix} 0 & \mathbf{0}_{1,3} & 1 & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3,1} & \mathbf{I}_{3 \times 3} \\ 1 & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{I}_{3 \times 3} & \mathbf{0}_{1 \times 3} & \mathbf{0}_{3,3} \end{bmatrix}. \quad (14)$$

The derivative of (13) is then

$$\dot{V} = \hat{\zeta}^T (\hat{\omega}^b - \hat{q}_e^* \otimes \hat{\omega}_d^d \otimes \hat{q}_e) + \hat{z}^T \hat{K} \hat{M} \dot{\hat{z}}.$$

Inserting (11) and (12) gives us

$$\begin{aligned} \dot{V} = & \hat{\zeta}^T \left(\hat{z} - k_1 \hat{\zeta} \right) + \\ & \hat{z}^T \hat{K} \left(\hat{f}_u^b - \hat{f}_G^b - \hat{\omega}^b \times \hat{M} \hat{\omega}^b - \hat{M} \hat{\dot{\omega}}_c^b \right). \end{aligned} \quad (15)$$

Choosing the input dual force as

$$\hat{f}_u^b = \hat{f}_G^b + \hat{\omega}^b \times \hat{M} \hat{\omega}^b + \hat{M} \hat{\dot{\omega}}_c^b - \hat{K}^{-1} \left(k_2 \hat{z} + \hat{\zeta} \right)$$

and inserting it into (15) yields

$$\dot{V} = -\hat{\zeta}^T k_1 \hat{\zeta} - \hat{z}^T k_2 \hat{z}.$$

D. Stability analysis

Theorem 1. Consider a rigid-body with its kinematics and dynamics described by (3) and (5) together with Assumption 2. Given a desired time-varying trajectory \hat{q}_d , $\hat{\omega}_d^d$ and $\hat{\omega}_d^d$ in accordance with Assumption 3, define the error kinematics as in (6) and (7) together with Assumption 1. If the input dual force is given by

$$\hat{f}_u^b = \hat{f}_G^b + \hat{\omega}^b \times \hat{M} \hat{\omega}^b + \hat{M} \hat{\dot{\omega}}_c^b - \hat{K}^{-1} \left(k_2 \hat{z} + \hat{\zeta} \right) \quad (16)$$

Then $(\hat{q}, \hat{\omega}^b) \rightarrow (\hat{q}_d, \hat{\omega}_d^b)$ as $t \rightarrow \infty$ for any initial condition.

Proof: From (13) we have that $V(\hat{q}_e, \hat{z}) > 0$ for $\hat{q}_e \in \mathbb{H}_u \setminus \{\hat{q}_I\}$ and $\hat{z} \in \mathbb{H}_v \setminus \{\hat{0}\}$ and that $V(\hat{q}_I, \hat{0}) = 0$. There exist a $\rho_M > \rho_m > 0$ such that $\rho_m \|\chi\|^2 \leq V(\hat{q}_e, \hat{z}) \leq \rho_M \|\chi\|^2$ where $\chi = [\zeta_p \quad \zeta_d \quad z_p \quad z_d]^T$. From (15) it is seen that $\dot{V} < -\rho_1 \|\chi\|^2$ for some $\rho_1 > 0$ which implies that $\hat{q}_e \rightarrow \hat{q}_I$ and $\hat{z} \rightarrow \hat{0}$ as $t \rightarrow \infty$. This further implies that $\hat{q} \rightarrow \hat{q}_d$ and therefore $\zeta \rightarrow \hat{0}$. Since $\hat{z} \rightarrow \hat{0}$ we have from (11) that $\hat{\omega}^b \rightarrow \hat{\omega}_c^b$ which implies that $\hat{\omega}^b \rightarrow \hat{\omega}_d^b$ since $\zeta \rightarrow \hat{0}$ in (9). Therefore based on standard Lyapunov arguments [19] (Theorem 4.9) it can be concluded that the equilibrium point (\hat{q}_e, \hat{z}) is uniformly asymptotically stable.

Remark Since the primary part of a dual-quaternion is a unit-quaternion there are two equilibrium points for \hat{q}_e namely \hat{q}_I and $-\hat{q}_I$ since they represent the same physical orientation. Therefore a choice has to be made of which equilibrium point to stabilize. This is a well known problem and several solutions have been proposed for instance [20], [21] and references therein.

IV. SIMULATION

In this section we detail the simulation of a rigid-body using the dual-quaternion formulation derived in Section II-D. The parameters of the rigid-body can be found in Table II. We define a desired trajectory for the rigid-body to follow

$$\begin{aligned} \mathbf{p}_d^i &= [0 \quad r \sin(\omega_0 t) \quad r \cos(\omega_0 t) \quad 10] \\ \dot{\mathbf{p}}_d^i &= [0 \quad \omega_0 r \cos(\omega_0 t) \quad -\omega_0 r \sin(\omega_0 t) \quad 0] \\ \ddot{\mathbf{p}}_d^i &= [0 \quad -\omega_0^2 r \sin(\omega_0 t) \quad -\omega_0^2 r \cos(\omega_0 t) \quad 0] \end{aligned}$$

where $\omega_0 = 0.35$ and $r = 10$ which describes a circle with radius 10 meters at an altitude of 10 meters. The desired trajectory for the orientation is

$$\begin{aligned} \omega_{n,d}^d &= [0 \quad 0.2 \quad -0.1 \quad 0.5] \\ \dot{\omega}_{n,d}^d &= [0 \quad 0 \quad 0 \quad 0] \end{aligned}$$

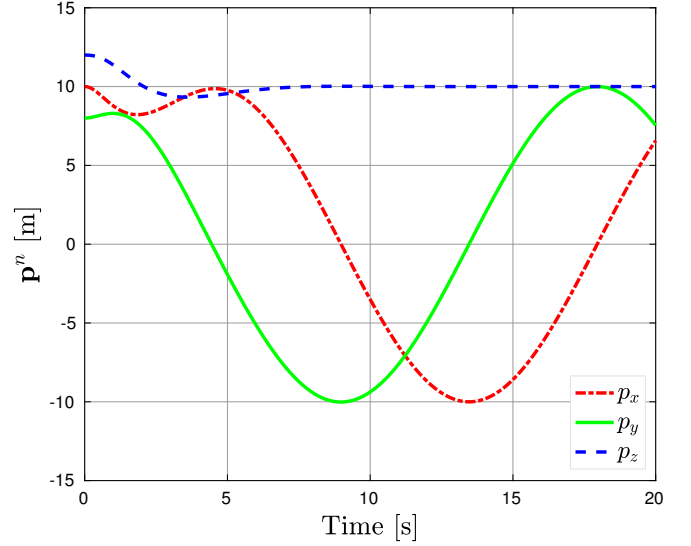


Fig. 1. Inertial position.

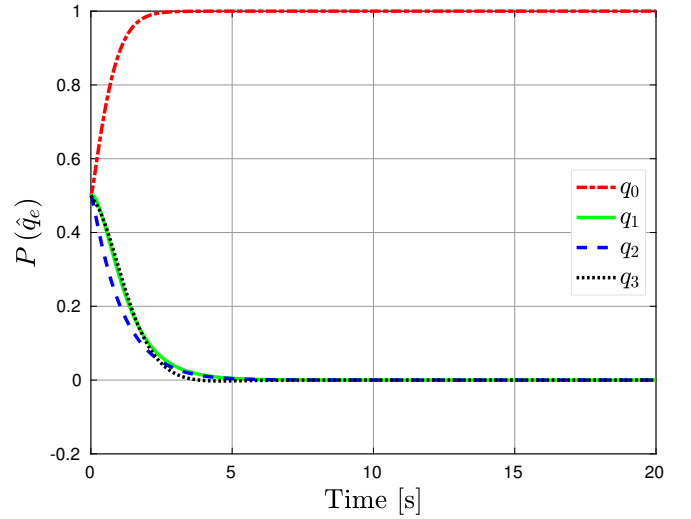


Fig. 2. Primary part of the error dual-quaternion.

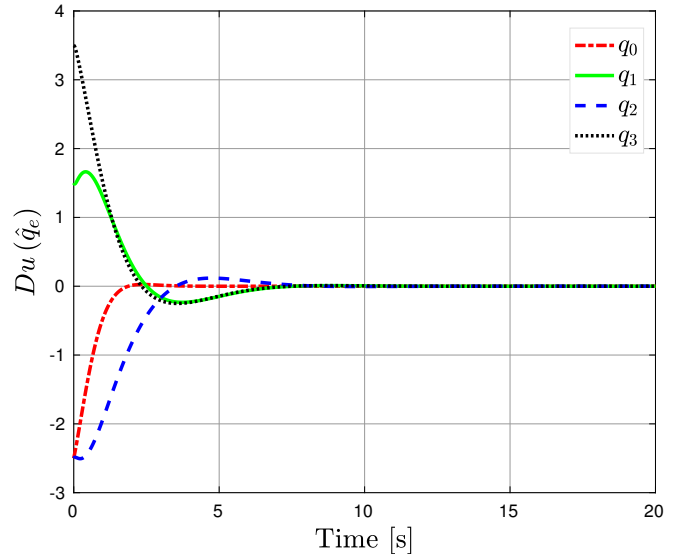


Fig. 3. Dual part of the error dual-quaternion.

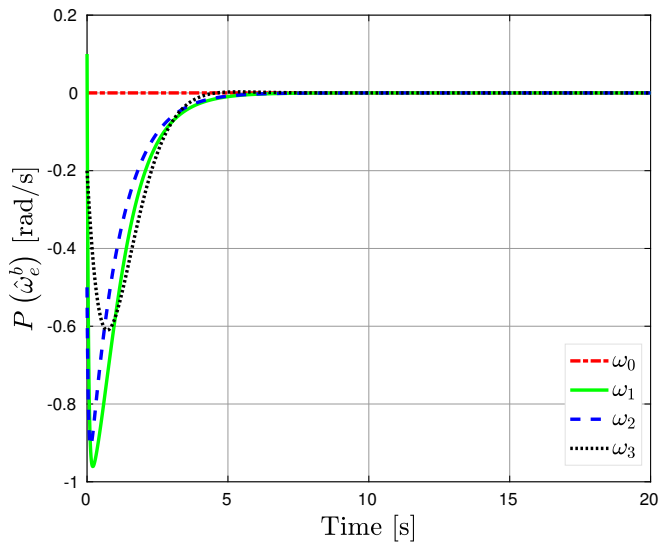


Fig. 4. Primary part of the error dual-velocity.

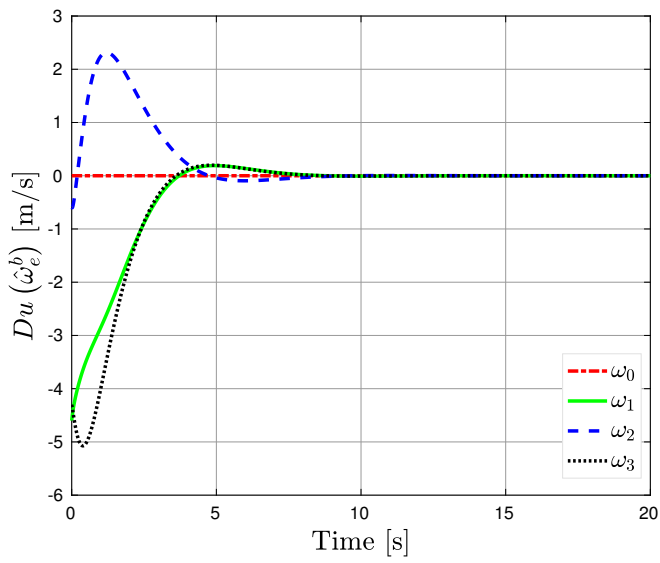


Fig. 5. Dual part of the error dual-velocity.

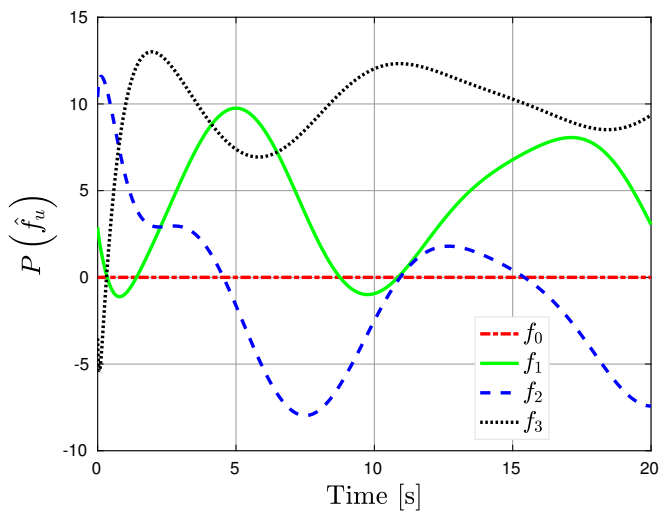


Fig. 6. Primary part of the actuator dual force.

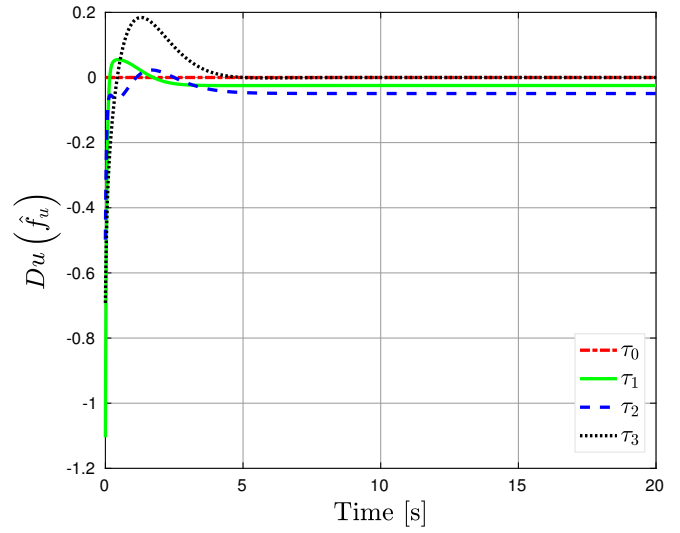


Fig. 7. Dual part of the actuator dual force.

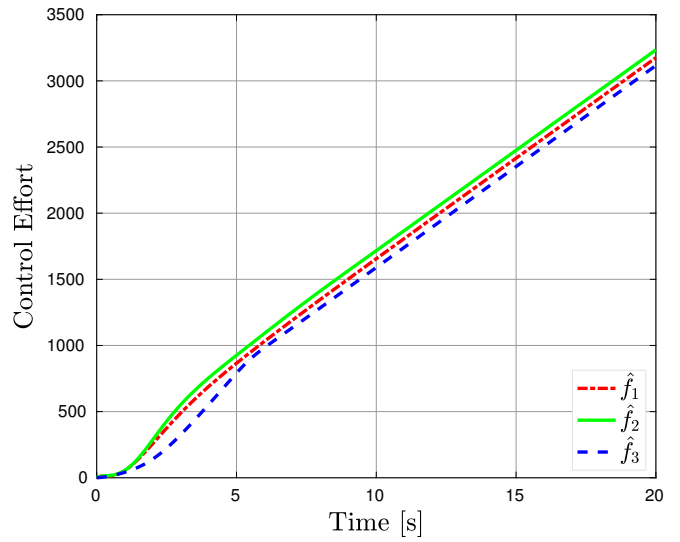


Fig. 8. Control effort comparison with controller \hat{f}_2 from and \hat{f}_3 .

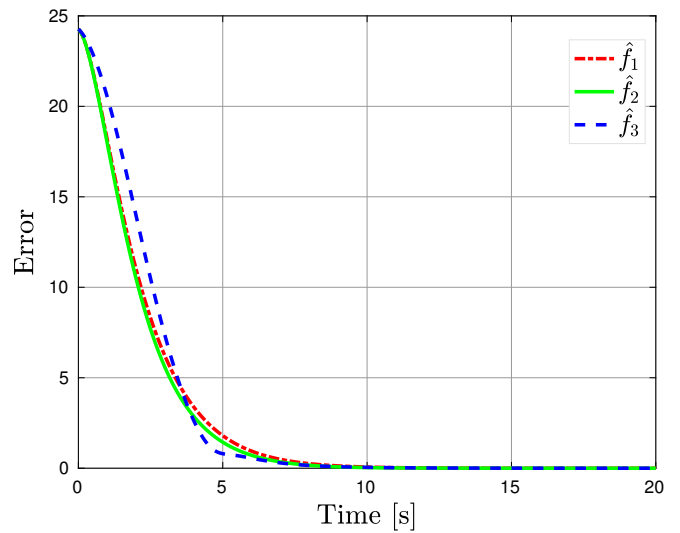


Fig. 9. Combined translational and rotational error comparison with controller \hat{f}_2 from and \hat{f}_3 .

TABLE I
TOTAL CONTROL EFFORT FROM FIGURE 8.

Controller	Total Control Effort
\hat{f}_1	3175 [N ² m ² s]
\hat{f}_2	3234 [N ² m ² s]
\hat{f}_3	3116 [N ² m ² s]

TABLE II
PARAMETERS FOR A RIGID-BODY

Parameter	Value	Unit
Rigid-body mass, m	1.2463	kg
X-axis inertia, J_{xx}	0.047316	kg·m ²
Y-axis inertia, J_{yy}	0.047316	kg·m ²
Z-axis inertia, J_{zz}	0.539632	kg·m ²

The desired dual-quaternion and dual-velocity is constructed based on (2) and (3) in Section II-C. The initial condition for the rigid-body is

$$\hat{q} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} + \epsilon \begin{bmatrix} -7.5 \\ 1.5 \\ 2.5 \\ 3.5 \end{bmatrix}, \quad \hat{\omega}^b = \mathbf{0} + \epsilon \mathbf{0}.$$

The controller gains k_1 and k_2 are both set to 1 while \hat{K} is defined as in (14). As can be seen in Figure 1 the rigid-body quickly converges to the desired translational trajectory and follows it, the same can be inferred from Figure 2 since the primary part of the error dual-quaternion is $P(\hat{q}_e) = \mathbf{q}_e$ which is a unit-quaternion it is seen that $\mathbf{q}_e \rightarrow \mathbf{q}_I$ and therefore $\mathbf{q}_{i,b} \rightarrow \mathbf{q}_{i,d}$. In Figure 8 and 9 the backstepping controller \hat{f}_1 in (16) is compared to the controller \hat{f}_2 in [14] and the controller \hat{f}_3 in [11]. The dual-quaternion filter [11] has been excluded since it is assumed that the velocity measurements are available and the adaptive part in [14] is omitted since the true values of the dual inertia matrix is known and there is no disturbance force. In Figure 9 the norm of the dual-quaternion tracking error is shown, where the controllers have been tuned to have approximately the same settling time (the actual convergence is in the order \hat{f}_2 , \hat{f}_1 then \hat{f}_3) and it can be seen that \hat{f}_1 has similar performance to that of \hat{f}_2 , however in Figure 8 which plots the control effort in accordance with [13] shows that \hat{f}_1 has less control effort than \hat{f}_2 . Compared to \hat{f}_3 the backstepping controller initially converges faster which results in a larger control effort in the beginning as can be seen in Figure 8. The total control effort is provided in Table I and it can be seen that \hat{f}_1 is in the middle of \hat{f}_2 and \hat{f}_3 . The seemingly high total control error comes from the fact that the reference is time-varying so there is always a control force being applied to the rigid-body to track the reference. After an initial convergence the control effort of all three controllers have an equal slope implying that the control effort is equal for the three controllers.

V. CONCLUSIONS

In this paper the trajectory tracking problem for a fully actuated rigid-body was solved in a dual-quaternion framework using a backstepping controller. A dynamic model for

a general rigid-body was presented using dual-quaternions. A backstepping controller was presented and proved to be uniformly asymptotically stable. A numerical example was provided through a simulation which showed fast convergence and zero tracking error, a comparison to some existing dual-quaternion controllers also showed that the proposed controller shows similar performance. Future work includes extending the result here to composite underactuated rigid-bodies such as quadrotors and fixed-wing UAVs with gimbaled payloads and extending the backstepping controller to the adaptive case.

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