UiT THE ARCTIC UNIVERSITY OF NORWAY

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Studies of some Operators of Harmonic Analysis in certain Function Spaces with Applications to PDEs

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To my family

Abstract

The study in this PhD thesis aims at development of certain mathematical methods used in applications, in particular, in the study of regularity properties of solutions in various mathematical models described by Partial Differential Equations (PDEs). To this end, we study various operators of harmonic analysis in certain function spaces, since solutions to many PDEs may be expressed in terms of such operators.

This PhD thesis consists of four papers (papers A–D) and an Introduction.

In Paper A we introduce a version of weighted anisotropic mixed norm Morrey spaces and anisotropic Hardy operators. We derive conditions for boundedness of these operators in such spaces. We also reveal the role of these operators in the solving of some degenerate hyperbolic PDEs of some class.

In Paper B we prove the boundedness of potential operators in weighted generalised Morrey space in terms of Matuszewska-Orlicz indices of weights and apply this result to the Helmholtz equation in $\mathbb{R}^3$ with a free term in such a space. We also give a short overview of some typical situations when potential type operators arise when solving PDEs.

In Paper C we study the boundedness of some multi-dimensional Hardy type operators in Hölder spaces and derive some new results of interest also in the theory of inequalities.

In Paper D we prove some differentiation formulas for weighted singular integrals, which we suppose to apply in our future studies concerning the solution of some integral equations of the first kind.

These new results are put into a more general frame in an Introduction, where also crucial parts of previous research by the candidate (e.g. published in two Licentiate theses) are briefly described. Note, in particular, that this PhD thesis may be regarded as a more theoretically based continuation of the Licentiate thesis in Wood Technology. This important link is carefully described in the Introduction.

Preface

This PhD thesis in Applied Mathematics and Computational Engineering is composed of four papers (**A**-**D**). These publications are reflected and put into a more general frame in an Introduction. Moreover, this Introduction contains an overview about some applied problems, which are of importance as background of the studies in this PhD thesis.

A S. Lundberg and N. Samko, *On some hyperbolic type equations and weighted anisotropic Hardy operators.* Math. Meth. Appl. Sci., **40** (2017), no. 5, 1414-1421.

B E. Burtseva, S. Lundberg, L.-E. Persson and N. Samko, *Potential type operators in PDEs and their applications.* AIP Conference Proceedings, **1798**, 020178, 11 pp, (2017).

C E. Burtseva, S. Lundberg, L.-E. Persson and N. Samko, *Multi-dimensional Hardy type inequalities in Hölder spaces.* J. Math. Inequal., **12** (2018), no. 3, 719-729.

D S. Lundberg, *On precise differentiation formula for weighted singular integrals of Sobolev functions.* AIP Conference Proceedings, **1637**, 621, 6 pp, (2014).

REMARK 0.1. The candidate is also author of the following Licentiate theses:

L1 S. Lundberg, *Experimental Investigations in Wood Machining related to Cutting Forces, Sawdust Gluing and Surface Roughness*, Licentiate thesis, Luleå University of Technology, 1994.

L2 S. Lundberg, *On Adjoint Symmetries and Reciprocal Bäcklund Transformations of Evolution Equations*, Licentiate thesis, Luleå University of Technology, 2009.

In particular, these Licentiate theses include the following Journal publications:

1 S. Lundberg and B. Porankiewicz, *Studies of non-contact methods for roughness measurements on wood surfaces*, Holz als Roh- und Werkstoff , **53** (1995), 309-314.

2 B. O. M. Axelsson, S. Lundberg and J. A. Grönlund, *Studies of the main cutting force at and near a cutting edge*, European Journal of Wood and Wood Products, **51**, no. 1, (1995), 43-48.

3 M. Euler, N. Euler and S. Lundberg, *Reciprocal Bäcklund transformations for autonomous evolution equations.* Theoret. Math. Phys., **159** (2009), no. 3, 770-778.

Since these publications constitute the content of my Licentiate theses [**L1**] and [**L2**], they are not included into this PhD thesis.

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Last but not least, hugs to my dear family. Helén, my profound thanks for your never-ending love and support.

Staffan Lundberg
Narvik, September, 2018.

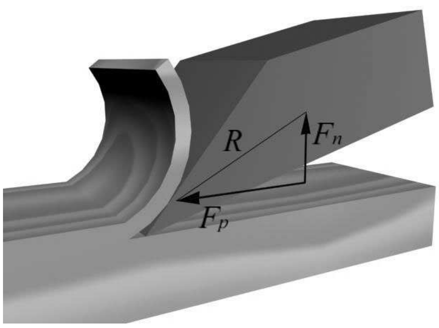
Introduction

This PhD thesis in Applied Mathematics and Computational Engineering is devoted to the development of some mathematical methods known to be widely used in applied sciences and to applications of this development in the theory of Partial Differential Equations (PDEs).

Before we proceed to the description of the main topics and results of this PhD thesis, I find it natural to present the background in my previous studies, which led me to investigations realised in this dissertation.

I have defended two Licentiate theses, [**L1**] and [**L2**], (c.f. [51] and [52]). In [**L1**], I studied the cutting forces on a cutting tool when cutting frozen and non-frozen wood at full speed and with all cutting edges of the tool. The research in this Licentiate thesis is connected to some investigations in this PhD thesis. In fact, some research in this PhD thesis may be regarded as a more theoretically based continuation of the practically based research in [**L1**].

FIGURE 1. Cutting forces. F_p: Main cutting force, F_n: Normal cutting force, R: Total cutting force.



In [**L2**], I studied some methods to obtain conservation laws and transformations between nonlinear PDEs and, moreover, to classify nonlinear PDEs with respect to these methods.

To better illustrate the above mentioned background which is essential for my studies in this PhD thesis, I find it reasonable to shortly describe the research questions and to characterise some main results obtained in my both Licentiate theses.

1. Short description of [L1] and [L2]

In [**L1**], the research was related to an investigation of the cutting forces on a cutting tool when cutting frozen and non-frozen wood at full industrial feed speed and with all (three) cutting edges of the tool. The results from the investigations showed that the main cutting force increased with increasing moisture content.

As a special issue, investigations related to the sawdust gluing phenomenon – a serious problem for sawmills in the northern part of the globe – were performed. These investigations showed that the heartwood/sapwood ratio was a determining factor for the amount of sawdust glued to the sawn surfaces.

An application, close to wood machining, was also studied, namely non-contact surface roughness measurements on sawn wood. The results indicate that a measurement approach, based on a laser scan principle, can measure surface roughness at industrial feed speeds with a sufficient degree of accuracy.

Remark 1.1. The research in [**L1**] was, to a great extent, experimental, so this type of research could be much supported by some complementary theoretical research. Parts of the research in this PhD thesis may be regarded as such a theoretical continuation of some results in [**L1**]. In particular, the following Journal publications were included in [**L1**]: [2] and [53].

In [**L2**], we discussed special transformations and so-called adjoint symmetries of nonlinear PDEs. The main emphasis was on adjoint symmetries and transformations of evolutions equations. In particular, we studied the adjoint symmetries and the construction of reciprocal Bäcklund transformations for evolution equations.

The obtained results show that by using integrating factors, together with corresponding conservation laws, we are able to construct reciprocal Bäcklund transformations for evaluation equations. Moreover, the achievements indicate the possibility to construct and classify a family of third-order evolution equations with respect to adjoint symmetries up to second-order, by means of an algorithmic procedure, so that the work, obtaining adjoint symmetries, can be substantially simplified.

Remark 1.2. The Journal publication [16] was included in [**L2**].

2. The link to the new results in this PhD thesis

As mentioned above, my Licentiate thesis [**L1**], was related to a study of the cutting forces on a cutting tool when cutting frozen and non-frozen wood at full speed and with all (three) cutting edges of the tool. Earlier studies of these phenomena have been performed under low speed conditions. By our study, the feed speed can be increased up to normal industrial conditions, yet obtaining results with a sufficient grade of accuracy.

One of the conclusions being that the main cutting force grows with increasing moisture content, after that study my interests turned to the question - how moisture transfer in wood in general influences on the wood production processes? Such studies can be found in literature, see for instance [39] and the references therein.

The study of the problem of moisture transfer is important for various other applications, for instance it is essential for better understanding the durability of materials. In general, the role of temperature and moisture is essential for most of material properties, when dealing with building materials. The process of temperature and moisture transfer in materials depends in particular on the environment climate and the geometry of the structure. Thus, it is difficult to overestimate the importance of studies of heat and moisture transfer in various branches of technology, industrial and civil engineering, chemical technology etc.

For various applied studies related to the role of heat and moisture transfer, we refer in particular to [97] with respect to the role of composition of materials, [98] for coupled thermal and moisture fields with application to tailoring of composites, [67] for isothermal moisture transport in various porous building materials, [104] for heat and moisture transfer in the special case of concrete. See also [17], [18], [32], [46], [66], [80], [105] and [107].

Mathematically, moisture transfer as well as heat transfer is described by parabolic-hyperbolic type PDEs. Recall the classification of types of PDEs, in the case of two independent variables. Let

$$(1)\quad A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+F(x,y,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})=0$$

be partial differential equation of the second order, linear with respect to the second order derivatives and with discriminant $D = D(x,y)$ defined by

$$(2)\qquad D(x,y) := B^2(x,y) - A(x,y)C(x,y).$$

The equation (1) is called elliptic, hyperbolic or parabolic at a point (x_0, y_0) if

$$(3)\qquad D(x_0,y_0) < 0, D(x_0,y_0) > 0 \text{ or } D(x_0,y_0) = 0,$$

respectively. It is called elliptic, hyperbolic or parabolic in a domain in $\mathbb{R}^2$ if it is elliptic, hyperbolic or parabolic at every point of this domain.

Sometimes in applied sciences there appear mixed type or degenerate hyperbolic partial differential equations of the form

$$(4)\quad y^m\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}+a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+C(x,y)u=f(x,y),$$

where the equation is of elliptic or hyperbolic type for $y < 0$ or $y > 0$, respectively, when m is odd, and of hyperbolic type in both the half planes when m is even, with the line $y = 0$ of parabolic degeneration in both the cases. The famous Tricomi equation

$$(5)\qquad y\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

which is used, in particular, to describe near-sonic flows of gas, is a particular case of (4) of mixed type. The moisture transport equation

$$y^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} = 0, \tag{6}$$

which was obtained by the well-known thermophysicist A. Luikov [48] for the density of moisture flux in a colloidal capillary-porous media, is another particular case of (4), this time with parabolic degeneration. An equation of type (6) in fact was earlier considered as a theoretical object by A.V.Bitsadze, see the book [8], who studied the Cauchy problem for such an equation. Because of this, the moisture transport equation (6) is also referred to as Bitsadze-Luikov equation.

For partial differential equations appearing in the study of heat and moisture transfer we refer to the book [48] by A. Luikov, widely known to experts in the field, and also [19], [60] and [73].

Mostly heat and moisture transfer is described by parabolic equations. In cases of more complicated media structure the governing equation may be of hyperbolic type with degeneracy to the parabolic type on the boundary of the domain or on some specific lines in the domain. Such hyperbolic type differential equations are known to appear in the study of moisture transfer in capillary-porous bodies, see e.g. [48], Section 1.6. Note that the history of degenerate hyperbolic equations goes back to the classical Tricomi equation, see for instance the book [102], the papers [4], [5], [6], [13], [23], [24], [25], [77], [78], and the references therein.

Differential equations in general are very effective mathematical models for the study of various phenomena in applied sciences. Several problems of physics and other natural sciences supply new ideas to the theory of PDEs via many applications, from which the rich content of the theory grows. Conversely, it also happens that a mathematical study, born within the mathematics itself, may lead to solving some specific physical problems in the process of their more profound study, although after maybe considerable time. Thus, the Tricomi problem for equations of mixed type, after more than a quarter of a century after its solution, found important applications in the problem of modern gas dynamics in the study of supersonic gas flows, see [61] and the references therein.

One of the features of the modern theory of differential equations is its deep connection with functional analysis and harmonic analysis.

My studies in this PhD thesis were highly influenced by the effectiveness of interplay between mathematical theories and their applications. We concentrate ourselves on the study of the following operators of harmonic analysis: Potential type operators and Hardy type operators, which are known to play a crucial role in applications to PDEs. We study these operators in the setting of generalised or modified Morrey type and Hölder function spaces, both popular in PDEs. This is motivated by the needs in applications to have properties of solutions inherited from prescribed properties of the data, see e.g. the Figure below.

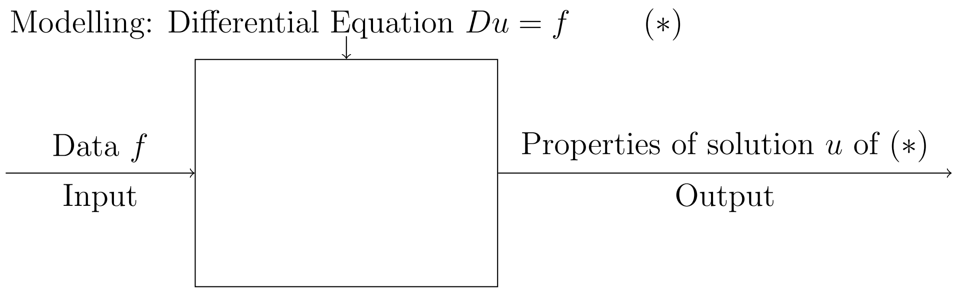


Figure 2. The relation between properties of data and inherited solutions.

3. Short description of the main research questions and results in this PhD thesis

The classical versions of function spaces in this PhD thesis:

In this PhD thesis we deal with Morrey- and Hölder-type spaces and their modifications and/or generalisations. We present here the definitions of the classical versions of these spaces and later, in the parts where descriptions of the main results will be presented, we will give some of their modifications and/or generalisations.

Morrey space:

The classical Morrey space $\mathcal{L}^{p,\lambda}$ is defined as follows:

$$\mathcal{L}^{p,\lambda} = \{f \in \mathcal{L}^p_{loc}(\Omega) : \|f\|_{p,\lambda} < \infty\}, \quad 1 \le p < \infty, \ 0 \le \lambda < n, \tag{7}$$

where $\Omega \subseteq \mathbb{R}^n$, and $\mathcal{L}^p_{loc}(\Omega)$ is the set of functions such that $f \in \mathcal{L}^p(B \cap \Omega)$ for every ball $B \subset \mathbb{R}^n$. Equipped with the norm

$$\|f\|_{p,\lambda} = \sup_{x\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(t)|^p \, dt \right)^{\frac{1}{p}} = \sup_{x\in\Omega, r>0} \frac{\|f\|_{\mathcal{L}^p(B(x,r))}}{r^{\frac{\lambda}{p}}}, \tag{8}$$

where $B(x, r) = \{y \in \Omega : |y - x| < r\}$, it is a Banach space.

The approach to measure regularity properties of solutions to PDEs by means of the property

$$\int_{B(x,r)} |f(t)|^p dt \le cr^\lambda$$

is due to C. B. Morrey [62]. The set of functions with this property as a function space $\mathcal{L}^{p,\lambda}$ with the corresponding norm appeared first in [10] and is called Morrey space since then.

Such spaces are known to be used often in PDEs, since Morrey spaces describe local regularity of solutions more precisely than Lebesgue spaces, and in the last decades they became also widely popular in harmonic analysis. We refer, for instance to the books [1], [20], [38], [41] and [103]. Various properties of functions in Morrey spaces are well studied and may be found in these books.

Many operators of harmonic analysis, e.g.singular, maximal and potential type operators and their commutators, have been intensively studied in Morrey spaces. We refer to the book [38], where a lot of references may be found.

Hölder space:

The classical Hölder space $C^\lambda(\Omega),\ 0<\lambda\leq 1$, where Ω is an open set in $\mathbb{R}^n, \Omega\subseteq\mathbb{R}^n,\ n\geq 1$, is defined by the seminorm

$$[f]_\lambda := \sup_{\substack{x,x+h\in\Omega\\ |h|<1}} \frac{|f(x+h)-f(x)|}{|h|^\lambda} < \infty. \tag{9}$$

Equipped with the norm

$$\|f\|_{C^\lambda} = \sup_{x\in\Omega}|f(x)| + [f]_\lambda \tag{10}$$

$C^\lambda(\Omega)$ is a Banach space.

Hölder spaces adjoin in a sense to Morrey space and together with Morrey spaces constitute the scale of so called Morrey-Campanato spaces, see for instance [20] and [41]. Hölder spaces are also known to be widely used in applications, in particular in PDEs. See, for instance [20].

Some operators of harmonic analysis studied in this PhD thesis:

Among the operators studied in this PhD thesis, the main are Hardy- and Potential-type operators. The classical Hardy operators H^α and $\mathcal{H}^\alpha$ for functions of one variable are defined as follows:

$$H^\alpha f(x) := x^{\alpha-1}\int_0^x f(y)dy \text{ and} \tag{11}$$

$$\mathcal{H}^\alpha f(x) := x^\alpha \int_x^\infty \frac{f(y)}{y}dy,\ \alpha\geq 0.$$

Their multidimensional versions are also known in the forms

$$H^\alpha f(x) := |x|^{\alpha-n} \int\limits_{|y|<|x|} f(y)dy \text{ and} \tag{12}$$

$$\mathcal{H}^\alpha f(x) := |x|^\alpha \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n} dy, \ \alpha \geq 0, \ x \in \mathbb{R}^n$$

For more information on Hardy type operators and inequalities, see the recent book [43] by A. Kufner, L. E. Persson and N. Samko. We also consider anisotropic Hardy operators

$$H^{\overline{\alpha}} = H^{\overline{\alpha}}(x_1, x_2), \quad \overline{\alpha} = (\alpha_1, \alpha_2),$$

of functions of two variables, defined by

$$H^{\overline{\alpha}} f(x, y) := x^{\alpha_1 - 1} y^{\alpha_2 - 1} \int\limits_0^x \int\limits_0^y f(t_1, t_2)\, dt_1\, dt_2. \tag{13}$$

As regards Potential operators, the classical potential operator I^α, known also under the name of Riesz fractional integral, has the form

$$I^\alpha f(x) := \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)\, dt}{|x-y|^{n-\alpha}}, \ x \in \mathbb{R}^n, \ 0 < \alpha < n,$$

where $\gamma_n(\alpha)$ is a certain normalising constant. In the case $\alpha = 2$ (when $n > 2$) this is also referred to as the Newton potential.

We also study weighted modifications of the above operators.

3.1. Main results obtained in Paper A. We recall that the degenerate hyperbolic equation (6) of the form

$$y^2 \frac{\partial^2 u}{\partial x \partial x} - \frac{\partial^2 u}{\partial y \partial y} + a \frac{\partial u}{\partial x} = f(x, y)$$

is known as an equation describing moisture and temperature transfer in porous media, as it was mentioned above. This equation, by the transformation

$$\xi = x - \frac{y^2}{2}, \quad \eta = x + \frac{y^2}{2},$$

reduces (see for instance [15], [79]) to the equation in the following form:

$$(\xi - \eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \text{ lower terms } = g(\xi, \eta).$$

The degenerate hyperbolic equation, related to the use of the anisotropic Hardy operators (13) introduced in Paper A, has the form

$$xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + u(x,y) = f(x,y). \tag{14}$$

We study a possibility to find a solution of this equation within the frame of weighted Morrey spaces, when the right-hand side of the equation is in such spaces well suited for their use in PDEs. Such a possibility is based on the boundedness of the weighted Hardy operators in the corresponding spaces. To this end, we introduce a version of weighted anisotropic Morrey spaces, and prove a theorem on the boundedness of the weighted anisotropic double Hardy operator in the framework of anisotropic Morrey spaces which are defined below.

We find conditions for the boundedness of these operators in weighted anisotropic Morrey spaces, with an emphasis on the role of the function spaces used in the solving process.

Some definitions:

We consider Morrey spaces defined above by (7)-(8) on $\mathbb{R}^n$. The weighted Morrey spaces $\mathcal{L}^{p,\lambda}$ are treated in the usual sense:

$$\mathcal{L}^{p,\lambda}(\Omega, w) := \{f : \quad wf \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)\},$$

$$\text{equipped with the norm } \|f\|_{\mathcal{L}^{p,\lambda}(\Omega,w)} := \|wf\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)}.$$

Below we present the definitions of the anisotropic Morrey spaces.

Anisotropic Morrey space $\mathcal{L}^{p,\lambda_1,\lambda_2}(\mathbb{R}^2_+)$

is defined in [69] by the norm

(15)

$$\|f\|_{p,\lambda_1,\lambda_2} := \sup_{\substack{x>0,y>0\\ r_1>0,r_2>0}} \left(\frac{1}{r_1^{\lambda_1} r_2^{\lambda_2}} \int\limits_{(x-r_1)_+}^{x+r_1} \int\limits_{(y-r_2)_+}^{y+r_2} |f(t_1,t_2)|^p \, dt_1 \, dt_2 \right)^{\frac{1}{p}} =$$

$$= \sup_{x,r\in\mathbb{R}^2_+} \frac{\|f\|_{\mathcal{L}^p(Q(x,r))}}{r_1^{\frac{\lambda_1}{p}} r_2^{\frac{\lambda_2}{p}}} \quad ,$$

where $(x_i - r_i)_+ = \begin{cases} x_i - r_i, & \text{if } x_i - r_i \geq 0 \\ 0, & \text{if } x_i - r_i < 0, \end{cases}, i = 1, 2,$

$$Q(x,r) = \big\{t = (t_1,t_2) \in \mathbb{R}^2_+ : \ (x_i - r_i)_+ < t_i < x_i + r_i, \ i = 1,2\big\} =$$
$$= I_{x,r_1} \times I_{y,r_2}, \ x = (x,y), r = (r_1,r_2), \text{ and}$$
$$I_{x_i,r_i} = ((x_i - r_i)_+, x_i + r_i), \ i = 1,2.$$

Anisotropic mixed norm Morrey space $\mathcal{L}^{\overline{p},\overline{\lambda}}(\mathbb{R}^2_+)$

is defined by the norm

(16)
$$\|f\|_{\overline{p},\overline{\lambda}} := \sup_{x,r\in\mathbb{R}^2_+} \frac{\|f\|_{\mathcal{L}^{\overline{p}}(Q(x,r))}}{r_1^{\frac{\lambda_1}{p_1}} r_2^{\frac{\lambda_2}{p_2}}},$$

where $\overline{p} = (p_1,p_2), \overline{\lambda} = (\lambda_1,\lambda_2)$, with the mixed norm $\|f\|_{\mathcal{L}^{\overline{p}}(Q(x,r))}$ over the rectangle $Q(x,r)$, where

(17)
$$\|f\|_{\mathcal{L}^{\overline{p}}(Q(x,r))} := \left(\int\limits_{I_{x,r_1}} \left(\int\limits_{I_{y,r_2}} |f(t_1,t_2)|^{p_2} \, dt_2 \right)^{p_1/p_2} dt_1 \right)^{1/p_1} =$$
$$= \big\| \|f(t_1,\cdot)\|_{\mathcal{L}^{p_2}(I_2)} \big\|_{\mathcal{L}^{p_1}(I_1)} \, ,$$

where "$\cdot$" stands for the variable in which the inner norm is applied (we refer to [7] for more information about mixed norm Lebesgue spaces).

Weighted anisotropic mixed norm Morrey space $\mathcal{L}^{\overline{p},\overline{\lambda}}(\mathbb{R}^2_+, w_1w_2)$

is defined by

$$\mathcal{L}^{\overline{p},\overline{\lambda}}(\mathbb{R}^2_+, w_1w_2) := \{f : \quad w_1(x)w_2(y)f(x,y) \in \mathcal{L}^{\overline{p},\overline{\lambda}}(\mathbb{R}^2_+).$$

We consider the **weighted two-dimensional Hardy operators** $H^{\overline{\alpha},w}$, defined by

(18)
$$H^{\overline{\alpha},w}f(x,y) := x^{\alpha_1-1}y^{\alpha_2-1}w_1(x)w_2(y)\int\limits_0^x\int\limits_0^y \frac{f(t_1,t_2)}{w_1(t_1)w_2(t_2)}\,dt_1\,dt_2,$$

where $\overline{\alpha} = (\alpha_1, \alpha_2)$ and $w = w(x,y) = w_1(x)\cdot w_2(y)$.

We may assume that $f \geq 0$. If the double integral (18) converges, then by Fubini's theorem it coincides with the also convergent iterated integrals:

$$H^{\overline{\alpha},w}f = H_1^{\alpha_1,w_1}H_2^{\alpha_2,w_2}f = H_2^{\alpha_2,w_2}H_1^{\alpha_1,w_1}f, \tag{19}$$

where
(20)
$$H_1^{\alpha_1,w_1}H_2^{\alpha_2,w_2}f(x,y) = \frac{w_1(x)w_2(y)}{x^{1-\alpha_1}y^{1-\alpha_2}}\int\limits_0^x \frac{1}{w_1(t_1)}\left(\int\limits_0^y \frac{f(t_1,t_2)}{w_2(t_2)}dt_2\right)dt_1,$$

and
(21)
$$H_2^{\alpha_2,w_2}H_1^{\alpha_1,w_1}f(x,y) = \frac{w_1(x)w_2(y)}{x^{1-\alpha_1}y^{1-\alpha_2}}\int\limits_0^y \frac{1}{w_2(t_2)}\left(\int\limits_0^x \frac{f(t_1,t_2)}{w_1(t_1)}dt_1\right)dt_2,$$

so that we can use any one of the forms in (19). Thus, we can interpret our anisotropic Hardy operator as a composition of the one-dimensional Hardy operators applied in the corresponding variable.

In Theorem 3.1 below on the boundedness of double Hardy type operator in the mixed norm anisotropic case, which is one of the main results of this paper, we use the notion of Zygmund classes of almost monotonic functions on $\mathbb{R}_+$, which are defined as follows:

(i) By $W = W(\mathbb{R}_+)$ we denote the class of functions φ continuous and positive on $\mathbb{R}_+$ such that there exists the finite limit $\lim\limits_{x\to 0} \varphi(x)$.

(ii) By $W_0 = W_0(\mathbb{R}_+)$ we denote the class of functions $\varphi \in W$ almost increasing on $(\mathbb{R}_+)$.

(iii) By $\overline{W} = \overline{W}(\mathbb{R}_+)$ we denote the class of functions $\varphi \in W$ such that $x^a\varphi(x) \in W_0$ for some $a = a(\varphi) \in \mathbb{R}$.

We say that a function $\varphi \in \overline{W}$ belongs to the Zygmund class $\mathbb{Z}_\gamma$, $\gamma \in \mathbb{R}^1$, if

$$\int_r^\infty \frac{\varphi(t)}{t^{1+\gamma}} dt \le c\frac{\varphi(r)}{r^\gamma}, \qquad r \in (0,\infty). \tag{22}$$

Let $\varphi \in W$. The following numbers $M(\varphi)$ and $M_\infty(\varphi)$ are known as upper Matuszewska-Orlicz indices of the function φ, at the origin and infinity, respectively:

$$M(\varphi) = \sup_{r>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)}\right)}{\ln r} = \lim_{r\to\infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)}\right)}{\ln r}$$

$$M_\infty(\varphi) = \inf_{r>1} \frac{\ln\ \left[\limsup\limits_{h\to \infty} \frac{\varphi(rh)}{\varphi(h)}\right]}{\ln\ r}.$$

The following theorem on weighted Hardy type inequality was conjectured in [69]:

THEOREM 3.1. *Let* $0 \le \lambda_i < 1, 0 \le \alpha_i < 1 - \lambda_i$, $1 < p_i < \frac{1-\lambda_i}{\alpha_i}$, $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha_i}{1-\lambda_i}$ *and* $w_i \in \overline{W}(\mathbb{R}_+)$, $i = 1, 2$. *For the weighted Hardy type inequality*

$$\left\| H^{\overline{\alpha}, w} f \right\|_{\overline{p}, \overline{\lambda}} \le C \left\| f \right\|_{\overline{p}, \overline{\lambda}} \tag{23}$$

to hold, the condition $w_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i} + \frac{1}{p_i'}}(\mathbb{R}_+)$ *is sufficient, and the condition* $w_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i} + \frac{1}{p_i'} + \varepsilon}(\mathbb{R}_+)$ *with an arbitrary* $\varepsilon > 0$, *is necessary,* $i = 1, 2$.

The detailed proof of Theorem 3.1 was given in Paper A. Moreover, based on the boundedness of weighted Hardy operators provided by Theorem 3.1, we stated and proved the following result for solutions in weighted Morrey space of the inhomogeneous equation (14):

THEOREM 3.2. *Let* $f \in \mathcal{L}^{\overline{p}, \overline{\lambda}}(\mathbb{R}^2_+, w_1 w_2)$, *where* $1 < p_i < \infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = 1, 0 \le \lambda_i < 1$, $i = 1, 2$. *Then there exists in* $\mathcal{L}^{\overline{p}, \overline{\lambda}}(\mathbb{R}^2_+, w_1 \cdot w_2)$ *a particular solution* $u(x, y)$ *of the equation* (14) *given by the Hardy operator*

$$u(x, y) = \frac{1}{xy} \int_0^x \int_0^y f(t_1, t_2)\, dt_1\, dt_2$$

for all weights w_1 *and* w_2 *such that*

$$w_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i} + \frac{\lambda_i}{p_i'}}(\mathbb{R}_+), \tag{24}$$

or, equivalently,

$$\max\left(M(w_i), M_\infty(w_i)\right) < \frac{\lambda}{p_i} + \frac{1}{p_i'}, \; i = 1, 2.$$

If we consider the case of power weights, i.e. when $w_1(x) = x^{\theta_1}$ and $w_2(y) = y^{\theta_2}$, we can formulate the following statement:

COROLLARY 3.3. *In the case of power weights, i.e. when* $w_1(x) = x^{\theta_1}$ *and* $w_2(y) = y^{\theta_2}$, *the condition (24) is reduced to the condition*

$$\max(\theta_i) < \frac{\lambda_i}{p_i} + \frac{1}{p_i'}, \ i = 1, 2, \tag{25}$$

which means that Theorem 3.2 in this case holds with (24) replaced by the simpler condition (25).

The results in Paper A are related to the following publications: [3], [7], [15], [34], [36], [48], [49], [50], [69], [71], [72], [75], [79], [86], [87], [88], [97] and [98].

3.2. Main results obtained in Paper B. It is well known that Potential type operators arise in the study of for instance the Poisson and Helmholtz equations. Such equations occur quite often in a variety of applied problems of science and engineering.

In this paper we prove the boundedness of Potential operators in weighted generalised Morrey space in terms of Matuszewska-Orlicz indices of weights and apply this result to the Helmholtz equation in $\mathbb{R}^3$ with a free term in such a space. We do an emphasis on the role of the function space used in the solving process. We also give a short overview of some typical situations when Potential type operators arise when solving PDEs.

We start with some definitions and assumptions.

Let $\overline{W}$ be the class of quasi-monotonic functions on $\mathbb{R}_+$ defined in the above overview of Paper A.

Besides this we also need the class $\underline{W}$ defined as follows. To underline separate roles of Matuszewska-Orlicz indices at the origin and infinity, we give here the definition of $\underline{W}$ via the corresponding classes on $[0, 1]$ and $[1, \infty]$.

Definition 3.4.

(i) By $W = W([0, 1])$ we denote the class of continuous and positive functions φ on $(0, 1]$ such that there exists finite or infinite limit $\lim_{r\to 0} \varphi(r)$.

(ii) By $\underline{W} = \underline{W}([0, 1])$ we denote the class of functions $\varphi \in W$ such that $t^b\varphi(t)$ is almost decreasing for some $b \in \mathbb{R}^1$.

DEFINITION 3.5.

(i) By $W_\infty = W_\infty([1,\infty])$ we denote the class of functions φ which are continuous and positive and almost increasing on $[1,\infty)$ and which have the finite or infinite limit $\lim_{r\to\infty}\varphi(r)$.

(ii) By $\underline{W}_\infty = \underline{W}_\infty([1,\infty))$ we denote the class of functions $\varphi \in W_\infty$ such that $t^b\varphi(t) \in W_\infty$ for some $b = b(\varphi) \in \mathbb{R}^1$.

By $\underline{W}(\mathbb{R}_+)$ we denote the set of functions on $\mathbb{R}_+$ whose restrictions onto $(0,1)$ are in $\underline{W}([0,1])$ and restrictions onto $[1,\infty)$ are in $\underline{W}_\infty([1,\infty))$. The set $\overline{W}(\mathbb{R}_+)$ is interpreted similarly.

Generalised Morrey space

DEFINITION 3.6. Let $\varphi(r)$ be a non-negative function on $[0,\ell]$, positive on $(0,\ell]$, and $1 \le p < \infty$. The generalised Morrey space $\mathcal{L}^{p,\varphi}(\Omega)$ is defined as the space of functions $f \in L^p_{\mathrm{loc}}(\Omega)$ such that

$$\|f\|_{p,\varphi} := \sup_{x\in\Omega, r>0} \left(\frac{1}{\varphi(r)} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty. \tag{26}$$

The classical Morrey space

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$$

corresponds to the case $\varphi(x,r) \equiv r^\lambda,\ 0 < \lambda < n$.

Everywhere in the sequel it is assumed that the functions φ and ψ, defining the generalised Morrey spaces are non-negative, are almost increasing functions and continuous in a neighborhood of the origin, such that $\varphi(0) = 0,\ \varphi(r) > 0$, for $r > 0$, and $\varphi \in \overline{W} \bigcap \underline{W}$, and similarly for ψ.

For the function $\varphi(r)$, we will make use of the following conditions:

$$\varphi(r) \ge cr^n \tag{27}$$

for $0 < r \le 1$, which makes the spaces $\mathcal{L}^{p,\varphi}(\Omega)$ non-trivial, see [70, Corollary 3.4],

$$\int_r^\infty \frac{\varphi^{\frac{1}{p}}(t)}{t^{\frac{n}{p}+1}}\,dt \le C\,\frac{\varphi^{\frac{1}{p}}(r)}{r^{\frac{n}{p}}} \tag{28}$$

and

$$\int_r^\infty \frac{\varphi^{\frac{1}{p}}(t)}{t^{\frac{n}{p}-\alpha+1}}dt \le Cr^{-\frac{\alpha p}{q-p}}. \tag{29}$$

We will consider the action of the Potential operator from one Morrey space $\mathcal{L}^{p,\varphi}$ to another Morrey space $\mathcal{L}^{q,\psi}$.

The **weighted generalised Morrey spaces** are treated in the usual sense:

$$\mathcal{L}^{p,\varphi}(\Omega, w) := \{f : \quad wf \in \mathcal{L}^{p,\varphi}(\Omega)\},\ \Omega \subseteq \mathbb{R}^n,$$
$$\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n,w)} := \|wf\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)}.$$

For the weights w we use the classes $\overline{W}(\mathbb{R}_+), \underline{W}(\mathbb{R}_+)$, and $\mathbf{V}^\mu_\pm$ defined as follows:

DEFINITION 3.7. Let $0 < \mu \le 1$. By $\mathbf{V}^\mu_\pm$, we denote the classes of functions w non-negative on $[0, \infty)$ and positive on $(0, \infty)$, defined by the conditions:

(30)
$$\mathbf{V}^\mu_+ : \qquad \frac{|w(t) - w(\tau)|}{|t-\tau|^\mu} \le C\frac{w(t_+)}{t^\mu_+},$$

(31)
$$\mathbf{V}^\mu_- : \qquad \frac{|w(t) - w(\tau)|}{|t-\tau|^\mu} \le C\frac{w(t_-)}{t^\mu_+},$$

where $t, \tau \in (0, \infty), t \ne \tau$, and $t_+ = \max(t, \tau)$, $t_- = \min(t, \tau)$.

Besides the upper Matuszewska-Orlicz indices defined in the above overview of Paper A, here we also need lower Matuszewska-Orlicz indices $m(\varphi)$ and $m_\infty(\varphi)$ for $\varphi \in W$:

$$m(\varphi) = \sup_{0<r<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hr)}{\varphi(h)}\right)}{\ln r} = \lim_{r\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(hr)}{\varphi(h)}\right)}{\ln r}$$

and

$$m_\infty(\varphi) = \sup_{r>1} \frac{\ln\left[\liminf\limits_{h\to\infty} \frac{\varphi(rh)}{\varphi(h)}\right]}{\ln\ r}.$$

One main result from Paper B reads:

THEOREM 3.8. *Let* $0 < \alpha < n,\ 1 < p < \frac{n}{\alpha},\ q > p,$ *and*

$w \in [\overline{W}(\mathbb{R}_+) \cap \underline{W}(\mathbb{R}_+)] \cap [\mathbf{V}_-^\mu(\mathbb{R}_+) \cup \mathbf{V}_+^\mu(\mathbb{R}_+)], \quad \mu = \min\{1, n-\alpha\}.$

Suppose also that the functions φ *and* ψ *satisfy the assumptions* (32)

$$M(\varphi), M_\infty(\varphi) < n - \alpha p, \quad \varphi(r) \le cr^{n - \frac{\alpha}{\frac{1}{p}-\frac{1}{q}}} \quad \text{and} \quad \frac{\varphi^{1/p}(|y|)}{|y|^{\frac{n}{p}-\alpha}} \in \mathcal{L}^{q,\psi}.$$

Under the conditions

(33) $$\alpha - \frac{n - M(\varphi)}{p} < m(w) \le M(w) < \frac{n}{p'} + \frac{m(\varphi)}{p},$$

and

(34) $$\alpha - \frac{n - M_\infty(\varphi)}{p} < m_\infty(w) \le M_\infty(w) < \frac{n}{p'} + \frac{m_\infty(\varphi)}{p},$$

the weighted Riesz potential operator $wI^\alpha \frac{1}{w}$ *is bounded from* $\mathcal{L}^{p,\varphi}(\mathbb{R}^n)$ *to* ${}^{q,\psi}(\mathbb{R}^n)$.

The above theorem leads us to the following result for the Helmholtz equation, in the case $n = 3,\ \alpha = 2$. In this application we consider Morrey spaces imbedded into the corresponding weighted Lebesgue spaces, i.e. $\mathcal{L}^{p,\varphi}(\mathbb{R}^3, w) \hookrightarrow L^p(\mathbb{R}^3, w)$. To this end, it suffices to assume that $\varphi(r)$ is a bounded function.

THEOREM 3.9. *Let* $1 < p < \frac{3}{2},\ q > p,$ *and*

$$w \in [\overline{W}(\mathbb{R}_+) \cap \underline{W}(\mathbb{R}_+)] \cap [\mathbf{V}_-^1(\mathbb{R}_+) \cup \mathbf{V}_+^1(\mathbb{R}_+)].$$

Let also the functions φ *and* ψ *satisfy the assumptions*

(35) $$M(\varphi) < 3 - 2p, \varphi(r) \le cr^{3 - \frac{2}{\frac{1}{p}-\frac{1}{q}}} \quad \text{and} \quad \frac{\varphi^{1/p}}{r^{\frac{3}{p}-2}} \in \mathcal{L}^{q,\psi}.$$

Under the conditions

$$2 - \frac{3 - M(\varphi)}{p} < m(w) \leq M(w) < \frac{3}{p'} + \frac{m(\varphi)}{p}, \tag{36}$$

and

$$2 - \frac{3 - M_\infty(\varphi)}{p} < m_\infty(w) \leq M_\infty(w) < \frac{3}{p'} + \frac{m_\infty(\varphi)}{p}, \tag{37}$$

for every $f \in \mathcal{L}^{p,\varphi}(\mathbb{R}^3, w)$*, there exists a twice Sobolev differentiable particular solution* $u \in \mathcal{L}^{q,\psi}(\mathbb{R}^3, w)$ *of the Helmholtz equation*

$$(\Delta + k^2 I)u(x) = f(x).$$

In the case of classical Morrey spaces, i.e. when $\varphi(r) = r^\lambda, 0 < r < n$, the statement of Theorem 3.9 holds in a more precise form as given in the following theorem.

THEOREM 3.10. *Let* $1 < p < \frac{3}{2},\ q > p, \lambda < 3 - 2p$ *and*

$$w \in [\overline{W}(\mathbb{R}_+) \cap \underline{W}(\mathbb{R}_+)] \cap [\mathbf{V}^1_-(\mathbb{R}_+) \cup \mathbf{V}^1_+(\mathbb{R}_+)].$$

Under the conditions

$$2 - \frac{3 - \lambda}{p} < \min(m(w), m_\infty(w)) \tag{38}$$

and

$$\max(M(w), M_\infty(w)) < \frac{3}{p'} + \frac{\lambda}{p} \tag{39}$$

for every $f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^3, w)$*, there exists a twice Sobolev differentiable particular solution* $u \in \mathcal{L}^{q,\lambda}(\mathbb{R}^3, w)$ *of the Helmholtz equation*

$$(\Delta + k^2 I)u(x) = f(x),$$

where $\frac{1}{q} = \frac{1}{p} - \frac{2}{3-\lambda}$.

The results in Paper B are related to the following publications: [3], [11], [12], [29], [30], [31], [33], [34], [35], [40], [44], [47], [55], [56], [57], [58], [59], [63], [64], [68], [70], [71], [72], [74], [76], [82], [84], [87], [88], [90], [92], [93], [94], [96], [99], [101], [106] and [108].

3.3. Main results obtained in Paper C. In this paper we study mapping properties of the multi-dimensional Hardy type operators H^α and $\mathcal{H}^\alpha$ (we write $H = H^\alpha$ and $\mathcal{H} = \mathcal{H}^\alpha$ in the case $\alpha = 0$) defined above in (12) as

$$H^\alpha f(x) := |x|^{\alpha-n} \int\limits_{|y|<|x|} f(y)dy$$

and

$$\mathcal{H}^\alpha f(x) := |x|^\alpha \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n} dy, \ \alpha \geq 0,$$

in Hölder spaces $C^\lambda(\Omega)$ defined above in (9)-(10). We deal with $\Omega = B_R$, where $B_R = B(0,R) := \{x \in \mathbb{R}^n : |x| < R\}, \ 0 < R \leq \infty$.

We will also use the subspaces $C_0^\lambda(B_R)$ of $C^\lambda(B_R)$, defined by

$$C_0^\lambda(B_R) := \left\{f \in C^\lambda(B_R) : f(0) = 0\right\},$$

and we deal also with the space $\tilde{C}_0^\lambda(B_R)$ consisting of functions f for which $[f]_\lambda < \infty$ and $f(0) = 0$. This space contains functions which are unbounded in the case $R = \infty$. Note that $[f]_\lambda$ is a norm in $C_0^\lambda(B_R)$.

In Paper C we also consider Hölder spaces of the functions on the whole space $\mathbb{R}^n$, i.e. in the case $R = \infty$ with the requirement that functions have also Hölder type behaviour at the infinite point, i.e. we deal with a compactification of $\mathbb{R}^n$ by a single infinite point, which we denote as $\dot{\mathbb{R}}^n$.

The space $C^\lambda(\dot{\mathbb{R}}^n)$ is defined by the norm

$$\|f\|_{C^\lambda(\dot{\mathbb{R}}^n)} := \|f\|_{C(\dot{\mathbb{R}}^n)} + \sup_{x,y\in\mathbb{R}^n} |f(x) - f(y)| \left(\frac{(1+|x|)(1+|y|)}{|x-y|}\right)^\lambda.$$

The operator $H^\alpha, \alpha = 0$, may be considered both with and without compactification, but a consideration of $\mathcal{H}$ requires the choice of the space $C^\lambda(\dot{\mathbb{R}}^n)$ instead of the space $C^\lambda(\mathbb{R}^n)$ due to the needed convergence of integrals at infinity. We prove the theorem for the operator $H^\alpha, \alpha \geq 0$, without compactification, and for both the operators H and $\mathcal{H}$ with compactification. We also show that in the setting of the spaces with compactification we may consider only the case $\alpha = 0$.

Our first main result in Paper C is the following theorem for the operator H^α :

THEOREM 3.11. *Let* $\alpha \geq 0$, $\lambda > 0$, $\lambda + \alpha \leq 1$ *and* $0 < R \leq \infty$. *In the case* $\alpha = 0$ *the Hardy operator* H^α *is bounded in* $C^\lambda(B_R)$ *and* $[H^\alpha f|_{\alpha=0}]_\lambda \leq C[f]_\lambda$. *In the case* $\alpha > 0$ *the operator* H^α *is bounded from* $\tilde{C}_0^\lambda(B_R)$ *into* $\tilde{C}_0^{\lambda+\alpha}(B_R)$.

We also consider the generalised Hölder space $C^{\omega(\cdot)}(\Omega)$.

The space $C^{\omega(\cdot)}(\Omega)$ is defined as the set of functions, continuous in Ω, having the finite norm

$$\|f\|_{C^{\omega(\cdot)}} := \sup_{x\in\Omega} |f(x)| + [f]_{\omega(\cdot)}$$

with the seminorm

$$[f]_{\omega(\cdot)} = \sup_{\substack{x,x+h\in\Omega \\ |h|<1}} \frac{|f(x+h) - f(x)|}{\omega(|h|)},$$

where $\omega : [0,1] \to \mathbb{R}_+$ is a non-negative increasing function in $C([0,1])$ such that $\omega(0) = 0$ and $\omega(t) > 0$ for $0 < t \leq 1$. Such spaces are known in the literature, see for instance [36, Section 13.6].

The classes $C_0^{\omega(\cdot)}(B_R)$ and $\tilde{C}_0^\lambda(B_R)$ are defined similarly to the above case $\omega(t) = t^\lambda$.

The following statement is a generalisation of Theorem 3.11 for the case of $\omega = \omega(t)$, defined in this paper.

THEOREM 3.12. *Let* $\omega \in C([0,1])$ *be positive on* $(0,1]$, *increasing and such that* $\omega(0) = 0$ *and* $\frac{\omega(t)}{t^{1-\alpha}}$ *is almost decreasing. In the case* $\alpha = 0$ *the operator* $H^\alpha|_{\alpha=0}$ *is bounded in* $C^{\omega(\cdot)}(B_R)$. *When* $\alpha > 0$, *it is bounded from* $\tilde{C}_0^{\omega(\cdot)}(B_R)$ *into* $\tilde{C}_0^{\omega_\alpha(\cdot)}(B_R)$, *where* $\omega_\alpha(t) = t^\alpha\omega(t)$.

In the setting of the spaces $C^\lambda(\dot{\mathbb{R}}^n)$ we consider only the case $\alpha = 0$, and our main results in this case for H and $\mathcal{H}$ read:

THEOREM 3.13. *Let* $0 \leq \lambda < 1$. *Then the operator* H *is bounded in* $C^\lambda(\dot{\mathbb{R}}^n)$.

To formulate the corresponding result for the operator $\mathcal{H}$ we need to consider the following subspaces:

$$C_0^\lambda(\dot{\mathbb{R}}^n) := \left\{ f \in C^\lambda(\dot{\mathbb{R}}^n) : f(0) = 0 \right\},$$
$$C_\infty^\lambda(\dot{\mathbb{R}}^n) := \left\{ f \in C^\lambda(\dot{\mathbb{R}}^n) : f(\infty) = 0 \right\},$$

and

$$C_{\infty,0}^\lambda := C_\infty^\lambda \cap C_0^\lambda.$$

THEOREM 3.14. *Let* $0 < \lambda < 1$. *Then the operator* $\mathcal{H}$ *is bounded from* $C_{\infty,0}^\lambda(\dot{\mathbb{R}}^n)$ *to* $C_\infty^\lambda(\dot{\mathbb{R}}^n)$.

REMARK 3.15. When $\alpha > 0$, Theorems 3.13 and 3.14 may not be extended to the setting $C^\lambda(\dot{\mathbb{R}}^n) \longrightarrow C^{\lambda+\alpha}(\dot{\mathbb{R}}^n)$, in which we require the Hölder behaviour of functions also at the infinite point, in contrast to the situation in Theorem 3.11.

The main results in Paper C are also cited and described in the recent book [43] by A. Kufner, L. E. Persson and N. Samko.

The results in Paper C are also related to the following publications: [9], [26], [27], [28], [36], [37], [42], [45], [50], [54], [65], [69], [70], [83], [86], [88], [95] and [100].

3.4. Main results obtained in Paper D. Besides the Hardy and Potential operators, singular operators play an important role in various applications, e.g. connected to problems related to PDEs. One-dimensional singular operators S, defined by

$$Sf(t) := \frac{1}{\pi} \int_a^b \frac{f(t)}{\tau - x} d\tau, x \in (a, b)$$

have various applications e.g. in aerodynamics and elasticity theory. In particular, the integral equation $Sf = g$ is known as the famous Söngen equation in aerodynamics. Sometimes it is also called thev Tricomi equation. More generally, equations of the form

$$a(t)f(t) + b(t)Sf(t) = g(t),$$

where in general (a, b) is replaced by an arbitrary closed or open curve are known as *singular integral equations*. Due to numerous applications the theory of these equations was intensively and comprehensively developed in the middle decades of the previous century. In the process of solving such equations there appear singular integrals with power weights T^μ, defined by

$$(40) \quad \left(T^\mu f\right)(x) := (x-a)^{\mu_1}(b-x)^{\mu_2} \int_a^b \frac{f(t)\,dt}{(t-a)^{\mu_1}(b-t)^{\mu_2}(t-x)}$$

(written for the case of the interval (a, b)), where $a < x < b$, $\mu = (\mu_1, \mu_2)$, the numbers μ_1 and μ_2 may be complex and $\operatorname{Re}(\mu_1) < 1, \operatorname{Re}(\mu_2) < 1$.

On the other hand it is known that integral equations of the first kind with logarithmic kernel, have various applications. In particular, many applied problems, where logarithmic kernels and potentials are used, can be dscribed and reduced to singular integral equations via differentiation. Consequently, there arises a problem of differentiation of the weighted singular integral $\left(T^\mu f\right)(x)$. Direct differentiation in x in the form as $\left(T^\mu f\right)(x)$ is written, leads to a cumbersome and non-applicable results with strong singularities of the so obtained results at the endpoints of the interval. This happens because such a direct differentiation does not use differentiability properties of the function f itself. Meanwhile the problem to study here is to show that if $\frac{df}{dt}$ belongs to some class, then $\frac{d}{dx}\left(T^\mu f\right)(x)$ belongs to the same class. Results of such a type were known in some specific setting for the class of derivatives. Here we solve the problem of justification of the differentiation formula for such a weighted singular integral $\left(T^\mu f\right)(x)$ in the framework of weighted Sobolev spaces $W^{p,1} = W^{p,1}(w)$, defined by

$$W^{p,1}(w) := \left\{f \in L^p(w, [a, b]) : df/dx \in L^p(w, [a, b])\right\}.$$

Here the derivative is understood as usual in the weak sense.

The weighted space $L^p(w,[a,b]) =: L^p(w), 1 \le p < \infty$, is defined by

$$L^p(w) := \left\{ \varphi : \|\varphi\|_{L^p(w)} := \int_a^b |\varphi(x)w(x)|^p dx < \infty \right\}.$$

We also use the notations:

$$f_\mu := \int_a^b \frac{f(t)\,dt}{(t-a)^{\mu_1}(b-t)^{\mu_2}},$$

$$\varrho_{1-\mu}(x) := \frac{1}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}} \qquad \text{and } D = d/dx.$$

One main result in Paper D is the following:

THEOREM 3.16. *Let $f \in W^{p,1}(w,[a,b])$, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $w = (x-a)^{\alpha_1}(b-x)^{\alpha_2}$. Under the assumption that*

$$-1/p \le \alpha_1 + \mathrm{Re}(\mu_1 - 1) \le 1/p'$$

and

$$-1/p \le \alpha_2 + \mathrm{Re}(\mu_2 - 1) \le 1/p',$$

the following differentiation formula is valid:

$$\frac{d}{dx} T^\mu f(x) =$$

$$= \frac{1}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}} \int_a^b (t-a)^{1-\mu_1}(b-t)^{1-\mu_2} \frac{f'(t)\,dt}{(t-x)} + \tag{41}$$

$$+ \frac{(\mu_1+\mu_2-1) f_\mu}{(x-a)^{1-\mu_1}(b-x)^{1-\mu_2}},$$

or in short form

$$(DT^\mu f)(x) = \left(T^{\mu-1} Df\right)(x) + (\mu_1+\mu_2-1) f_\mu \cdot \varrho_{1-\mu}(x). \tag{42}$$

Similar differentiation results are also obtained when there is admitted an additional logarithmic behavior at the endpoints of the interval, i.e. when $f(t)$ is replaced by $f(t)\ln(t-a)$ or $f(t)\ln(b-t)$, but $f(t)$ still belongs to $W^{p,1}(w,[a,b])$.

The results in Paper D are related to the following publications: [14], [21], [22], [81], [85], [89] and [91].

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Paper A

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On some hyperbolic type equations and weighted anisotropic Hardy operators

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We introduce a version of weighted anisotropic Morrey spaces and anisotropic Hardy operators. We find conditions for boundedness of these operators in such spaces. We also reveal the role of these operators in solving some classes of degenerate hyperbolic partial differential equations. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: weighted Hardy operators, anisotropic Hardy operators, Morrey spaces, weighted anisotropic Morrey spaces

1. Introduction

It is well known that many operators of harmonic analysis such as potential type operators, singular operators, and others are widely used in PDEs. In this paper, the authors present their results on anisotropic double Hardy type operators arising in relation to some degenerate hyperbolic type PDEs, with an emphasis on the role of the function space used in the solving process.

Degenerate hyperbolic equations arise in various applied problems, for instance, in the study of distribution of heat and moisture transfer in capillary-porous media. We do not provide any historical overview: this would lead us too far away. We just present here some references, see, for example, [1–5], and references therein.

We also refer to some papers where the degenerate hyperbolic equation of the form

$$y^2\frac{\partial^2 u}{\partial x\partial x}-\frac{\partial^2 u}{\partial y\partial y}+a\frac{\partial u}{\partial x}=f(x,y) \tag{1.1}$$

was studied, see, for instance, [1, 3], and references therein, which by the transformation

$$\xi=x-\frac{y^2}{2},\quad \eta=x+\frac{y^2}{2}$$

reduces to the equation in the following form:

$$(\xi-\eta)\frac{\partial^2 u}{\partial\xi\partial\eta}+\text{ lower terms }=g(\xi,\eta).$$

The degenerate hyperbolic equation related to the use of the anisotropic weighted Hardy operator has the form

$$xy\frac{\partial^2 u}{\partial x\partial y}+ax\frac{\partial u}{\partial x}+by\frac{\partial u}{\partial y}+cu(x,y)=f(x,y). \tag{1.2}$$

We study a possibility to find a solution of the equation within the frame of weighted Morrey spaces well suited for their use in PDEs.

For Morrey spaces and investigation of various operators of harmonic analysis in such spaces, related to studies here, we refer, for instance, to [6–8]. See also [9] and references therein. Precise definitions of the spaces are given in Section 2.

We will study this equation in the weighted anisotropic Morrey space, when the right-hand side of the equation is in that space. Such a possibility is based on the boundedness of weighted Hardy operators in the corresponding spaces. To this end, we prove a theorem on the boundedness of weighted anisotropic double Hardy operator in the frameworks of anisotropic Morrey spaces.

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Anisotropic Morrey spaces were introduced in [10] .

In order not to overload the exposition with details, and for reader's convenience, we present all necessary definitions and properties of the weights in the Appendix.

2. Preliminaries

2.1. Morrey space

Morrey space $\mathcal{L}^{p,\lambda}$ is defined as follows:

$$\mathcal{L}^{p,\lambda} = \left\{ f \in \mathcal{L}^p_{loc}(\Omega) : \|f\|_{p,\lambda} < \infty \right\}, \quad 1 \le p < \infty,\ 0 \le \lambda < n, \tag{2.1}$$

where $\Omega \subseteq \mathbb{R}^n$. Equipped with the norm

$$\|f\|_{p,\lambda} = \sup_{x\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(t)|^p\, dt \right)^{\frac{1}{p}} = \sup_{x\in\Omega, r>0} \frac{\|f\|_{\mathcal{L}^p(B(x,r))}}{r^{\frac{\lambda}{p}}}, \tag{2.2}$$

where $B(x,r) = \{y \in \Omega : |y - x| < r\}$, it is a Banach space.

We consider Morrey space on $\mathbb{R}^n$, and weighted Morrey spaces are treated in the usual sense:

$$\mathcal{L}^{p,\lambda}(\Omega, w) := \{f : \ wf \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)\}, \qquad \|f\|_{\mathcal{L}^{p,\lambda}(\Omega,w)} := \|wf\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)}.$$

Next, we present the definitions of the anisotropic Morrey spaces.

Anisotropic Morrey space $\mathcal{L}^{p,\lambda_1,\lambda_2}(\mathbb{R}^2_+)$ is defined in [10] by the norm

$$\|f\|_{p,\lambda_1,\lambda_2} = \sup_{\substack{x>0,y>0\\ r_1>0,r_2>0}} \left(\frac{1}{r_1^{\lambda_1} r_2^{\lambda_2}} \int_{(x-r_1)_+}^{x+r_1} \int_{(y-r_2)_+}^{y+r_2} |f(t_1,t_2)|^p\, dt_1\, dt_2 \right)^{\frac{1}{p}} = \sup_{x,r\in\mathbb{R}^2_+} \frac{\|f\|_{\mathcal{L}^p(Q(x,r))}}{r_1^{\frac{\lambda_1}{p}} r_2^{\frac{\lambda_2}{p}}}, \tag{2.3}$$

where $(x_i - r_i)_+ = \begin{cases} x_i - r_i, & \text{if } x_i - r_i \ge 0 \\ 0, & \text{if } x_i - r_i < 0, \end{cases}$ $, i = 1,2,\ Q(x,r) = \{t = (t_1,t_2) \in \mathbb{R}^2_+ :\ (x_i - r_i)_+ < t_i < x_i + r_i,\ i = 1,2\} = I_{x,r_1} \times I_{y,r_2},\ x = (x,y), r = (r_1,r_2)$, and $I_{x_i,r_i} = ((x_i - r_i)_+, x_i + r_i),\ i = 1,2$.

Anisotropic mixed norm Morrey space $\mathcal{L}^{\bar{p},\bar{\lambda}}(\mathbb{R}^2_+)$ is defined by the norm

$$\|f\|_{\bar{p},\bar{\lambda}} = \sup_{x,r\in\mathbb{R}^2_+} \frac{\|f\|_{\mathcal{L}^{\bar{p}}(Q(x,r))}}{r_1^{\frac{\lambda_1}{p_1}} r_2^{\frac{\lambda_2}{p_2}}}, \tag{2.4}$$

where $\bar{p} = (p_1,p_2), \bar{\lambda} = (\lambda_1,\lambda_2)$, with the mixed norm $\|f\|_{\mathcal{L}^{\bar{p}}(Q(x,r))}$ over the rectangle $Q(x,r)$, where

$$\|f\|_{\mathcal{L}^{\bar{p}}(Q(x,r))} = \left(\int_{I_{x,r_1}} \left(\int_{I_{y,r_2}} |f(t_1,t_2)|^{p_2}\, dt_2 \right)^{p_1/p_2} dt_1 \right)^{1/p_1} = \| \|f(t_1,\cdot)\|_{\mathcal{L}^{p_2}(I_2)} \|_{\mathcal{L}^{p_1}(I_1)}, \tag{2.5}$$

where "$\cdot$" stands for the variable in which the inner norm is applied (we refer to [11] for mixed norm Lebesgue spaces).

Weighted anisotropic mixed norm Morrey space $\mathcal{L}^{\bar{p},\bar{\lambda}}(\mathbb{R}^2_+, w_1w_2)$ is defined as

$$\mathcal{L}^{\bar{p},\bar{\lambda}}\left(\mathbb{R}^2_+, w_1w_2\right) := \{f :\ w_1(x)w_2(y)f(x,y) \in \mathcal{L}^{\bar{p},\bar{\lambda}}\left(\mathbb{R}^2_+\right).$$

3. On Hardy operators appearing in partial differential equations

3.1. Non-weighted case

The main facts in this section concern the application of double Hardy operators of functions of two variables.

We will consider the well-known one-dimensional Hardy operators

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad \mathcal{H}f(x) = \int_x^\infty \frac{f(t)dt}{t}, \quad x > 0 \tag{3.1}$$

and the following double Hardy operators of two variables

$$H_1H_2f = \frac{1}{xy}\int_0^x\int_0^y f(t_1,t_2)dt_1dt_2, \tag{3.2}$$

$$H_1\mathcal{H}_2f = \frac{1}{x}\int_0^x\int_y^\infty \frac{f(t_1,t_2)}{t_2}dt_2dt_1, \tag{3.3}$$

$$\mathcal{H}_1H_2f = \frac{1}{y}\int_x^\infty \frac{1}{t_1}\int_0^y f(t_1,t_2)dt_2dt_1 \tag{3.4}$$

and

$$\mathcal{H}_1\mathcal{H}_2f = \int_x^\infty\int_y^\infty \frac{f(t_1,t_2)}{t_1t_2}dt_1dt_2. \tag{3.5}$$

It is not hard to check that in terms of these operators, we obtain particular solutions of the partial differential Equation (1.2) with certain values of the coefficients a, b, and c, namely,

if $a = b = c = 1$, then the Equation (1.2), that is, the equation

$$xy\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial x}u(x,y) + y\frac{\partial u}{\partial y} + u(x,y) = f(x,y) \tag{3.6}$$

has the solution

$$u = H_1H_2f;$$

if $a = c = 0,\ b = 1$, then the Equation (1.2) has the solution

$$u = H_1\mathcal{H}_2f;$$

if $a = 1,\ b = c = 0$, then the Equation (1.2) has the solution

$$u = \mathcal{H}_1H_2f;$$

if $a = b = c = 0$, then the Equation (1.2) has the solution

$$u = \mathcal{H}_1\mathcal{H}_2f.$$

3.2. *Weighted case*

Consider the double-weighted Hardy operator, defined by

$$H_1^{w_1}H_2^{w_2}f(x) = \frac{w_1(x)w_2(y)}{xy}\int_0^x\int_0^y \frac{f(t_1,t_2)}{w_1(t_1)w_2(t_2)}dt_1\,dt_2,\ \ x = (x,y). \tag{3.7}$$

The function $u(x,y) = H_1^{w_1}H_2^{w_2}f(x)$ satisfies the differential equation:

$$xyu''_{xy} + xa_2(y)u'_x + ya_1(x)u'_y + a_1(x)a_2(y)u = f(x,y), \tag{3.8}$$

where

$$a_1(x) = 1 - x\frac{\partial}{\partial x}(\ln w_1),$$
$$a_2(y) = 1 - y\frac{\partial}{\partial y}(\ln w_2),$$

and the weights $w_1(x)$ and $w_2(y)$ can be expressed as

$$w_1(x) = \exp\left(\int \frac{1-a_1(x)}{x}dx\right),$$
$$w_2(y) = \exp\left(\int \frac{1-a_2(y)}{y}dy\right),$$

respectively.

In the case of power weights, that is, when $w_1(x) = x^{\theta_1}$ and $w_2(y) = y^{\theta_2}$; $a_1(x) = 1 - \theta_1$ and $a_2(y) = 1 - \theta_2$, the function

$$u(x,y) = H_1^{\theta_1} H_2^{\theta_2} f(x) = x^{\theta_1 - 1} y^{\theta_2 - 1} \int_0^x \int_0^y \frac{f(t_1,t_2)}{t_1^{\theta_1} t_2^{\theta_2}} dt_1\, dt_2, \ \ x = (x,y),$$

satisfies the differential equation:

$$xyu''_{xy} + x(1-\theta_2)u'_x + y(1-\theta_1)u'_y + (1-\theta_1)(1-\theta_2)u = f(x,y).$$

3.3. Application of weighted boundedness of two-dimensional Hardy operators to the study of partial differential equations

In this section, based on the mapping properties of the two-dimensional weighted Hardy operator in anisotropic mixed norm Morrey space, we study Morrey-type properties of a particular solution of the inhomogenious Equation (3.6).

We consider the weighted double Hardy operator

$$H^{\overline{\alpha},w} f(x,y) := x^{\alpha_1 - 1} y^{\alpha_2 - 1} w_1(x) w_2(y) \int_0^x \int_0^y \frac{f(t_1,t_2)}{w_1(t_1) w_2(t_2)} dt_1\, dt_2, \tag{3.9}$$

where $\overline{\alpha} = (\alpha_1, \alpha_2)$, $w = w_1(x) \cdot w_2(y)$.

We may assume that $f \geq 0$. Then, if the double integral Equation (3.9) converges then by Fubini's theorem, it coincides with the also convergent iterated integrals:

$$H^{\overline{\alpha},w} f = H_1^{\alpha_1,w_1} H_2^{\alpha_2,w_2} f = H_2^{\alpha_2,w_2} H_1^{\alpha_1,w_1} f, \tag{3.10}$$

where

$$H_1^{\alpha_1,w_1} H_2^{\alpha_2,w_2} f(x,y) = \frac{w_1(x) w_2(y)}{x^{1-\alpha_1} y^{1-\alpha_2}} \int_0^x \frac{1}{w_1(t_1)} \left(\int_0^y \frac{f(t_1,t_2)}{w_2(t_2)} dt_2 \right) dt_1, \tag{3.11}$$

and

$$H_2^{\alpha_2,w_2} H_1^{\alpha_1,w_1} f(x,y) = \frac{w_1(x) w_2(y)}{x^{1-\alpha_1} y^{1-\alpha_2}} \int_0^y \frac{1}{w_2(t_2)} \left(\int_0^x \frac{f(t_1,t_2)}{w_1(t_1)} dt_1 \right) dt_2, \tag{3.12}$$

so that we can use any one of the forms in Equation (3.10). Thus, we can interpret our anisotropic Hardy operator as a composition of the one-dimensional Hardy operators applied in the corresponding variable. However, with respect to the notation used in Equations (3.11) and (3.12), note that $H_1^{\overline{\alpha},w}$ is an operator defined on functions of two variables, while the notation H^{α_i,w_i} stands for operators defined on functions of one variable. So to interpret, for example, $H_1^{\alpha_1,w_1} H_2^{\alpha_2,w_2}$ as a composition, we should write $H_1^{\alpha_1,w_1} \otimes I$ instead of $H_1^{\alpha_1,w_1}$ and $I \otimes H_2^{\alpha_2,w_2}$ instead of $H_2^{\alpha_2,w_2}$, where I is the identity operator and $\otimes$ stands for the tensor product of one-dimensional operators (see e.g., [12, Chapter 24] for this notion). However, to avoid complications on writing, we keep the simple notation $H_1^{\alpha_1,w_1}$ and $H_2^{\alpha_2,w_2}$ without danger of confusion of notation.

To formulate and prove our Theorem 3.2 on the boundedness of double Hardy type operator in the mixed norm anisotropic case, we need the following result:

Theorem 3.1 ([8], Theorem 4.5)
Let $0 \leq \lambda < 1, 0 \leq \alpha < 1 - \lambda, 1 < p < \frac{1-\lambda}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{1-\lambda}$ and $w \in \overline{W}(\mathbb{R}_+)$. For the weighted Hardy type inequality

$$\left\| x^{\alpha-1} w(x) \int_0^x \frac{f(t)}{w(t)} dt \right\|_{\mathcal{L}^{q,\lambda}(\mathbb{R}_+)} \leq C \, \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}_+)} \tag{3.13}$$

to hold, condition

$$w \in \mathbb{Z}_{\frac{\lambda}{p}+\frac{1}{p'}}(\mathbb{R}_+), \text{ or equivalently } \max\left(M(w), M_\infty(w)\right) < \frac{\lambda}{p} + \frac{1}{p'}, \tag{3.14}$$

is sufficient, and the condition

$$w \in \mathbb{Z}_{\frac{\lambda}{p}+\frac{1}{p'}+\varepsilon}(\mathbb{R}_+), \text{ or equivalently } \max\left(M(w), M_\infty(w)\right) \leq \frac{\lambda}{p} + \frac{1}{p'}, \tag{3.15}$$

with an arbitrary $\varepsilon > 0$, is necessary.

Theorem 3.1 was proved in [13] for the case $\alpha = 0$ and in [8] for $\alpha > 0$.

The following theorem on weighted Hardy type inequality was formulated without proof, in [10]. We give here its complete proof.

Theorem 3.2
Let $0 \le \lambda_i < 1, 0 \le \alpha_i < 1-\lambda_i, 1 < p_i < \frac{1-\lambda_i}{\alpha_i}$ and $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha_i}{1-\lambda_i}$ and $w_i \in \overline{W}(\mathbb{R}_+)$, $i = 1, 2$. For the weighted Hardy type inequality

$$\left\| H^{\overline{\alpha},w} f \right\|_{\overline{p},\overline{\lambda}} \le C \, \|f\|_{\overline{p},\overline{\lambda}} \tag{3.16}$$

to hold, the condition $w_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i}+\frac{1}{p_i'}}(\mathbb{R}_+)$ is sufficient, and the condition $w_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i}+\frac{1}{p_i'}+\varepsilon}(\mathbb{R}_+)$ with an arbitrary $\varepsilon > 0$ is necessary, $i = 1, 2$.

The proof of Theorem 3.2 will be obtained from two theorems: Theorems 3.4 and 3.5 proved next for the operators

$$A_1 f = H^{\alpha_1, w_1} \otimes If = \frac{w_1(x)}{x^{1-\alpha_1}} \int_0^x \frac{f(t,y)}{w_1(t)} dt$$

$$A_2 f = I \otimes H^{\alpha_2, w_2} f = \frac{w_2(y)}{y^{1-\alpha_2}} \int_0^y \frac{f(x,\tau)}{w_2(\tau)} d\tau.$$

The operators A_1, A_2 behave like the identical operators in the variables y, x, respectively, so they keep the behavior of functions with respect to the variables y, x, respectively, and they will be considered in the following setting:

$$A_1 : \mathcal{L}^{\overline{p},\overline{\lambda}} \to \mathcal{L}^{\overline{q^1},\overline{\lambda}}, \ \overline{q^1} = (q_1, p_2); \quad A_2 : \mathcal{L}^{\overline{p},\overline{\lambda}} \to \mathcal{L}^{\overline{q^2},\overline{\lambda}}, \ \overline{q^2} = (p_1, q_2).$$

Clearly, our anisotropic Hardy operator is their composition:

$$H^{\overline{\alpha},w} = A_1 \cdot A_2 f = A_2 \cdot A_1 f.$$

We need also the following lemma.

Lemma 3.3
For the norm (2.4), the equality

$$\|f\|_{\overline{p},\overline{\lambda}} \left(\mathbb{R}^2_+\right) = \left\| \|f(t_1,\cdot)\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}_+)} \right\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}_+)} \tag{3.17}$$

holds.

Proof
By (2.2) and (2.5), we have

$$\|f\|_{\mathcal{L}^{\overline{p},\overline{\lambda}}}(\mathbb{R}^2_+) = \sup_{x,r \in \mathbb{R}^2_+} \frac{1}{r_1^{\frac{\lambda_1}{p_1}}} \left\| \frac{1}{r_2^{\frac{\lambda_2}{p_2}}} \|f(t_1,\cdot)\|_{\mathcal{L}^{p_2}(I_{y,r_2})} \right\|_{\mathcal{L}^{p_1}(I_{x,r_1})}. \tag{3.18}$$

Because $\sup\limits_{u,v} g(u,v) = \sup\limits_{u} \sup\limits_{v} g(u,v)$ for non-negative functions $g(u,v)$, from Equation (3.18), we obtain Equation (3.17). □

Theorem 3.4
Let $0 \le \lambda_i < 1,\ 0 \le \alpha_1 < 1-\lambda_1,\ i = 1, 2, 1 < p_1 < \frac{1-\lambda_1}{\alpha_1}$ and $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha_1}{1-\lambda_1}, 1 < p_2 < \infty$ and $w_1 \in \overline{W}(\mathbb{R}_+)$. For the boundedness

$$\|A_1 f\|_{\mathcal{L}^{\overline{q^1},\overline{\lambda}}} \le C \, \|f\|_{\mathcal{L}^{\overline{p},\overline{\lambda}}}, \tag{3.19}$$

where $\overline{q^1} = (q_1, p_2)$, it is sufficient that $w_1 \in \mathbb{Z}_{\frac{\lambda_1}{p_1}+\frac{1}{p_1'}}(\mathbb{R}_+)$ and necessary that $w_1 \in \mathbb{Z}_{\frac{\lambda_1}{p_1}+\frac{1}{p_1'}+\varepsilon}(\mathbb{R}_+)$ with an arbitrary $\varepsilon > 0$.

Proof
By Lemma 3.3, we have to estimate

$$\| \|A_1 f\|_{\mathcal{L}^{p_2,\lambda_2}} \|_{\mathcal{L}^{q_1,\lambda_1}}.$$

By Minkowski inequality, we have

$$\|A_1 f\|_{\mathcal{L}^{p_2,\lambda_2}} \le \frac{w_1(x)}{x^{1-\alpha_1}} \int_0^x \frac{\|f(t,\cdot)\|_{\mathcal{L}^{p_2,\lambda_2}}}{w_1(t)} dt$$

(the validity of Minkowski inequality for Morrey spaces follows from its validity for Lebesgue spaces). Then

$$\| \|A_1 f\|_{\mathcal{L}^{p_2,\lambda_2}} \|_{\mathcal{L}^{q_1,\lambda_1}} \le \left\| \frac{w_1(x)}{x^{1-\alpha_1}} \int_0^x \frac{\|f(t,\cdot)\|_{\mathcal{L}^{p_2,\lambda_2}}}{w_1(t)} dt \right\|_{\mathcal{L}^{q_1,\lambda_1}},$$

it remains to apply Theorem 3.1 in the sufficiency part. To cover the necessity part, it suffices to observe that the boundedness of the operator A_1 in particular on functions f of the form $f(x,y) = f_1(x) \cdot f_2(y)$, where $f_1 \in \mathcal{L}^{p_1,\lambda_1}(\mathbb{R}_+)$ and $f_2 \in \mathcal{L}^{p_2,\lambda_2}(\mathbb{R}_+)$. Taking f_2 fixed and f_1 running the space $\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}_+)$, it remains to refer again to Theorem 3.1. □

Theorem 3.5
Let $0 \le \lambda_i < 1, 0 \le \alpha_1 < 1 - \lambda_1$, $i = 1, 2$, $1 < p_2 < \frac{1-\lambda_2}{\alpha_2}$ and $\frac{1}{q_2} = \frac{1}{p_2} - \frac{\alpha_2}{1-\lambda_2}$, $1 < p_1 < \infty$ and $w_2 \in \overline{W}(\mathbb{R}_+)$. For the boundedness

$$\|A_2 f\|_{\mathcal{L}^{\overline{q^2}, \overline{\lambda}}} \le C \, \|f\|_{\mathcal{L}^{\overline{p}, \overline{\lambda}}}, \tag{3.20}$$

where $\overline{q^2} = (p_1, q_2)$, it is sufficient that $w_2 \in \mathbb{Z}_{\frac{\lambda_2}{p_2} + \frac{1}{p_2'}}(\mathbb{R}_+)$ and necessary that $w_2 \in \mathbb{Z}_{\frac{\lambda_2}{p_2} + \frac{1}{p_2'} + \varepsilon}(\mathbb{R}_+)$ with an arbitrary $\varepsilon > 0$.

Proof
The proof is similar to that of Theorem 3.4 with the only difference that now we have to estimate the norm

$$\| \|A_2 f\|_{\mathcal{L}^{q_2, \lambda_2}} \|_{\mathcal{L}^{p_1, \lambda_1}}$$

and here is enough to apply Theorem 3.1. The necessity part is similarly proved. □

Proof of Theorem 3.2 itself:

Sufficiency part. For

$$H^{\overline{\alpha}, w} = A_1 \cdot A_2$$

we have that

$$A_2 : \mathcal{L}^{\overline{p}, \overline{\lambda}} \to \mathcal{L}^{\overline{q^2}, \overline{\lambda}}$$

by Theorem 3.5 and then

$$A_1 : \mathcal{L}^{\overline{q^2}, \overline{\lambda}} \to \mathcal{L}^{\overline{q}, \overline{\lambda}}$$

by Theorem 3.4.

Necessity part. Use the familiar argument with the passage $f = f_1(x) \cdot f_2(y)$ and fixing the function f_2 when obtaining necessary condition in the first variable and fixing the function f_1 in the case of the second variable.

Based on the boundedness of weighted Hardy operators provided by Theorem 3.2, we obtain the following result for solutions in weighted Morrey space of the inhomogeneous Equation (3.6).

Theorem 3.6
Let $f \in \mathcal{L}^{\overline{p}, \overline{\lambda}}(\mathbb{R}^2_+, w_1 \cdot w_2)$, where $1 < p_i < \infty, 0 \le \lambda_i < 1$, $i = 1, 2$. Then there exists in $\mathcal{L}^{\overline{p}, \overline{\lambda}}(\mathbb{R}^2_+, w_1 \cdot w_2)$ a particular solution $u(x, y)$ of the Equation (3.6) given by the Hardy operator

$$u(x, y) = \frac{1}{xy} \int_0^x \int_0^y f(t_1, t_2) \, dt_1 \, dt_2$$

for all weights w_1 and w_2 such that

$$w_i \in \mathbb{Z}_{\frac{\lambda_i}{p_i} + \frac{1}{p_i'}}(\mathbb{R}_+), \text{ or equivalently } \max\left(M(w_i), M_\infty(w_i)\right) < \frac{\lambda}{p_i} + \frac{1}{p_i'}, \; i = 1, 2. \tag{3.21}$$

Proof
We need to see that the non-weighted Hardy operator is bounded in the weighted Morrey space $\mathcal{L}^{\overline{p}, \overline{\lambda}}(\mathbb{R}^2_+, w_1 w_2)$. This is equivalent to the boundedness of Hardy operator $H^{w_1} H^{w_2}$ in the non-weighted Morrey space $\mathcal{L}^{\overline{p}, \overline{\lambda}}(\mathbb{R}^2_+)$. For the latter, it suffices to apply Theorem 3.2 with $\alpha_1 = \alpha_2 = 0$. □

If we consider the case of power weights, that is, when $w_1(x) = x^{\theta_1}$ and $w_2(y) = y^{\theta_2}$, we formulate the following statement:

Corollary 3.7
In the case of power weights, that is, when $w_1(x) = x^{\theta_1}$ and $w_2(y) = y^{\theta_2}$, the condition 3.21 is reduced to the condition

$$\max(\theta_i) < \frac{\lambda}{p_i} + \frac{1}{p_i'}, \; i = 1, 2. \tag{3.22}$$

4. Appendix

4.1. *On some classes of quasi-monotone functions*

Next, we give the known definitions and properties of some classes of quasi-monotone functions. For more details and proofs, we refer to [14, 15], see also [16] and references therein.

Definition 4.1

1. By $W = W(\mathbb{R}_+)$, we denote the class of functions φ continuous and positive on $\mathbb{R}_+$ such that there exists the finite limit $\lim_{x \to 0} \varphi(x)$;
2. by $W_0 = W_0(\mathbb{R}_+)$, we denote the class of functions $\varphi \in W$ almost increasing on $(\mathbb{R}_+)$;

3. by $\overline{W} = \overline{W}(\mathbb{R}_+)$, we denote the class of functions $\varphi \in W$ such that $x^a\varphi(x) \in W_0$ for some $a = a(\varphi) \in \mathbb{R}$;

4. by $\underline{W} = \underline{W}(\mathbb{R}_+)$, we denote the class of functions $\varphi \in W$ such that there exist a number $b \in \mathbb{R}$ such that $\frac{f(t)}{t^b}$ is almost decreasing.

Zygmund–Bary–Stechkin classes and Matuszewska–Orlicz indices

Definition 4.2
We say that a function $\varphi \in \overline{W}$ belongs to the Zygmund class $\mathbb{Z}^\beta$, $\beta \in \mathbb{R}^1$, if

$$\int_0^r \frac{\varphi(t)}{t^{1+\beta}} dt \le c\frac{\varphi(r)}{r^\beta}, \qquad r \in (0, \infty), \tag{4.1}$$

and to the Zygmund class $\mathbb{Z}_\gamma$, $\gamma \in \mathbb{R}^1$, if

$$\int_r^\infty \frac{\varphi(t)}{t^{1+\gamma}} dt \le c\frac{\varphi(r)}{r^\gamma}, \qquad r \in (0, \infty). \tag{4.2}$$

We also denote

$$\Phi_\gamma^\beta(\mathbb{R}_+) := \mathbb{Z}^\beta(\mathbb{R}_+) \cap \mathbb{Z}_\gamma(\mathbb{R}_+), \tag{4.3}$$

the latter class being also known as Zygmund–Bary–Stechkin class [17].

It is known that the property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so-called Matuszewska–Orlicz indices. We refer, for instance, to [15] and later paper [18] and references therein, for the properties of the indices of such a type.

For a function $\varphi \in \underline{W} \bigcap \overline{W}$, such indices at the origin are defined as follows:

$$m(\varphi) = \sup_{0<r<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)}\right)}{\ln r} = \lim_{r\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)}\right)}{\ln r} \tag{4.4}$$

and

$$M(\varphi) = \sup_{r>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)}\right)}{\ln r} = \lim_{r\to \infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)}\right)}{\ln r}. \tag{4.5}$$

Similarly, there are introduced such indices at infinity:

$$m_\infty(\varphi) = \sup_{r>1} \frac{\ln\left[\liminf\limits_{h\to\infty} \frac{\varphi(rh)}{\varphi(h)}\right]}{\ln\, r}, \quad M_\infty(\varphi) = \inf_{r>1} \frac{\ln\left[\limsup\limits_{h\to\infty} \frac{\varphi(rh)}{\varphi(h)}\right]}{\ln\, r}. \tag{4.6}$$

The following properties of the indices of functions $u, v \in \underline{W} \bigcap \overline{W}$ are known, see for instance [4, 15].

$$m[r^a u(r)] = a + m(u), \quad M[r^a u(r)] = a + M(u), \qquad a \in \mathbb{R}^1, \tag{4.7}$$

$$m[(u)^a] = am(u), \quad M[(u)^a] = aM(u), \quad ,a \ge 0 \tag{4.8}$$

$$m\left(\frac{1}{u}\right) = -M(u), \qquad M\left(\frac{1}{u}\right) = -m(u). \tag{4.9}$$

$$m(uv) \ge m(u) + m(v), \qquad M(uv) \le M(u) + M(v). \tag{4.10}$$

$$c_1 r^{M(u)+\varepsilon} \le u(r) \le c_2 r^{m(u)-\varepsilon},\ 0 < r < 1, \tag{4.11}$$

hold with an arbitrarily small $\varepsilon > 0$ and $c_1 = c_1(\varepsilon), c_2 = c_2(\varepsilon)$.

Similarly,

$$m_\infty[r^a u(r)] = a + m_\infty(u), \quad M_\infty[r^a u(r)] = a + M_\infty(u), \qquad a \in \mathbb{R}^1, \tag{4.12}$$

$$m_\infty[(u)^a] = am_\infty(u), \quad M_\infty[(u)^a] = aM_\infty(u), \quad ,a \ge 0 \tag{4.13}$$

$$m_\infty\left(\frac{1}{u}\right) = -M_\infty(u), \qquad M_\infty\left(\frac{1}{u}\right) = -m_\infty(u). \tag{4.14}$$

$$m_\infty(uv) \ge m_\infty(u) + m_\infty(v), \qquad M_\infty(uv) \le M_\infty(u) + M_\infty(v). \tag{4.15}$$

$$c_1 r^{m_\infty(u)-\varepsilon} \leq u(r) \leq c_2 r^{M_\infty(u)+\varepsilon}, \qquad r \geq 1. \tag{4.16}$$

The properties (4.12) – (4.16) follow from the properties (4.7) – (4.11) in view of the equivalences:

$$u \in \mathbb{Z}^{\beta}([1,\infty)) \Longleftrightarrow u_* \in \mathbb{Z}_{-\beta}([0,1]), \quad u \in \mathbb{Z}_{\gamma}([1,\infty)) \Longleftrightarrow u_* \in \mathbb{Z}^{-\gamma}([0,1]), \tag{4.17}$$

where $u_*(t) = u\left(\frac{1}{t}\right)$.

We will also use the following known properties:

$$u \in \mathbb{Z}^{\beta} \Longleftrightarrow \min\{m(u), m_\infty(u)\} > \beta \quad \text{and} \quad u \in \mathbb{Z}_{\gamma} \Longleftrightarrow \max\{M(u), M_\infty(u)\} < \gamma. \tag{4.18}$$

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Paper B

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Paper C

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Paper D

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