ON THE ISOTYPIC DECOMPOSITION OF COHOMOLOGY MODULES OF SYMMETRIC SEMI-ALGEBRAIC SETS: POLYNOMIAL BOUNDS ON MULTIPLICITIES

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Abstract. We consider symmetric (under the action of products of finite symmetric groups) real algebraic varieties and semi-algebraic sets, as well as symmetric complex varieties in affine and projective spaces, defined by polynomials of degrees bounded by a fixed constant d. We prove that if a Specht module, $\mathbb{S}^\lambda$, appears with positive multiplicity in the isotypic decomposition of the cohomology modules of such sets, then the rank of the partition λ is bounded by $O(d)$. This implies a polynomial (in the dimension of the ambient space) bound on the number of such modules. Furthermore, we prove a polynomial bound on the multiplicities of those that do appear with positive multiplicity in the isotypic decomposition of the above mentioned cohomology modules.

We give some applications of our methods in proving lower bounds on the degrees of defining polynomials of certain symmetric semi-algebraic sets, as well as improved bounds on the Betti numbers of the images under projections of (not necessarily symmetric) bounded real algebraic sets, improving in certain situations prior results of Gabrielov, Vorobjov and Zell.

Contents

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1. Introduction

For any finite group G, a real or complex variety V equipped with a G-action, and a field of coefficients $\mathbb{F}$, the cohomology groups, $\mathrm{H}^*(V, \mathbb{F})$, of V inherit a structure of a G-module. In this paper, we consider the special case when G is a finite group, and more specifically a product of symmetric groups, $\mathfrak{S}_{\mathbf{k}} = \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_\omega}$, acting linearly on finite dimensional real and complex vector spaces by the standard action of permuting coordinates, and $\mathbb{F}$ a field of characteristic 0. (Note that the topological structure of varieties (also symmetric spaces) admitting actions of finite groups is a very well-studied topic. Here we concentrate on the action of finite *reflection* groups, in fact, exclusively products of finite symmetric groups, which seems to be a less developed field of study.) We study quantitatively, the $\mathfrak{S}_{\mathbf{k}}$-module structure of the cohomology groups of $\mathfrak{S}_{\mathbf{k}}$-symmetric algebraic varieties, and more generally semi-algebraic sets. We prove upper bounds on the multiplicities of the various irreducibles that appear in the isotypic decomposition of these modules, as well as restrictions on those that are allowed to appear with non-zero multiplicities. Our upper bounds (both on the multiplicities as well as on the number of irreducibles that are allowed) are polynomial in the number of variables, as long as the degrees of the polynomials defining the variety or semi-algebraic set are held fixed. We give a couple of applications of these results in proving lower bounds on degrees, as well as improving existing bounds on the Betti numbers of images of semi-algebraic sets (not necessarily symmetric) under polynomial maps.

We begin with some history and motivation behind studying these questions.

1.1. **History and motivation.** Throughout this paper R will denote a fixed real closed field and C the algebraic closure of R. We also fix a field $\mathbb{F}$ of characteristic 0. For any closed semi-algebraic set S we will denote by $b^i(S,\mathbb{F})$ the dimension of the i-th cohomology group, $\mathrm{H}^i(S,\mathbb{F})$, and by $b(S,\mathbb{F}) = \sum_{i\geq 0} b^i(S,\mathbb{F})$. (We refer the reader to [9, Chapter 6] for the definition of homology/cohomology groups of semi-algebraic sets defined over arbitrary real closed fields, noting that they are isomorphic to the singular homology/cohomology groups in the special case of $\mathrm{R} = \mathbb{R}$.)

1.1.1. *Non-equivariant bounds.* The problem of obtaining quantitative bounds on the topology measured by the the Betti numbers of real semi-algebraic as well as complex constructible sets in terms of the degrees and the number of defining polynomials is very well studied (see for example, [8] for a survey). For semi-algebraic (respectively, constructible) subsets of R^k (respectively, C^k) defined by s polynomials of degrees bounded by d, these bounds are typically exponential in k, and polynomial (for fixed k) in s and d.

More precisely, suppose that S is a semi-algebraic (resp. constructible) subset of R^k (resp. C^k) defined by a quantifier-free formula involving s polynomials in $\mathrm{R}[X_1,\ldots,X_k]$ (resp. $\mathrm{C}[X_1,\ldots,X_k]$) of degrees bounded by d.

Theorem 1.1 (Oleĭnik and Petrovskiĭ [25], Thom [31], Milnor [24], [21])**.**

$$b(S,\mathbb{F}) \leq (skd)^{O(k)}.$$

The single exponential dependence on k of the bound in Theorem 1.1 is unavoidable. In the real case it suffices to consider the real variety

$$V_k = \{1,\ldots,d\}^k \subset \mathrm{R}^k \tag{1.1}$$

defined by the polynomial

$$F_k = \sum_{i=1}^{k}\prod_{j=1}^{d}(X_i - j)^2.$$

It is easy to see that $\deg(F_k) = 2d$, and $b_0(V_k) = d^k$.

In the complex case, it follows from a classical formula of algebraic geometry that the sum of the Betti numbers of a non-singular hypersurface $V_k \subset \mathrm{C}^k$ of degree d equals $1 + (d-1)^k = (O(d))^k$ (this is well known, but for a precise reference see for example [14, Proposition 3.21]).

1.1.2. *Motivation for studying the equivariant case.* The problem of obtaining tighter estimates on the Betti numbers of semi-algebraic sets (motivated partly by applications in other areas of mathematics and theoretical computer science) has been considered by several authors [3, 21, 7]. The algorithmic problem of designing efficient algorithms for computing these invariants has attracted attention as well [10, 4]. Most of this work has concentrated on the real semi-algebraic case, since by separating real and imaginary parts any constructible subset $S \subset \mathrm{C}^k$ can be considered as a real semi-algebraic subset of R^{2k} defined by twice as many polynomials of the same degrees as those defining S in twice as many variables. However, the complex case has also been considered separately as well [29, 33]. From the point of view of algorithmic complexity, the problem of computing the Betti numbers is provably a hard problem – and so in its full generality a polynomial time algorithm for solving this problem probably does not exist, except in special situations (see

[6, 5] for some of these exceptional cases). However, an algorithm with even a singly exponential complexity is not known for computing all the Betti numbers.

It is a (unproven) *meta-theorem* in algorithmic semi-algebraic geometry – that the worst-case topological complexity of a class of semi-algebraic sets (measured by the Betti numbers for example) serve as a rough lower bound for the complexity of algorithms for computing topological invariants or deciding topological properties of this class of sets. For example, the best complexity known for algorithms for determining whether a general semi-algebraic set is empty or connected is singly exponential, reflecting the singly exponential behavior of the topological complexity of such sets as exhibited by the example given in Example 1.19. This is true even if the degrees of the polynomials describing the given set is bounded by some constant > 2). On the other hand there are certain classes of semi-algebraic sets where the situation is better. For example, for semi-algebraic sets defined by few (i.e. any constant number of) quadratic inequalities, we have polynomial upper bounds on the Betti numbers [2], as well as algorithms with polynomial complexities for computing them [6].

It is intuitively clear that the symmetry imposes strong restrictions on the topology of such sets. Nevertheless, as shown in Example 1.19 below, the Betti numbers of such sets can be exponentially large. However, when the degrees of the defining polynomials are fixed, a polynomial bound is proved on the *equivariant* Betti numbers of such sets in [11]. (These bounds have been subsequently tightened using different methods in [13], but these tighter estimates are not relevant for the current paper.)

On the algorithmic side, an algorithm with polynomially bounded complexity is given in [12] for computing the (generalized) Euler-Poincaré characteristics of symmetric semi-algebraic sets and their quotients by the action of the symmetric group using techniques developed in [11]. An algorithm with polynomially bounded complexity for computing the Betti numbers of the quotients of such sets is given in [13].

Thus, from the point of view of the *meta-theorem* mentioned above, symmetric semi-algebraic sets pose a dilemma. On the one hand their Betti numbers can be exponentially large in the worst case, on the other hand there are reasons to believe that their topological invariants (when the degree is fixed) has some structure allowing for efficient computation. The polynomial bound on the equivariant Betti numbers proved in [11] is the first indication of such a structure.

1.2. **Summary of the main contributions.** We summarize here the main contributions of the current paper.

1. We consider real varieties and semi-algebraic sets, on which a product of symmetric groups acts linearly permuting coordinates. This setting is similar to, but more general than that considered in [11, 13] in that we let the symmetric group act by permuting blocks of variables at a time (in [11, 13] the size of such blocks was limited to one). This extra generality is essential in some applications (see below). The key technical result which makes this generality possible is Proposition 3.8, which generalizes similar results in [11, 27, 32] to blocks of sizes larger than one (see also [23] for an algorithmic application of this result). On a few occasions we will consider symmetric complex varieties as well to contrast with the real case, but the study of symmetric complex varieties is not the central theme of this paper.

2. Instead of studying the cohomology of the quotients, $V/\mathfrak{S}_k$, where V is a symmetric real variety, or a semi-algebraic set in k-dimensional affine or projective space, we study the isotypic decomposition of the $\mathfrak{S}_k$-module $\mathrm{H}^*(V,\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0.

   The Betti numbers of the quotients, i.e. $\dim_{\mathbb{F}} \mathrm{H}^*(V/\mathfrak{S}_k,\mathbb{F})$, can then be recovered from the multiplicity of the trivial representation in the isotypic decomposition of $\mathrm{H}^*(V,\mathbb{F})$ (which is also the dimension of the invariant subspace $\mathrm{H}^*(V,\mathbb{F})^{\mathfrak{S}_k}$). We prove (Theorems 2.5, 2.8, 2.9) polynomial bounds on the multiplicities of *all* irreducibles in the isotypic decomposition, thus generalizing the results in [11, 13] where polynomial bounds were proved only on the dimension of the trivial representation. Note that unlike the trivial representation which is of dimension one, the other irreducible representations of $\mathfrak{S}_{\mathbf{k}}$ can have dimensions which are exponentially large (as is unavoidable since the dimension of $\mathrm{H}^*(V,\mathbb{F})$ can be exponentially large as in Example 1.19).

   Moreover, we prove (see Remark 4.3) that the number of irreducibles that are allowed to appear is polynomially bounded (and hence a negligible fraction as $k \to \infty$) of all irreducibles (which are in bijection with the set of partitions of k). Thus, while the Betti numbers of symmetric semi-algebraic sets can be exponentially large, they can be expressed as a sum of polynomially many numbers (the dimensions of the isotypic components), and each of these numbers is a product of a multiplicity (which is polynomially bounded) and the dimension of a Specht module (which can be exponentially large, but efficiently computable due to the hook formula (cf. Theorem 3.13)).
3. In the special case of the multiplicity of the trivial representations, or equivalently the Betti numbers of the quotients, the bounds proved in the current paper still generalizes those in [11], since we consider more general actions (permuting blocks of size greater than one). This extra generality is useful in several applications, and we give two applications. In the first application, this added flexibility allows us to treat the case of symmetric complex projective varieties (Theorem 2.8), with the symmetric group permuting blocks of size 2 (the real and imaginary parts). Secondly, we are able to generalize a result in [11] on bounding the Betti numbers of the image under projection of a real variety, from the case considered in [11] where the projection was along one variable, to more general projections (Theorem 2.14). The crucial new ingredient is the generalization of the results in [11] to the case of block size greater than one.
4. Finally, we ask a question and make a conjecture suggested by the results in this paper. The question (Question 5.1) is motivated by similar representational stability results in the theory of finitely generated FI-modules [17] and asks whether the multiplicities of the irreducible corresponding to some fixed partition should ultimately stabilize to a polynomial for certain naturally defined sequences of varieties. We also make the algorithmic conjecture (Conjecture 5.5), stating that the ordinary Betti numbers of symmetric varieties defined by polynomials of fixed degrees should be computable with polynomially bounded complexity. We give some evidence in favor of these conjectures.

Remark 1.2 (Homology versus cohomology). Note that since $\mathbb{F}$ is a field of characteristic 0, $\mathrm{H}^*(S,\mathbb{F}) \cong \hom(\mathrm{H}_*(S,\mathbb{F}),\mathbb{F})$ as vector spaces. Moreover, from the basic property of $\mathfrak{S}_k$ that the conjugacy class of an element equals that of its inverse it follows that for any finite-dimensional representation W of $\mathfrak{S}_k$, $\hom(W,\mathbb{F}) \cong W$

as $\mathfrak{S}_k$-modules. Taken together this implies that $\mathrm{H}_*(S,\mathbb{F}), \mathrm{H}^*(S,\mathbb{F})$ for any symmetric semi-algebraic set $S \subset \mathrm{R}^k$, are isomorphic as $\mathfrak{S}_k$-modules, and thus for the purposes of determining the multiplicities of irreducible representations it does not matter whether we consider homology or cohomology modules.

1.3. **Notation and definitions.** In this section we introduce notation and definitions that we will use for the rest of the paper.

Notation 1.3 (Zeros)**.** For $P \in \mathrm{R}[X_1,\ldots,X_k]$ (respectively $P \in \mathrm{C}[X_1,\ldots,X_k]$) we denote by $\mathrm{Z}(P,\mathrm{R}^k)$ (respectively $\mathrm{Z}(P,\mathrm{C}^k)$) the set of zeros of P in R^k(respectively C^k). More generally, for any finite set $\mathcal{P} \subset \mathrm{R}[X_1,\ldots,X_k]$ (respectively $\mathcal{P} \subset \mathrm{C}[X_1,\ldots,X_k]$), we denote by $\mathrm{Z}(\mathcal{P},\mathrm{R}^k)$ (respectively $\mathrm{Z}(\mathcal{P},\mathrm{C}^k)$) the set of common zeros of $\mathcal{P}$ in R^k(respectively C^k). For a homogeneous polynomial $P \in \mathrm{R}[X_0,\ldots,X_{k-1}]$ (respectively $P \in \mathrm{C}[X_0,\ldots,X_{k-1}]$) we denote by $\mathrm{Z}(P,\mathbb{P}^{k-1}_{\mathrm{R}})$ (respectively $\mathrm{Z}(P,\mathbb{P}^{k-1}_{\mathrm{C}})$) the set of zeros of P in $\mathbb{P}^{k-1}_{\mathrm{R}}$(respectively $\mathbb{P}^{k-1}_{\mathrm{C}}$). And, more generally, for any finite set of homogeneous polynomials $\mathcal{P} \subset \mathrm{R}[X_0,\ldots,X_{k-1}]$ (respectively $\mathcal{P} \subset \mathrm{C}[X_0,\ldots,X_{k-1}]$), we denote by $\mathrm{Z}(\mathcal{P},\mathbb{P}^{k-1}_{\mathrm{R}})$ (respectively $\mathrm{Z}(\mathcal{P},\mathbb{P}^{k-1}_{\mathrm{C}})$) the set of common zeros of $\mathcal{P}$ in $\mathbb{P}^{k-1}_{\mathrm{R}}$ (respectively $\mathbb{P}^{k-1}_{\mathrm{C}}$).

Notation 1.4 (Sign conditions, realizations, $\mathcal{P}$- and $\mathcal{P}$-closed semi-algebraic sets)**.** For any finite family of polynomials $\mathcal{P} \subset \mathrm{R}[X_1,\ldots,X_k]$, we call an element $\sigma \in \{0,1,-1\}^{\mathcal{P}}$, a *sign condition* on $\mathcal{P}$. For any semi-algebraic set $Z \subset \mathrm{R}^k$, and a sign condition $\sigma \in \{0,1,-1\}^{\mathcal{P}}$, we denote by $\mathcal{R}(\sigma,Z)$ the semi-algebraic set defined by

$$\{\mathbf{x} \in Z \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P), P \in \mathcal{P}\},$$

and call it the *realization* of σ on Z. More generally, we call any Boolean formula Φ with atoms, $P\{=,>,<\}0, P \in \mathcal{P}$, to be a *$\mathcal{P}$-formula*. We call the realization of Φ, namely the semi-algebraic set

$$\mathcal{R}\left(\Phi,\mathrm{R}^k\right) \quad = \quad \left\{\mathbf{x} \in \mathrm{R}^k \mid \Phi(\mathbf{x})\right\}$$

a *$\mathcal{P}$-semi-algebraic set*. Finally, we call a Boolean formula without negations, and with atoms $P\{\geq,\leq\}0$, $P \in \mathcal{P}$, to be a *$\mathcal{P}$-closed formula*, and we call the realization, $\mathcal{R}\left(\Phi,\mathrm{R}^k\right)$, a *$\mathcal{P}$-closed semi-algebraic set*.

The notion of *partitions* of a given integer will play an important role in the representation theory of the symmetric group, which necessitates the following notation that we fix for the remainder of the paper.

Notation 1.5 (Partitions)**.** We denote by $\mathrm{Par}(k)$ the set of *partitions* of k, where each partition $\lambda \in \mathrm{Par}(k)$ (also denoted $\lambda \vdash k$) is a tuple $(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1$, and $\lambda_1+\lambda_2+\cdots+\lambda_\ell = k$. We call ℓ the length of the partition λ, and denote $\mathrm{length}(\lambda) = \ell$.

More generally, for any tuple $\mathbf{k} = (k_1,\ldots,k_\ell) \in \mathbb{Z}^\ell_{>0}$, we will denote by $\mathrm{Par}(\mathbf{k}) = \mathrm{Par}(k_1)\times\cdots\times\mathrm{Par}(k_\ell)$, and for each $\boldsymbol{\lambda} = (\lambda^{(1)},\ldots,\lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k})$, we denote by $\mathrm{length}(\boldsymbol{\lambda}) = \sum_{i=1}^{\ell}\mathrm{length}(\lambda^{(i)})$. We also denote for each $\mathbf{p} = (p_1,\ldots,p_\ell) \in \mathbb{N}^\ell$,

$$\begin{aligned} |\mathbf{p}| &= p_1+\cdots+p_\ell, \\ F(\mathbf{k},\mathbf{p}) &= \mathrm{card}(\{\boldsymbol{\pi} = (\pi^{(1)},\ldots,\pi^{(\ell)}) \mid \mathrm{length}(\pi^{(i)}) = p_i, 1 \leq i \leq \ell\}). \end{aligned}$$

Notation 1.6 (Transpose of a partition and partitions of bounded lengths)**.** For a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell) \vdash k$, we will denote by $\tilde{\lambda}$ the *transpose* of λ. More precisely, $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\tilde{\ell}})$, where $\tilde{\lambda}_j = \mathrm{card}(\{i \mid \lambda_i \geq j\})$. For $k, d \geq 0$, we denote

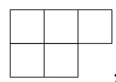
$$\mathrm{Par}(k,d) := \{\lambda \in \mathrm{Par}(k) \mid \mathrm{length}(\lambda) \leq d\}.$$

More generally, for $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{d} = (d_1, \ldots, d_\ell)$ we denote

$$\mathrm{Par}(\mathbf{k}, \mathbf{d}) := \{\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \mid \lambda^{(i)} \in \mathrm{Par}(k_i), \mathrm{length}(\lambda^{(i)}) \leq d_i, 1 \leq i \leq \ell\}.$$

When $\mathbf{d} = (d, \ldots, d)$, we will also use $\mathrm{Par}(\mathbf{k}, d)$ to denote $\mathrm{Par}(\mathbf{k}, \mathbf{d})$.

Definition 1.7 (Young diagrams)**.** Partitions are often identified with Young diagrams. We follow the English convention and associate the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with the Young diagram with its i-th row consisting of λ_i boxes. Thus, the Young diagram corresponding to the partition $\lambda = (3, 2)$ is



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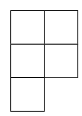
the Young diagram associated to its transpose, $\tilde{\lambda} = (2, 2, 1)$, is



(note that the Young diagram of $\tilde{\lambda}$ is obtained by reflecting the Young diagram of λ about its diagonal).

Definition 1.8 (Dominance order)**.** For any two partitions $\mu = (\mu_1, \mu_2, \ldots), \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathrm{Par}(k)$, we say that $\mu \trianglerighteq \lambda$, if for each $i \geq 0$, $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$. This is a partial order on $\mathrm{Par}(k)$. More generally, for $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{Z}^\ell_{>0}$, and $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)}), \boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k})$, we denote $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ if and only if $\mu^{(i)} \trianglerighteq \lambda^{(i)}$ for each $i, 1 \leq i \leq \ell$.

Notation 1.9 (Products of symmetric groups)**.** For each $k \in \mathbb{N}$, we denote by $\mathfrak{S}_k$ the symmetric group on k letters (or equivalently the Coxeter group A_{k-1}). For $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{Z}^\ell_{>0}$ we denote by $\mathfrak{S}_{\mathbf{k}}$ the product group $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_\ell}$, and we will usually denote $k = |\mathbf{k}| = \sum_{i=1}^{\ell} k_i$.

We introduce more notation which is used in the representation theory of the symmetric group.

Notation 1.10 (Young subgroups of product of symmetric groups)**.** For

$$\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathrm{Par}(k),$$

we will denote by $\mathfrak{S}_{\boldsymbol{\lambda}} \cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_d}$ the subgroup of $\mathfrak{S}_k$ which is the direct product of the subgroups $G_i \cong \mathfrak{S}_{\lambda_i}$, where G_i is the subgroup of permutations of $[1, k]$ fixing $[1, k] \setminus [\lambda_1 + \cdots + \lambda_{i-1} + 1, \lambda_1 + \cdots + \lambda_i]$.

More generally, for $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{Z}^\ell_{>0}$, $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k})$, we denote by $\mathfrak{S}_{\boldsymbol{\lambda}}$ the subgroup $\mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(\ell)}}$ of $\mathfrak{S}_{\mathbf{k}}$, where for $1 \leq j \leq \ell$, $\mathfrak{S}_{\lambda^{(j)}}$ is the subgroup of $\mathfrak{S}_{k_j}$ defined above.

Definition 1.11 (Young module)**.** For $\lambda \vdash k$, we will denote

$$M^\lambda = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}(\mathbf{1}_{\mathfrak{S}_\lambda})$$

(where $\mathbf{1}_{\mathfrak{S}_\lambda}$ denotes the trivial one-dimensional representation of $\mathfrak{S}_\lambda$).

Notation 1.12 (Irreducible representations (Specht modules) of symmetric groups)**.** For $\lambda \in \mathrm{Par}(k)$, we will denote by $\mathbb{S}^\lambda$ the irreducible representation (over the field $\mathbb{F}$) of $\mathfrak{S}_k$ corresponding to λ (see [26] for definition). Note that $\mathbb{S}^{(k)}$ is the trivial representation, which corresponding to the partition $(k) \in \mathrm{Par}(k)$. This representation will also be denoted by $\mathbf{1}_{\mathfrak{S}_k}$. Further, $\mathbb{S}^{(1^k)}$ is the sign representation, which we will also denote by $\mathbf{sign}_k$. It is a well known fact that for any $\lambda \in \mathrm{Par}(k)$,

$$\mathbb{S}^{(\tilde{\lambda})} \cong \mathbb{S}^{(\lambda)} \otimes \mathbf{sign}_k.$$

For $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{Z}^\ell_{>0}$, $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k})$, we denote by $\mathbb{S}^{\boldsymbol{\lambda}}$ the irreducible representation $\mathbb{S}^{\lambda^{(1)}} \boxtimes \cdots \boxtimes \mathbb{S}^{\lambda^{(\ell)}}$ of $\mathfrak{S}_{\mathbf{k}}$.

The Kostka numbers which are defined below serve as an important link between combinatorics and the representation theory of the symmetric group.

Definition 1.13 (Kostka numbers)**.** Let $\mu \vdash k$. A *semi-standard* Young tableau of shape μ is a filling of the boxes in the Young diagram (cf. Definition 1.7) associated to μ with integers between 1 and k, in such a way, that the rows are weakly increasing and the columns are strictly increasing. Given a semi-standard Young tableau T, the content of T is the array $\lambda := (\lambda_1, \ldots, \lambda_\ell)$, where λ_i equals the number of times i appears in T. (The reader unfamiliar with these notations can find detailed information on semi-standard Young tableaux, and also their shape and weight for example, in [26]).

For a pair $\lambda, \mu \vdash k$ one denotes by $K(\mu, \lambda)$ the number of semi-standard Young tableaux of shape μ and content λ.

The use of the Kostka numbers is in particular due to the following fact, which is a basic statement in the representation theory of the symmetric group. (see for example [16, Theorem 3.6.11] or [26, page 541, §7.3]).

Proposition 1.14 (Young's rule)**.** *Let $k \in \mathbb{N}$, and $\lambda \in \mathrm{Par}(k)$. Then,*

$$\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k} \left(\mathbb{S}^{(\lambda_1)} \boxtimes \cdots \boxtimes \mathbb{S}^{(\lambda_{\mathrm{length}(\lambda)})}\right) \cong \bigoplus_{\mu \trianglerighteq \lambda} K(\mu, \lambda) \mathbb{S}^\mu.$$

The following proposition is classical and is a direct consequence of Schur's Lemma (see for example Lemma 1.7 and Proposition 1.8 in [20]). Note that we can extend the statement - given in [20] only for complex representations - to finite dimensional representations over any field of characteristic 0, since the Specht modules are defined over $\mathbb{Q}$.

Proposition 1.15 (Isotypic decomposition)**.** *Let V be a finite dimensional $\mathfrak{S}_k$ representation (or equivalently a $\mathfrak{S}_k$-module) defined over $\mathbb{F}$. Then, for every $\lambda \vdash k$ there exists unique $m_\lambda \in \mathbb{N}$ and $\mathfrak{S}_k$-submodules V_λ of V, such that*

$$V = \bigoplus_{\lambda \vdash k} V_\lambda, \tag{1.2}$$

and each V_λ is isomorphic as a $\mathfrak{S}_k$-module to

$$\bigoplus_{i=1}^{m_\lambda} \mathbb{S}^\lambda.$$

More generally, if $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{Z}^\ell_{>0}$, and V a finite dimensional $\mathfrak{S}_{\mathbf{k}}$ representation defined over $\mathbb{F}$, then for every $\boldsymbol{\lambda} \in \mathrm{Par}(\mathbf{k})$ there exist sunique $m_{\boldsymbol{\lambda}} \in \mathbb{N}$

and $\mathfrak{S}_{\mathbf{k}}$*-submodules* $V_{\boldsymbol{\lambda}}$ *of* V*, such that*

$$V = \bigoplus_{\boldsymbol{\lambda}\in \mathrm{Par}(\mathbf{k})} V_{\boldsymbol{\lambda}}, \tag{1.3}$$

and each $V_{\boldsymbol{\lambda}}$ *is isomorphic as a* $\mathfrak{S}_{\mathbf{k}}$*-module to*

$$\bigoplus_{i=1}^{m_{\boldsymbol{\lambda}}} \mathbb{S}^{\boldsymbol{\lambda}}.$$

(The submodule V_λ *(respectively,* $V_{\boldsymbol{\lambda}}$*) is called the* isotypic component *of* V *corresponding to* λ *(respectively,* $\boldsymbol{\lambda}$*), and the decompositions* (1.2) *and* (1.3) *are called the* isotypic decomposition *of the module* V*.)*

Definition 1.16 ($\mathfrak{S}_{\mathbf{k}}$-symmetric polynomials)**.** Let $\mathbb{K}$ be the field R or C. Suppose that $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell_{>0}$, and let $P \in \mathbb{K}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$ where for $1 \leq h \leq \ell$, $\mathbf{X}^{(h)} = \left(X^{(h)}_{i,j}\right)_{1\leq i\leq k_h, 1\leq j\leq m_h}$.

The group $\mathfrak{S}_{\mathbf{k}}$, acts on $\mathbb{K}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$ by permuting for each $i, 1 \leq i \leq \ell$, the rows of $\mathbf{X}^{(h)}$ by the group $\mathfrak{S}_{k_h}$. For $\boldsymbol{\pi} \in \mathfrak{S}_{\mathbf{k}}$, and $P \in \mathbb{K}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$, we denote the by $\boldsymbol{\pi} \cdot P$ the image of P under $\boldsymbol{\pi}$. We say that P *is* $\mathfrak{S}_{\mathbf{k}}$*-symmetric* if it is invariant under the action of $\mathfrak{S}_{\mathbf{k}}$, i.e. if $\boldsymbol{\pi} \cdot P = P$ for every $\boldsymbol{\pi} \in \mathfrak{S}_{\mathbf{k}}$.

For $\mathbf{d} = (d_1, \ldots, d_\ell) \in \mathbb{Z}^\ell_{>0}$, we will denote by $\mathbb{K}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]^{\mathfrak{S}_{\mathbf{k}}}_{\leq \mathbf{d}}$, the finite dimensional subspace of $\mathbb{K}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$ consisting of $\mathfrak{S}_{\mathbf{k}}$-symmetric polynomials whose degree in $\mathbf{X}^{(i)}$ is bounded by d_i for $1 \leq i \leq \ell$.

Similarly, we say that a subset $S \subset \mathbb{K}^K, K = \sum_{1\leq i\leq \ell} k_i m_i$, is $\mathfrak{S}_{\mathbf{k}}$-symmetric if it is stable under the above action of $\mathfrak{S}_{\mathbf{k}}$.

When $\ell = 1, m_1 = 1$, and $K = k_1 m_1 = k$, the action defined above is the usual action of $\mathfrak{S}_k$ on $\mathbb{K}^k$ permuting coordinates.

Remark 1.17*.* Note in case $\mathbb{K} = \mathrm{C}$, the action of $\mathfrak{S}_{\mathbf{k}}$ on C^K defined above in Definition 1.16 can also be seen as the action of $\mathfrak{S}_{\mathbf{k}}$ on R^{2K} (considering $\mathrm{C} = \mathrm{R} \oplus i\mathrm{R}$), replacing $\mathbf{m}$ by $2\mathbf{m}$.

Remark 1.18*.* Many of the results proved in this paper hold for the action of a product of symmetric groups, $\mathfrak{S}_{\mathbf{k}}$, acting on R^K or C^K (as in Definition 1.16 above) permuting *blocks* of variables, where the block sizes are allowed to be greater than one, and unless otherwise stated this is the case. Rarely, in some special situations we state results that hold only when the block sizes equal one and we make a remark preceding each such result whenever this is the case.

1.4. **Basic example.** Before proceeding further, we discuss an example which is our guiding example for the rest of the paper. While explaining the example we will assume a certain familiarity with the representation theory of symmetric groups. For the convenience of the reader we have included all the facts from the representation theory of symmetric groups that we need in §3.5 (and which the reader can consult if needed).

Example 1.19 (Real affine case)**.** Let

$$F_k = \sum_{i=1}^{k} X_i^2 (X_i - 1)^2 - \varepsilon,$$

and

$$V_k = \mathrm{Z}(F_k, \mathrm{R}^k). \tag{1.4}$$

Then, for all $\varepsilon, 0 < \varepsilon \ll 1$, V_k is a closed and bounded non-singular hypersurface in R^k, (in fact also in $\mathbb{P}^k_{\mathrm{R}}$), the semi-algebraic set S_k defined by $F_k \leq 0$ is homotopy equivalent to the finite set of points $\{0,1\}^k$, and is bounded by V_k.

Clearly, $b_0(V_k, \mathbb{F}) = 2^k$, and it follows from Poincaré duality applied to V_k that $b_{k-1}(V_k, \mathbb{F}) = 2^k$ as well. It also follows from Alexander-Lefshetz duality that $\mathrm{H}^i(V_k, \mathbb{F}) = 0$ for $0 < i < k-1$.

The real algebraic variety V_k is symmetric under the standard action of the symmetric group $\mathfrak{S}_k$ on R^k permuting the coordinates. This action induces an $\mathfrak{S}_k$-module structure on $\mathrm{H}^*(V_k, \mathbb{F})$, and it is interesting to study the isotypic decomposition (cf. Proposition) of this representation into its isotypic components corresponding to the various irreducible representations of $\mathfrak{S}_k$, namely the Specht modules $\mathbb{S}^\lambda$ indexed by different partitions $\lambda \vdash k$ (see for example [26] for the definition of Specht modules).

We now describe this decomposition.

$$\mathrm{H}^0(V_k, \mathbb{F}) \cong \bigoplus_{0 \leq i \leq k} \mathrm{H}^0(V_{k,i}, \mathbb{F}),$$

where for $0 \leq i \leq k$, $V_{k,i}$ is the $\mathfrak{S}_k$-orbit of the connected component of V_k infinitesimally close (as a function of ε) to the point $\mathbf{x}^i = (\underbrace{0,\ldots,0}_{i}, \underbrace{1,\ldots,1}_{k-i})$, and $\mathrm{H}^0(V_{k,i}, \mathbb{F})$ is a sub-representation of $\mathrm{H}^0(V_k, \mathbb{F})$.

It is also clear that the isotropy subgroup of the class in $\mathrm{H}^0(V_k, \mathbb{F})$ corresponding to $V_{k,i}$ is isomorphic to $\mathfrak{S}_i \times \mathfrak{S}_{k-i}$, and hence,

$$\begin{aligned}
\mathrm{H}^0(V_{k,i}, \mathbb{F}) &\cong \mathrm{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_{k-i}}^{\mathfrak{S}_k} (\mathbb{S}^{(i)} \boxtimes \mathbb{S}^{(k-i)}) \\
&\cong M^{(i,k-i)} \text{ if } i \geq k-i, \\
&\cong M^{(k-i,i)} \text{ otherwise.}
\end{aligned}$$

where for any $\lambda \vdash k$, we denote by M^λ the Young module corresponding to λ (see Definition 1.11).

Also, observe that $\mathrm{H}^0(V_{k,i}, \mathbb{F})$ and $\mathrm{H}^0(V_{k,k-i}, \mathbb{F})$ are isomorphic as $\mathfrak{S}_k$-modules. In the following, for partitions $\mu, \lambda \vdash k$, we will denote by $K(\mu, \lambda)$ the corresponding *Kostka number* (see Definition 1.13 below). For this example, it is sufficient to observe that if $\mu \trianglerighteq \lambda$ (see Definition 1.8 for the definition of the *dominance order* $\trianglerighteq$ on the set of partitions), and if μ has at most 2 rows, then $K(\mu, \lambda) = 1$. It now follows from Proposition 1.14 that for k odd,

$$\begin{aligned}
\mathrm{H}^0(V_k,\mathbb{F}) &\cong \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\leq 2}} (M^\lambda \oplus M^\lambda)\\
&\cong \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\leq 2}} \bigoplus_{\mu\trianglerighteq\lambda} 2K(\mu,\lambda)\mathbb{S}^\mu\\
&\cong \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\leq 2}} \bigoplus_{\mu\trianglerighteq\lambda} 2\mathbb{S}^\mu\\
&\cong \bigoplus_{\substack{\mu\vdash k\\ \ell(\mu)\leq 2}} m_\mu\mathbb{S}^\mu,
\end{aligned}$$

where for each $\mu=(\mu_1,\mu_2)\vdash k$,

$$\begin{aligned}
m_\mu &= 2(\mu_1-\lfloor k/2\rfloor)\\
&= 2\mu_1-k+1.
\end{aligned}$$

For k even we have,

$$\begin{aligned}
\mathrm{H}^0(V_k,\mathbb{F}) &\cong \left(\bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\leq 2\\ \lambda\neq(k/2,k/2)}} (M^\lambda \oplus M^\lambda)\right)\bigoplus M^{(k/2,k/2)}\\
&\cong \left(\bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\leq 2\\ \lambda\neq(k/2,k/2)}} \bigoplus_{\mu\trianglerighteq\lambda} 2K(\mu,\lambda)\mathbb{S}^\mu\right)\oplus\left(\bigoplus_{\mu\trianglerighteq(k/2,k/2)} K(\mu,(k/2,k/2))\mathbb{S}^\mu\right)\\
&\cong \bigoplus_{\substack{\mu\vdash k\\ \ell(\mu)\leq 2}} m_\mu\mathbb{S}^\mu,
\end{aligned}$$

where for each $\mu=(\mu_1,\mu_2)\vdash k$,

$$\begin{aligned}
m_\mu &= 2(\mu_1-k/2)+1\\
&= 2\mu_1-k+1.
\end{aligned}$$

We deduce for all k,

$$\begin{aligned}
m_\mu &= 2\mu_1-k+1\\
&\leq k+1.
\end{aligned}$$

For $\mu=(\mu_1,\mu_2)\vdash k$, by the hook-length formula (Eqn. (3.1)) we have,

$$\dim\ \mathbb{S}^\mu = \frac{k!\,(\mu_1-\mu_2+1)}{(\mu_1+1)!\mu_2!}. \tag{1.5}$$

This completes the description of the isotypic decomposition of $\mathrm{H}^0(V_k,\mathbb{F})$.

In particular for $k=2,3$ we have:

$$\begin{aligned}\mathrm{H}^0(V_2,\mathbb{F}) &\cong 3\mathbb{S}^{(2)} \oplus \mathbb{S}^{(1,1)},\\ \mathrm{H}^0(V_3,\mathbb{F}) &\cong 4\mathbb{S}^{(3)} \oplus 2\mathbb{S}^{(2,1)}.\end{aligned}$$

The isotypic decomposition of $\mathrm{H}^{k-1}(V_k,\mathbb{F})$ requires one further ingredient – namely, an $\mathfrak{S}_k$-equivariant version of the classical Poincaré duality theorem for oriented manifolds. We include a proof of this result (Theorem 3.23) in §3.6.

We note that V_k is a closed and bounded real orientable manifold, by Poincaré duality theorem there exists an isomorphism between $\mathrm{H}^0(V,\mathbb{F})$ and $\mathrm{H}^{k-1}(V,\mathbb{F})$. This isomorphism is not necessarily a $\mathfrak{S}_k$-module isomorphism. However, it follows from Theorem 3.23 that the isotypic representation of $\mathrm{H}^{k-1}(V_k,\mathbb{F})$ is isomorphic (as an $\mathfrak{S}_k$-module) to $\mathrm{H}^0(V_k,\mathbb{F}) \otimes \mathbf{sign}_k$.

Thus, denoting for each $\lambda \vdash k$, the *transpose* of the partition λ by $\tilde{\lambda}$,

$$\mathrm{H}^{k-1}(V_k,\mathbb{F}) \cong \bigoplus_{\substack{\mu\vdash k\\ \ell(\mu)\leq 2}} m_\mu \mathbb{S}^{\tilde{\mu}},$$

where for each $\mu=(\mu_1,\mu_2)\vdash k$, m_μ is defined above in (1.5). In particular for $k=2,3$ we have:

$$\begin{aligned}\mathrm{H}^1(V_2,\mathbb{F}) &\cong 3\mathbb{S}^{(1,1)} \oplus \mathbb{S}^{(2)},\\ \mathrm{H}^2(V_3,\mathbb{F}) &\cong 4\mathbb{S}^{(1,1,1)} \oplus 2\mathbb{S}^{(2,1)}.\end{aligned}$$

Notice that the multiplicity m_{1^k} of the Specht module $\mathbb{S}^{1^k}=\mathbf{sign}_k$ in $\mathrm{H}^0(V_k,\mathbb{F})$ is equal to 0 for $k>2$. This implies that the multiplicity of the trivial representation $\mathbb{S}^{(k)}$ is equal to 0 in $\mathrm{H}^{k-1}(V_k,\mathbb{F})$, and thus $\mathrm{H}^{k-1}_{\mathfrak{S}_k}(V_k,\mathbb{F})=0$ as well (for $k>2$).

Also, notice that the multiplicity of each Specht-module, $\mathbb{S}^\mu, \mu\vdash k$, in the isotypic decomposition of $\mathrm{H}^*(V_k,\mathbb{F})$ is bounded polynomially (in fact, linearly) in k, but the dimension of $\mathrm{H}^*(V_k,\mathbb{F})$ itself is exponentially large in k.

Note that since $\dim \mathrm{H}^0(V_k,\mathbb{F})=2^k$, we obtain as a consequence (from (1.5) and (1.5)) the identity

$$k!\left(\sum_{\substack{\mu_1\geq\mu_2\geq 0\\ \mu_1+\mu_2=k}} \frac{(\mu_1-\mu_2+1)^2}{(\mu_1+1)!\mu_2!}\right) = 2^k$$

(which can also be proved easily by more elementary means).

Example 1.20 (Projective case)**.** Let

$$P=\sum_{0\leq i<j\leq k-1}(X_i^2-X_j^2)^2,$$

and let $W_k=\mathrm{Z}(P,\mathbb{P}^{k-1}_{\mathrm{R}})$. Then,

$$W_k=\{(x_0:\cdots:x_{k-1}) \mid x_i=\pm 1, 0\leq i\leq k-1\},$$

and is symmetric under the action of $\mathfrak{S}_k$ on $\mathbb{P}^{k-1}_{\mathrm{R}}$ permuting the homogeneous coordinates.

It is clear that

$$\mathrm{H}^0(W_k,\mathbb{F})\cong \mathrm{H}^0(V_k,\mathbb{F}),$$

where V_k is the real affine variety defined in (1.4), and the stated isomorphism is an isomorphism of $\mathfrak{S}_k$-modules.

1.5. **Equivariant cohomology.** We recall also the definition of *equivariant cohomology groups* of a G-space for an arbitrary compact Lie group G. For G any compact Lie group, there exists a *universal principal G-space*, denoted EG, which is contractible, and on which the group G acts freely on the right. The *classifying space* BG, is the orbit space of this action, i.e. $BG = EG/G$.

Definition 1.21 (Equivariant cohomology)**.** (Borel construction) Let X be a space with a left action of the group G. Then, G acts diagonally on the space $EG \times X$ by $g(z, x) = (z \cdot g^{-1}, g \cdot x)$. For any field of coefficients $\mathbb{F}$, the *G-equivariant cohomology groups of X* with coefficients in $\mathbb{F}$, denoted by $\mathrm{H}^*_G(X, \mathbb{F})$, is defined by $\mathrm{H}^*_G(X, \mathbb{F}) = \mathrm{H}^*(EG \times X/G, \mathbb{F})$.

In the situation of interest in the current paper, where $G = \mathfrak{S}_{\mathbf{k}}$ acting on a $\mathfrak{S}_{\mathbf{k}}$-symmetric semi-algebraic subset $S \subset \mathrm{R}^k$, and $\mathbb{F}$ is a field with characteristic equal to 0, the following isomorphisms follow directly from the Borel-Serre spectral sequence, and the fact that finite groups have trivial cohomology in the case the field of coefficients $\mathbb{F}$ has characteristic 0:

$$\mathrm{H}^*(S/\mathfrak{S}_{\mathbf{k}}, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^*_{\mathfrak{S}_{\mathbf{k}}}(X, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^*(S, \mathbb{F})^{\mathfrak{S}_{\mathbf{k}}}. \tag{1.6}$$

1.6. **Prior work.** The problem of bounding the equivariant Betti numbers of symmetric semi-algebraic subsets of R^k was investigated in [11]. We recall in this section a few results from [11] that are generalized in the current paper.

We recall some definitions and notation from [11].

Notation 1.22 (Equivariant Betti numbers)**.** For any $\mathfrak{S}_{\mathbf{k}}$ symmetric semi-algebraic subset $S \subset \mathrm{R}^k$ with $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell$, with $k = \sum_{i=1}^{\ell} k_i$, and any field $\mathbb{F}$, we denote

$$\begin{aligned} b^i_{\mathfrak{S}_{\mathbf{k}}}(S, \mathbb{F}) &= b_i(S/\mathfrak{S}_{\mathbf{k}}, \mathbb{F}), \\ b_{\mathfrak{S}_{\mathbf{k}}}(S, \mathbb{F}) &= \sum_{i \geq 0} b^i_{\mathfrak{S}_{\mathbf{k}}}(S, \mathbb{F}). \end{aligned}$$

The following theorem is proved in [11]. Note that the block sizes are equal to one in the following theorem.

Theorem 1.23. [11, Theorem 6] *Let $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell$,with $k = \sum_{i=1}^{\ell} k_i$. Suppose that $P \in \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$, where each $\mathbf{X}^{(i)}$ is a block of k_i variables, is a non-negative polynomial, such that $V = \mathrm{Z}(P, \mathrm{R}^k)$ is stable under the action of $\mathfrak{S}_{\mathbf{k}}$ permuting each block $\mathbf{X}^{(i)}$ of k_i coordinates. Let $\deg_{\mathbf{X}^{(i)}}(P) \leq d$ for $1 \leq i \leq \ell$. Then, for any field of coefficients $\mathbb{F}$, the sum of the equivariant Betti numbers can be bounded by*

$$b(V/\mathfrak{S}_{\mathbf{k}}, \mathbb{F}) \leq \sum_{\mathbf{p}=(p_1,\ldots,p_\ell), 1 \leq p_i \leq \min(2d, k_i)} F(\mathbf{k}, \mathbf{p}) d(2d-1)^{|\mathbf{p}|+1}$$

(where $F(\mathbf{k}, \mathbf{p})$ is defined in Notation 1.5). If for each $i, 1 \leq i \leq \ell$, $2d \leq k_i$, then

$$b(V/\mathfrak{S}_{\mathbf{k}}, \mathbb{F}) \leq (k_1 \cdots k_\ell)^{2d} (O(d))^{2\ell d+1}.$$

More generally, the following bound holds for symmetric semi-algebraic sets.

Note that the block sizes are equal to one in the following theorem.

Theorem 1.24. [11, Theorem 7] *Let $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell$, with $k = \sum_{i=1}^{\ell} k_i$, and let $\mathcal{P} \subset \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$ be a finite set of polynomials, where each $\mathbf{X}^{(i)}$ is a block of $k^{(i)}$ variables, and such that each $P \in \mathcal{P}$ is symmetric in each block of variables $\mathbf{X}^{(i)}$. Let $S \subset \mathrm{R}^k$ be a $\mathcal{P}$-closed-semi-algebraic set. Suppose that $\deg(P) \leq d$ for each $P \in \mathcal{P}$, $\mathrm{card}(\mathcal{P}) = s$, and let $D = D(\mathbf{k}, d) = \sum_{i=1}^{\ell} \min(k_i, 5d)$. Then, for any field of coefficients $\mathbb{F}$, the sum of the equivariant Betti numbers can be bounded by*

$$b(S/\mathfrak{S}_{\mathbf{k}}, \mathbb{F}) \quad \leq \quad \sum_{i=0}^{D-1} \sum_{j=1}^{D-i} \binom{2s+1}{j} 6^j G(\mathbf{k}, 2d)$$

where

$$G(\mathbf{k}, d) \quad = \quad \sum_{\mathbf{p}=(p_1,\ldots,p_\ell), 1 \leq p_i \leq \min(2d, k_i)} F(\mathbf{k}, \mathbf{p}) d(2d-1)^{|\mathbf{p}|+1}$$

(and $F(\mathbf{k}, \mathbf{p})$ is defined in Notation 1.5).

Remark 1.25. In the particular case, when $\ell = 1$, $d = O(1)$, the bound in Theorem 1.24 takes the following asymptotic (for $k \gg 1$) form.

$$b(S/\mathfrak{S}_k, \mathbb{F}) \quad \leq \quad O(s^{5d} k^{4d-1}).$$

The rest of the paper is organized as follows. In §2 we state the new results proved in this paper. In §3 we prove or recall certain preliminary facts that will be needed in the proofs of the main theorems. In §4 we prove the main theorems, and finally in §5 we end with some open problems.

2. Main Results

In view of the isomorphisms (1.6), Theorem 1.23 (respectively, Theorem 1.24) gives a bound (which is polynomial for fixed d) on the multiplicity of the trivial representation in the $\mathfrak{S}_{\mathbf{k}}$-module $\mathrm{H}^*(V, \mathbb{F})$ (respectively, $\mathrm{H}^*(S, \mathbb{F})$). In the current paper we generalize both Theorems 1.23 and 1.24 by proving a polynomial bound on the multiplicities of every irreducible representation appearing in the isotypic decomposition of $\mathrm{H}^*(V, \mathbb{F})$ and $\mathrm{H}^*(S, \mathbb{F})$. Note that as Example 1.19 shows, the dimensions of $\mathrm{H}^*(V, \mathbb{F})$, where V is a symmetric real variety in R^k defined by polynomials of degree bounded by d can be exponentially large in k. We also extend these basic results in several directions – including more general actions of the symmetric group, and as a particular case symmetric varieties in C^k, as well as symmetric projective varieties.

2.1. Affine algebraic case.

We first state our results for symmetric real algebraic subvarieties of real affine space.

Notation 2.1. Let $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}_{>0}^\ell$, and $K = \sum_{i=1}^{\ell} k_i m_i$. For any $\mathfrak{S}_{\mathbf{k}}$-symmetric semi-algebraic subset $S \subset \mathrm{R}^K$, any field $\mathbb{F}$, and $\boldsymbol{\lambda} \in \mathrm{Par}(\mathbf{k})$, we denote

$$\begin{aligned} m_{i,\boldsymbol{\lambda}}(S, \mathbb{F}) &= \dim_{\mathbb{F}} \hom_{\mathfrak{S}_{\mathbf{k}}}(\mathbb{S}^{\boldsymbol{\lambda}}, \mathrm{H}^i(S, \mathbb{F})) \\ &= \mathrm{mult}(\mathbb{S}^{\boldsymbol{\lambda}}, \mathrm{H}^i(S, \mathbb{F})), \\ m_{\boldsymbol{\lambda}}(S, \mathbb{F}) &= \sum_i m_{i,\boldsymbol{\lambda}}(S, \mathbb{F}). \end{aligned}$$

Note that in the particular case when $\boldsymbol{\lambda} = ((k_1),\ldots,(k_\ell))$ (i.e. when $\mathbb{S}^{\boldsymbol{\lambda}}$ is the trivial representation of $\mathfrak{S}_{\mathbf{k}}$),

$$\begin{aligned} m_{i,\boldsymbol{\lambda}}(S,\mathbb{F}) &= b_i(S/\mathfrak{S}_{\mathbf{k}},\mathbb{F}),\\ m_{\boldsymbol{\lambda}}(S,\mathbb{F}) &= b(S/\mathfrak{S}_{\mathbf{k}},\mathbb{F}). \end{aligned}$$

Remark 2.2. Note that it follows from Proposition 1.19 that the numbers $m_{i,\boldsymbol{\lambda}}$ are well defined.

Notation 2.3 (Ordered tuple of degrees raised to some power)**.** For

$$\mathbf{d} = (d_1,\ldots,d_\ell), \mathbf{m} = (m_1,\ldots,m_\ell) \in \mathbb{Z}^\ell_{>0},$$

we denote by

$$\mathbf{d}^{\mathbf{m}} = (d_1^{m_1},\ldots,d_\ell^{m_\ell}).$$

Definition 2.4 (Rank of a partition)**.** For any partition $\mu \in \mathrm{Par}(k)$, we denote by $\mathrm{rank}(\mu)$ the length of the main diagonal in the Young diagram (cf. Definition 1.7) of μ. Equivalently, $\mathrm{rank}(\mu)$ is the side length of the largest square with a vertex at the origin (also called the *Durfee square* of μ) that fits inside the Young diagram of μ (see for example [30, Page 65]).

More generally, for $\mathbf{k} \in \mathbb{Z}^\ell_{>0}$, and $\boldsymbol{\mu} = (\mu^{(1)},\ldots,\mu^{(\ell)}) \in \mathrm{Par}(\mathbf{k})$, we define

$$\mathrm{rank}(\mathbf{k}) = (\mathrm{rank}(\mu^{(1)}),\ldots,\mathrm{rank}(\mu^{(\ell)})).$$

We are now ready to state the main theorem of this section, which gives restrictions on the irreducible representation contributing to the isotypic decomposition of $\mathrm{H}^*(V,\mathbb{F})$ and further bounds the multiplicities $m_{\boldsymbol{\mu}}$ (see Notation 2.1).

Theorem 2.5. *Let* $\mathbf{k} = (k_1,\ldots,k_\ell), \mathbf{m} = (m_1,\ldots,m_\ell), \mathbf{d} = (d,\ldots,d) \in \mathbb{Z}^\ell_{>0}$, *and* $K = \sum_{i=1}^\ell k_i m_i$. *Let* $P \in \mathrm{R}[\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(\ell)}]^{\mathfrak{S}_{\mathbf{k}}}_{\le \mathbf{d}}$ *be a non-negative polynomial and let* $V = \mathrm{Z}(P,\mathrm{R}^K)$.

1. *Then, for all partitions* $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$ *the assumption* $m_{\boldsymbol{\mu}}(V,\mathbb{F}) > 0$ *implies that*
$$\mathrm{rank}(\boldsymbol{\mu}) \le (2\mathbf{d})^{\mathbf{m}}.$$
2. *Further, the following bound in the multiplicities in the isotypic decomposition holds:*
$$\begin{aligned} m_{\boldsymbol{\mu}}(V,\mathbb{F}) &\le \prod_{1\le i\le \ell}\left(\sum_{j=0}^{(2d)^{m_i}} k_i^{O(j^2)}\binom{k_i-1}{j-1}(O(d))^{m_i j}\right)\\ &\le \prod_{1\le i\le \ell}\left(k_i^{O((2d)^{2m_i})}(O(d))^{m_i(2d)^{m_i}}\right). \end{aligned}$$

In the particular case, when $\ell = 1$, *and* $d_1 = d$ *and* $m_1 = m$ *are fixed, the above bound is polynomial in* $k_1 = k$.

Remark 2.6. Note that the restriction on the Specht modules that are allowed to appear in the cohomology module $\mathrm{H}^*(V,\mathbb{F})$ that are implied by Part (1) of Theorem 2.5 *does not* follow only from dimension considerations, and the Oleĭnik-Petrovskiĭ-Thom-Milnor bound (Theorem 1.1) on $b(V,\mathbb{F})$.

For example, let $\ell = 1, m_1 = 1, k_1 = k = 2^p - 1$, and let $\lambda \vdash k$ be the partition $(2^{p-1}, 2^{p-2},\ldots,1)$. In this case:

$$\begin{aligned}\dim_{\mathbb{F}} \mathbb{S}^{\lambda} &\leq \dim_{\mathbb{F}} \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k} (\text{ since } K(\lambda,\lambda) = 1 (\text{Definition 1.13 and Proposition 1.14}) \\ &= \binom{k}{2^{p-1},\ldots,1} \\ &\leq O(1)^k \text{ using Stirling's approximation.}\end{aligned}$$

Thus, if V is defined by a polynomial of degree bounded by d, and k is large enough, $\mathbb{S}^\lambda$ is not ruled out from appearing with positive multiplicity in $\mathrm{H}^*(V,\mathbb{F})$ just on the basis of the upper bound in Theorem 1.1. On the other hand, it follows from Part (1) of Theorem 2.5 that for all k large enough, and fixed d,

$$m_\lambda(V,\mathbb{F}) = 0.$$

We will also need the following somewhat special form of Theorem 2.5, which yields a bound on the multiplicity of the trivial representation in $\mathrm{H}^*(V,\mathbb{F})$ in a special case. It follows easily from the proof of Theorem 2.5 and will be used later in the proof of Theorem 2.14. Following the same notation as above:

Theorem 2.7. *Suppose that* $\mathbf{k} = (\underbrace{1,\ldots 1}_{\ell-1}, k)$*,* $\mathbf{m} = (\underbrace{1,\ldots 1}_{\ell-1}, m)$*, and* $(2d)^m \leq k$*. We consider* $\boldsymbol{\mu} = ((1),\ldots,(1),(k))$ *(i.e.* $\mathbb{S}^{\boldsymbol{\mu}}$ *is the trivial representation). Then*

$$\begin{aligned} m_{\boldsymbol{\mu}}(V,\mathbb{F}) &= b(V/\mathfrak{S}_{\mathbf{k}},\mathbb{F}) \\ &\leq k^{(2d)^m}(O(d))^{m(2d)^m+\ell}.\end{aligned}$$

Notice that Theorem 2.7 generalizes Corollary 3 in [11] to the case $m > 1$. We have the following theorem for symmetric complex affine varieties.

Theorem 2.8 (Symmetric complex affine varieties)**.** *Let* $\mathbf{k} = (k_1,\ldots,k_\ell), \mathbf{m} = (m_1,\ldots,m_\ell), \mathbf{d} = (d,\ldots,d) \in \mathbb{Z}_{>0}^\ell$*, and* $K = \sum_{i=1}^\ell k_i m_i$*. Let*

$$\mathcal{P} \subset \mathrm{C}[\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(\ell)}]^{\mathfrak{S}_{\mathbf{k}}}_{\leq \mathbf{d}}$$

be a finite set of polynomials. Let $V = \mathrm{Z}(\mathcal{P},\mathrm{C}^K)$*.*

1. *Then, for all partitions* $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$*, the assumption* $m_{\boldsymbol{\mu}}(V,\mathbb{F}) > 0$ *implies that:*

$$\mathrm{rank}(\boldsymbol{\mu}) \leq (4\mathbf{d})^{2\mathbf{m}}.$$

2. *Further, we have the following bound on the multiplicities in the isotypic decomposition:*

$$\begin{aligned} m_{\boldsymbol{\mu}}(V,\mathbb{F}) &\leq k_i^{O(d^2)} \prod_{1\leq i\leq \ell} \left(\sum_{j=0}^{(4d)^{2m_i}} \binom{k_i}{j} (O(d))^{2m_i j}\right) \\ &\leq \prod_{1\leq i\leq\ell} \left(k_i^{O((4d)^{4m_i})} (O(d))^{2m_i(4d)^{2m_i}}\right).\end{aligned}$$

2.2. **Affine semi-algebraic case.** We now state our results in the semi-algebraic case, which again yield restrictions on the irreducible representation contributing to the isotypic decomposition of $\mathrm{H}^*(V,\mathbb{F})$ bounds for the multiplicities $m_{\boldsymbol{\mu}}$ (see Notation 2.1).

Theorem 2.9 (Symmetric affine semi-algebraic sets)**.** *Let* $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{m} = (m_1, \ldots, m_\ell), \mathbf{d} = (d, \ldots, d) \in \mathbb{Z}_{>0}^\ell$, $K = \sum_{i=1}^\ell k_i m_i$. *Let* $\mathcal{P} \subset \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]_{\leq \mathbf{d}}^{\mathfrak{S}_{\mathbf{k}}}$ *be a finite set of polynomials, and let* $\mathrm{card}(\mathcal{P}) = s$. *Let* $S \subset \mathrm{R}^K$ *be a* $\mathcal{P}$*-closed semi-algebraic set.*

1. *Then, for all partitions* $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$, *the assumption* $m_{\boldsymbol{\mu}}(S, \mathbb{F}) > 0$ *implies that:*

$$\mathrm{rank}(\boldsymbol{\mu}) \leq (4\mathbf{d})^{\mathbf{m}}.$$

2. *Further, the following bound for the multiplicities in the isotypic decomposition holds:*

$$m_{\boldsymbol{\mu}}(S, \mathbb{F}) \quad \leq \quad O(s)^D \prod_{1 \leq i \leq \ell} \left(k_i^{O((4d)^{2m_i})} (O(d))^{2m_i (4d)^{m_i}} \right),$$

where

$$D = D(\mathbf{k}, \mathbf{m}, d) = \sum_{i=1}^{\ell} \min(k_i m_i, d^{m_i}).$$

In the particular case, when $\ell = 1$, *and* $d_1 = d$ *and* $m_1 = m$ *are fixed, both bounds are polynomial in* s *and* $k_1 = k$.

2.3. **Projective case.** We can apply our results obtained in the previous section to study the topology of symmetric projective varieties as well. We state one such result below.

Note that the block size is equal to one in the following theorem.

Theorem 2.10 (Symmetric complex projective varieties)**.** *Let* $V \subset \mathbb{P}_{\mathrm{C}}^k$ *be defined by a finite set of homogeneous polynomials in* $\mathrm{C}[X_0, \ldots, X_k]_{\leq d}^{\mathfrak{S}_{k+1}}$. *Then, the following holds:*

1. *For all partitions* $\mu \in \mathrm{Par}(k+1)$, *the assumption* $m_{\boldsymbol{\mu}}(V, \mathbb{F}) > 0$ *implies that:*

$$\mathrm{rank}(\mu) \leq (4d).$$

2. *The multiplicities in the isotypic decomposition can be bounded as follows:*

$$m_{\boldsymbol{\mu}}(S, \mathbb{F}) \quad \leq \quad k^{O(d^4)} d^{O(d)}.$$

Remark 2.11. Suppose $V \subset \mathbb{P}_{\mathrm{C}}^k$ be defined by symmetric homogeneous polynomials in $\mathrm{C}[X_0, \ldots, X_k]$ of degrees bounded by d. Unlike in the affine case, it is not true that dimensions of equivariant cohomology, $\dim_{\mathbb{F}} \mathrm{H}^*_{\mathfrak{S}_{k+1}}(V, \mathbb{F})$, are bounded by a function of d independent of k. For example,

$$\mathrm{H}^*(\mathbb{P}_{\mathrm{C}}^k, \mathbb{F}) \cong \mathrm{H}^*(\mathbb{P}_{\mathrm{C}}^k / \mathfrak{S}_{k+1}, \mathbb{F}),$$

and thus

$$\dim_{\mathbb{F}} \mathrm{H}^*(\mathbb{P}_{\mathrm{C}}^k / \mathfrak{S}_{k+1}, \mathbb{F}) = k + 1,$$

which clearly grows linearly with k. This is not especially surprising, as the same is true for the non-equivariant Betti numbers as well – namely, $\dim_{\mathbb{F}} \mathrm{H}^*(\mathrm{C}^k, \mathbb{F}) = 1$, while $\dim_{\mathbb{F}} \mathrm{H}^*(\mathbb{P}_{\mathrm{C}}^k, \mathbb{F}) = k + 1$.

2.4. **Application to bounding topological complexity of images of polynomial maps.** In this section we discuss an application of Theorem 2.7 to bounding the Betti numbers of images of real algebraic varieties under linear projections. In [11], similar results were proved in the very special case of projections of the form $\pi : \mathrm{R}^{k+1} \to \mathrm{R}^k$. In this paper, since we consider more general actions of the symmetric group, we are able to handle projections along more than one variables, and so are able to strengthen as well as generalize the results in [11]. In order to state our results more precisely we first introduce some notation.

Let $P \in \mathrm{R}[Y_1, \ldots, Y_k, X_1, \ldots, X_m]$ be a non-negative polynomial with $\deg(P) \leq d$. Let

$$\pi : \mathrm{R}^{m+k} \longrightarrow \mathrm{R}^k$$

be the projection map to the first k co-ordinates, and let $V = \mathrm{Z}(P, \mathrm{R}^{m+k})$. We consider the problem of bounding the Betti numbers of the image $\pi(V)$. Bounding the complexity of the image under projection of semi-algebraic sets is a very important and well-studied problem related to quantifier elimination in the first order theory of the reals, and has many ramifications – including in computational complexity theory.

There are two different approaches. One can first obtain a semi-algebraic description of the image $\pi(V)$ with bounds on the degrees and the number of polynomials appearing in this description (via results in effective quantifier elimination in the the first order theory of the reals), and then apply known bounds on the Betti numbers of semi-algebraic sets in terms of these parameters. Another approach (due to Gabrielov, Vorobjov and Zell [22]) is to use the "descent spectral sequence" of the map $\pi|_V$ which abuts to the cohomology of $\pi(V)$ and bound the Betti numbers of $\pi(V)$ by bounding the dimensions of the E_1-terms of this spectral sequence. For this approach it is essential that the map π is proper (which is ensured by requiring that V is bounded) since in the general case the spectral sequence might not converge to $\mathrm{H}^*(S, \mathbb{F})$. The second approach produces a slightly better bound. The following theorem (in the special case of algebraic sets) whose proof uses the second approach appears in [22].

Theorem 2.12. [22] *With the same notation as above, the following bound in the Betti numbers of the projection $\pi(V)$ holds:*

$$b(\pi(V), \mathbb{F}) \quad = \quad (O(d))^{(m+1)k}. \tag{2.1}$$

Notice that in the exponent of the bound in (2.1), there is a factor of $(m + 1)$ which is linear in the dimension of the fibers of the projection π. This factor is also present if one uses effective quantifier elimination method to bound the Betti numbers of $\pi(V)$. Using Theorem 2.9 we are able to remove this *multiplicative* factor of $(m + 1)$ in the exponent of the bound in (2.1) at the expense of an extra additive term that depends just on d and m.

We now state the result more precisely. In [11], the following bound on the Betti numbers of the image under projection to a subspace of dimension one less than that of the ambient space of real algebraic varieties (i.e. with $m = 1$), as well as of semi-algebraic sets (not necessarily symmetric).

Theorem 2.13. [11, Theorem 10] *Let $P \in \mathrm{R}[Y_1, \ldots, Y_k, X]$ be a non-negative polynomial, with $\deg(P) \leq d$, and let $V = \mathrm{Z}\left(P, \mathrm{R}^{k+1}\right)$ be bounded. Consider the* *projection map* $\pi : \mathrm{R}^k \times \mathrm{R} \longrightarrow \mathrm{R}^k$ *to the first* k *coordinates. Then,*

$$b(\pi(V), \mathbb{F}) \quad \leq \quad \binom{k}{d}^{2d} (O(d))^{k+2d+1}.$$

In this paper we generalize the above results to the case $m > 1$. We prove the following theorem.

Theorem 2.14. *Let* $P \in \mathrm{R}[Y_1, \ldots, Y_k, X_1, \ldots, X_m]$ *be a non-negative polynomial, with* $\deg(P) \leq d$. *Suppose that* $V = \mathrm{Z}(P, \mathrm{R}^{k+m})$ *is bounded, and consider the projection map* $\pi : \mathrm{R}^k \times \mathrm{R}^m \longrightarrow \mathrm{R}^k$ *to the first* k *coordinates. Then,*

$$b(\pi(V), \mathbb{F}) \quad \leq \quad k^{(2d)^m} (O(d))^{k+m(2d)^m+1}. \tag{2.2}$$

Remark 2.15. For every fixed d and m and $d, m \geq 1$, the bound in inequality (2.2) in Theorem 2.14 is better than the one in (2.1) in Theorem 2.12, for all large enough k, since in this case

$$k + m(2d)^m + 1 \ll (m+1)k.$$

2.5. **Application to proving lower bounds on degrees.** The upper bounds in the theorems stated above can be potentially applied to prove lower bounds on the degrees of polynomials needed to define symmetric varieties having certain prescribed geometry. We describe one such example.

Example 2.16. Let $k = 2^p - 1$, and let $\tilde{V}_k$ be any non-empty closed and bounded semi-algebraic set contained in the subset of R^k defined by

$$\begin{array}{rcl}
X_1 = & \cdots & = X_{2^{p-1}} \\
 & \neq & \\
X_{2^{p-1}+1} = & \cdots & = X_{2^{p-1}+2^{p-2}} \\
 & \neq & \\
X_{2^{p-1}+2^{p-2}+1} = & \cdots & = X_{2^{p-1}+\cdots+2^2+1} X_{2^{p-1}+\cdots+2^1} \\
 & \neq & \\
 & X_{2^{p-1}+\cdots+2^1+1} & .
\end{array}$$

Then, the stabilizer of $\tilde{V}_k$ under the action of $\mathfrak{S}_k$ on R^k, is the Young subgroup $\mathfrak{S}_{\lambda^{(k)}}$, where $\lambda^{(k)} = (2^{p-1}, 2^{p-2}, \ldots, 1)$. Let V_k be the orbit of $\tilde{V}_k$ under the action of $\mathfrak{S}_k$. In other words,

$$V_k = \mathfrak{S}_k \cdot \tilde{V}_k.$$

Then,

$$\begin{aligned}
b_0(V_k, \mathbb{F}) &= b_0(\tilde{V}_k, \mathbb{F}) \cdot \binom{k}{2^{p-1}, 2^{p-2}, \ldots, 2^0} \\
&= b_0(\tilde{V}_k, \mathbb{F}) \cdot (\Theta(1))^k \text{ using Stirling's approximation.}
\end{aligned}$$

We claim that that for any constant d_0, for all k large enough, V_k cannot be described as the set of real zeros of a polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$ with $\deg(P) \leq d_0$. To see this observe that

$$\mathrm{H}^0(V_k, \mathbb{F}) \cong_{\mathfrak{S}_k} b_0(\tilde{V}_k, \mathbb{F}) \cdot M^{\lambda^{(k)}} \text{ (cf. Definition 1.11)},$$

with $\lambda^{(k)} = (2^{p-1}, 2^{p-2}, \ldots, 1)$, and it follows that $m_{0,\lambda^{(k)}}(V_k, \mathbb{F}) > 0$. However, clearly $\mathrm{rank}(\lambda^{(k)})$ is a strictly increasing function of k, and hence it follows from

Theorem 2.5 that V_k cannot be defined by polynomials with degrees bounded by d_0.

Note that in the case, when $\tilde{V}_k$ is a finite set of points, the same result can also be deduced from Proposition 3.8.

3. Preliminaries

Before proving the new theorems stated in the previous section we need some preliminary results. These are described in the following subsections.

3.1. **Real closed extensions and Puiseux series.** We prove the main theorems of this paper for symmetric semi-algebraic sets or varieties defined over arbitrary real closed fields. The main advantage of proving results in this generality (rather than just over the field of real numbers), is that it allows us to make arguments using infinitesimals which necessarily belong to non-Archimedean real closed extensions of the ground field. These arguments would be much more cumbersome to do if we restricted ourselves just to just the Archimedean fields. To this end, in this section we recall some basic facts about real closed fields and real closed extensions.

We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [9] for further details.

Notation 3.1 (Field of Puiseux series)**.** For R a real closed field we denote by $\mathrm{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in R. We use the notation $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$ to denote the real closed field $\mathrm{R}\langle\varepsilon_1\rangle\langle\varepsilon_2\rangle\cdots\langle\varepsilon_m\rangle$. Note that in the unique ordering of the field $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$, $0<\varepsilon_m\ll\varepsilon_{m-1}\ll\cdots\ll\varepsilon_1\ll 1$.

Notation 3.2 (Extensions)**.** If R' is a real closed extension of a real closed field R, and $S\subset\mathrm{R}^k$ is a semi-algebraic set defined by a first-order formula with coefficients in R, then we will denote by $\mathrm{Extn}(S,\mathrm{R}')\subset\mathrm{R}'^k$ the semi-algebraic subset of R'^k defined by the same formula. It is well-known that $\mathrm{Extn}(S,\mathrm{R}')$ does not depend on the choice of the formula defining S [9].

Notation 3.3 (Balls)**.** For $x\in\mathrm{R}^k$ and $r\in\mathrm{R}$, $r>0$, we will denote by $B_k(x,r)$ the open Euclidean ball centered at x of radius r. If R' is a real closed extension of the real closed field R and when the context is clear, we will continue to denote by $B_k(x,r)$ the extension $\mathrm{Extn}(B_k(x,r),\mathrm{R}')$. This should not cause any confusion.

In some proofs involving Morse theory (see for example the proof of Lemma 3.5), where integration of gradient flows is used in an essential way, we first restrict to the case $\mathrm{R}=\mathbb{R}$. After having proved the result over $\mathbb{R}$, we use the Tarski-Seidenberg transfer theorem to extend the result to all real closed fields. We refer the reader to [9, Chapter 2] for an exposition of the Tarski-Seidenberg transfer principle.

3.2. **Equivariant Morse theory.** In this section we recall some basic results from the classical topic of equivariant Morse theory that we will need for the proof of Theorem 4.4. These results are known in far greater generality (for general smooth manifolds with group actions) than what we need (see for example [34, §4] and [1]). However, for the sake of completeness and also to fix notation we state and sketch the proofs of two results in the special case that we need in this paper. These are Lemma 3.4 and Proposition 3.6 below, which correspond to the equivariant versions of the usual Morse Lemmas A and B respectively.

Let G be a finite group acting on a closed and bounded semi-algebraic $S\subset\mathrm{R}^k$, defined by $Q\leq 0$, and $W=\mathrm{Z}(Q,\mathrm{R}^k)=\partial S$ a bounded non-singular real algebraic hypersurface. Let $e : W \to \mathrm{R}$ be a G-equivariant regular function with isolated non-degenerate critical points on W. For each such critical point $\mathbf{x}$, we will denote by $\mathrm{ind}^-(\mathbf{x})$ the dimension of the negative eigenspace of the Hessian of e at $\mathbf{x}$. More precisely, the Hessian $\mathrm{Hess}(e)(\mathbf{x})$ is a symmetric, non-degenerate quadratic form on the tangent space T_pW, and $\mathrm{ind}^-(\mathbf{x})$ is the number of negative eigenvalues of $\mathrm{Hess}(e)(\mathbf{x})$.

Consider the set of critical points, $\mathcal{C}$, of the function e restricted to V. For any subset $I \subset \mathrm{R}$, we will denote by $S_I = S \cap e^{-1}(I)$. If $I = [\infty, c]$ we will denote $S_I = S_{\leq c}$.

In the next two lemmas we will set $\mathrm{R} = \mathbb{R}$ since we will use properties of gradient flows.

Lemma 3.4. *Let $v_1 < \cdots < v_N$ be the critical values of e restricted to W. Then, for $1 \leq i < N$, and for each $v \in [v_i, v_{i+1})$, $S_{\leq v_i}$ is a deformation retract of $S_{\leq v}$, and the retraction can be chosen to be G-equivariant.*

Proof. See for example the proof of Theorem 7.5 (Morse Lemma A) in [9] with $W = \mathrm{Z}(Q, \mathrm{R}^k)$, $a = v_i$ and $b = v$, noting since W is symmetric and e is symmetric, the retraction of $W_{\leq v}$ to $W_{\leq v_i}$ that is constructed in the proof of Theorem 7.5 in [9] is symmetric as well. $\square$

We also need the following equivariant version of Morse Lemma B.

Lemma 3.5. *Let $v \in e(\mathcal{C})$ be a critical value of e. Let $\mathcal{C}_v^+, \mathcal{C}_v^-, \mathcal{C}_v \subset \mathcal{C}$ be defined by*

$$\begin{aligned} \mathcal{C}_v^+ &= \{\mathbf{x} \in \mathcal{C} \mid e(\mathbf{x}) = v,\ \langle \mathrm{grad}(e), \mathrm{grad}(Q)\rangle(\mathbf{x}) > 0\}, \\ \mathcal{C}_v^- &= \{\mathbf{x} \in \mathcal{C} \mid e(\mathbf{x}) = v,\ \langle \mathrm{grad}(e), \mathrm{grad}(Q)\rangle(\mathbf{x}) < 0\}, \\ \mathcal{C}_v &= \mathcal{C}_v^+ \mathbin{\mathring{\cup}} \mathcal{C}_v^- \end{aligned}$$

($\mathring{\cup}$ denotes disjoint union). Then for all $0 < \varepsilon \ll 1$, $S_{\leq v+\varepsilon}$ retracts G-equivariantly to a space

$$S_{\leq v-\varepsilon} \cup_B A,$$

where

$$(A, B) = \coprod_{\mathbf{y} \in \mathcal{C}_v} (A_{\mathbf{y}}, B_{\mathbf{y}}),$$

and for each $\mathbf{y} \in \mathcal{C}_v$, $(A_{\mathbf{y}}, B_{\mathbf{y}})$ is G-equivariantly homotopy equivalent to the pair

$$(\mathbf{D}^{\mathrm{ind}^-(\mathbf{y})} \times [0,1], \partial\mathbf{D}^{\mathrm{ind}^-(\mathbf{y})} \times [0,1] \cup \mathbf{D}^{\mathrm{ind}^-(\mathbf{y})} \times \{1\})$$

if $\mathbf{y} \in \mathcal{C}_v^+$, or to the pair

$$(\mathbf{D}^{\mathrm{ind}^-(\mathbf{y})}, \partial\mathbf{D}^{\mathrm{ind}^-(\mathbf{y})}),$$

if $\mathbf{y} \in \mathcal{C}_v^-$.

Proof. See proof of Proposition 7.19 in [9], noting again that the retraction constructed in that proof is symmetric in case Q is a symmetric polynomial and the Morse function e is symmetric as well. $\square$

Now, let $\overline{\mathcal{C}}$ be a set containing a unique representative from each G-orbit of $\mathcal{C}$.

Let $\overline{\mathcal{C}}_i \subset \overline{\mathcal{C}}$ be the set of representatives of the different orbits corresponding to the critical value v_i – in other words, $e(\overline{\mathcal{C}}_i) = \{v_i\}$. Note that the cardinality of $\overline{\mathcal{C}}_i$ can be greater than one. For each $\mathbf{x} \in \overline{\mathcal{C}}_i$, let $G_{\mathbf{x}} \subset G$ denote the stabilizer subgroup of $\mathbf{x}$.

Let also for each $i, 0 \leq i \leq N$, and $0 < \varepsilon \ll 1$, $j_{i,\varepsilon}$ denote inclusion $S_{\leq v_i - \varepsilon} \hookrightarrow S_{\leq v_i + \varepsilon}$, and $j^*_{i,\varepsilon} : \mathrm{H}^*(S_{\leq v_i+\varepsilon}, \mathbb{F}) \to \mathrm{H}^*(S_{\leq v_i - \varepsilon}, \mathbb{F})$ the induced homomorphism (which is in fact a homomorphism of the corresponding G-modules).

Let $\overline{\mathcal{C}}_i = \{\mathbf{x}^{i,1}, \ldots, \mathbf{x}^{i,N_i}\}$ (choosing an arbitrary order).

Proposition 3.6. *The homomorphism $j^*_{i,\varepsilon}$ factors through N_i homomorphisms as follows:*

$$\mathrm{H}^*(S_{\leq v_i+\varepsilon}, \mathbb{F}) = M_0 \xrightarrow{j^*_{i,\varepsilon,1}} M_1 \xrightarrow{j_{i,\varepsilon,2,*}} \cdots \xrightarrow{j_{i,\varepsilon,N_i,*}} M_{N_i+1} = \mathrm{H}^*(S_{\leq v_i-\varepsilon}, \mathbb{F}),$$

where each M_h is a finite dimensional G-module, and for each $h, 1 \leq h \leq N_i$, either

(a) $j^*_{i,\varepsilon,h}$ is injective, and

$$M_{h+1} \cong M_h \oplus \mathrm{Ind}^G_{G_{\mathbf{x}^{i,h}}}(W_{\mathbf{x}^{i,h}}),$$

for some one-dimensional representation $W_{\mathbf{x}^{i,h}}$ of $G_{\mathbf{x}^{i,h}}$, or

(b) the homomorphism $j^*_{i,\varepsilon,h}$ is surjective, and

$$M_h \cong M_{h+1} \oplus \mathrm{Ind}^G_{G_{\mathbf{x}^{i,h}}}(W_{\mathbf{x}^{i,h}}),$$

for some one-dimensional representation $W_{\mathbf{x}^{i,h}}$ of $G_{\mathbf{x}^{i,h}}$.

Proof. We first assume that $\mathrm{R} = \mathbb{R}$. Using Lemma 3.5 (equivariant Morse Lemma B), we have that $S_{\leq v_i+\varepsilon}$ can be retracted G-equivariantly to a semi-algebraic set

$$\tilde{S}_i = S_{\leq v_i-\varepsilon} \coprod_{\substack{1 \leq j \leq N_i, \\ \langle \mathrm{grad}(e), \mathrm{grad}(Q)\rangle(\mathbf{x}^{i,j})<0}} \coprod_{\mathbf{y} \in G\cdot\mathbf{x}^{i,j}} \mathbf{D}^{\mathrm{ind}^-(\mathbf{x}^{i,j})} / \sim,$$

where the identification $\sim$ identifies the boundaries of the disks $\mathbf{D}^{\mathrm{ind}^-(\mathbf{x}^{i,j})}$ with spheres of the same dimension in $S_{\leq v_i-\varepsilon}$.

Since the different balls $\mathbf{D}^{\mathrm{ind}^-(\mathbf{x}^{i,j})}$ are disjoint, we can decompose the gluing process by gluing disks belonging to each orbit successively, choosing the order arbitrarily, and thus obtain a filtration,

$$S|_{e \leq v_i-\varepsilon} = S_{i,0} \subset S_{i,1} \subset \cdots \subset S_{i,N'_i} = \tilde{S}_i,$$

where $S_{i,j} = S_{i,j-1} \coprod \left(\coprod_{\mathbf{y} \in \mathrm{orbit}(\mathbf{x}^{i,j'})} \mathbf{D}^{\mathrm{ind}^-(\mathbf{y})} / \sim \right)$ for some $j', 1 \leq j' \leq N_i$.

Let $D_{i,j'}$ denote the disjoint union of the balls $\mathbf{D}^{\mathrm{ind}^-(\mathbf{x}^{i,j'})}$, and $C_{i,j'} \subset D_{i,j'}$ the disjoint union of their boundaries.

We have the Mayer-Vietoris exact sequence

$$\cdots \to \mathrm{H}^{p-1}(C_{i,j'}, \mathbb{F}) \to \mathrm{H}^p(S_{i,j}, \mathbb{F}) \to \mathrm{H}^p(D_{i,j'}, \mathbb{F}) \oplus \mathrm{H}^p(S_{i,j-1}, \mathbb{F}) \to \mathrm{H}^p(C_{i,j'}, \mathbb{F}) \to \cdots$$

which is also equivariant. Let $n = \mathrm{ind}^-(\mathbf{x}^{i,j'})$, and assume $n \neq 0$. In this case, $\mathrm{H}^p(C_{i,j'}, \mathbb{F}) = 0$ unless $p = 0, n-1$. Now, $\mathrm{H}^{n-1}(C_{i,j'}, \mathbb{F})$ is a direct sum of $\mathrm{card}(\mathrm{orbit}(\mathbf{x}^{i,j'}))$, each summand is stable under the action of a subgroup of G each isomorphic to $G_{\mathbf{x}^{i,j'}}$, and is thus a one-dimensional representation of $G_{\mathbf{x}^{(i,j')}}$ which we denote by by $W_{\mathbf{x}^{i,j'}}$. It follows that the representation $\mathrm{H}^{n-1}(C_{i,j'}, \mathbb{F})$ is the induced representation $\mathrm{Ind}^G_{G_{\mathbf{x}^{i,j'}}}(W_{\mathbf{x}^{i,j'}})$. From the Mayer-Vietoris sequence it is evident that either

(i)
$$\mathrm{H}^n(S_{i,j}, \mathbb{F}) = \mathrm{H}^n(S_{i,j-1}, \mathbb{F}) \oplus \mathrm{H}^{n-1}(C_{i,j'}, \mathbb{F}),$$

or

(ii)
$$\mathrm{H}^{n-1}(S_{i,j},\mathbb{F}) \oplus \mathrm{H}^{n-1}(C_{i,j'},\mathbb{F}) \cong \mathrm{H}^{n}(S_{i,j-1},\mathbb{F}).$$

These two cases correspond to (a) and (b) respectively. Finally, we extend the proof to general R using the Tarski-Seidenberg transfer principle in the usual way (see [9, Chapter 7] for example). □

3.3. **Structure of critical points of a symmetric Morse function on a symmetric hypersurface of small degree in R^k.** In this section we prove an important proposition (Proposition 3.8) that forms the basis of all our quantitative results. It generalizes to the multi-symmetric case (i.e. for $\mathbf{m}$ not necessarily equal to $(1,\ldots,1)$) a similar result proved earlier (see [27, 32, 11]).

Notation 3.7. Let $\mathbf{k} = (k_1,\ldots,k_\ell), \mathbf{m} = (m_1,\ldots,m_\ell), \mathbf{p} = (p_1,\ldots,p_\ell) \in \mathbb{Z}^\ell_{>0}$, and let
$$K = \sum_{1 \le i \le \ell} k_i m_i.$$

We denote by $A^{\mathbf{p}}_{\mathbf{k},\mathbf{m}}$ the subset of R^k defined by
$$A^{\mathbf{p}}_{\mathbf{k},\mathbf{m}} \quad = \quad \left\{ \mathbf{x} = (\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(\ell)}) \mid \mathrm{card}\left(\bigcup_{j=1}^{k_i} \{\mathbf{x}^{(i)}_j\}\right) = p_i \right\}.$$

In the special case when $\ell = 1, m = 1$, we define for $p \le k = k_1$,
$$A^{p}_{k} \quad = \quad \left\{ \mathbf{x} = (x_1,\ldots,x_k) \mid \mathrm{card}\left(\bigcup_{j=1}^{k} \{x_j\}\right) = p \right\}.$$

Let $\mathbf{k},\mathbf{m},\mathbf{d}, \in \mathbb{Z}^\ell_{>0}$, $K = \sum_{i=1}^{\ell} k_i m_i$, and $P \in \mathrm{R}[\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(\ell)}]^{\mathfrak{S}_{\mathbf{k}}}_{\le \mathbf{d}}$.

The following proposition generalizes (to the case $\mathbf{m} \ne (1,\ldots,1)$) Proposition 5 in [11].

Proposition 3.8. *Let*
$$e = \sum_{1 \le h \le \ell} \sum_{1 \le i \le m_h} \sum_{1 \le j \le k_h} X^{(h)}_{i,j}.$$

Let $\mathcal{C}$ denote the set of critical points of e restricted to $W = \mathrm{Z}(P,\mathrm{R}^K)$, and suppose that $\mathcal{C}$ is a finite set. Then,
$$\mathcal{C} \subset \bigcup_{\mathbf{p} \le \mathbf{d}^{\mathbf{m}}} A^{\mathbf{p}}_{\mathbf{k},\mathbf{m}}.$$

Proof. For $\mathbf{m} = \mathbf{1} := (1,\ldots,1)$, the proposition follows immediately from [11, Proposition 5]. Suppose that $\mathbf{x} = (\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(\ell)}) \in \mathcal{C}$. For $\mathbf{x} = (\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(\ell)}) \in \mathrm{R}^K$, and $\mathbf{i} = (i_1,\ldots,i_\ell) \in [1,m_1] \times \cdots \times [1,m_\ell]$ denote by $\bar{\mathbf{x}}_{\mathbf{i}} = (\mathbf{x}^{(1)}_{i_1},\ldots,\mathbf{x}^{(\ell)}_{i_\ell}) \in \mathrm{R}^{K'}$, where $K' = \sum_{i=1}^{\ell} k_i$. It follows from the case $\mathbf{1} = (1,\ldots,1)$ ([11, Proposition 5]), that for each $\mathbf{i} \in [1,m_1] \times \cdots \times [1,m_\ell]$,
$$\mathbf{x}_{\mathbf{i}} \in \bigcup_{\mathbf{p} \le \mathbf{d}} A^{\mathbf{p}}_{\mathbf{k},\mathbf{1}}.$$

This proves that for each $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(\ell)}) \in \mathcal{C}$, each row of each $(m_h \times k_h)$-matrix $\mathbf{x}^{(h)}$ has at most d distinct entries, and this implies that the matrix $\mathbf{x}^{(h)}$ has at most d^{m_h} distinct columns. This implies that

$$\mathbf{x} \in \bigcup_{\mathbf{p} \leq \mathbf{d^m}} A^{\mathbf{p}}_{\mathbf{k},\mathbf{m}},$$

which proves the proposition. □

3.4. **Deformation.** In this section we recall from [11] an important technique for equivariantly deforming a given real variety, such that the deformed variety has good algebraic and topological properties. The results that we are going to use later are Propositions 3.10 and 3.11 (both of which are reproduced here from [11] for the reader's convenience)

Let $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell_{>0}, K = \sum_{i=1}^{\ell} k_i m_i$, and $d \geq 0$. Following the notation introduced previously,

Notation 3.9 (Deformation)**.** For any $P \in \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$ we denote

$$\mathrm{Def}(P, \zeta, d) = P - \zeta \left(1 + \sum_{1 \leq h \leq \ell} \sum_{1 \leq i \leq m_h} \sum_{1 \leq j \leq k_h} (X^{(h)}_{i,j})^d \right),$$

where ζ is a new variable.

Notice that if P is $\mathfrak{S}_{\mathbf{k}}$-symmetric, then so is $\mathrm{Def}(P, \zeta, d)$.

The following two propositions appear in [11]. We restate them here for the ease of the reader.

Proposition 3.10. [11, Proposition 3] *Let d be even, P be $\mathfrak{S}_{\mathbf{k}}$- symmetric and non-negative polynomial, and suppose that $V = \mathrm{Z}\left(P, \mathrm{R}^K\right)$ is bounded. The variety $\mathrm{Extn}\left(V, \mathrm{R}\langle\zeta\rangle^K\right)$ is a semi-algebraic deformation retract of the (symmetric) semi-algebraic subset S of $\mathrm{R}\langle\zeta\rangle^K$ consisting of the union of the semi-algebraically connected components of the semi-algebraic set defined by the inequality*

$$\mathrm{Def}(P, \zeta, d) \leq 0,$$

which are bounded over R, and hence is semi-algebraically homotopy equivalent to S.

Proposition 3.11. [11, Proposition 4] *Let $P \in \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$, and d be an even number with $\deg(P) < d = p + 1$, with p a prime. Let*

$$e = \sum_{1 \leq h \leq \ell} \sum_{1 \leq i \leq m_h} \sum_{1 \leq j \leq k_h} X^{(h)}_{i,j},$$

and

$$V_\zeta = \mathrm{Z}\left(\mathrm{Def}(P, \zeta, d), \mathrm{R}\langle\zeta\rangle^K\right).$$

Suppose also that $\gcd(p, K) = 1$. Then, the critical points of e restricted to V_ζ are finite in number, and each critical point is non-degenerate.

3.5. **Representation theory of products of symmetric groups.** In this section we recall some well known facts from the representation theory of symmetric groups, and prove one new result (Proposition 3.16) that will be used later. The following classical formula (due to Frobenius) gives the dimensions of the representations $\mathbb{S}^\lambda$ in terms of the *hook lengths* of the partition λ defined below.

Definition 3.12 (Hook lengths)**.** Let $B(\lambda)$ denote the set of boxes in the Young diagram (cf. Definition 1.7) corresponding to a partition $\lambda \vdash k$. For a box $b \in B(\lambda)$, the length of the hook of b, denoted h_b is the number of boxes strictly to the right and below b plus 1.

Theorem 3.13 (Hook length formula)**.** *Let $\lambda \vdash k$. Then,*

$$\dim_{\mathbb{F}} \mathbb{S}^\lambda = \frac{k!}{\prod_{b\in B(\lambda)} h_b}. \tag{3.1}$$

We also need the definition the so called Littlewood-Richardson coefficients.

Definition 3.14 (Littlewood-Richardson coefficients)**.** For $\lambda \vdash m, \mu \vdash n, \nu \vdash m+n$, $c^\nu_{\lambda,\mu}$ is the multiplicity of the irreducible representation $\mathbb{S}^\nu$ in $\mathrm{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(\mathbb{S}^\lambda \boxtimes \mathbb{S}^\mu)$.

In order to state the main new result in this section (Proposition 3.16 below) we need one more notation.

Notation 3.15 (Induced representations and multiplicities)**.** For each $\lambda \vdash k$, we denote by $\overline{\mathrm{Par}}(\lambda)$ the set of partitions $\mu \vdash k$ such that, there exists a decomposition $\lambda = \lambda' \coprod \lambda''$, $\lambda' = (\lambda'_1, \ldots, \lambda'_{\ell'}), \lambda'' = (\lambda''_1, \ldots, \lambda''_{\ell''}), \ell' + \ell'' = \ell = \mathrm{length}(\lambda)$,such that $\mathbb{S}^\mu$ occurs with positive multiplicity in the representation

$$\mathbb{S}_{\lambda',\lambda''} := \mathrm{Ind}_{\mathfrak{S}_{\lambda'}\times\mathfrak{S}_{\lambda''}}^{\mathfrak{S}_k}\left(\left(\boxtimes_{i=1}^{\ell'}\mathbb{S}^{(\lambda'_i)}\right)\boxtimes\left(\boxtimes_{j=1}^{\ell''}\mathbb{S}^{(1^{\lambda''_j})}\right)\right),$$

and we denote the multiplicity of $\mathbb{S}^\mu$ in $\mathbb{S}_{\lambda',\lambda''}$ by $m^\mu_{\lambda',\lambda''}$.

More generally, for $\mathbf{k} \in \mathbb{Z}^\ell_{>0}$, and $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k})$, we denote by $\overline{\mathrm{Par}}(\boldsymbol{\lambda}) = \overline{\mathrm{Par}}(\lambda^{(1)}) \times \cdots \times \overline{\mathrm{Par}}(\lambda^{(\ell)})$.

Proposition 3.16. *Let $k, d > 0$, $\lambda \in \mathrm{Par}(k, d)$ such that $\lambda = \lambda' \coprod \lambda''$, and $\mu \in \overline{\mathrm{Par}}(\lambda)$. Then,*

1.

$$\mathrm{rank}(\mu) \leq d,$$

2.

$$m^\mu_{\lambda',\lambda''} = \sum_{\substack{\nu'\vdash|\lambda'|,\nu' \trianglerighteq \lambda' \\ \nu''\vdash|\lambda''|,\nu'' \trianglerighteq \widetilde{\lambda''}}} K(\nu', \lambda') \cdot K(\nu'', \widetilde{\lambda''}) \cdot c^\mu_{\nu',\nu''}, \tag{3.2}$$

3.

$$\sum_{\mu\vdash k} m^\mu_{\lambda',\lambda''} \leq k^{O(d^2)}.$$

Remark 3.17. It is well known that $K(\mu, \mu) = 1$ for all $\mu \in \mathrm{Par}(k)$, $K(\mu, \lambda) = 0$ unless $\mu \trianglerighteq \lambda$. Finally, if μ is the maximal element in the dominance ordering $\trianglerighteq$ on $\mathrm{Par}(k)$, that is $\mu = (k)$, then $K(\mu, \lambda) = 1$ for all $\lambda \in \mathrm{Par}(k)$. In particular, in conjunction with Schur's lemma the above fact implies, that the trivial representation, $\mathbb{S}^{(k)}$ occurs with multiplicity equal to $1(= K((k), \lambda))$ in $\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}\big(\boxtimes_{j=1}^{\mathrm{length}(\lambda)}\mathbb{S}^{(\lambda_j)}\big)$.

Remark 3.18. Note also that the representation $\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}(\boxtimes_{j=1}^{\mathrm{length}(\lambda)}\mathbb{S}^{(\lambda_j)})$ is isomorphic to the permutation representation of $\mathfrak{S}_k$ on the set of cosets $\mathfrak{S}_k/\mathfrak{S}_\lambda$, and in particular

$$\dim_{\mathbb{F}} \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}(\boxtimes_{j=1}^{\mathrm{length}(\lambda)}\mathbb{S}^{(\lambda_j)}) = \frac{k!}{\prod_{1\le j\le \mathrm{length}(\lambda)} \lambda_j!}.$$

In order to prove Proposition 3.16 we need the following definition and results which are all well known.

Definition 3.19 (Skew partitions, horizontal and vertical strips)**.** For any two partitions, $\lambda = (\lambda_1,\lambda_2,\ldots) \vdash m$, $\mu = (\mu_1,\mu_2,\ldots)\vdash n$, $m \le n$, we say that $\lambda \subset \mu$, if $\lambda_i \le \mu_i$ for all i.

Identifying λ,μ with their respective Young diagrams, we say that the *skew partition* μ/λ is a *horizontal strip* if no two cells of μ/λ belong to the same *column*. We say that μ/λ is a *vertical strip* if no two cells of μ/λ belong to the same *row*.

Proposition 3.20 (Pieri's rule)**.** *For $\lambda\vdash m$, and $n\ge 0$, we have the two following relations.*

$$\begin{aligned}
\mathrm{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(\mathbb{S}^\lambda\boxtimes\mathbb{S}^{(n)}) &\cong \bigoplus_{\substack{\mu\vdash m+n\\ \mu/\lambda \text{ is a horizontal strip}}} \mathbb{S}^\mu,\\
\mathrm{Ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(\mathbb{S}^\lambda\boxtimes\mathbb{S}^{1^n}) &\cong \bigoplus_{\substack{\mu\vdash m+n\\ \mu/\lambda \text{ is a vertical strip}}} \mathbb{S}^\mu.
\end{aligned}$$

We also have the following associativity relationship that allows us to apply Pieri's rule (Proposition 3.20) iteratively.

Proposition 3.21. *Let $n = m_1+\cdots+m_\ell$, where for each $i, 1\le i\le \ell$. Then,*

$$\mathrm{Ind}_{\mathfrak{S}_{m_1}\times\cdots\times\mathfrak{S}_{m_\ell}}^{\mathfrak{S}_n}(V_1\boxtimes\cdots\boxtimes V_\ell)$$

is isomorphic to

$$\mathrm{Ind}_{\mathfrak{S}_{m_1+\ldots+m_{\ell-1}}\times\mathfrak{S}_{m_\ell}}^{\mathfrak{S}_n}(\mathrm{Ind}_{\mathfrak{S}_{m_1}\times\cdots\times\mathfrak{S}_{m_{\ell-1}}}^{\mathfrak{S}_{m_1+\cdots+m_{\ell-1}}}(V_1\boxtimes\cdots\boxtimes V_{\ell-1})\boxtimes V_\ell),$$

where for each $i, 1\le i\le\ell$, V_i is an $\mathfrak{S}_{m_i}$-module.

Proof of Proposition 3.16. We first prove (2). Let $k' = |\lambda'|$ and $k'' = |\lambda''|$. Then, using Young's rule (Proposition 1.14)

$$\begin{aligned}
\mathrm{Ind}_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_{k'}}\left(\boxtimes_{i=1}^{\ell'}\mathbb{S}^{(\lambda'_i)}\right) &\cong \bigoplus_{\nu'\vdash k',\nu'\,\trianglerighteq\,\lambda'} K(\nu',\lambda')\mathbb{S}^{\nu'},\\
\mathrm{Ind}_{\mathfrak{S}_{\lambda''}}^{\mathfrak{S}_{k''}}\left(\boxtimes_{i=1}^{\ell''}\mathbb{S}^{1^{\lambda''_i}}\right) &\cong \bigoplus_{\nu''\vdash k',\nu''\,\trianglerighteq\,\widetilde{\lambda''}} K(\nu'',\widetilde{\lambda''})\mathbb{S}^{\nu''}.
\end{aligned}$$

It follows that

$$\mathrm{Ind}_{\mathfrak{S}_{\lambda'}\times\mathfrak{S}_{\lambda''}}^{\mathfrak{S}_{k'}\times\mathfrak{S}_{k''}}\left(\left(\boxtimes_{i=1}^{\ell'}\mathbb{S}^{(\lambda'_i)}\right)\boxtimes\left(\boxtimes_{j=1}^{\ell''}\mathbb{S}^{(1^{\lambda''_j})}\right)\right)$$

is isomorphic to

$$\bigoplus_{\substack{\nu'\vdash k',\nu'\,\trianglerighteq\,\lambda'\\ \nu''\vdash k'',\nu''\,\trianglerighteq\,\widetilde{\lambda''}}} K(\nu',\lambda')K(\nu'',\widetilde{\lambda''})\mathbb{S}^{\nu'}\boxtimes\mathbb{S}^{\nu''}.$$

Eqn. (3.2) then follows from the isomorphism

$$\mathbb{S}_{\lambda',\lambda''} \quad\cong\quad \mathrm{Ind}_{\mathfrak{S}_{k'}\times\mathfrak{S}_{k''}}^{\mathfrak{S}_k}\mathrm{Ind}_{\mathfrak{S}_{\lambda'}\times\mathfrak{S}_{\lambda''}}^{\mathfrak{S}_{k'}\times\mathfrak{S}_{k''}}\left(\left(\boxtimes_{i=1}^{\ell'}\mathbb{S}^{(\lambda'_i)}\right)\boxtimes\left(\boxtimes_{j=1}^{\ell''}\mathbb{S}^{(1^{\lambda''_j})}\right)\right)$$

and the definition of the Littlewood-Richardson's coefficients, $c^{\mu}_{\nu',\nu''}$ (Definition 3.14).

An alternative way of obtaining the multiplicities $m^{\mu}_{\lambda',\lambda''}$ is by applying Pieri's rule iteratively at most d times using Propositions 3.21 and 3.20. Let $\mathrm{length}(\lambda') = \ell', \mathrm{length}(\lambda'') = \ell''$, so that $\ell' + \ell'' = \mathrm{length}(\lambda) \le d$.

Let for $1 \le i \le \ell'$,

$$M_i = \mathrm{Ind}_{\mathfrak{S}_{\lambda'_1+\ldots+\lambda'_{i-1}}\times\mathfrak{S}_{\lambda'_i}}^{\mathfrak{S}_{\lambda'_1+\ldots+\lambda'_i}}(M_{i-1}\boxtimes\mathbb{S}^{(\lambda'_i)}),$$

with the convention that $M_0 = \mathbf{1}$. For $\nu \vdash \lambda'_1 + \ldots + \lambda'_i$, let m^{ν}_i denote the multiplicity of $\mathbb{S}^{\nu}$ in M_i, and $m_i = \sum_{\nu\vdash\lambda'_1+\ldots+\lambda'_i} m^{\nu}_i$. We prove by induction on i the following two statements.

(a)

$$m_i \le m_{i-1}\cdot\binom{\lambda'_i + i - 1}{i-1}.$$

(b) For each $\nu \vdash \lambda'_1 + \ldots + \lambda'_i$, such that $m^{\nu}_i > 0$, $\mathrm{length}(\nu) \le i$.

Assuming statements (a) and (b) hold for $i-1$ we prove them for i. By induction for each $\nu' \vdash \lambda'_1 + \ldots + \lambda'_{i-1}$ with $m^{\nu'}_{i-1} > 0$, $\mathrm{length}(\nu') \le i-1$. Applying Pieri's rule (Proposition 3.20) we obtain

$$\mathrm{Ind}_{\mathfrak{S}_{\lambda'_1+\cdots+\lambda'_{i-1}}\times\mathfrak{S}_{\lambda'_i}}^{\mathfrak{S}_{\lambda'_1+\cdots+\lambda'_i}}(\mathbb{S}^{\nu'}\boxtimes\mathbb{S}^{(\lambda'_i)}) \quad\cong\quad \bigoplus_{\substack{\nu\vdash\lambda'_1+\cdots+\lambda'_i\\ \nu/\nu' \text{ is a horizontal strip}}} \mathbb{S}^{\nu}. \tag{3.3}$$

Observe that each choice of $\nu \vdash \lambda'_1 + \cdots + \lambda'_i$ such that $\nu/\nu'$is a horizontal strip, corresponds uniquely to a composition of λ'_i into at most $\mathrm{length}(\nu')$ parts, and the number of such compositions is clearly bounded by

$$\binom{\lambda'_i + \mathrm{length}(\nu') - 1}{\mathrm{length}(\nu') - 1} \le \binom{\lambda'_i + i - 1}{i-1},$$

since $\mathrm{length}(\nu') \le i-1$ by induction hypothesis.

This proves part (a). Part (b) also follows from (3.3) noting that the length of each μ that occurs on the right is at most $\mathrm{length}(\mu') + 1$ which is $\le i$ using the induction hypothesis. This completes the proof of parts (a) and (b).

Now let for $1 \le j \le \ell''$,

$$N_j = \mathrm{Ind}_{\mathfrak{S}_{\ell'+\lambda''_1+\ldots+\lambda''_{i-1}}\times\mathfrak{S}_{\lambda''_j}}^{\mathfrak{S}_{\ell'+\lambda''_1+\cdots+\lambda''_j}}(N_{j-1}\boxtimes\mathbb{S}^{(\lambda''_j)}),$$

with the convention that $N_0 = M_{\ell'}$. For $\nu \vdash \lambda''_1 + \ldots + \lambda''_j$, let n^{ν}_j denote the multiplicity of $\mathbb{S}^{\nu}$ in N_j, and $n_j = \sum_{\nu\vdash\lambda''_1+\cdots+\lambda''_j} n^{\nu}_j$.

The following two statements are easily proved using induction on j. The proofs are very similar to the proofs of (a) and (b) above and are omitted.

(c)

$$n_j \le n_{j-1}\cdot\binom{\lambda''_j + \ell' + j - 1}{\ell' + j - 1}.$$

(d) For each $\nu \vdash \lambda''_1 + \cdots + \lambda''_j$, such that $n^{\nu}_j > 0$, $\mathrm{length}(\widetilde{\nu}) \le \ell' + j$.

It follows from (a), (b), (c), and (d), that

$$\sum_{\mu \vdash k} m^{\mu}_{\lambda',\lambda''} \leq k^{O(d^2)},$$

which proves (3). Finally, it is easy to check that for each μ with $m^{\mu}_{\lambda',\lambda''} > 0$ that arises in the above process satisfies

$$\mathrm{card}(\{i \mid \mu_i \geq d\}) \leq d, \mathrm{card}(\{j \mid \tilde{\mu}_j \geq d\}) \leq d,$$

which proves (1). □

Remark 3.22. The following particular case of Proposition 3.16 will be of interest. If $\mu = (k)$,

$$m^{\mu}_{\lambda',\lambda''} = 1.$$

3.6. **Equivariant Poincaré duality.** In this section, we derive an equivariant version of Poincaré duality for real algebraic hypersurfaces (Theorem 3.23) that was used in analyzing Example 1.19.

Let $\mathrm{Z}(P, \mathrm{R}^k)$ be a non-singular, bounded real algebraic hypersurface. Then, V is an orientable submanifold of R^k. We pick the the orientation of V which is determined by the choice of normal vector $\mathrm{grad}(P)(x) = (\frac{\partial P}{\partial X_1}(x), \ldots, \frac{\partial P}{\partial X_k}(x))^t$ at each point $x \in \mathrm{Z}(P, \mathrm{R}^k)$, and refer to this orientation as the *canonical orientation* of $\mathrm{Z}(P, \mathrm{R}^k)$.

Note that the block size is equal to one in the following theorem.

Theorem 3.23. *Let $V = \mathrm{Z}(P, \mathrm{R}^k) \subset \mathrm{R}^k$ be a bounded non-singular real algebraic hypersurface which is stable under the standard action of $\mathfrak{S}_k$ on R^k. Then, for each $p, 0 \leq p \leq k$, there is an $\mathfrak{S}_k$-module isomorphism*

$$\mathrm{H}^p(V, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^{k-p-1}(V, \mathbb{F}) \otimes \mathbf{sign}_k.$$

Proof. If M is a C^0-manifold of dimension ℓ, then the following sheaf-theoretic statement of Poincaré duality is well known (see for example [28, Corollary 5.5.6]).

$$\hom_{\mathbb{F}}(\mathrm{H}^*_c(M; \mathbb{F}_M), \mathbb{F}) \cong \mathrm{H}^*(M; \mathrm{or}_M)[\ell]. \tag{3.4}$$

In our case, taking $M = V$, the $\mathfrak{S}_k$-action on the ambient space R^k, induces an $\mathfrak{S}_k$-module structure on both sides of (3.4) making the isomorphism in (3.4) an isomorphism between $\mathfrak{S}_k$-modules.

Notice that choosing a everywhere non-vanishing global section $s \in \Gamma(\mathrm{or}_V)$ gives rise to an isomorphism $\phi_s : \mathrm{H}^p(V; \mathrm{or}_V) \to \mathrm{H}^p(V; \mathbb{F}_V)$. The isomorphism ϕ_s need not be $\mathfrak{S}_k$-equivariant (if V is not connected). However, if we pick s to be the canonical orientation of $\mathrm{Z}(P, \mathrm{R}^k)$ defined previously, then since for every transposition (and thus every odd permutation) π, the action of π on R^k is orientation reversing, we get the following $\mathfrak{S}_k$-isomorphism for each $p \geq 0$,

$$\mathrm{H}^p(V; \mathrm{or}_V) \cong \mathrm{H}^p(V; \mathbb{F}_V) \otimes \mathbf{sign}_k. \tag{3.5}$$

The theorem follows from (3.4) and (3.5), after noting that since V is assumed to be bounded

$$\hom_{\mathbb{F}}(\mathrm{H}^*(V, \mathrm{C}), \mathbb{F}) \cong \mathrm{H}^*_c(V, \mathbb{F}) \cong \mathrm{H}^*(V, \mathbb{F})$$

where all isomorphisms are $\mathfrak{S}_k$-module isomorphisms. □

3.7. **Equivariant Mayer-Vietoris inequalities.** In this section we derive equivariant versions of Mayer-Vietoris inequalities that we will need to obtain bounds on the multiplicities of the various Specht-modules in the cohomology modules of symmetric varieties and semi-algebraic sets that we consider. We will use Propositions 3.24 and 3.25.

Suppose that $S_1, S_2 \subset \mathrm{R}^K$ are $\mathfrak{S}_{\mathbf{k}}$-symmetric closed semi-algebraic sets. Then $S_1 \cup S_2$, and $S_1 \cap S_2$ are also $\mathfrak{S}_{\mathbf{k}}$-symmetric closed semi-algebraic sets, and there is the classical Mayer-Vietoris exact sequence,

$$\cdots \to \mathrm{H}^i(S_1 \cup S_2, \mathbb{F}) \to \mathrm{H}^i(S_1, \mathbb{F}) \oplus \mathrm{H}^i(S_2, \mathbb{F}) \to \mathrm{H}^i(S_1 \cap S_2, \mathbb{F}) \to \mathrm{H}^{i+1}(S_1 \cup S_2, \mathbb{F}) \to \cdots$$

where all the homomorphisms are $\mathfrak{S}_{\mathbf{k}}$-equivariant. Denoting by $\mathrm{H}^*(S, \mathbb{F})_{\boldsymbol{\mu}}$ the isotypic component of $\mathrm{H}^*(S, \mathbb{F})$ corresponding to $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$ for any $\mathfrak{S}_{\mathbf{k}}$-symmetric closed semi-algebraic set $S \subset \mathrm{R}^k$, we obtain using Schur's lemma for each $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$, an exact sequence,

$$\cdots \to \mathrm{H}^i(S_1 \cup S_2, \mathbb{F})_{\boldsymbol{\mu}} \to \mathrm{H}^i(S_1, \mathbb{F})_{\boldsymbol{\mu}} \oplus \mathrm{H}^i(S_2, \mathbb{F})_{\boldsymbol{\mu}} \to \mathrm{H}^i(S_1 \cap S_2, \mathbb{F})_{\boldsymbol{\mu}} \to \mathrm{H}^{i+1}(S_1 \cup S_2, \mathbb{F})_{\boldsymbol{\mu}} \to \cdots$$

The following inequalities follow from the above exact sequence (the proofs are similar to the non-equivariant case and can be found in [9]).

Let $S_1, \ldots, S_s \subset \mathrm{R}^K$, $s \geq 1$, be $\mathfrak{S}_{\mathbf{k}}$-symmetric closed semi-algebraic sets of R^K, contained in a $\mathfrak{S}_{\mathbf{k}}$-symmetric closed semi-algebraic set T.

For $1 \leq t \leq s$, let $S_{\leq t} = \bigcap_{1 \leq j \leq t} S_j$, and $S^{\leq t} = \bigcup_{1 \leq j \leq t} S_j$. Also, for $J \subset \{1, \ldots, s\}$, $J \neq \emptyset$, let $S_J = \bigcap_{j \in J} S_j$, and $S^J = \bigcup_{j \in J} S_j$. Finally, let $S^\emptyset = T$.

Proposition 3.24. *(a) For $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$ and $i \geq 0$,*

$$m_{i,\boldsymbol{\mu}}(S^{\leq s}, \mathbb{F}) \leq \sum_{j=1}^{i+1} \sum_{\substack{J \subset \{1,\ldots,s\} \\ \mathrm{card}(J)=j}} m_{i-j+1,\boldsymbol{\mu}}(S_J, \mathbb{F}).$$

(b) For $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$ and $0 \leq i \leq K$,

$$m_{i,\boldsymbol{\mu}}(S_{\leq s}, \mathbb{F}) \leq \sum_{j=1}^{K-i} \sum_{\substack{J \subset \{1,\ldots,s\} \\ \mathrm{card}(J)=j}} m_{i+j-1,\boldsymbol{\mu}}(S^J, \mathbb{F}) + \binom{s}{K-i} m_{K,\boldsymbol{\mu}}(S^\emptyset, \mathbb{F}).$$

Proof. Follows from the proof of [9, Proposition 7.33] and Schur's lemma. □

Proposition 3.25. *If S_1, S_2 are $\mathfrak{S}_{\mathbf{k}}$-symmetric closed semi-algebraic sets, then for $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$, any field $\mathbb{F}$ and every $i \geq 0$*

$$m_{i,\boldsymbol{\mu}}(S_1, \mathbb{F}) + m_{i,\boldsymbol{\mu}}(S_2, \mathbb{F}) \quad \leq \quad m_{i,\boldsymbol{\mu}}(S_1 \cup S_2, \mathbb{F}) + m_{i,\boldsymbol{\mu}}(S_1 \cap S_2, \mathbb{F}).$$

Proof. It follows from the proof of [9, Proposition 6.44] and Schur's lemma. □

3.8. **Descent spectral sequence.** In this section we derive an improvement on an inequality first obtained in [22] by taking advantage of the symmetry of the fibered products. The main result of this section that we will use later is Theorem 3.26 (which will be used in the proof of Theorem 2.14).

Suppose that $V \subset \mathrm{R}^{k+m}$ is a closed and bounded semi-algebraic set, and $\pi : V \to Y = \pi(V)$ is the projection on the first k coordinates restricted to V. Following

[22] we define for each $p \geq 0$,
(3.6)
$$W_\pi^{(p)}(V) = \underbrace{X \times_\pi \cdots \times_\pi X}_{p+1} = \{(\mathbf{y}, \mathbf{x}_0, \ldots, \mathbf{x}_p) \in \mathrm{R}^{k+(p+1)m} \mid (\mathbf{y}, \mathbf{x}_i) \in V, 0 \leq i \leq p\}.$$

Notice that $W_\pi^{(p)}(V)$ is $\mathfrak{S}_{\mathbf{k}(p)}$-symmetric semi-algebraic set, where

$$\mathbf{k}(p) = (\underbrace{1, \ldots, 1}_{k}, p+1),$$

and $\mathfrak{S}_{\mathbf{k}(p)}$ acts by permuting the blocks $\mathbf{x}_0, \ldots, \mathbf{x}_p$, and by identity on the remaining coordinates.

We have the following theorem. Following the same notation as above:

Theorem 3.26.

$$b(\pi(V), \mathbb{F}) \quad \leq \quad \sum_{0 \leq p < k} b_{\mathfrak{S}_{\mathbf{k}(p)}}(W_\pi^{(p)}(V), \mathbb{F}).$$

Proof. It is proved in [22], that there exists a spectral sequence whose E^1 term is given by $E^1_{p,q} = \mathrm{H}_q(W_\pi^{(p)}(X), \mathbb{F})$, and such that it converges to $\mathrm{H}_{p+q}(Y, \mathbb{F})$ in a finite number of steps. Note that from the definition of the spectral sequence it is easy to see that $E^1_{p,q}$ has the structure of an $\mathfrak{S}_{p+1,1^k}$-module, and so does each $E^r_{p,q}$ which are all sub-quotients of $E^1_{p,q}$. There is an isomorphism,

$$F_n : \bigoplus_{p+q=n} E^\infty_{p,q} \to \mathrm{H}_n(Y, \mathbb{F}),$$

and F_n restricted to $E^\infty_{p,q}$ is an $\mathfrak{S}_{\mathbf{k}(p)}$-module isomorphism onto its image, where the $\mathfrak{S}_{\mathbf{k}(p)}$-module structure on the image is the trivial one. This implies by Schur's lemma that

$$E^\infty_{p,q} = (E^\infty_{p,q})^{\mathfrak{S}_{\mathbf{k}(p)}},$$

and also that

$$\dim_{\mathbb{F}}(E^\infty_{p,q}) \leq \dim_{\mathbb{F}}(E^1_{p,q})^{\mathfrak{S}_{\mathbf{k}(p)}}.$$

Finally, observe that

$$(E^1_{p,q})^{\mathfrak{S}_{\mathbf{k}(p)}} \cong \mathrm{H}_q(W_\pi^{(p)}(X), \mathbb{F})^{\mathfrak{S}_{\mathbf{k}(p)}}.$$

The theorem follows after observing that

$$\dim_{\mathbb{F}}(\mathrm{H}_q(W_\pi^{(p)}(X), \mathbb{F})^{\mathfrak{S}_{\mathbf{k}(p)}}) = b_{\mathfrak{S}_{\mathbf{k}(p)}}(W_\pi^{(p)}(X), \mathbb{F}).$$

□

4. Proofs of the main theorems

We first prove a structural result that will be used in the proofs of the main theorem.

4.1. **Structural result.** We define for $\mathbf{k}, \mathbf{d}, \mathbf{m} \in \mathbb{Z}_{>0}^{\ell}$, a subset $\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$ of $\mathrm{Par}(\mathbf{k})$, and then prove (cf. Proposition 4.2) that $\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$ has the property that, only the irreducible representations of $\mathfrak{S}_{\mathbf{k}}$ associated to the elements from $\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$ can appear with positive multiplicity in the cohomology modules of symmetric varieties in R^K defined by a non-negative polynomial having degree bounded by $\mathbf{d}$.

Notation 4.1 (Definition of $\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$). For $\mathbf{k} \in \mathbb{Z}_{>0}^{\ell}, \boldsymbol{\lambda} \in \mathrm{Par}(\mathbf{k})$, we denote (cf. Notation 3.15)

$$\mathcal{I}(\boldsymbol{\lambda}) = \bigcup_{\substack{\lambda^{(i)} = \lambda^{(i)\prime} \coprod \lambda^{(i)\prime\prime} \\ 1 \leq i \leq \ell}} \{\boldsymbol{\mu} \mid \boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathrm{Par}(\mathbf{k}), m^{\mu^{(i)}}_{\lambda^{(i)\prime}, \lambda^{(i)\prime\prime}} > 0\}. \tag{4.1}$$

For $\mathbf{k}, \mathbf{d}, \mathbf{m} \in \mathbb{Z}_{>0}^{\ell}$, we denote

$$\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m}) := \bigcup_{\boldsymbol{\lambda} \in \mathrm{Par}(\mathbf{k}, (2\mathbf{d})^{\mathbf{m}})} \mathcal{I}(\boldsymbol{\lambda}). \tag{4.2}$$

If $\ell = 1$, $\mathbf{k} = (k), \mathbf{d} = (d), \mathbf{m} = (m)$, we will denote $\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$ by $\mathcal{I}(k, d, m)$. Notice that for $\mathbf{k}, \mathbf{d}, \mathbf{m} \in \mathbb{Z}_{>0}^{\ell}$,

$$\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m}) \quad = \quad \prod_{i=1}^{\ell} \mathcal{I}(k_i, d_i, m_i). \tag{4.3}$$

The following Proposition follows directly from Part (1) of Proposition 3.16 and the definition of $\mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$ (cf. Notation 4.1).

Proposition 4.2. *For $\mathbf{k}, \mathbf{d}, \mathbf{m} \in \mathbb{Z}_{>0}^{\ell}$, and $\boldsymbol{\mu} \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$,*

$$\mathrm{rank}(\boldsymbol{\mu}) \leq (2\mathbf{d})^{\mathbf{m}}.$$

Remark 4.3. Note that Proposition 4.2 implies that the Young diagram for each $\mu \in \mathcal{I}(k, d, m)$ is contained in the union of $(2d)^m$ rows and$(2d)^m$ columns. This is shown in Figure 1 for fixed d, m and large k. The shaded area inside the $k \times k$ sized box contains all possible Young diagrams of partitions of k. The darker part contains the partitions belonging to $\mathcal{I}(k, d, m)$.

Since for every $d \leq k$, it is clear that

$$\mathrm{card}(\{\mu \in \mathrm{Par}(k) | \mathrm{rank}(\mu) = d\}) \leq 2k^d,$$

it follows immediately from Proposition 4.2 that for every fixed d, m, $\mathrm{card}(\mathcal{I}(k, d, m))$ is bounded by a polynomial in k.

The main result of this section is the following.

Theorem 4.4. *Let $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{m} = (m_1, \ldots, m_\ell), \mathbf{d} = (d, \ldots, d) \in \mathbb{Z}_{>0}^{\ell}$, and $K = \sum_{i=1}^{\ell} k_i m_i$. Let $P \in \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$ be a non-negative $\mathfrak{S}_{\mathbf{k}}$-symmetric polynomial, with $\deg(P) \leq d$. Let $V = \mathrm{Z}(P, \mathrm{R}^K)$. Then, for all partitions $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$, the assumption $m_{\boldsymbol{\mu}}(V, \mathbb{F}) > 0$ implies that*

$$\boldsymbol{\mu} \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m}). \tag{4.4}$$

Moreover, for each $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$,

$$m_{\boldsymbol{\mu}}(V, \mathbb{F}) \quad \leq \sum_{\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k}, (2\mathbf{d})^{\mathbf{m}})} G(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{d}, \mathbf{m}), \tag{4.5}$$

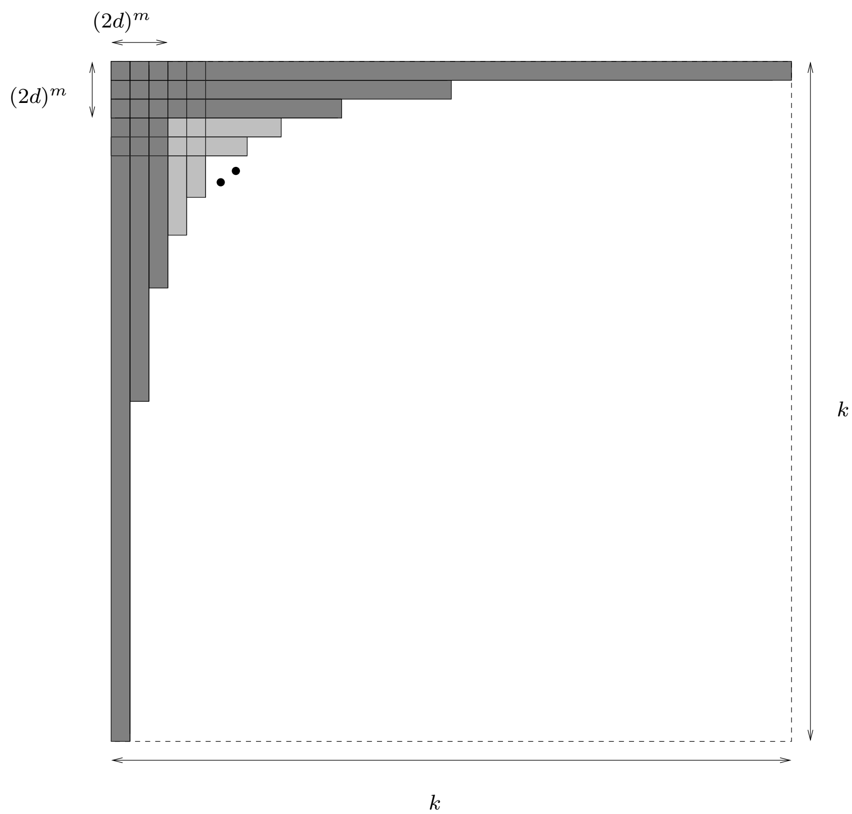


FIGURE 1. The shaded area contains all Young diagrams of partitions in $\mathrm{Par}(k)$, while the darker area contains the Young diagrams of the partitions in the subset $\mathcal{I}(k,d,m) \subset \mathrm{Par}(k)$ for fixed d,m and large k.

where

$$(4.6)\ G(\boldsymbol{\mu},\boldsymbol{\lambda},\mathbf{d},\mathbf{m}) \quad = \quad \prod_{1\leq i\leq \ell}\left((2d)^{(m_i \mathrm{length}(\lambda^{(i)}))} \max_{\lambda^{(i)}={\lambda^{(i)}}' \coprod {\lambda^{(i)}}''} m^{\mu^{(i)}}_{{\lambda^{(i)}}',{\lambda^{(i)}}''}\right)$$

(the maximum on the right hand side is taken over all decompositions $\lambda^{(i)} = {\lambda^{(i)}}' \coprod {\lambda^{(i)}}''$*).*

Proof of Theorem 4.4. We first assume that V is bounded. We replace V by the set S defined as the union of semi-algebraically connected components of the set defined by $\mathrm{Def}(P,d',\zeta) \leq 0$ which are bounded over R, where d' is the least even number such that $d' > d$ and where $d'-1$ is prime. It follows from Bertrand's postulate that $d' \leq 2d$. Using Proposition 3.10, Proposition 3.11, Lemma 3.4, Proposition 3.6, and Proposition 3.8, we see that for $\mathbb{S}^{\boldsymbol{\mu}}, \boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$ to occur with positive multiplicity in the isotypic decomposition of $\mathrm{H}^*(V,\mathbb{F})$, there must exist $\boldsymbol{\lambda} \in \mathrm{Par}(\mathbf{k},(2\mathbf{d})^{\mathbf{m}})$ such that $\boldsymbol{\mu} \in \overline{\mathrm{Par}}(\boldsymbol{\lambda})$, where $\overline{\mathrm{Par}}(\boldsymbol{\lambda})$ is defined in Notation 3.15. Eqn. (4.4) now follows from Eqn. (4.2), and inequality (4.5) follows from Part (3) of Proposition 3.16.

More generally, introduce a new block of one variable, Y, and let

$$\begin{aligned}
\widetilde{P} &= P + \left(\varepsilon^2 \cdot \sum_{1\leq h\leq \ell} \sum_{1\leq i\leq m_h} \sum_{1\leq j\leq k_h} (X_{i,j}^{(h)})^2 + Y^2 - 1\right)^2, \\
\widetilde{Q} &= P + \left(\varepsilon^2 \cdot \sum_{1\leq h\leq \ell} \sum_{1\leq i\leq m_h} \sum_{1\leq j\leq k_h} (X_{i,j}^{(h)})^2 - 1\right)^2,
\end{aligned}$$

and let

$$\begin{aligned}
\widetilde{V} &= \mathrm{Z}(\widetilde{P}, \mathrm{R}\langle\varepsilon\rangle^{K+1}), \\
\widetilde{W} &= \mathrm{Z}(\widetilde{Q}, \mathrm{R}\langle\varepsilon\rangle^{K}), \\
\widetilde{T} &= \mathrm{Extn}(V, \mathrm{R}\langle\varepsilon\rangle) \cap \overline{B_K(0, 1/\varepsilon)}.
\end{aligned}$$

We let $\mathbf{k}' = (\mathbf{k}, 1)$. Clearly, $\widetilde{V}$ is $\mathfrak{S}_{\mathbf{k}'}$-symmetric and bounded over $\mathrm{R}\langle\varepsilon\rangle$, and $\widetilde{W}$ is $\mathfrak{S}_{\mathbf{k}}$-symmetric and also bounded over $\mathrm{R}\langle\varepsilon\rangle$.

It follows from the local conical structure at infinity of semi-algebraic sets [15, Corollary 9.3.7], that:

i $\mathrm{Extn}(V, \mathrm{R}\langle\varepsilon\rangle)$ is semi-algebraically homeomorphic to $\widetilde{T}$; and
ii $\widetilde{V} = \widetilde{V}_+ \cup \widetilde{V}_-$, where for $\sigma \in \{+,-\}$, $\widetilde{V}_\sigma$ is the intersection of $\widetilde{V}$ with the half-space defined by $\sigma Y \geq 0$;
iii $\widetilde{T}$, and hence $\mathrm{Extn}(V, \mathrm{R}\langle\varepsilon\rangle)$, is semi-algebraically homeomorphic to each of $\widetilde{V}_+, \widetilde{V}_-$;
iv $\widetilde{W} = \widetilde{V}_+ \cap \widetilde{V}_-$.

It follows from Proposition 3.25 that

$$m_{\boldsymbol{\mu}}(V, \mathbb{F}) \leq \frac{1}{2}(m_{\boldsymbol{\mu}'}(\widetilde{V}, \mathbb{F}) + m_{\boldsymbol{\mu}}(\widetilde{W}, \mathbb{F})).$$

The theorem now follows from the bounded case proved before, noticing that the result in the bounded case implies that,

$$m_{\boldsymbol{\mu}'}(\widetilde{V}, \mathbb{F}), m_{\boldsymbol{\mu}}(\widetilde{W}, \mathbb{F})$$

are both bounded by

$$\sum_{\boldsymbol{\lambda}=(\lambda^{(1)},\ldots,\lambda^{(\ell)})\in \mathrm{Par}(\mathbf{k},(2\mathbf{d})^{\mathbf{m}})} G(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{d}, \mathbf{m}).$$

□

4.2. **Proofs of Theorems 2.5, 2.7, and 2.8.**

Proof of Theorem 2.5. Theorem 2.5 follows from Theorem 4.4, Proposition 4.2, and Part (3) of Proposition 3.16. □

Proof of Theorem 2.7. Theorem 2.7 follows from Theorem 4.4 and Remark 3.22. □

Proof of Theorem 2.8. Substituting $\mathbf{X}^{(j)} = \mathbf{Y}^{(j)} + i\mathbf{Z}^{(j)}, 1 \leq j \leq \ell$ in $\mathcal{P}$ and separating the real and imaginary parts, obtain another family of polynomials, $\mathcal{Q} \subset \mathrm{R}[\mathbf{Y}^{(1)}, \mathbf{Z}^{(1)}, \ldots, \mathbf{Y}^{(\ell)}, \mathbf{Z}^{(\ell)}]$ with $\deg_{\mathbf{Y}^{(j)}}(Q), \deg_{\mathbf{Z}^{(j)}}(Q) \leq d, 1 \leq j \leq \ell$, such that the polynomials in $\mathcal{Q}$ are $\mathfrak{S}_{\mathbf{k}}$-symmetric.

Now apply Theorem 2.5 with $\mathbf{k} = (k_1, \ldots, k_\ell)$, $\mathbf{m} = (2m_1, \ldots, 2m_\ell)$, and $\mathbf{d} = (d, \ldots, d)$. □

4.3. **Proof of Theorem 2.9.** We first prove the following result from which Theorem 2.9 will follow easily.

Theorem 4.5. *Let $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{m} = (m_1, \ldots, m_\ell), \mathbf{d} = (d, \ldots, d) \in \mathbb{Z}^\ell_{>0}$, and $K = \sum_{i=1}^\ell k_i m_i$. Let $\mathcal{P} \subset \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]^{\mathfrak{S}_\mathbf{k}}_{\leq \mathbf{d}}$ be a finite set of of polynomials, and let $\mathrm{card}(\mathcal{P}) = s$. Let $S \subset \mathrm{R}^K$ be a $\mathcal{P}$-closed semi-algebraic set.*

Then, for all partitions $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$, the assumption $m_{\boldsymbol{\mu}}(S, \mathbb{F}) > 0$ implies that

$$\boldsymbol{\mu} \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m}).$$

Moreover, let $D = D(\mathbf{k}, \mathbf{m}, d) = \sum_{i=1}^\ell \min(k_i m_i, d^{m_i})$. Then, for each

$$\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m}),$$

$$m_{\boldsymbol{\mu}}(S, \mathbb{F}) \quad \leq \quad \sum_{i=0}^{D-1} \sum_{j=1}^{D-i} \binom{2s+1}{j} 6^j \cdot \left(\sum_{\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k}, (4\mathbf{d})^{\mathbf{m}})} G(\boldsymbol{\mu}, \boldsymbol{\lambda}, 2\mathbf{d}, \mathbf{m}) \right),$$

where $G(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{d}, \mathbf{m})$ is defined in Eqn. (4.6).

Before proving Theorem 4.5 we first need a few preliminary definitions and results.

Definition 4.6 (ℓ-general position)**.** For any finite family $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ and $\ell \geq 0$, we say that $\mathcal{P}$ is in ℓ-general position with respect to a semi-algebraic set $V \subset \mathrm{R}^k$ if for any subset $\mathcal{P}' \subset \mathcal{P}$, with $\mathrm{card}(\mathcal{P}') > \ell$, $\mathrm{Z}(\mathcal{P}', V) = \emptyset$.

Let $\mathbf{k} = (k_1, \ldots, k_\ell), \mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell_{>0}$, and $K = \sum_{i=1}^\ell k_i m_i$. Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]^{\mathfrak{S}_\mathbf{k}}_{\leq d}$ be a finite set of polynomials, and let $S \subset \mathrm{R}^K$ be a $\mathcal{P}$-closed semi-algebraic set. Let $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s)$ be a tuple of new variables, and let $\mathcal{P}_{\bar{\varepsilon}} = \bigcup_{1 \leq i \leq s} \{P_i \pm \varepsilon_i\}$. We have the following two lemmas.

Lemma 4.7. *Let*

$$D(\mathbf{k}, \mathbf{m}, d) \quad = \quad \sum_{i=1}^\ell \min(k_i m_i, d^{m_i}).$$

The family $\mathcal{P}_{\bar{\varepsilon}} \subset \mathrm{R}'[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(\ell)}]$ is in D-general position with respect to any semi-algebraic subset $Z' \subset \mathrm{R}'^K$, where $\mathrm{R}' = \mathrm{R}\langle \bar{\varepsilon} \rangle$ (cf. Notation 3.1), and where $Z' = \mathrm{Extn}(Z, \mathrm{R}'^K)$ (cf. Notation 3.2), and $Z \subset \mathrm{R}^K$ is a semi-algebraic set stable under the action of $\mathfrak{S}_\mathbf{k}$.

Proof. The lemma follows from the fact that the ring of multi-symmetric polynomials is generated by the multi-symmetric power sum polynomials [18, Theorem 1.2], and the cardinality of the set of multi-symmetric power sum polynomials in the variables $X^{(i)}$ of degree bounded by d is bounded by d^{m_i}. □

Let Φ be a $\mathcal{P}$-closed formula, and let $S = \mathcal{R}(\Phi, V)$ be bounded over R.

Notation 4.8 (Multiplicities)**.** For $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$ and $i \geq 0$, we will denote

$$\begin{aligned} m_{i,\boldsymbol{\mu}}(\Phi, \mathbb{F}) &= m_{i,\boldsymbol{\mu}}(S, \mathbb{F}), \\ m_{\boldsymbol{\mu}}(\Phi, \mathbb{F}) &= m_{\boldsymbol{\mu}}(S, \mathbb{F}). \end{aligned}$$

Let $\Phi_{\overline{\varepsilon}}$ be the $\mathcal{P}_{\overline{\varepsilon}}$-closed formula obtained from Φ be replacing for each $i, 1 \leq i \leq s$,

i. each occurrence of $P_i \leq 0$ by $P_i - \varepsilon_i \leq 0$, and
ii. each occurrence of $P_i \geq 0$ by $P_i + \varepsilon_i \geq 0$.

Let $\mathrm{R}' = \mathrm{R}\langle \varepsilon_1, \ldots, \varepsilon_s \rangle$, and $S_{\overline{\varepsilon}} = \mathcal{R}(\Phi_{\overline{\varepsilon}}, \mathrm{R}'^K)$.

Lemma 4.9. *For any $r > 0$, $r \in \mathrm{R}$, the semi-algebraic set set $\mathrm{Extn}(S \cap \overline{B_K(0,r)}, \mathrm{R}')$ is contained in $S_{\overline{\varepsilon}} \cap \overline{B_K(0,r)}$, and the inclusion $\mathrm{Extn}(S \cap \overline{B_K(0,r)}, \mathrm{R}') \hookrightarrow S_{\overline{\varepsilon}} \cap \overline{B_K(0,r)}$ is a semi-algebraic homotopy equivalence. The induced isomorphism,*

$$\mathrm{H}(S_{\overline{\varepsilon}} \cap \overline{B_K(0,r)}, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^*(\mathrm{Extn}(S \cap \overline{B_K(0,r)}, \mathrm{R}'), \mathbb{F})$$

is an isomorphism of $\mathfrak{S}_{\mathbf{k}}$-modules.

Proof. The proof is similar to the one of Lemma 16.17 in [9]. □

Remark 4.10. In view of Lemmas 4.7 and 4.9 we can assume (at the cost of doubling the number of polynomials) after possibly replacing $\mathcal{P}$ by $\mathcal{P}_{\overline{\varepsilon}}$, and R by $\mathrm{R}\langle \varepsilon_1, \ldots, \varepsilon_s \rangle$, that the family $\mathcal{P}$ is in $D(\mathbf{k}, \mathbf{m}, d)$-general position.

Now, let $\delta_1, \cdots, \delta_s$ be new infinitesimals, and let $\mathrm{R}' = \mathrm{R}\langle \delta_1, \ldots, \delta_s \rangle$.

Notation 4.11 (Infinitesimal thickening)**.** We define $\mathcal{P}_{>i} = \{P_{i+1}, \ldots, P_s\}$ and

$$\begin{aligned} \Sigma_i &= \{P_i = 0, P_i = \delta_i, P_i = -\delta_i, P_i \geq 2\delta_i, P_i \leq -2\delta_i\}, \\ \Sigma_{\leq i} &= \{\Psi \mid \Psi = \bigwedge_{j=1,\ldots,i} \Psi_i, \Psi_i \in \Sigma_i\}. \end{aligned}$$

Note that for each $\Psi \in \Sigma_i$, $\mathcal{R}(\Psi, \mathrm{R}\langle \delta_1, \ldots, \delta_i \rangle^K)$ is symmetric with respect to the action of $\mathfrak{S}_{\mathbf{k}}$, and for $\Psi \neq \Psi'$, $\Psi, \Psi' \in \Sigma_{\leq i}$,

$$\mathcal{R}\left(\Psi, \mathrm{R}\langle \delta_1, \ldots, \delta_i \rangle^K\right) \cap \mathcal{R}\left(\Psi', \mathrm{R}\langle \delta_1, \ldots, \delta_i \rangle^K\right) = \emptyset. \tag{4.7}$$

If Φ is a $\mathcal{P}$-closed formula, we denote

$$\mathcal{R}_i(\Phi) = \mathcal{R}(\Phi, \mathrm{R}\langle \delta_1, \ldots, \delta_i \rangle^K),$$

and

$$\mathcal{R}_i(\Phi \wedge \Psi) = \mathcal{R}(\Psi, \mathrm{R}\langle \delta_1, \ldots, \delta_i \rangle^K) \cap \mathcal{R}_i(\Phi).$$

The proof of the following proposition is very similar to Proposition 7.39 in [9] where it is proved in the non-symmetric case.

Proposition 4.12. *For every $\mathcal{P}$-closed formula Φ, and $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$, such that $\mathcal{R}(\Phi)$ is bounded,*

$$m_{\boldsymbol{\mu}}(\Phi, \mathbb{F}) \leq \sum_{\substack{\Psi \in \Sigma_{\leq s} \\ \mathcal{R}_s(\Psi, \mathrm{R}'^K) \subset \mathcal{R}_s(\Phi, \mathrm{R}'^K)}} m_{\boldsymbol{\mu}}(\Psi, \mathbb{F}).$$

Proof. The symmetric spaces $\mathcal{R}(\Psi, \mathrm{Extn}(V, \mathrm{R}'))$, $\Psi \in \Sigma_{\leq s}$ are disjoint by (4.7). The proposition now follows from Schur's lemma, and the proof of Proposition 7.39 in [9]. □

Proposition 4.13. *Suppose for $\boldsymbol{\mu} \in \mathrm{Par}(\mathbf{k})$ and $i \geq 0$, $m_{i,\boldsymbol{\mu}}(S, \mathbb{F}) > 0$. Then,*

$$\boldsymbol{\mu} \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m}), \tag{4.8}$$

where $\mathbf{d} = (d, \ldots, d)$. For $i \geq 0$, and $\boldsymbol{\mu} \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$,

$$\sum_{\Psi \in \Sigma_{\leq s}} m_{i,\boldsymbol{\mu}}(\Psi, \mathbb{F}) \leq \sum_{j=0}^{D(\mathbf{k},\mathbf{m},d)} \binom{s}{j} 6^j F(\boldsymbol{\mu}, \mathbf{k}, \mathbf{m}, 2d),$$

where

$$F(\boldsymbol{\mu}, \mathbf{k}, \mathbf{m}, d) = \sum_{\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathrm{Par}(\mathbf{k}, (2\mathbf{d})^{\mathbf{m}})} G(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{d}, \mathbf{m}), \tag{4.9}$$

and $G(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{d}, \mathbf{m})$ is defined in (4.6).

In order to prove Proposition 4.13 we first need the following lemmas.
Let for $1 \leq i \leq s$, $Q_i = P_i^2(P_i^2 - \delta_i^2)^2(P_i^2 - 4\delta_i^2)$.
For $j \geq 1$ let,

$$\begin{aligned} V_j' &= \mathcal{R}(\bigvee_{1 \leq i \leq j} Q_i = 0, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K), \\ W_j' &= \mathcal{R}(\bigvee_{1 \leq i \leq j} Q_i \geq 0, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K). \end{aligned}$$

Lemma 4.14. *Let $I \subset [1, s]$, $\sigma = (\sigma_1, \ldots, \sigma_s) \in \{0, \pm 1, \pm 2\}^s$ and let $\mathcal{P}_{I,\sigma} = \bigcup_{i \in I}\{P_i + \sigma_i \delta_i\}$. Then, $\mathrm{Z}\left(\mathcal{P}_{I,\sigma}, \mathrm{R}'^K\right) = \emptyset$, whenever $\mathrm{card}(I) > D$.*

Proof. This follows from the fact that $\mathcal{P}$ is in D-general position by Remark 4.10. □

Lemma 4.15. *For each $\boldsymbol{\mu} \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$, and $i \geq 0$,*

$$m_{i,\boldsymbol{\mu}}(V_j', \mathbb{F}) \leq \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p F(\boldsymbol{\mu}, \mathbf{k}, \mathbf{m}, 2d)$$

(see (4.9) above for the definition of $F(\boldsymbol{\mu}, \mathbf{k}, \mathbf{m}, d)$).

Proof. The set $\mathcal{R}((P_j^2(P_j^2 - \delta_j^2)^2(P_j^2 - 4\delta_j^2) = 0), \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K)$ is the disjoint union of

$$\begin{gathered} \mathcal{R}(P_i = 0, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K), \\ \mathcal{R}(P_i = \delta_i, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K), \\ \mathcal{R}(P_i = -\delta_i, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K), \\ \mathcal{R}(P_i = 2\delta_i, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K), \\ \mathcal{R}(P_i = -2\delta_i, \mathrm{R}\langle \delta_1, \ldots, \delta_j \rangle^K). \end{gathered} \tag{4.10}$$

It follows from part (a) of Proposition 3.24 that $m_{i,\boldsymbol{\mu}}(V_j', \mathbb{F})$ is bounded by the sum for $1 \leq p \leq i+1$, of the multiplicities of $\mathbb{S}^{\boldsymbol{\mu}}$ in the $(i-p+1)$-th cohomology module of all possible non-empty sets obtained by the intersection of p distinct sets from amongst amongst the sets listed in (4.10). Because of the fact that the set of polynomials $\mathcal{P}$ is in D-general position it follows that all such intersections will be empty if $p > D$ or $p > j$. Hence, the total number of non-empty intersections that we need to consider is bounded by

$$\sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p.$$

It now follows from Theorem 4.4 applied to the non-negative symmetric polynomials $P_i^2, (P_i \pm \delta_i)^2, (P_i \pm 2\delta_i)^2$, and noting that the degrees of these polynomials are bounded by $2d$, that

$$m_{i,\boldsymbol{\mu}}(V'_j, \mathbb{F}) \quad \leq \quad \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p F(\boldsymbol{\mu}, \mathbf{k}, \mathbf{m}, 2d).$$

□

Lemma 4.16. *For each* $\boldsymbol{\mu} \in \mathcal{I}(\mathbf{k}, \mathbf{d}, \mathbf{m})$*, and* $i \geq 0$*,*

$$m_{i,\boldsymbol{\mu}}(W'_j, \mathbb{F}) \leq \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p F(\boldsymbol{\mu}, \mathbf{k}, \mathbf{m}, 2d) + m_{i,\boldsymbol{\mu}}(\mathrm{R}\langle\delta_1, \ldots, \delta_j\rangle^K, \mathbb{F}).$$

Proof. Let

$$T = \mathcal{R}\left(\bigwedge_{1 \leq i \leq j} Q_i \leq 0 \vee \bigvee_{1 \leq i \leq j} Q_i = 0, \mathrm{Extn}(Z, \mathrm{R}\langle\delta_1, \ldots, \delta_i\rangle)\right).$$

Now, from the fact that

$$W'_j \cup T = \mathrm{R}\langle\delta_1, \ldots, \delta_j\rangle^k, W'_j \cap T = V'_j,$$

and Proposition 3.25 it follows that

$$\begin{aligned} m_{i,\boldsymbol{\mu}}(W'_j, \mathbb{F}) &\leq m_{i,\boldsymbol{\mu}}((W'_j \cap T), \mathbb{F}) + m_{i,\boldsymbol{\mu}}((W'_j \cup T), \mathbb{F}) \\ &= m_{i,\boldsymbol{\mu}}(V'_j, \mathbb{F}) + m_{i,\boldsymbol{\mu}}(\mathrm{R}\langle\delta_1, \ldots, \delta_j\rangle^K, \mathbb{F}). \end{aligned}$$

We conclude using Lemma 4.15. □

Proof of Proposition 4.13. Using part (b) of Proposition 3.24 we get that

$$\begin{aligned} \sum_{\Psi \in \Sigma_{\leq s}} m_{i,\boldsymbol{\mu}}(\Psi, \mathbb{F}) \quad &\leq \quad \sum_{j=1}^{\min(D,K-i)} \quad \sum_{\substack{J \subset \{1,\ldots,s\} \\ \mathrm{card}(J) = j}} m_{i+j-1,\boldsymbol{\mu}}(S^J, \mathbb{F}) \\ &\quad + \binom{s}{K-i} m_{K,\boldsymbol{\mu}}(S^\emptyset, \mathbb{F}). \end{aligned}$$

It follows from Lemma 4.16 that,

$$m_{i+j-1,\boldsymbol{\mu}}(S^J) \quad \leq \quad \sum_{p=1}^{\min(j,D)} \binom{j}{p} 5^p F(\boldsymbol{\mu}, \mathbf{k}, \mathbf{m}, 2d) + m_{K,\boldsymbol{\mu}}(\mathrm{R}^K, \mathbb{F}).$$

Hence,

$$\begin{aligned}\sum_{\Psi\in\Sigma_{\leq s}} m_{i,\boldsymbol{\mu}}(\Psi,\mathbb{F}) &\leq \sum_{j=1}^{D}\sum_{\substack{J\subset\{1,\ldots,s\}\\ \mathrm{card}(J)=j}} m_{i+j-1,\boldsymbol{\mu}}(S^J,\mathbb{F}) + \binom{s}{K-i} m_{K,\boldsymbol{\mu}}(S^{\emptyset},\mathbb{F}) \\ &\leq \sum_{j=1}^{D}\binom{s}{j}\left(\sum_{p=1}^{\min(j,D)}\binom{j}{p}5^p F(\boldsymbol{\mu},\mathbf{k},\mathbf{m},2d)\right) \\ &\leq \sum_{j=1}^{D}\binom{s}{j}6^j F(\boldsymbol{\mu},\mathbf{k},\mathbf{m},2d).\end{aligned}$$

□

Proof of Theorem 4.5. We first add an extra polynomial, $\delta(X_1^2+\cdots+X_K^2)-1$ to the set $\mathcal{P}$, replace the field R, by $\mathrm{R}\langle\delta\rangle$, and replace the given formula $\mathcal{P}$-closed formula Φ by the formula $\Phi\wedge(\delta(X_1^2+\cdots+X_K^2)-1\leq 0)$. Notice that the new set $\mathrm{Reali}(\Phi)$ is bounded in $\mathrm{R}\langle\delta\rangle^k$ and is $\mathfrak{S}_{\mathbf{k}}$-symmetric. The theorem now follows from Propositions 4.12 and 4.13. □

Proof of Theorem 2.9. Follows immediately from Theorem 4.5 and Proposition 3.16. □

4.4. **Proof of Theorem 2.10.**

Proof of Theorem 2.10. Let $\mathbf{S}^{2k+1}\subset \mathrm{C}^{k+1}$ denote the unite sphere defined by $|Z_0|^2+\cdots+|Z_k|^2=1$. Consider the Hopf fibration $\phi:\mathbf{S}^{2k+1}\to\mathbb{P}^k_{\mathrm{C}}$, defined by $(z_0,\ldots,z_k)\mapsto(z_0:\cdots:z_k)$. We denote by $\tilde{V}=\phi^{-1}(V)$. We have the following commutative diagram:

$$\begin{array}{ccc}\tilde{V} & \xrightarrow{i} & \mathbf{S}^{2k+1}\\ \downarrow{\scriptstyle \phi|_{\tilde{V}}} & & \downarrow{\scriptstyle\phi}\\ V & \xrightarrow{i} & \mathbb{P}^k_{\mathrm{C}}\end{array}$$

Note that $\tilde{V}$ is a $\mathbf{S}^1$-bundle over V, and there is a $\mathfrak{S}_{k+1}$-equivariant spectral sequence degenerating at its E_3 term converging to the cohomology of $\tilde{V}$.

The E_3-term of the spectral sequence is given by

$$\begin{aligned}E_2^{p,q} &\cong \mathrm{H}_p(V,\mathbb{F}),\ \text{if } q=0,1,\\ E_2^{p,q} &= 0,\ \text{else },\end{aligned}$$

and the differentials $d_2^{p,q}:E_2^{p,q}\to E_2^{p+2,q-1}$ shown below.

$$\begin{tikzcd}[column sep=small]
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 \arrow[drr, "d_2^{-1,2}"] & 0 & 0 & \cdots & 0 \arrow[drr, "d_2^{i,2}"] & 0 & 0 \\
0 \arrow[drr, "d_2^{-1,1}"] & \mathrm{H}^0(V,\mathbb{F}) & \mathrm{H}^1(V,\mathbb{F}) & \cdots & \mathrm{H}^i(V,\mathbb{F}) \arrow[drr, "d_2^{i,1}"] & \mathrm{H}^{i+1}(V,\mathbb{F}) & \mathrm{H}^{i+2}(V,\mathbb{F}) \\
0 & \mathrm{H}^0(V,\mathbb{F}) & \mathrm{H}^1(V,\mathbb{F}) & \cdots & \mathrm{H}^i(V,\mathbb{F}) & \mathrm{H}^{i+1}(V,\mathbb{F}) & \mathrm{H}^{i+2}(V,\mathbb{F})
\end{tikzcd}$$

Fix $\lambda \vdash k+1$, and recall that we denote for each $i \geq 0$, $m_{i,\lambda}(V,\mathbb{F})$ (resp. $m_{i,\lambda}(\tilde{V},\mathbb{F})$) the multiplicity of $\mathbb{S}^\lambda$ in $\mathrm{H}^i(V,\mathbb{F})$ (resp. $\mathrm{H}^i(\tilde{V},\mathbb{F})$).

We observe that since $\mathrm{H}^0(V,\mathbb{F}) \cong_{\mathfrak{S}_{k+1}} \mathrm{H}^0(\tilde{V},F)$, we have for all $\lambda \vdash k+1$,

$$m_{0,\lambda}(V,\mathbb{F}) = m_{\lambda,0}(\tilde{V},\mathbb{F}). \tag{4.11}$$

Also, note that it follows from the fact that the spectral sequence $E_r^{p,q}$ degenerates at its E_3 term that,

$$\mathrm{H}^1(V,\mathbb{F}) \oplus \ker(d_2^{0,1}) \cong_{\mathfrak{S}_{k+1}} \mathrm{H}^1(\tilde{V},\mathbb{F}),$$

and we obtain from the fact that the spectral sequence $E^r_{p,q}$ is $\mathfrak{S}_{k+1}$-equivariant that

$$m_{1,\lambda}(V,\mathbb{F}) \leq m_{1,\lambda}(\tilde{V},\mathbb{F}). \tag{4.12}$$

More generally, we have from the E^2-term of the spectral sequence that

$$\mathrm{H}^i(\tilde{V},\mathbb{F}) \cong_{\mathfrak{S}_{k+1}} \mathrm{coker}(d_2^{i-2,1}) \oplus \ker(d_2^{i-1,1}).$$

For $\lambda \vdash k+1, i \geq 0$, and any finite dimensional $\mathbb{F}$-representation W of $\mathfrak{S}_{k+1}$, we denote by $\mathrm{mult}_\lambda(W,\mathbb{F})$ the multiplicity of $\mathbb{S}^\lambda$ in W.

Since,

$$\mathrm{H}^i(V,\mathbb{F}) \cong_{\mathfrak{S}_{k+1}} \mathrm{Im}(d_2^{i-2,1}) \oplus \mathrm{coker}(d_2^{i-2,1}),$$

we have for all $\lambda \vdash k+1, i \geq 0$,

$$\mathrm{mult}_\lambda(\mathrm{coker}(d_2^{i-2,1}),\mathbb{F}) = m_{i,\lambda}(V,\mathbb{F}) - \mathrm{mult}_\lambda(\mathrm{Im}(d_2^{i-2,1}),\mathbb{F}),$$

and we also have for $i \geq 2$,

$$\mathrm{mult}_\lambda(\mathrm{Im}(d_2^{i-2,1}),\mathbb{F}) \leq m_{i-2,\lambda}(V,\mathbb{F}). \tag{4.13}$$

This implies that for all $\lambda \vdash k+1, i \geq 0$

$$m_{i,\lambda}(\tilde{V},\mathbb{F}) = (m_{i,\lambda}(V,\mathbb{F}) - \mathrm{mult}_\lambda(\mathrm{Im}(d_2^{i-2,1}),\mathbb{F})) + \mathrm{mult}_\lambda(\mathrm{coker}(d_2^{i-1,1}),\mathbb{F}).$$

It follows that

$$\begin{aligned}
m_{i,\lambda}(V,\mathbb{F}) &= m_{i,\lambda}(\tilde{V},\mathbb{F}) + \mathrm{mult}_\lambda(\mathrm{Im}(d_2^{i-2,1}),\mathbb{F})) - \mathrm{mult}_\lambda(\mathrm{coker}(d_2^{i-1,1}),\mathbb{F}) \\
&\leq m_{i,\lambda,i}(\tilde{V},\mathbb{F}) + \mathrm{mult}_\lambda(\mathrm{Im}(d_2^{i-2,1}),\mathbb{F}) \\
&\leq m_{i,\lambda}(\tilde{V},\mathbb{F}) + m_{i-2,\lambda}(V,\mathbb{F}) \text{ using (4.13).}
\end{aligned}$$

Finally we have shown that for each $\lambda \vdash k+1$ and $i \geq 2$,

$$
\begin{aligned}
m_{i,\lambda}(V,\mathbb{F}) &\leq m_{i,\lambda}(\tilde{V},\mathbb{F}) + m_{i-2,\lambda}(V,\mathbb{F}) \\
&\leq \sum_{0\leq j\leq \lfloor \frac{i}{2}\rfloor} m_{i-2j,\lambda}(\tilde{V},\mathbb{F}) \text{ using induction.}
\end{aligned} \tag{4.14}
$$

The theorem follows from applying Theorem 2.5 to the set $\tilde{V}$, and inequalities (4.11), (4.12), and (4.14). □

Remark 4.17. Note that in the proof of Theorem 2.10 it is possible to replace the spectral sequence argument altogether by an argument using the equivariant version of the Gysin exact sequence.

4.5. **Proof of Theorem 2.14.**

Proof of Theorem 2.14. Let $P^{(p)} \in \mathrm{R}[\mathbf{X},\mathbf{Y}_0,\ldots,\mathbf{Y}_p]$ be defined by

$$
\begin{aligned}
P^{(p)} &= P(\mathbf{X},\mathbf{Y}_0) + \cdots + P(\mathbf{X},\mathbf{Y}_p), \\
V^{(p)} &= \mathrm{Z}(P^{(p)},\mathrm{R}^{k+(p+1)m}),
\end{aligned}
$$

and

$$
\mathbf{k}(p) = (\underbrace{1,\ldots,1}_{k}, p+1).
$$

Notice that since V is bounded, so is $V^{(p)} = \mathrm{Z}(P,\mathrm{R}^{k+(p+1)m})$, and moreover, $V^{(p)}$ is semi-algebraically homeomorphic to $W_\pi^{(p)}(V)$ (cf. Eqn. (3.6)). Moreover, $\deg(P^{(p)}) = \deg(P)$, and $P^{(p)}$ is symmetric in $(\mathbf{Y}_0,\ldots,\mathbf{Y}_p)$, and is thus $\mathfrak{S}_{\mathbf{k}(p)}$-symmetric.

By Theorem 3.26,

$$
b(\pi(V),\mathbb{F}) \leq \sum_{0\leq p<k} b_{\mathfrak{S}_{\mathbf{k}(p)}}(V^{(p)},\mathbb{F}).
$$

Now using Theorem 2.7,

$$
b_{\mathfrak{S}_{\mathbf{k}(p)}}(V^{(p)},\mathbb{F}) \leq (p+1)^{(2d)^m}(O(d))^{k+m(2d)^m+1},
$$

and hence,

$$
\begin{aligned}
b(\pi(V),\mathbb{F}) &\leq \sum_{0\leq p<k} b_{\mathfrak{S}_{\mathbf{k}(p)}}(V^{(p)},\mathbb{F}) \\
&\leq \sum_{0\leq p<k} (p+1)^{(2d)^m}(O(d))^{k+m(2d)^m+1} \\
&\leq k^{(2d)^m}(O(d))^{k+m(2d)^m+1}.
\end{aligned}
$$

This completes the proof of the theorem. □

5. Conclusion and open problems

In this paper we have proved polynomial bounds on the number and the multiplicities of the irreducible representations of the symmetric group (or more generally product of symmetric groups) that appear in the cohomology modules of symmetric real algebraic and more generally real semi-algebraic sets. We have given several applications of the main results, including to improve existing bounds on the topological complexity of sets defined as images of semi-algebraic maps, and proving lower bounds on the degrees etc. We end with some open problems and future research directions.

5.1. **Representational Stability Question.** The bounds on the multiplicities that we prove in this paper are all polynomial in the number of variables (for fixed degrees). Motivated by the recently developed theory of FI-modules [17] it makes sense to ask whether it is possible to prove some stability result as $k \to \infty$. We formulate one such question below.

Let $\mathbb{K}$ be a field, and let $A(\mathbb{K})$ denote the polynomial ring $\mathbb{K}[(X_i)_{i\in\mathbb{N}}]$ in the denumerable set of variables $\{X_1, X_2, \ldots\}$.

Let $\mathfrak{S}_\infty$ denote the infinite symmetric group, whose elements are bijections $\mathbb{N} \to \mathbb{N}$ which keep all but finitely many elements of $\mathbb{N}$ fixed.

We say that an ideal $I \subset A(\mathbb{K})$ is symmetric it is stable under the natural action of $\mathfrak{S}_\infty$ permuting the variables. We say that a symmetric ideal $I \subset A(\mathbb{K})$ is finitely generated, if there exists a finite subset $\mathcal{F} \subset A(\mathbb{K})$ such that I is generated by the orbits of the polynomials in $\mathcal{F}$ under the action of $\mathfrak{S}_\infty$.

Given a symmetric ideal I, we denote for each $k > 0$, $I_k = I \cap \mathbb{K}[X_1, \ldots, X_k]$, and $V_k(I) = \mathrm{Z}(I_k, \mathbb{K}^k)$. Clearly, $V_k(I)$ is $\mathfrak{S}_k$-symmetric.

Also, let $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0$ be any fixed partition, and for all $k \geq k_0 + \mu_1$, let

$$\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k. \tag{5.1}$$

It is a consequence of the hook-length formula (Eqn.(3.1)) that

$$\dim_{\mathbb{F}}(\mathbb{S}^{\{\mu\}_k}) = \frac{\dim_{\mathbb{F}}(\mathbb{S}_\mu)}{|\mu|!} P_\mu(k), \tag{5.2}$$

where $P_\mu(T)$ is a monic polynomial having distinct integer roots, and $\deg(P_\mu) = |\mu|$ (see [19, 7.2.2]).

Finally, for a fixed number $p \geq 0$ we pose the following question.

Question 5.1. *Let $I \subset A(\mathrm{R})$ be a finitely generated symmetric ideal. Does there exist a polynomial $P_{I,p,\mu}(k)$ such that for all sufficiently large k, $m_{p,\{\mu\}_k}(V_k(I), \mathbb{F}) = P_{I,p,\mu}(k)$? In conjunction with (5.2), a positive answer would imply that*

$$\dim_{\mathbb{F}}(\mathrm{H}^p(V_k(I), \mathbb{F}))_{\{\mu\}_k} = \frac{\dim_{\mathbb{F}}(\mathbb{S}_\mu)}{|\mu|!} P_{I,p,\mu}(k) P_\mu(k)$$

is also given by a polynomial for all large enough k.

In particular, taking $\mu = ()$ to be the empty partition, is it true that

$$m_{p,\{\mu\}_k}(V_k(I), \mathbb{F}) = b^p_{\mathfrak{S}_k}(V_k(I), \mathbb{F})$$

(that is the p-th equivariant Betti number of $V_k(I)$ cf. Notation 1.22) is given by a polynomial in k ?

A stronger question is to ask for a bound on the degree of $P_{I,p,\mu}(k)$ as a function of d, μ and p, where d is the maximum of the degrees of the generators of I.

Remark 5.2. Note that it follows from the results of this paper (Theorem 2.5) that there exists a polynomial $P_{I,p,\mu}(k)$ of degree $O(d^2)$ (where, d is the maximum of the degrees of the generators of I) with the property that

$$m_{p,\{\mu\}_k}(V_k(I), \mathbb{F}) \leq P_{I,p,\mu}(k)$$

for all $k \geq 0$.

Remark 5.3. Question 5.1 has a positive answer for the ideal $I \subset A(\mathrm{R})$, generated by the polynomial

$$f = X_1(X_1 - 1).$$

It is clear from the definition that in this case for each $k > 0$, $I_k = (X_1(X_1 - 1), \ldots, X_k(X_k - 1))$, and $V_k(I) = \{0,1\}^k$.

From the discussion in Example 1.19 we deduce that for each $p > 0, \mu \vdash k_0$, and for all large enough k,

$$m_{p,\{\mu\}_k}(V_k(I), \mathbb{F}) = 0.$$

For $p = 0$, and partitions μ with $\mathrm{length}(\mu) > 1$, we again have for all large enough k,

$$m_{p,\{\mu\}_k}(V_k(I), \mathbb{F}) = 0.$$

Finally, for $p = 0$, and any partition (k_0) of length ≤ 1, and for all $k \geq 2k_0$,

$$\begin{aligned} m_{0,\{\mu\}_k}(V_k(I), \mathbb{F}) &= 2(k - k_0) - k + 1 \text{ (using (5.1) and (1.5))} \\ &= k - 2k_0 + 1. \end{aligned}$$

Thus, $m_{p,\{\mu\}_k}(V_k(I), \mathbb{F})$ is given by a polynomial for all large k, for any fixed p and μ. Notice also that the degree of this polynomial is bounded by 1.

Remark 5.4. We point out one crucial difference between the stability asked for in Question 5.1 and what is usually meant by *representational stability* in the FI-module context. In the case of finitely generated FI-modules [17], the multiplicities $m_{p,\{\mu\}_k}$ are ultimately constant, and in topological applications of the theory, this leads to dimensions of homology groups of each fixed dimension (for example, those of the configuration spaces of some fixed manifold) stabilizing to some polynomial (this phenomenon is usually called homological stability). In our case however the multiplicities, $m_{p,\{\mu\}_k}$, can grow (albeit polynomially), and the dimensions of the homology groups can grow exponentially as seen in Example 1.19.

5.2. **Algorithmic Conjecture.** As mentioned in the Introduction, a polynomial bound on any topological invariant of a class of semi-algebraic sets usually implies also that there exists an algorithm with polynomially bounded complexity for computing it. Since we have proved that the multiplicities of the irreducible representations of $\mathfrak{S}_k$ appearing in the cohomology group of a symmetric $\mathcal{P}$-semi-algebraic set $S \subset \mathrm{R}^k$, where $\deg(P), P \in \mathcal{P}$ is bounded by a constant, is bounded by a polynomial function of $\mathrm{card}(\mathcal{P})$ and k, the mentioned principle implies that these multiplicities should be computable by an algorithm with polynomially bounded complexity (for fixed d). If this holds, then since the number of irreducibles that are allowed to appear with positive multiplicity is also polynomially bounded, and their respective dimensions are polynomially computable using the hook length formula (cf. Eqn. (3.13)), we deduce that once these multiplicities are computed, the dimensions of the cohomology groups of S (with coefficients in $\mathbb{Q}$) can be computed with polynomially bounded complexity.

This leads us to make the following algorithmic conjecture.

Conjecture 5.5. *For any fixed $d > 0$, there is an algorithm that takes as input the description of a symmetric semi-algebraic set $S \subset \mathrm{R}^k$, defined by a $\mathcal{P}$-closed formula, where $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k}_{\leq d}$ is a finite set of polynomials, and computes $m_{i,\lambda}(S, \mathbb{Q})$, for each $\lambda \vdash k$, and $m_{i,\lambda}(S, \mathbb{Q}) > 0$, as well as all the Betti numbers $b_i(S, \mathbb{Q})$, with complexity which is polynomial in* $\mathrm{card}(\mathcal{P})$ *and k.*

Remark 5.6. We note that Conjecture 5.5 is not completely unreasonable, since an analogous result for computing the generalized Euler-Poincaré characteristic of symmetric semi-algebraic sets has been proved in [12]. However, computing the Betti numbers of a semi-algebraic set is usually a much harder task than computing the Euler-Poincaré characteristic. More recently, an algorithm with polynomially bounded complexity has also been given for computing the multiplicities of the trivial representation (i.e. the numbers $m_{i,(k)}(S, \mathbb{Q})$ using the notation in Conjecture 5.5) [13].

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