A SHARP BOUNDEDNESS RESULT FOR RESTRICTED MAXIMAL OPERATORS OF VILENKIN-FOURIER SERIES ON MARTINGALE HARDY SPACES

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Abstract. The restricted maximal operators of partial sums with respect to bounded Vilenkin systems are investigated. We derive the maximal subspace of positive numbers, for which this operator is bounded from the Hardy space H_p to the Lebesgue space L_p for all $0<p\leq 1$. We also prove that the result is sharp in a particular sense.

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1. Introduction

Pointwise convergence problems are of fundamental importance in Fourier analysis, and as it is well known they are closely related to studying boundedness of associated maximal operators. In the present paper we will deal with maximal operators. Let us first recall in brief a historical development of this theory.

It is well-known (for details see e.g. [2] and [11]) that Vilenkin systems do not form bases in the space $L_1(G_m)$. Moreover, (for details see e.g. [18, 19]) there is a function in the martingale Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in $L_1(G_m)$-norm, but Watari [17] (see also Gosselin [10] and Young [20]) proved that there exist absolute constants c and c_p such that, for $n=1,2,...,$

$$\begin{aligned} \left\|S_nf\right\|_p &\leq c_p\left\|f\right\|_p, \quad f\in L_p(G_m), \quad 1<p<\infty, \\ \sup_{\lambda>0}\lambda\mu\left(\left|S_nf\right|>\lambda\right) &\leq c\left\|f\right\|_1, \quad f\in L_1(G_m), \quad \lambda>0. \end{aligned}$$

In [14] it was proved that there exists a martingale $f\in H_p(G_m)\,(0<p<1)$, such that

$$\sup_{n\in\mathbb{N}}\left\|S_{M_n+1}f\right\|_{L_{p,\infty}}=\infty.$$

The reason of divergence of $S_{M_n+1}f$ is that the Fourier coefficients of $f\in H_p(G_m)$ $(0<p<1)$ are not bounded (see Tephnadze [13]).

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Uniform and point-wise convergence and some approximation properties of the partial sums in $L_1(G_m)$ norms were investigated by Goginava [8, 9] and Avdispahić, Memić [1]. Some related results can also be found in the recent PhD thesis by Tephnadze [15]. Moreover, Fine [4] obtained sufficient condition for the uniform convergence it is in complete analogy with the Dini-Lipschitz condition. Guličev [12] estimated the rate of uniform convergence of a Walsh-Fourier series using Lebesgue constants and modulus of continuity. Uniform convergence of subsequences of partial sums was also investigated by Goginava and Tkebuchava [7]. This problem was considered for the Vilenkin group G_m by Fridli [5], Blahota [3] and Gát [6].

In [14] the following maximal operator was considered:

$$\widetilde{S}_p^* f := \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}\log^{[p]}(n+1)}, \quad 0<p\leq 1$$

(where $[x]$ denotes integer part of x). It was proved that the maximal operator $\widetilde{S}_p^*$ is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$. Moreover, if $0<p\leq 1$ and $\varphi:\mathbb{N}_+\to[1,\infty)$ is a non-decreasing function satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-1}\log^{[p]}(n+1)}{\varphi(n)} = +\infty,$$

then

$$\sup_{n\in\mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_{L_{p,\infty}(G_m)} = \infty, \text{ for } 0<p<1,$$

and

$$\sup_{n\in\mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_1 = \infty.$$

It is also known (for details see e.g. Weisz [19]) that

$$\|S_{n_k} f\|_1 \leq c\|f\|_1$$

holds if and only if

$$\sup_{k\in\mathbb{N}} \|D_{n_k}\|_1 < c < \infty,$$

where D_{n_k} denotes the n_kth Dirichlet kernel with respect to Vilenkin system. Moreover, the corresponding subsequence S_{n_k} of the partial sums S_n are bounded from the Hardy space $H_p(G_m)$ to the Hardy space $H_p(G_m)$, for all $p>0$.

It is also well-known (for details see e.g. Weisz [19] and Tephnadze [15]) that the following restricted maximal operator

$$S^{\#} f := \sup_{n\in\mathbb{N}} |S_{M_n} f|$$

is bounded from the martingale Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$, for all $p>0$.

In this paper we find the maximal subspace of positive numbers, for which the restricted maximal operator of partial sums with respect to Vilenkin systems in this subspace is bounded from the Hardy space H_p to the Lebesgue space L_p for all $0 < p \leq 1$. As applications, both some well-known and new results are pointed out.

The paper is organized as follows: Some Preliminaries (definitions, notations and basic facts) are presented in Section 2. The main result (Theorem 1) and some of its consequences (Corollaries 1-5) are presented and discussed in Section 3. Theorem 1 is proved in Section 5. For this proof we need some Lemmas, one of them is new and of independent interest (see Section 4).

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ and assume that $m := (m_0, m_1, \ldots)$ is a sequence of positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_k} with the product of the discrete topologies of Z_{m_k}'s.

The product measure μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is a Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence $m := (m_0, m_1, \ldots)$ is bounded, then G_m is called a bounded Vilenkin group, otherwise it is called an unbounded one. In the present paper we deal only with bounded Vilenkin groups.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \quad (x_k \in Z_{m_k}).$$

A base for the neighbourhood of G_m can be given as follows:

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m,\ n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

It is evident that

$$\overline{I_N} = \bigcup_{s=0}^{N-1} I_s \backslash I_{s+1}. \tag{1}$$

The generalized number system based on m is defined in the following way

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

Every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j, \qquad \text{where} \qquad n_j \in Z_{m_j}\ (j \in \mathbb{N})$$

and only a finite number of n_j's differ from zero.

Let

$$\langle n\rangle := \min\{j\in\mathbb{N} : n_j\neq 0\} \quad\text{and}\quad |n| := \max\{j\in\mathbb{N} : n_j\neq 0\},$$

that is $M_{|n|}\leq n\leq M_{|n|+1}$. Set

$$\rho\left(n\right) := |n| - \langle n\rangle\,, \quad \text{for all} \quad n\in\mathbb{N}.$$

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\left\|f\right\|_p := \left(\int_{G_m}\left|f\right|^p d\mu\right)^{1/p} \qquad \left(0<p<\infty\right).$$

The space $L_{p,\infty}\left(G_m\right)$ consists of all measurable functions f for which

$$\left\|f\right\|_{L_{p,\infty}} := \sup_{\lambda>0}\lambda\mu\left(f>\lambda\right)^{1/p} < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called Vilenkin system.

First, we define the complex valued function $r_k\left(x\right) : G_m\to\mathbb{C}$, the generalized Rademacher functions as

$$r_k\left(x\right) := \exp\left(2\pi\imath x_k/m_k\right) \qquad \left(\imath^2=-1,\ x\in G_m,\ k\in\mathbb{N}\right).$$

Let us define the Vilenkin system $\psi := (\psi_n : n\in\mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}\left(x\right) \qquad \left(n\in\mathbb{N}\right).$$

Specifically, we call this system the Walsh-Paley one if $m\equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2\left(G_m\right)$ (see e.g. [2, 16]).

Now, we present the usual definitions in Fourier analysis.

If $f\in L_1\left(G_m\right)$ we can establish Fourier coefficients, the partial sums of Fourier series, Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\begin{aligned}
\widehat{f}\left(k\right) &:= \int_{G_m} f\overline{\psi}_k d\mu, \quad \left(k\in\mathbb{N}\right),\\
S_n f &:= \sum_{k=0}^{n-1}\widehat{f}\left(k\right)\psi_k, \quad \left(n\in\mathbb{N}_+,\ S_0 f := 0\right),\\
D_n &:= \sum_{k=0}^{n-1}\psi_k, \quad \left(n\in\mathbb{N}_+\right).
\end{aligned}$$

Recall that (see [2])

$$D_{M_n}\left(x\right) = \begin{cases} M_n, & \text{if } x\in I_n,\\ 0, & \text{if } x\notin I_n,\end{cases} \tag{2}$$

and

$$D_n(x)=\psi_n(x)\left(\sum_{j=0}^{\infty}D_{M_j}(x)\sum_{u=m_j-n_j}^{m_j-1}r_j^u(x)\right). \tag{3}$$

The σ-algebra generated by the intervals $\{I_n(x): x\in G_m\}$ will be denoted by $\digamma_n$ $(n\in\mathbb{N})$. Let us denote a martingale with respect to $\digamma_n$ $(n\in\mathbb{N})$ by $f=(f_n: n\in\mathbb{N})$ (for details see e.g. [18]). The maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbb{N}}\left|f^{(n)}\right|.$$

In the case $f\in L_1(G_m)$, the maximal function is also given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{\mu(I_n(x))}\left|\int_{I_n(x)}f(u)\,d\mu(u)\right|$$

For $0<p<\infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

A bounded measurable function a is a p-atom, if there exists an interval I, such that

$$\int_I a\,d\mu=0,\quad \|a\|_\infty\le\mu(I)^{-1/p},\quad \operatorname{supp}(a)\subseteq I.$$

The Hardy martingale spaces $H_p(G_m)$ have an atomic characterization for $0<p\le1$. In fact the following theorem is true (see e.g. Weisz [18, 19]):

Theorem W. *A martingale $f=(f_n: n\in\mathbb{N})\in H_p(G_m)$ $(0<p\le1)$ if and only if there exists a sequence $(a_k:\ k\in\mathbb{N})$ of p-atoms and a sequence $(\mu_k:\ k\in\mathbb{N})$ of real numbers, such that for every $n\in\mathbb{N}$,*

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=f_n \tag{4}$$

and

$$\sum_{k=0}^{\infty}|\mu_k|^p<\infty.$$

Moreover,

$$\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (4).

If $f\in L_1(G_m)$, then it is easily shown that the sequence $(S_{M_n}f: n\in\mathbb{N})$ is a martingale.

If $f=(f_n,n\in\mathbb{N})$ is a martingale, then Vilenkin-Fourier coefficients are defined in a slightly different manner:

$$\widehat{f}(i):=\lim_{k\to\infty}\int_{G_m}f_k(x)\overline{\psi}_i(x)\,d\mu(x).$$

Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as the martingale $(S_{M_n}f:n\in\mathbb{N})$ obtained from f.

3. The Main Result

Our main theorem reads as follows:

Theorem 1. *a) Let $0<p\leq 1$ and $\{\alpha_k:k\in\mathbb{N}\}$ be a subsequence of positive natural numbers, such that*

$$\sup_{k\in\mathbb{N}}\rho(\alpha_k)=:\varkappa<\infty. \tag{5}$$

Then the maximal operator

$$\widetilde{S}^{*,\triangle}f:=\sup_{k\in\mathbb{N}}|S_{\alpha_k}f|$$

is bounded from the Hardy space H_p to the Lebesgue space L_p.

b) Let $0<p<1$ and $\{\alpha_k:k\in\mathbb{N}\}$ be a subsequence of positive natural numbers satisfying the condition

$$\sup_{k\in\mathbb{N}}\rho(\alpha_k)=\infty. \tag{6}$$

Then there exists a martingale $f\in H_p$ such that

$$\sup_{k\in\mathbb{N}}\|S_{\alpha_k}f\|_{L_{p,\infty}}=\infty.$$

Remark 1. *Since $L_p\subset L_{p,\infty}$ part b) means in particular that the statement in part a) is sharp in a special sense for the case $0<p<1$.*

We also mention the following well-known consequences (for details see e.g. the books [18, 19] and [14]):

Corollary 1 (Tephnadze [14])**.** *Let $0<p\leq 1$ and $f\in H_p$. Then the maximal operator*

$$\sup_{n\in\mathbb{N}_+}|S_{M_n+1}f|$$

is not bounded from the Hardy space H_p to the space L_p.

In fact, we only have to notice that

$$|M_n+1|=n,\quad \langle M_n+1\rangle=0,\quad \rho(M_n+1)=n.$$

The second part of Theorem 1 implies our Corollary.

Corollary 2. *Let $p>0$ and $f\in H_p$. Then the maximal operator*

$$\sup_{n\in\mathbb{N}_+}\left|S_{M_n+M_{n-1}}f\right|$$

is bounded from the Hardy space H_p to the space L_p.

We notice that

$$|M_n + M_{n-1}| = n, \quad \langle M_n + M_{n-1}\rangle = n-1, \quad \rho\left(M_n + M_{n-1}\right) = 1.$$

Thus, the second part of Theorem 1 gives again Corollary 2.

Corollary 3. *Let $p>0$ and $f\in H_p$. Then the maximal operator*

$$S^{\#}f := \sup_{n\in\mathbb{N}} |S_{M_n} f|$$

is bounded from the Hardy space H_p to the space L_p.

We find that $|M_n| = \langle M_n\rangle = n$, $\rho\left(M_n\right)=0$. Using part b) of Theorem 1, we immediately get Corollary 3.

Since $S_nP = P$ for every $P\in\mathcal{P}$, where $\mathcal{P}$ is the set of all Vilenkin polynomials. The set $\mathcal{P}$ is dense in the space $L_1(G_m)$. Combining Lemma 1 and part a) of Theorem 1, we obtain that under condition (5) the restricted maximal operator of partial sums is bounded from the space $L_1(G_m)$ to the space $weak-L_1(G_m)$ It follows that

Corollary 4. *Let $f\in L_1$ and $\{\alpha_k : k\in\mathbb{N}\}$ be a subsequence of positive natural numbers, satisfying condition (5). Then*

$$S_{\alpha_k} f \to f \ \textit{a.e. when} \ \ k\to\infty.$$

Corollary 5. *Let $f\in L_1$. Then*

$$S_{M_n} f \to f \quad \textit{a.e.} \quad \textit{when} \ \ n\to\infty.$$

4. Lemmas

First, we note the following well-known result, which was proved in Weisz [18, 19]:

Lemma 1. *Suppose that an operator T is sub-linear and, for some $0<p\le 1$*

$$\int\limits_{\overline{I}} |Ta|^p \, d\mu \le c_p < \infty$$

for every p-atom a, where I denotes the support of the atom a. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \le c_p \|f\|_{H_p}.$$

Moreover, if $p<1$, then we have weak (1,1) type estimate, i.e. it holds that

$$\lambda\mu\left\{x\in G_m : \ |Tf(x)| > \lambda\right\} \le \|f\|_1$$

for all $f\in L_1$.

The next Lemma can be found in Tephnadze [13]:

Lemma 2. *Let $n\in\mathbb{N}$ and $x\in I_s\backslash I_{s+1}$, $0\le s\le N-1$. Then*

$$\int_{I_N} \left|D_n\left(x-t\right)\right| d\mu\left(t\right) \le \frac{cM_s}{M_N}.$$

We also need the following estimate of independent interest:

Lemma 3. *Let $n \in \mathbb{N}$, $|n| \neq \langle n\rangle$ and $x \in I_{\langle n\rangle+1}\left(e_{\langle n\rangle}\right)$ where $e_k := (0,\dots,0,1,0,\dots) \in G_m$ (only the k-th coordinate is one, the others are zero). Then*

$$\left|D_n(x)\right| = \left|D_{n-M_{|n|}}(x)\right| \geq M_{\langle n\rangle}.$$

Proof. Let $x \in I_{\langle n\rangle+1}\left(e_{\langle n\rangle}\right)$. Since

$$n = n_{\langle n\rangle}M_{\langle n\rangle} + \sum_{j=\langle n\rangle+1}^{|n|-1} n_j M_j + n_{|n|}M_{|n|}$$

and

$$n - M_{|n|} = n_{\langle n\rangle}M_{\langle n\rangle} + \sum_{j=\langle n\rangle+1}^{|n|-1} n_j M_j + \left(n_{|n|}-1\right)M_{|n|},$$

Applying (2) and (3) we can conclude that

$$\begin{aligned}
\left|D_{n-M_{|n|}}\right| &\geq \left|\psi_{n-M_n} D_{M_{\langle n\rangle}} \sum_{s=m_{\langle n\rangle}-n_{\langle n\rangle}}^{m_{\langle n\rangle}-1} r^s_{\langle n\rangle}\right| - \left|\psi_{n-M_n}\sum_{j=\langle n\rangle+1}^{|n|} D_{M_j}\sum_{s=m_j-n_j}^{m_j-1} r_j^s\right| \\
&= \left|D_{M_{\langle n\rangle}} \sum_{s=m_{\langle n\rangle}-n_{\langle n\rangle}}^{m_{\langle n\rangle}-1} r^s_{\langle n\rangle}\right| \\
&= \left|D_{M_{\langle n\rangle}} r_{\langle n\rangle}^{m_{\langle n\rangle}-n_{\langle n\rangle}} \sum_{s=0}^{n_{\langle n\rangle}-1} r^s_{\langle n\rangle}\right| \\
&= D_{M_{\langle n\rangle}} \left|\sum_{s=0}^{n_{\langle n\rangle}-1} r^s_{\langle n\rangle}\right|.
\end{aligned}$$

Let $x_n = 1$. Then we readily get for $s_n < m_n$ that

$$\begin{aligned}
\left|\sum_{u=0}^{s_n-1} r_n^u(x)\right| &= \left|\frac{r_n^{s_n}(x)-1}{r_n(x)-1}\right| \\
&= \frac{\sin(\pi s_n x_n/m_n)}{\sin(\pi x_n/m_n)} \\
&= \frac{\sin(\pi s_n/m_n)}{\sin(\pi/m_n)} \geq 1.
\end{aligned}$$

It follows that

$$\left|D_{n-M_{|n|}}(x)\right| \geq D_{M_{\langle n\rangle}}(x) \geq M_{\langle n\rangle}.$$

Moreover, by using the same arguments as above it is easily seen that

$$\left|D_n(x)\right| = \left|D_{n-M_{|n|}}(x)\right|, \quad \text{for } x \in I_{\langle n\rangle+1}\left(e_{\langle n\rangle}\right),\ |n| \neq \langle n\rangle,\ n \in \mathbb{N}.$$

The proof is complete. □

5. Proof of the main theorem

Proof of Theorem 1. First, we prove part a). Combining (2) and (3) we easily conclude that if condition (5) holds, then

$$\begin{aligned}\|D_{\alpha_k}\|_1 &\leq \sum_{j=\langle\alpha_k\rangle}^{|\alpha_k|} \left\|D_{M_j}\right\|_1 m_j \\ &\leq c\sum_{j=\langle\alpha_k\rangle}^{|\alpha_k|} 1 = c(\rho\left(\alpha_k\right)+1) \leq c < \infty.\end{aligned}$$

It follows that $\widetilde{S}^{*,\triangle}$ is bounded from L_∞ to L_∞. By Lemma 1 we obtain that the proof of part a) will be complete if we show that

$$\int\limits_{\overline{I_N}} \left|\widetilde{S}^{*,\triangle} a\right|^p d\mu \leq c < \infty$$

for every p-atom a with support $I = I_N$. Since $S_{\alpha_k}(a) = 0$ when $\alpha_k \leq M_N$, we can suppose that $\alpha_k > M_N$. (That is, $|\alpha_k| \geq N$.)

Let $t \in I_N$ and $x \in I_s \backslash I_{s+1}$, $1 \leq s \leq N-1$. If $\langle\alpha_k\rangle \geq N$, we get that $s < N \leq \langle\alpha_k\rangle$ and since $x - t \in I_s\backslash I_{s+1}$, by combining (2) and (3) we obtain that

$$D_{\alpha_k}(x-t) = 0. \tag{7}$$

Analogously, by combining again (2) and (3) we can conlude that (7) holds, for $s < \langle\alpha_k\rangle \leq N-1$.

It follows that

$$|S_{\alpha_k} a(x)| = 0, \quad \text{either} \quad \langle\alpha_k\rangle \geq N, \quad \text{or} \quad s < \langle\alpha_k\rangle \leq N-1. \tag{8}$$

Let $0 < p \leq 1$, $t \in I_N$ and $x \in I_s\backslash I_{s+1}$, $\langle\alpha_k\rangle \leq s \leq N-1$. Applying the fact that $\|a\|_\infty \leq M_N^{1/p}$ and Lemma 2 we find that

$$|S_{\alpha_k}(a)| \leq M_N^{1/p}\int_{I_N} |D_{\alpha_k}(x-t)|\, d\mu(t) \leq c_p M_N^{1/p-1} M_s.$$

Let us set $\varrho := \min_{k\in\mathbb{N}} \langle\alpha_k\rangle$. Then, in view of (8) and (9) we can conclude that

$$\left|\widetilde{S}^{*,\triangle} a(x)\right| = 0, \quad \text{for} \quad x \in I_s\backslash I_{s+1},\ 0 \leq s \leq \varrho \tag{9}$$

and

$$\left|\widetilde{S}^{*,\triangle} a(x)\right| \leq c_p M_N^{1/p-1} M_s, \text{ for } \ x \in I_s\backslash I_{s+1},\ \varrho < s \leq N-1. \tag{10}$$

By the definition of ϱ there exists at least one index $k_0 \in \mathbb{N}_+$ such that $\varrho = \langle \alpha_{k_0} \rangle$. By using contition (5) we can conclude that

$$\begin{aligned} N - \varrho &= N - \langle \alpha_{k_0} \rangle \leq |\alpha_{k_0}| - \langle \alpha_{k_0} \rangle \\ &\leq \sup_{k \in \mathbb{N}} \rho\left(\alpha_k\right) = \varkappa < c < \infty. \end{aligned}$$

Let us set $m_* := \sup_k m_k$.

Let $0 < p < 1$. According to (1) and using (9), (10) and (11) we obtain that

$$\begin{aligned} \int_{\overline{I_N}} \left| \widetilde{S}^{*,\vartriangle} a\left(x\right) \right|^p d\mu\left(x\right) &= \sum_{s=\varrho+1}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \widetilde{S}^{*,\vartriangle} a\left(x\right) \right|^p d\mu\left(x\right) \\ &\leq c_p M_N^{1-p} \sum_{s=\varrho+1}^{N-1} \frac{M_s^p}{M_s} \\ &= c_p M_N^{1-p} \sum_{s=\varrho+1}^{N-1} \frac{1}{M_s^{1-p}} \\ &\leq \frac{c_p M_N^{1-p}}{M_\varrho^{1-p}} \leq c_p m_*^{\varkappa(1-p)} \leq c_p < \infty. \end{aligned}$$

Let $p = 1$. We combine (9)-(11) and invoke identity (1) to obtain that

$$\begin{aligned} \int_{\overline{I_N}} \left| \widetilde{S}^{*,\vartriangle} a\left(x\right) \right| d\mu\left(x\right) &= \sum_{s=\varrho+1}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \widetilde{S}^{*,\vartriangle} a\left(x\right) \right| d\mu\left(x\right) \\ &\leq c \sum_{s=\varrho+1}^{N-1} \frac{M_s}{M_s} \\ &= c \sum_{s=\varrho+1}^{N-1} 1 \leq c\varkappa \leq c < \infty. \end{aligned}$$

The proof of part a) is complete.

Now, we prove the second part of our theorem. Since,

$$\frac{M_{|\alpha_k|}}{M_{\langle \alpha_k \rangle}} \geq 2^{\rho(\alpha_k)},$$

under condition (6), there exists an increasing subsequence $\{n_k : \ k \in \mathbb{N}\} \subset \{\alpha_k : \ k \in \mathbb{N}\}$ such that $n_0 \geq 3$ and

$$\lim_{k \to \infty} \frac{M_{|n_k|}^{(1-p)/2}}{M_{\langle n_k \rangle}^{(1-p)/2}} = \infty \tag{11}$$

and

$$\sum_{k=0}^{\infty} \frac{M_{\langle n_k \rangle}^{(1-p)/2}}{M_{|n_k|}^{(1-p)/2}} < c < \infty. \tag{12}$$

Let $f = (f_n : n \in \mathbb{N})$ be a martingale defined by

$$f_n := \sum_{\{k:\ |n_k| < n\}} \lambda_k a_k,$$

where

$$a_k := \frac{M_{|n_k|}^{1/p-1}}{m_*} \left(D_{M_{|n_k|+1}} - D_{M_{|n_k|}} \right)$$

and

$$\lambda_k = \frac{m_* M_{\langle n_k \rangle}^{(1/p-1)/2}}{M_{|n_k|}^{(1/p-1)/2}}. \tag{13}$$

It is easily seen that a is a p-atom. Under condition (12) we can conclude that $f \in H_p$. (Theorem W immediately yields that $\|f\|_{H_p} \le c_p < \infty$.)

According to (13) we readily see that

$$\widehat{f}(j) = \begin{cases} M_{\langle n_k \rangle}^{(1/p-1)/2} M_{|n_k|}^{(1/p-1)/2}, & j \in \left\{ M_{|n_k|}, ..., M_{|n_k|+1} - 1 \right\},\ k \in \mathbb{N}, \\ 0, & j \notin \bigcup\limits_{k=0}^{\infty} \left\{ M_{|n_k|}, ..., M_{|n_k|+1} - 1 \right\}. \end{cases}$$

Since, $M_{|n_k|} < n_k$, we get

$$\begin{aligned} S_{n_k} f &= \sum_{j=0}^{M_{|n_k|}-1} \hat{f}(j) \psi_j + \sum_{j=M_{|n_k|}}^{n_k-1} \hat{f}(j) \psi_j \\ &= S_{M_{|n_k|}} f + M_{\langle n_k \rangle}^{(1/p-1)/2} M_{|n_k|}^{(1/p-1)/2} \psi_{M_{|n_k|}} D_{n_k - M_{|n_k|}} \\ &:= I + II. \end{aligned}$$

According to part a) of Theorem 1 for I we have that

$$\|I\|_{L_{p,\infty}}^p \le \left\| S_{M_{|n_k|}} f \right\|_{L_{p,\infty}}^p \le c_p \|f\|_{H_p}^p \le c_p < \infty.$$

Moreover, under condition (6) we can conclude that

$$\langle n_k \rangle \ne |n_k| \ \text{ and } \ \left\langle n_k - M_{|n_k|} \right\rangle = \langle n_k \rangle .$$

Let $x \in I_{\langle n_k \rangle + 1} \left(e_{\langle n_k \rangle} \right)$. Applying Lemma 3 we obtain that

$$\left| D_{n_k - M_{|n_k|}} \right| \ge M_{\langle n_k \rangle}$$

Thus, we immediately have

$$\begin{aligned} |II| &= M_{\langle n_k\rangle}^{(1/p-1)/2} M_{|n_k|}^{(1/p-1)/2} \left| D_{n_k - M_{|n_k|}} \right| \\ &\geq M_{\langle n_k\rangle}^{(1/p+1)/2} M_{|n_k|}^{(1/p-1)/2}. \end{aligned}$$

It follows that

$$\begin{aligned} & \|II\|_{L_{p,\infty}}^p \\ \geq\ & c_p \left(M_{\langle n_k\rangle}^{(1/p+1)/2} M_{|n_k|}^{(1/p-1)/2} \right)^p \mu \left\{ x \in G_m :\ |II| \geq c_p M_{\langle n_k\rangle}^{(1/p+1)/2} M_{|n_k|}^{(1/p-1)/2} \right\} \\ \geq\ & c_p M_{|n_k|}^{(1-p)/2} M_{\langle n_k\rangle}^{(1+p)/2} \mu \left\{ I_{\langle n_k\rangle+1} \left(e_{\langle n_k\rangle} \right) \right\} \geq \frac{c_p M_{|n_k|}^{(1-p)/2}}{M_{\langle n_k\rangle}^{(1-p)/2}}. \end{aligned}$$

Hence, for large enough k,

$$\begin{aligned} & \|S_{n_k} f\|_{L_{p,\infty}}^p \\ \geq\ & \|II\|_{L_{p,\infty}}^p - \|I\|_{L_{p,\infty}}^p \geq \frac{1}{2} \|II\|_{L_{p,\infty}}^p \\ \geq\ & \frac{c_p M_{|n_k|}^{(1-p)/2}}{M_{\langle n_k\rangle}^{(1-p)/2}} \to \infty, \quad \text{when} \quad k \to \infty. \end{aligned}$$

The proof is complete. □

References

[1] M. Avdispahić and N. Memić, On the Lebesgue test for convergence of Fourier series on unbounded Vilenkin groups, *Acta Math. Hungar.,* 129 (2010), no. 4, 381-392.

[2] G. N. Agaev, N. Ya. Vilenkin, G. M. Dzafarly and A. I. Rubinshtein, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, Baku, Elm, 1981 (in Russian).

[3] I. Blahota, Approximation by Vilenkin-Fourier sums in $L_p(G_m)$, *Acta Acad. Paed. Nyiregy.,* 13(1992), 35-39.

[4] N. I. Fine, On Walsh function, *Trans. Amer. Math. Soc.,* 65 (1949), 372-414.

[5] S. Fridli, Approximation by Vilenkin-Fourier series, *Acta Math. Hungar.,* 47 (1-2), (1986), 33-44.

[6] G. Gát, Best approximation by Vilenkin-like systems, *Acta Acad. Paed. Nyiregy.,* 17(2001), 161-169.

[7] U. Goginava and G. Tkebuchava, Convergence of subsequence of partial sums and logarithmic means of Walsh-Fourier series, *Acta Sci. Math. (Szeged),* 72 (2006), 159-177.

[8] U. Goginava, On uniform convergence of Walsh-Fourier series, *Acta Math. Hungar.,* 93 (1-2) (2001), 59-70.

[9] U. Goginava, On approximation properties of partial sums of Walsh-Fourier series, *Acta Sci. Math. (Szeged),* 72 (2006), 569-579.

[10] J. A. Gosselin, Almost everywhere convergence of Vilenkin-Fourier series, *Trans. Amer. Math. Soc.,* 185 (1973), 345-370.

[11] B. I. Golubov, A. V. Efimov and V. A. Skvortsov, Walsh series and transforms. (Russian) Nauka, Moscow, 1987, English transl. in Mathematics and its Applications (Soviet Series), 64. Kluwer Academic Publishers Group, Dordrecht, 1991.

[12] N. V. Guličev, Approximation to continuous functions by Walsh-Fourier series, *Anal. Math.,* 6(1980), 269-280.

[13] G. Tephnadze, On the Vilenkin-Fourier coefficients, *Georgian Math. J.,* 20 (1), (2013), 169-177.

[14] G. Tephnadze, On the partial sums of Vilenkin-Fourier series, *J. Contemp. Math. Anal.,* 49 (2014), no. 1, 23-32.

[15] G. Tephnadze, Martingale Hardy Spaces and Summability of the One Dimensional Vilenkin-Fourier Series, PhD thesis, Department of Engineering Sciences and Mathematics, Luleå University of Technology, Oct. 2015 (ISSN 1402-1544).

[16] N. Ya. Vilenkin, A class of complete orthonormal systems, *Izv. Akad. Nauk. U.S.S.R., Ser. Mat.,* 11 (1947), 363-400, (in Russian).

[17] C. Watari, On generalized WalshFourier series, *Thoku Math. J.,* (2) 10 (1958), 211-241.

[18] F. Weisz, Martingale Hardy spaces and their applications in Fourier Analysis, Springer, Berlin-Heidelberg-New York, 1994.

[19] F. Weisz, Summability of multi-dimensional Fourier series and Hardy spaces, Mathematics and its Applications, 541. Kluwer Academic Publishers, Dordrecht, 2002.

[20] W. S. Young, Mean convergence of generalized Walsh-Fourier series, *Trans. Amer. Math. Soc.,* 218 (1976), 311-320.

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