UiT THE ARCTIC UNIVERSITY OF NORWAY

Department of Mathematics and Statistics

The EOS Formulation for Linear and Nonlinear Transient Scattering Problems

Aihua Lin

A dissertation for the degree of Philosophiae Doctor – December 2018

The EOS Formulation for Linear and Nonlinear Transient Scattering Problems

Aihua Lin

December, 2018

Acknowledgements

I would like to express my sincere gratitude to my supervisor Prof. Per Kristen Jakobsen. This thesis would never come into being without his guidance. Thanks for continuous support and patience to help me grow both academically and personally. Thanks for offering me the opportunity to start a new life in Norway. Words are far from expressing my thanks.

I am deeply thankful to Prof. Einar Mjølhus for all his help and guidance at my first course. Thanks to Tor Flå for kind and nice talks and for his care. I greatly appreciate it.

Many thanks to Elinor Ytterstad for all the joyful memories we have shared together. Many thanks to Hugues Verdure for always being available to help me with IT problems no matter where I was located. Deep thanks to them for their generous help in my daily life.

My sincere thanks go to Trygve Johnsen, Martin Rypdal and Helge Johansen for their kind understanding and financial support.

Many thanks to Jan Nyquist Roksvold, Ragnar Soleng, Sigrunn Holbek Sørbye, Marius Overholt and Boris Kruglikov for a lot of interesting conversations.

Great thanks to my parents, my husband and my sister for all their support and for being by my side all the time. I am particularly grateful to having my lovely son, Oliver, without whom I may have been suffering from depression during this long period of doctoral study.

Last but not least, my infinite heartfelt gratitude goes to my grandma, for everything.

Thank you all very much!

Contents

1 Introductions

1.1 Background and motivations

The formulation of partial differential equations in unbounded domains in terms of boundary integral equations has a long history. The roots stretch back at least to the ground breaking publication of a certain set of integral identities by George Green in 1828 [1]. These ideas, which became well known only after his death, became generalized into the notion of Green's functions which are the key elements in all investigations involving boundary integral equations, and many other places in pure and applied science. It would for example be hard to imagine solid state physics and particle physics without Green's functions.

Using computational methods to solve PDEs by solving their corresponding boundary integral equations, also has fairly deep roots, but not nearly as deep as the boundary integral equations themselves. This is because such methods only became feasible in the 1960s after the invention and widespread availability of electronic computers. This approach to solving PDEs numerically became known as the Boundary Element method(BEM) [2].

The boundary element method has several attractive features that makes it well suited to take on any computational problem involving some kind of wave scattering from compact objects in an infinite domain. Many modeling problems in science and engineering are of this type.

First and foremost, the solution of the original PDEs is changed to the solution of an integral equation defined only on the boundaries of the scattering objects. Thanks to this, one whole space dimension is removed from the problem, and no numerical grids outside the scattering objects are needed. This greatly reduces the computational cost and complexity, and therefore BEM is more efficient compared to a traditional domain-based method. The reduction of the number spatial dimensions in the problem is the main advantage of the BEM. For a domain-based method, all spatial dimensions are retained and must be discretize in a numerical solutions algorithm. The discretizations of the outside domain is usually resolved by introducing a box big enough to include all the scattering objects. This domain can easily become very large and thus the computational box will have to be very large. For 3D scattering problems this translates into a large memory requirement for the numerical implementation. In addition, boundary conditions have to be imposed at the boundary of the computational box in such a way that reflections are minimized. For the case of electromagnetic scattering this problem was solved by the introduction of a perfectly matched layer(PML) [3,4]. This is a key element of the finite difference time domain method(FDTD) [5–7], which is the method of choice for simulating electromagnetic scattering. The introduction of PML however does not come free. There are both an increased computational cost, because the PML layer has to be thick enough in order to ensure minimal reflection, and an increased implementional complexity needed to counter the numerical instabilities that are originating from the PML layer. The PML approach has subsequently been simplified and generalized to all kinds of scattering problems [8, 9] using an approach where the physical space outside the computational box is complexified.

Secondly, there is usually a big difference in the material properties between the scattering objects and their surroundings, which in the mathematical model appears as discontinuous, or near-discontinuous, coefficients in the PDEs defining the model. It is not easy to represent this difference accurately using a domain-based method, like FDTD or the Finite Element method (FEM) [10–12]. This is usually resolved by introducing multiple, interlinked grids. However it is challenging to design accurate and stable numerical algorithm on such grids and it also adds to computation cost of these methods.

Thirdly, the surface localization of boundary integral equations also means that the boundary discretization, which leads to the BEM equations, can be optimized with respect to the geometry of each surface separately when there are more than one scattering object, which there usually is. For domain-based methods like Finite Element methods(FEM) [10–12] and FDTD, such an optimization can only be achieved by using non-uniform or multiple grids tailored to the geometrical shape of the objects. These kinds of efforts have met with some degree of success, but it does add new layers of complexity. Methods such as these, combined with Transformational Optics(TO) [13, 14] have an intuitive appeal and have recently met with some amount of success [15], but numerical stability issues are a major concern.

Lastly, BEM, is exceptionally well suited for modeling scattering problems where the sources are slow on relevant timescales. In this setting the boundary integral equations are derived in the spectral domain and the discretized equations defining BEM only needs to be solved for the small set of discrete frequencies that are required for an accurate representation of the time dependent source. For a domain-based method like FDTD, near-stationary sources are the worst possible case since FDTD is, as the name indicate, a time domain method, and slow sources mean that Maxwell's equations have to be solved for a long interval of time, which is costly in terms of computational resources. FEM, on the other hand, originated in the context of stationary problems in elasticity [16, 17], and as such is well suited to frequency domain problems. However for fast sources in 3D electromagnetic scattering the computational load builds quickly.

Given all these attractive features of BEM for solving scattering problems in electromagnetics, it is somewhat surprising that in a popularity contest, FEM and FDTD beats BEM hands down [2]. The reason for this is that in addition to all its attractive features, BEM has some real drawbacks too.

Firstly, reformulating the scattering problem for a system of PDEs in terms of boundary integral equations typically will involve classical but fairly intricate mathematical tools, as will the discretization of the resulting integral equations. In particular one will need to content with singularities which appear in the limits that always must be taken while deriving the boundary equations from the PDEs. Accurately representing these singularities in the ensuing discretization leading to BEM, is a major issue. FEM have some of the same issues, whereas FDTD is much simpler to implement using mathematical tools that are straight forward and known to all.

Secondly, BEM relies on Green's functions and such functions can only be defined for linear systems of PDEs. In our opinion this is one of the major reasons why BEM is significantly less popular than the major contenders FEM and FDTD. Nonlinearities are common in most areas of pure and applied science, and having to change your whole computational approach when nonlinearities are added to your model is a major nuisance. In some areas of application nonlinearities dominate and hence a computational approach based on Green's functions is out of the question, fluid mechanics is such an area. However in other areas one can frequently disregard nonlinearities and a computational approach based on Green's functions is feasible. Scattering of electromagnetic waves, where nonlinearities only come into play at very high field intensities, is such an area.

Thirdly, BEM, which is simplest to formulate in the spectral domain, is at it's best when we are looking at a near stationary situation with narrow band sources. For transient scattering with broad band sources one needs to formulate BEM in the time domain and this involves space-time Green's functions that in general tends to be more singular than the frequency domain ones. However, for the specific case of electromagnetic scattering, an integral formulation has been derived [18] whose singularities are fairly weak. This integral formulation of electromagnetics is the foundation of all time dependent BEM formulations to date, and is also the basis for our approach. However, the progress in developing these time dependent BEM schemes has been slow due to several drawbacks.

The time-domain integral equations are retarded and this means that in order to compute the solutions at a certain time, one needs to retain the solutions for an interval of previous times that in some cases can be very large. This leads to a large memory requirement that needs to be met using parallel processing. In today's computational environment parallel implementation of time dependent BEM is fairly standard, but the possibly limited efficiency due to the problem of load balancing is something that always must be contended with.

However the major obstacle that has prevented this method from being widely applied for electromagnetic scattering is the occurrence of numerical instabilities. These instabilities, whose source is not fully understood, occur not at early times, but at later times and have become known as the late time instability. Many efforts have been made and several techniques have been developed in order to improve the stabilities of BEM schemes for time dependent electromagnetics in the last several decades. Basically there are two directions that are pursued. One direction is focused on delaying or removing the late time instability by applying increasing accurate spatial integrations [19–25]. The other direction is aimed at designing more stable time discretization schemes [24, 25]. Some researches were reported to mitigate the instability by both making better approximations of the integrals and applying improved time derivatives [26, 27].

In this thesis we have developed a new hybrid approach for solving linear and nonlinear scattering problems, an approach which is aimed at a situation where a collection of compact scattering objects are located in a homogeneous unbounded space and where the scattering objects can have an inhomogeneous and/or nonlinear response. The basic idea is to combine a domain-based method and a boundary integral method in such a way that the domain-based method is used to propagate the equations governing the wave field inside the scattering objects forward in time while the boundary integral method is used to supply the domain-based method with the required boundary values. The boundary integral method is derived from a space-time integral formulation of the PDEs such that all the scattering and re-scattering outside the scattering objects will be taken into account automatically. As a result, for the numerical implementations, there is no need for grids outside the scattering objects, only grids on the inside and the boundaries of the scattering objects are needed. Thus the new approach combines the best features of both methods; the response inside the scattering objects, which can be caused by both material inhomogeneity and nonlinearities, is easily taken into account using the domain-based method, and the boundary conditions supplied by the boundary integral method makes it possible to confine the domain based method to the inside of each scattering object.

This kind of idea was firstly proposed in 1972 by E. Wolf and D. N. Pattanyak [28] for the stationary linear scattering of electromagnetic waves in frequency space. The approach was based on the Ewald-Oseen optical extinction theorem, and because of this we call our method the Ewald-Oseen Scattering(EOS) formulation.

In this thesis our aim is to calculate the scattering of waves from objects that are in general inhomogeneous and, additionally, may have a nonlinear response. Thus a space-time integral formulation of the PDEs of interest is needed.

Firstly, we explored the viability of our approach by applying it to two toy models of 1D linear and nonlinear transient wave scattering. These two problems are set up to be analogs of the 3D scattering of electromagnetic waves, whose mathematical incarnation are Maxwell's equations . Our investigation of these two 1D problems is detailed in Paper 1.

Secondly, after confirming the viability of our approach in Paper 1, we extended our approach to the real world highly relevant, but also mathematically highly complex, case of 3D electromagnetic scattering. This extension of our method was successful, and we take this as proof that our EOS formulation is a general approach that is applicable to a wide array of linear and nonlinear wave scattering problems. In Paper 2 we report on the main features of this extension and the problems that needed to be addressed in order for us to succeed in the application of our EOS formulation to the 3D Maxwell's equations.

In paper 3 we discuss, in the context of electromagnetic scattering, two issues that are relevant for the application of our EOS formulation to any wave scattering problem of real world relevance and complexity. This is the issues of stability of the numerically implemented EOS formulation, and the issue of how to handle the singular integrals that occurs as matrix elements in the numerical implementation of the boundary part of the EOS formulation.

1.2 Summary of the papers

This section summarizes the work and the main results of the three papers that are the core of the current thesis.

1.2.1 Paper 1

In this paper, we introduce a new method, which we have called the EOS formulations, for solving linear and nonlinear transient wave scattering problems. As stated in section 1, the method has been developed by combining a boundary integral representations and a domain-based method. This is done in such a way that the inside fields will be propagated forward in time by the domain-based method, while the needed boundary values will supplied by the boundary integral representations. The method is illustrated on two 1D toy models which are chosen as analogs of electromagnetic wave scattering.

Model 1 is governed by equations

$$\begin{aligned} \varphi_t &= c_1\varphi_x + j, \\ \rho_t &= -j_x, \\ j_t &= (\alpha - \beta\rho)\varphi - \gamma j, \quad a_0 < x < a_1, \end{aligned} \tag{1.1}$$

and model 2 is governed by

$$\begin{aligned} \varphi_t &= \mu_1\psi_x + j, \\ \psi_t &= \nu_1\varphi_x, \\ \rho_t &= -j_x, \\ j_t &= (\alpha - \beta\rho)\varphi - \gamma j, \quad a_0 < x < a_1, \end{aligned} \tag{1.2}$$

where α, β and γ are constants, $\varphi = \varphi(x,t)$ is the "electric field", $\psi(x,t)$ is the "magnetic" field, $j = j(x,t)$ is the "current density" and ρ is the "charge density", c_1 is the propagation speed inside the "material". μ_1, ν_1 are "material" parameters and under the translations of $\mu = \frac{1}{\varepsilon}$ and $\nu = \frac{1}{\mu}$, μ and ν are the analog of the electric permittivity and magnetic permeability. The charge density and current density are confined to the interval $[a_0, a_1]$, which is an analog to a compact scattering object in the electromagnetic situation. The fields φ and ψ are defined on the whole real axis. The equation for the "current density", j is a simplification of a real electromagnetic current density model [29].

Outside the interval the two models are respectively governed by

$$\varphi_t = c_0\varphi_x + j_s,$$

and

$$\begin{aligned} \varphi_t &= \mu_0\psi_x + j_s, \\ \psi_t &= \nu_0\varphi_x, \end{aligned}$$

where $j_s(x,t)$ is a given source that has its support in interval $x > a_1$. Model 1 describes a one way propagation with its speed c_0 outside the interval $[a_1, a_1]$ and model 2 describes a two way propagation with its speed $c_0 = \sqrt{\mu_0\nu_0}$ outside the interval. In order to derive the space-time integral formulation of the EOS formulations, we firstly need to derive an integral identity involving the operator

$$L_1 = \partial_t - v\partial_x,$$

for model 1 and

$$L_2 = \begin{pmatrix} \partial_t & -\mu\partial_x \\ -\nu\partial_x & \partial_t \end{pmatrix},$$

for model 2 and the needed advanced Green's functions of their adjoint operators. The integral identities on different intervals will be obtained by inserting the corresponding advanced Green's functions. Finally the integral representations on the boundaries are reached by taking the limit of the integral identities with x approaching the boundaries a_0 and a_1 of the interval. However, the boundary integral representations and the boundary conditions compose an overdetermined system for both model 1 and model 2. This is a general situation occurring when the EOS formulation is applied to systems of PDEs. An general approach to handling this kind of problem is introduced in this paper. This approach is subsequently applied to the case of Maxwells equations in paper2.

Equations (1.1) or (1.2) together with their corresponding boundary integral representations, by definition, is the EOS formulations of toy model 1 and toy model 2 respectively.

In order to get a second order accuracy numerical solution, we apply the Lax-Wendroof method to the first two equations of (1.1) and the first three equations of (1.2) and we apply the modified Euler's method to the equations of the current density j for the inside interval. For the boundary integrals, we use the mid-point rule. A space grid for the inside of the scattering objects, (a, b), need to support both this mid-point integration rule and also finite difference formulas for the partial derivatives, and it will for this reason be nonuniform close to the boundary. This occurrence of nonuniform internal spatial grids also occur for the case of electromagnetic scattering in paper 2 and is generic if one discretize the EOS formulation using finite difference methods.

For the two models in this paper we use the following spatial grid for the inside of the scattering object

$$x_i = a_0 + (i + \frac{1}{2})\frac{a_1 - a_0}{N}, \; i = 0, 1, \cdots, N-1. \tag{1.3}$$

The "electric field" and the "magnetic field" are continuous through the boundaries of the scattering objects. Their space derivatives are therefore approximated by finite difference formulas which involve values both from the inside and the outside of the scattering object. The current density j_1, and the charge density ρ_1, are entirely supported inside the scattering object, and for these quantities it is thus appropriate to approximate their space derivatives using only values from inside the scattering object. For the discretizations of the boundary values that is located between the grid points for the time grid, we choose to use a quadratic interpolation in order to maintain overall second order accuracy for our scheme.

For both toy models we verify the stability and accuracy of our EOS formulations using an approach based on the use of artificial sources. The idea is motivated by a fact that adding an arbitrary source to all the governing equations of the system normally will lead to only minor changes to the numerical scheme. The introduction does however change the model equations in an interesting and useful way. Extending the model equations by the addition of arbitrary sources means that *any* function is a solution to the extended equations for *some* choice of sources.

For example, model 1 extended by the addition of artificial sources is of the form

$$\begin{aligned}
\varphi_t &= c_1\varphi_x + j + g_1, \\
\rho_t &= -j_x + g_2, \\
j_t &= (\alpha - \beta\rho)\varphi - \gamma j + g_3,
\end{aligned} \tag{1.4}$$

where g_1, g_2, g_3, are the artificial sources. If we choose some functions $\hat{\varphi}, \hat{j}$ and

$\hat{\rho}$ and let

$$\begin{aligned}
\hat{g}_1 &= \hat{\varphi}_t - c_1 \hat{\varphi}_x - \hat{j}, \\
\hat{g}_2 &= \hat{\rho}_t + \hat{j}_x, \\
\hat{g}_3 &= \hat{j}_t - (\alpha - \beta \hat{\rho})\hat{\varphi} + \gamma \hat{j},
\end{aligned}$$

then model 1 with these source functions will have the functions $\hat{\varphi}, \hat{j}$ and $\hat{\rho}$ as solution. If we now use these sources in our numerical discretization of model 1, we *know* that the correct solution is given by $\hat{\varphi}, \hat{j}$ and $\hat{\rho}$, and we can now validate our numerical solution by comparing it to the known correct solution

The comparison between the EOS formulations and the exact solutions of the two toy models show that discretization of our EOS formulation give accurate numerical solutions, and thus shows that our new approach to wave scattering is viable. The solutions of a general scattering where the source is located outside the interval are also implemented for both toy models. The implementations of both the artificial source test and the general scattering are stable with a proper choice of the time step. We observe in this paper that the numerical implementation of the EOS formulations for model 1 and model 2 is stable if the time step is bounded both above and below, and thus belong to a bounded interval. We show in the paper that this stability interval is a consequence of the nonuniform spatial grid used inside the scattering object.

1.2.2 Paper 2

This paper is a continuation and extension of Paper 1 where we explore the possibility of applying the EOS formulations to 3D electromagnetic scattering problems. The goal of this paper is to derive, and implement numerically, the EOS formulation for the scattering of electromagnetic waves from a single scattering object. The extension to several such objects is conceptually trivial, although with respect to computational load, the extension will of course be nontrivial. The basic equations for the situation discussed in this paper are the Maxwell's equations

$$\nabla \times \mathbf{E}_j + \partial_t \mathbf{B}_j = 0, \tag{1.5a}$$

$$\nabla \times \mathbf{B}_j - \frac{1}{c_j^2}\partial_t \mathbf{E}_j = \mu_j \mathbf{J}, \tag{1.5b}$$

$$\partial_t \rho_j + \nabla \cdot \mathbf{J}_j = 0, \tag{1.5c}$$

with the index $j = 0$ representing the outside of the scattering object and $j = 1$ representing the inside of the object. $\mathbf{E}_j$ is the electric field, $\mathbf{B}_j$ is the magnetic field, $\mathbf{J}_j$ and ρ_j are the current density and the charge density respectively. The speed of light in a material with electric permittivity ε_j and magnetic permeability μ_j is given by $c_j = 1/(\varepsilon_j \mu_j)$. For the inside of the object, the dynamics of the current density is determined by the equation

$$\partial_t \mathbf{J}_1 = (\alpha - \beta \rho_1)\mathbf{E}_1 - \gamma \mathbf{J}_1 = F(\mathbf{E}_1, \rho_1, \mathbf{J}_1), \tag{1.6}$$

where α, β, γ are some constants. For the outside of the object, the current density J_0 and the electric density ρ_0 are some given sources that satisfy the continuity equation (1.5c).

The principle of the EOS formulations for 3D electromagnetic scattering is the same as in the 1D case in Paper 1. We solve the inside domain of the model using a domain-based method and use a integral identity on the surface to support the surface values needed by the inside updating in time. In order to derive the boundary integral representations, we firstly need to derive a space-time integral identities for the electric field and the magnetic field. The required integral identities can be derived by noting that each component of the electric and magnetic fields satisfy scalar wave equations. From the integral identities satisfied by solutions to the 3D wave equations one can, after highly nontrivial vector calculus manipulations, derive the integral representations of the electric field and the magnetic field in the form

$$
\begin{aligned}
&\mathbf{E}_j(\mathbf{x},t) = -\partial_t \frac{\mu_j}{4\pi}\int_{V_j} \mathrm{d}V' \frac{\mathbf{J}_j(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|} - \nabla \frac{1}{4\pi\varepsilon_j}\int_{V_j} \mathrm{d}V' \frac{\rho_j(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|} \\
&\mp \partial_t [\frac{1}{4\pi}\int_S \mathrm{d}S' \{ \frac{1}{c_j|\mathbf{x}'-\mathbf{x}|}(\mathbf{n}'\times\mathbf{E}_j(\mathbf{x}',T))\times\nabla'|\mathbf{x}'-\mathbf{x}| \\
&+ \frac{1}{c_j|\mathbf{x}'-\mathbf{x}|}(\mathbf{n}'\cdot\mathbf{E}_j(\mathbf{x}',T))\nabla'|\mathbf{x}'-\mathbf{x}| + \frac{1}{|\mathbf{x}'-\mathbf{x}|}\mathbf{n}'\times\mathbf{B}_j(\mathbf{x}',T)\}] \\
&\pm \frac{1}{4\pi}\int_S \mathrm{d}S' \{ (\mathbf{n}'\times\mathbf{E}_j(\mathbf{x}',T))\times\nabla'\frac{1}{|\mathbf{x}'-\mathbf{x}|} \\
&+ (\mathbf{n}'\cdot\mathbf{E}_j(\mathbf{x}',T))\nabla'\frac{1}{|\mathbf{x}'-\mathbf{x}|}\},
\end{aligned}
\tag{1.7}
$$

$$
\begin{aligned}
&\mathbf{B}_j(\mathbf{x},t) = \nabla\times\frac{\mu_j}{4\pi}\int_{V_j}\mathrm{d}V' \frac{\mathbf{J}_j(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|} \\
&+ \partial_t [\frac{1}{4\pi}\int_S \mathrm{d}S' \{ \frac{1}{c_j|\mathbf{x}'-\mathbf{x}|}(\mathbf{n}'\times\mathbf{B}_j(\mathbf{x}',T))\times\nabla'|\mathbf{x}'-\mathbf{x}| \\
&\mp \frac{1}{c_j|\mathbf{x}'-\mathbf{x}|}(\mathbf{n}'\cdot\mathbf{B}_j(\mathbf{x}',T))\nabla'|\mathbf{x}'-\mathbf{x}| - \frac{1}{c_j^2}\frac{1}{|\mathbf{x}'-\mathbf{x}|}\mathbf{n}'\times\mathbf{E}_j(\mathbf{x}',T)\}] \\
&\pm \frac{1}{4\pi}\int_S \mathrm{d}S' \{ (\mathbf{n}'\times\mathbf{B}_j(x',T))\times\nabla'\frac{1}{|\mathbf{x}'-\mathbf{x}|} \\
&+ (\mathbf{n}'\cdot\mathbf{B}_j(\mathbf{x}',T))\nabla'\frac{1}{|\mathbf{x}'-\mathbf{x}|}\},
\end{aligned}
\tag{1.8}
$$

for $\mathbf{x} \in V_j$. These are however not the only integral identities that follows from this procedure. We also find that from the two outer identities we get two additional inner identities and similarly from the two inner we get two additional outer identities. In optics, which is a subfield of electromagnetics, these extra identities are called the Ewald-Oseen extinction theorem. From a mathematical point of view they follow from a defining property of Dirac delta functions. The extra pair of integral identities are

$$
\begin{aligned}
0 = &-\partial_t \frac{\mu_{1-j}}{4\pi} \int_{V_{1-j}} \mathrm{d}V' \frac{\mathbf{J}_{1-j}(\mathbf{x}', T)}{|\mathbf{x}' - \mathbf{x}|} - \nabla \frac{1}{4\pi\varepsilon_{1-j}} \int_{V_{1-j}} \mathrm{d}V' \frac{\rho_{1-j}(\mathbf{x}', T)}{|\mathbf{x}' - \mathbf{x}|} \\
&\pm \partial_t [\frac{1}{4\pi} \int_S \mathrm{d}S' \{ \frac{1}{c_{1-j}|\mathbf{x}' - \mathbf{x}|} (\mathbf{n}' \times \mathbf{E}_{1-j}(\mathbf{x}', T)) \times \nabla' |\mathbf{x}' - \mathbf{x}| \\
&+ \frac{1}{c_{1-j}|\mathbf{x}' - \mathbf{x}|} (\mathbf{n}' \cdot \mathbf{E}_{1-j}(\mathbf{x}', T)) \nabla' |\mathbf{x}' - \mathbf{x}| + \frac{1}{|\mathbf{x}' - \mathbf{x}|} \mathbf{n}' \times \mathbf{B}_{1-j}(\mathbf{x}', T) \}] \\
&\mp \frac{1}{4\pi} \int_S \mathrm{d}S' \{ (\mathbf{n}' \times \mathbf{E}_{1-j}(\mathbf{x}', T)) \times \nabla' \frac{1}{|\mathbf{x}' - \mathbf{x}|} \\
&+ (\mathbf{n}' \cdot \mathbf{E}_{1-j}(\mathbf{x}', T)) \nabla' \frac{1}{|\mathbf{x}' - \mathbf{x}|} \},
\end{aligned} \tag{1.9}
$$

and

$$
\begin{aligned}
0 = &\nabla \times \frac{\mu_{1-j}}{4\pi} \int_{V_{1-j}} \mathrm{d}V' \frac{\mathbf{J}_{1-j}(\mathbf{x}', T)}{|\mathbf{x}' - \mathbf{x}|} \\
&\pm \partial_t [\frac{1}{4\pi} \int_S \mathrm{d}S' \{ \frac{1}{c_{1-j}|\mathbf{x}' - \mathbf{x}|} (\mathbf{n}' \times \mathbf{B}_{1-j}(\mathbf{x}', T)) \times \nabla' |\mathbf{x}' - \mathbf{x}| \\
&+ \frac{1}{c_{1-j}|\mathbf{x}' - \mathbf{x}|} (\mathbf{n}' \cdot \mathbf{B}_{1-j}(\mathbf{x}', T)) \nabla' |\mathbf{x}' - \mathbf{x}| \\
&- \frac{1}{c_{1-j}^2 |\mathbf{x}' - \mathbf{x}|} \mathbf{n}' \times \mathbf{E}_{1-j}(\mathbf{x}', T) \}] \\
&\mp \frac{1}{4\pi} \int_S \mathrm{d}S' \{ (\mathbf{n}' \times \mathbf{B}_{1-j}(x', T)) \times \nabla' \frac{1}{|\mathbf{x}' - \mathbf{x}|} \\
&+ (\mathbf{n}' \cdot \mathbf{B}_{1-j}(\mathbf{x}', T)) \nabla' \frac{1}{|\mathbf{x}' - \mathbf{x}|} \},
\end{aligned} \tag{1.10}
$$

for $\mathbf{x} \in V_j$. After taking the limits of (1.7)-(1.10) when $\mathbf{x}$ approaches the surfaces from both the inside and the outside of the scattering objects, we get four integral representations on the surfaces. We also have four electromagnetic interface conditions at the boundary of the scattering object.boundary conditions for the usual electromagnetics. We have thus four unknowns and six interface conditions, which is overdetermined. The technique for solving this overdetermined problem is explained in the main text of Paper 2. The final boundary integral identities takes the following compact form

$$(I + \frac{1}{2}(\frac{\varepsilon_1}{\varepsilon_0} - 1)\mathbf{n}\ \mathbf{n})\mathbf{E}_+(\mathbf{x}, t) = \mathbf{I}_e + \mathbf{O}_e + \mathbf{B}_e, \tag{1.11a}$$

$$(I + \frac{1}{2}(1 - \frac{\mu_0}{\mu_1})\mathbf{n}\ \mathbf{n})\mathbf{B}_+(\mathbf{x}, t) = \mathbf{I}_b + \mathbf{O}_b + \mathbf{B}_b, \tag{1.11b}$$

where I is a 3×3 identity matrix, $\mathbf{E}_+$ and $\mathbf{B}_+$ are the limits of the electric field and the magnetic field by letting $\mathbf{x}$ approach the surface from the inside of the scattering object, $\mathbf{I}_e$ and $\mathbf{I}_b$ are volume integrals of the inside current density and the charge density, $\mathbf{O}_e$ and $\mathbf{O}_b$ are fields on the surfaces generated by the source in the absence of the scattering objects, $\mathbf{B}_e$ and $\mathbf{B}_b$ are surface integrals of the historical values of the fields on the surface. Expressions of these integrals are presented in Paper 2. Equations (1.5) and (1.6) together with the boundary integral identities (1.11a) and (1.11b) compose the EOS formulations of the 3D model.

In Paper 2, our aim is focused on illustrating the EOS formulations with respect to its complexity and numerical stability, thus we choose a scattering object of the simplest possible shape.set the shape , a rectangular box. In order to get a second order accuracy numerical solutions, we use the Lax-Wendroff method on (1.5) and the modified Euler's method on (1.6). For the boundary integral identities, we use the mid-point rules for second order accuracy. Inside the scattering object we introduce, for the reasons explained in paper 1, a non-uniform spatial grid of the form

$$\begin{aligned} x_i &= x_a + (i+0.5)\Delta x, \qquad & i &= 0,1,\cdots N_x - 1,\\ y_j &= y_a + (j+0.5)\Delta y, \qquad & j &= 0,1,\cdots N_y - 1,\\ z_k &= z_a + (k+0.5)\Delta z, \qquad & k &= 0,1,\cdots N_z - 1, \end{aligned} \tag{1.12}$$

with

$$\begin{aligned} \Delta x &= \frac{x_b - x_a}{N_x},\\ \Delta y &= \frac{y_b - y_a}{N_y},\\ \Delta z &= \frac{z_b - z_a}{N_z}, \end{aligned}$$

where N_x, N_y and N_z are the number of grid points in x, y and z directions respectively.

For the inside grids, the discretized solutions of (1.5) at grid point (x_i, y_j, z_k) at time t_{n+1} are calculated by

$$\phi^{n+1}_{i,j,k} = \phi^n_{i,j,k} + \Delta t(\frac{\partial \phi}{\partial t})^n_{i,j,k} + \frac{1}{2}(\Delta t)^2(\frac{\partial^2 \phi}{\partial t^2})^n_{i,j,k}, \tag{1.13}$$

where ϕ represents e_i, b_i, ρ_1, $i = 1,2,3$. The expressions of $\frac{\partial \phi}{\partial t}$ and $\frac{\partial^2 \phi}{\partial t^2}$ will be expressed by the space derivatives through Lax-Wendroff method. The same as for the 1D models, due to the sharp changes of the properties of the material on the boundaries of the scattering objects, the space derivatives of the electric field and the magnetic field involve both the inside values and the outside values while the space derivatives of the current density and the charge density only involve the inside values.

After discretization the current equation (1.6), using the modified Euler's method, we get a numerical scheme defined by the difference equation

$$\begin{aligned} (\bar{j_l})^{n+1}_{i,j,k} &= (j_l)^n_{i,j,k} + \Delta t \cdot F((e_l)^n_{i,j,k}, (\rho_1)^n_{i,j,k}, (j_l)^n_{i,j,k}),\\ (j_l)^{n+1}_{i,j,k} &= \frac{1}{2}((j_l)^n_{i,j,k} + (\bar{j_l})^{n+1}_{i,j,k} + \Delta t \cdot F((e_l)^{n+1}_{i,j,k}, (\rho_1)^{n+1}_{i,j,k} (\bar{j_l})^{n+1}_{i,j,k})), \end{aligned} \tag{1.14}$$

where $l = 1,2,3$, representing the three components of the current density.

For the discretizations of the boundary integral identities, the final formulas takes the form

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \mathbf{E}^n_p \\ \mathbf{B}^n_p \end{pmatrix} = \begin{pmatrix} \mathbf{E}_R \\ \mathbf{B}_R \end{pmatrix}, \tag{1.15}$$

where $\mathbf{E}_p^n$ and $\mathbf{B}_p^n$ are the numerical solutions at grid point $\mathbf{x}_p$ at time t_n. $\mathbf{E}_R$ and $\mathbf{B}_R$ are the right hand of (1.11a) and (1.11b) respectively after moving the unknown into the right side of the equations. M_{11} and so on are 3×3 matrices. The numerical solution for a general case where the sources are located outside the scattering domain is implemented. In order to be easily integrated in space, we have set up the outside source J_0 and ρ_0 as a combination of a smooth localized function in time and a delta function in space.

In order to validate our method, we solve an extended system with artificial sources described by

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \tag{1.16a}$$

$$\frac{1}{c_1^2}\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mu_1 \mathbf{J}, \tag{1.16b}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_1}\rho, \tag{1.16c}$$

$$\partial_t \mathbf{J} = (\alpha - \beta\rho)\mathbf{E} - \gamma \mathbf{J} + \boldsymbol{\varphi}. \tag{1.16d}$$

If we choose some functions for $\tilde{\mathbf{E}}, \tilde{\mathbf{B}}$, then these choices will be a solution to model (1.16), if the artificial source are chosen to be

$$\boldsymbol{\varphi} = \partial_t \tilde{\mathbf{J}} - (\alpha - \beta\tilde{\rho})\tilde{\mathbf{E}} - \gamma\tilde{\mathbf{J}}, \tag{1.17}$$

with

$$\tilde{\mathbf{J}} = \frac{1}{\mu_1}(\nabla \times \tilde{\mathbf{B}} - \frac{1}{c_1^2}\partial_t \tilde{\mathbf{E}}),$$

and

$$\tilde{\rho} = \varepsilon_1 \nabla \cdot \tilde{\mathbf{E}}.$$

The comparison between the numerical implementations and the exact solutions show high accuracy without instabilities with proper choices of parameters and restricted time step. It's also noted in Paper 2 that the numerical solution is stable if the time step is contained in a certain bounded interval. This interval is determined by both the inside domain-based method and the boundary integral identities.

1.2.3 Paper 3

In this paper we discuss the three major issues involved in solving the 3D Maxwell's equations using the EOS formulations. We believe that these issues are representative for the kind of problems that must be overcome while using the EOS formulation to calculate transient scattering of waves.

The first issue to be discussed in this paper is numerical stability. As mentioned above, the numerical scheme for the EOS formulation derived in paper2 is stable for time steps in a certain bounded interval. In the two 1D toy models, this interval is purely determined by the domain-based method, namely the Lax-Wendroff method, in our case. For the 3D model of Maxwell's equations, our finding is that the instability, when it occur, shows up at late times and comes from two sources. One source is the domain-based method applied inside the scattering object and the other source is the integral identities applied on the surfaces of the scattering object which are supplying the boundary conditions that are needed by the domain-based method. Specifically, the lower limit of this interval comes from the boundary integral identities and are determined by the material of the scattering object through the relative electric permittivity and the relative magnetic permeability, while the upper limit is determined by the domain-based method applied inside the scattering object. Our conclusions are based on two tests.

In the first test we assume that there is no current density and the charge density inside the object, and that the inside and outside has the same material parameters. Thus $\mu_1 = \mu_0$, $\varepsilon_1 = \varepsilon_0$, $\mathbf{J}_1 = 0$ and $\rho_1 = 0$. In other words, there is no scattering object present in the material sense. Under these assumptions, the electric field inside the scattering object $\mathbf{E}_1$ can be calculated by three methods.

In method 1, the field inside the scattering object is produced directly by the outside sources, thus the exact solution is calculated by

$$\mathbf{E}_1(\mathbf{x},t) = -\partial_t \frac{\mu_0}{4\pi}\int_{V_0} \mathrm{d}V' \frac{\mathbf{J}_0(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|} - \nabla \frac{1}{4\pi\varepsilon_0}\int_{V_0} \mathrm{d}V' \frac{\rho_0(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|}, \tag{1.18}$$

where V_0 denotes the outside domain of the scattering object while $\mathbf{x}$ is located inside of the object.

Note that by using the formula (1.18), for the electric field, and the corresponding one for the magnetic field, we find that the boundary values for the electric and magnetic field are given by

$$\mathbf{E}_+(\mathbf{x},t) = -\partial_t \frac{\mu_0}{4\pi}\int_{V_0} \mathrm{d}V' \frac{\mathbf{J}_0(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|} - \nabla \frac{1}{4\pi\varepsilon_0}\int_{V_0} \mathrm{d}V' \frac{\rho_0(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|}, \tag{1.19}$$

and

$$\mathbf{B}_+(\mathbf{x},t) = \nabla \times \frac{\mu_0}{4\pi}\int_{V_0} \mathrm{d}V' \frac{\mathbf{J}_0(\mathbf{x}',T)}{|\mathbf{x}'-\mathbf{x}|}, \tag{1.20}$$

where $\mathbf{x}$ is located on the surface S of the scattering object. $\mathbf{E}_+(\mathbf{x},t)$ and $\mathbf{B}_+(\mathbf{x},t)$ respectively represent the limits of the electric filed $\mathbf{E}_+$ and the magnetic field $\mathbf{B}_+$ by letting $\mathbf{x}$ approach the surface from the inside of the scattering object.

Based on this we can now calculate the inside electric field $\mathbf{E}_+$ by two methods more.

In method 2, we calculate the inside electric field using the boundary integral identity of the inside solution,

$$\begin{aligned}
\mathbf{E}_1(\mathbf{x},t) = \partial_t [&\frac{1}{4\pi}\int_S \mathrm{d}S^{'} \{ \frac{1}{c_1|\mathbf{x}'-\mathbf{x}|}(\mathbf{n}' \times \mathbf{E}_+(\mathbf{x}',T)) \times \nabla'|\mathbf{x}'-\mathbf{x}| \\
&+ \frac{1}{c_1|\mathbf{x}'-\mathbf{x}|}(\mathbf{n}' \cdot \mathbf{E}_+(\mathbf{x}',T))\nabla'|\mathbf{x}'-\mathbf{x}| \\
&+ \frac{1}{|\mathbf{x}'-\mathbf{x}|}\mathbf{n}' \times \mathbf{B}_+(\mathbf{x}',T)\}] \\
&- \frac{1}{4\pi}\int_S \mathrm{d}S^{'} \{(\mathbf{n}' \times \mathbf{E}_+(\mathbf{x}',T)) \times \nabla' \frac{1}{|\mathbf{x}'-\mathbf{x}|} \\
&+ (\mathbf{n}' \cdot \mathbf{E}_+(\mathbf{x}',T))\nabla' \frac{1}{|\mathbf{x}'-\mathbf{x}|}\},
\end{aligned} \tag{1.21}$$

which involves the surface values $\mathbf{E}_+$ and $\mathbf{B}_+$ expressed by (1.19) and (1.20).

In method 3, we update the inside values using the Lax-Wendroff method with boundary values given by the same surface values $\mathbf{E}_+$ and $\mathbf{B}_+$ as in method 2.

The comparison of the results from these three method show high accuracy and no instabilities if the time step is confined in the stable interval.

If we break the upper limit, Method 2 works well but Method 3 has late time instabilities. Thus the upper limit is controlled by the domain-based part of the EOS method.

The second test is based on the artificial source model (1.16). Given exact solutions of $\mathbf{E}_1$ and $\mathbf{B}_1$, the artificial source can be calculated by (1.17). Then we numerically calculate $\mathbf{E}_1$ and $\mathbf{B}_1$ by two methods.

In method 1 we apply the EOS formulations developed in this thesis which combines a domain based method and the boundary integral representation.

In method 2 we update the inside values of the fields using the Lax-Wendroff method with boundary values given explicitly by our choice of the electric $\mathbf{E}_1$ field and magnetic $\mathbf{B}_1$ field. Results show that for time steps that breaks the lower limit of the stable interval, Method 1 shows the late time instability while Method 2 works perfectly. Thus the lower limit is determined by the boundary integral part of the EOS method

Considering test 1 together with test 2, we come to the conclusion that the lower limit is determined the boundary integral representations while the upper limit is determined by the inside domain-based method. Specifically, the lower limit is directly determined by the relative electric permittivity and magnetic permeability and the upper limit is determined by the introduction of the non-uniform grids of the inside domain-based method.

The second issue is how to handle the singular integrals that appear in matrix elements when we discretize the EOS formulation. These singular integrals come from the process taking limits of the integral representations when the observing point moves from the inside or the outside onto the surfaces of the scattering object. The integrals are calculated by splitting them in a regular part and a singular core. The regular part we calculate using midpoint rule or 3D Gaussian integration and the singular core we calculate exactly using certain integral theorems.

The third issue is parallelization. Because of the retardation in time which is an integral part of the EOS formulation, or any formulation using dimensional reduction based on space-time Greens functions, there is a large memory requirement for the algorithm. Implementation in a parallel computational environment will therefore be necessary for most nontrivial application of our method.

1.3 Discussions and future work

A new hybrid method for solving linear and nonlinear transient scattering problems is introduced in this thesis. The hybrid method combines a domain based method and boundary integral representations in the time domain.

In this thesis, our focus has been to provide proof of principle that our new method is viable as an approach to the numerical calculation of transient wave scattering. By detailing the mathematical formulation and numerical implementation of our approach for two models of 1D transient wave scattering, and the 3D transient scattering of electromagnetic waves, we have achieved what the goal of this thesis.

Beyond the proof of viability of our new approach, there are several topics relating to this new method that needs to be investigated.

Firstly, it would be very useful if an unconditional stable numerical implementation of our EOS formulation could be found. The numerical implementations we have introduced, through our proof of principle studies, are all only conditionally stable. For all cases the numerical implementations are only stable for the time step in a bounded interval. For the 1D case this interval is entirely determined by the inner domain based method whereas for the 3D electromagnetic case the upper limit is determined by the domain based part of the method whereas the lower limit is determined by the boundary part of the method. Specifically the lower limit depends on the difference in material properties outside and inside the scattering object. If this difference is too large, the lower limit become larger than the higher limit and thus the numerical implementation is unstable for all sizes of the step length and thus the scheme is useless. In applications of our scheme to antenna theory this situation is realized and this might be the whole explanation, or part of the explanation, for the late time instability that always, in one way or another, seems to appear in this area of application of boundary methods. A fully implicit implementation of the EOS formulation , if it can be found, would remove restriction on the time step for stable operation and, inn all probability, may remove the late time instability for good. During this thesis work we briefly investigated the possibility of an implicit implementation of the EOS formulation, but did not achieve anything worth reporting here. We did however gain enough insight into the problem to realize that this is a difficult, perhaps even impossible, thing to achieve. This is certainly a topic worth looking into in any future investigation of the EOS formulation.

Secondly, the fundamental integral equations underlying both the BEM and the EOS formulations, developed in this work, are always retarded in time. This is because the underlying equations can only be derived using space-time Green's functions. Thus the solutions at a specific grid point at a certain time will depend on a series of historical solutions of all other grid points of the scattering object. Therefore, these methods are memory intensive. This is in particular true for the EOS formulation, because it grids the inside of the scattering object as well as it boundary. Although this can be solved by parallel computing, whenever large scale parallel processing is needed, there are always the issues of load balancing and saturation to take into account. In our work, the EOS formulation of 3D Maxwell's model was implemented on a large cluster, but we were not focused on parallel issues in any systematic way and have not reported on any parallelization issues that came up during our investigations. Because of the memory intensive nature of the EOS formulation these are however important issues, and therefore must form the part of any future work aimed at making our approach to transient wave scattering into at practical and efficient tool in the toolbox of scientific computing.

Thirdly, there is the issue how the EOS formulation compare to other, more conventional approaches to transient wave scattering. The main contenders here are FDTD and FEM. On the surface of it, it would appear that the EOS formulation is a clear front runner in any such comparison. After all, using this method removes the need to grid most of the physical domain, only the inside and the boundaries of the actual scattering objects need to be discretized. Thus the EOS formulation requires much fewer spatial grid points than either of FDTD and FEM. However, the retardation of the equations defining the boundary part of the EOS formulation means that this method require many more temporal grid points than the two main contenders. It is appropriate to ask if anything has been gained with respect to memory usage compared to a fully domain-based method like the FDTD method?

The outcome of comparing the memory usage of FDTD and FEM with the EOS formulation is anything but obvious. The outcome of such an investigation most likely will not present us with a clear winner. The ranking will almost surely depend on the nature of the problems under investigation. If the EOS formulation is going to take its place in the toolbox of scientific computing investigations like the one described in this section is sorely needed.

However, even if the memory usage for purely domain based methods and our EOS approach are roughly the same for many problems of interest, our approach avoid many of the sources of problems that need to be considered while using purely domain based methods. These are problems like stair-casing at sharp interfaces defining the scattering objects, issues of accuracy, stability and complexity associated with the use of multiple grids in order to accommodate the possibly different geometric shapes of the scattering objects, and the need to minimize the reflection from the boundary of the finite computational box. The EOS approach is not subject to any of these problems.

2 Paper 1

Submitted to PLOS ONE and revised in the fall of 2018.

3 Paper 2

Submitted to Physica Scripta in the fall of 2018.

4 Paper 3

Submitted to PLOS ONE in the fall of 2018.

5 List of papers and contributions

1. **<u>Aihua Lin</u>**, Anastasiia Kuzmina, Per Kristen Jakobsen. A Boundary Integral Approach to Linear and Nonlinear Transient Wave Scattering.
Submitted.
2. **<u>Aihua Lin</u>**, Per Kristen Jakobsen. A 3D Nonlinear Maxwells Equations Solver Based On A Hybrid Numerical Method.
Submitted.
3. **<u>Aihua Lin</u>**, Per Kristen Jakobsen. On the EOS Formulation for Light Scattering. Stability, Singularity and Parallelization.
Submitted.

Aihua Lin, was responsible for the method derivations, numerical implementations, the analysis and interpretation of the results and the writings of the first draft of all the three papers.

References

[1] Georg Green. An Essay on the Application of mathematical Analysis to the theories of Electricity and Magnetism. Originally published as book in Nottingham, 1828. Reprinted in three parts in Journal f ur die reine und angewandte Mathematik Vol. 39, 1 (1850) p. 7389, Vol. 44, 4 (1852) p. 35674, and Vol. 47, 3 (1854) p. 161221.Transcribed version of the original published in ArXiv under the original title in 2008.

[2] Alexander H. H. Cheng and Daisy T. Cheng. Heritage and early history of the boundary element method. *Engineering Vnalysis with Boundary Elements*, 29:268–302, 2005.

[3] J. Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *Journal of computational physics*, 114(2):185–200, 1994.

[4] S. Gedney. An anisotropic perfectly matched layer absorbing media for the truncation of fdtd lattices. *IEEE Transactions on Antennas and Propagation*, 44(12):1630–1639, 1996.

[5] K. Yee. Numerical solution of initial boundary value problems involving maxwell's equations in isotropic media. *IEEE Transactions on Antennas and Propagation*, 14(3):302–307, May 1966.

[6] A. Taflove. Application of the finite-difference time-domain method to sinusoidal steady state electromagnetic penetration problems. *IEEE Transactions on Electromagnetic Compatibility*, 22(3):191–202, 1980.

[7] A. Taflove et. al. *Computational Electrodynamics: the Finite-Difference Time-Domain Method, 3rd Ed.* Artech House Publishers, 2005.

[8] W. C. Chew and W. H. Weedon. A 3d perfectly matched medium from modified maxwell's equations with stretched coordinates. *Microwave and Optical Tech. Lett.*, 7(13):599–604, 1994.

[9] Steven G. Johnson. Notes on perfectly matched layers. Technical report, MIT, 2007.

[10] S. Ahmed. Finite-element method for waveguide problems. *Electronics Letters*, 4(18):387–389, 1968.

[11] J. P. Webb. Application of the finite-element method to electromagnetic and electric topics. *Rep. Prog. Phys.*, 58:1673–1712, 1995.

[12] J. Jin. *The Finite Element Method in Electromagnetics, 2nd Edition.* Wiley-IEEE Press, 2002.

[13] J. B. Pendry, D. Schurig, and D. R. Smith. Controlling Electromagnetic Fields. *Science*, 312:1780–1782, 2006.

[14] Ulf Leonard. Optical Conformal Mapping. *Science*, 312:1777–1780, 2006.

[15] Jinjie Liu, Moysey Brio, and Jerome V. Moloney. Transformational optics based local mesh refinement for solving Maxwell's equations. *Journal of Computational Physics*, 258:359–370, 2014.

[16] Alexander Hrennikoff. Solution of problems of elasticity by the framework method. *Journal of Applied Mechanics*, 8.4:169–175, 1941.

[17] Alexander Hrennikoff. Variational methods for the solution of problems of equilibrium and vibration. *Bulletin of the American mathematical Society*, 49:1–23, 1943.

[18] D. S. Jones. *The Theory of Electromagnetism*, volume 47 of *International Series of Monographs on Pure and Applied mathematics*. Pergamon Press, 1964.

[19] Greeshma Pisharody and Daniel S. Weile. Electromagnetic scattering from homogeneous dielectric bodies using time-domain integral equations. *IEEE Transactions on Antennas and Propagation*, 54:687–697, 2006.

[20] D. S. Weile, I. Uluer, J. Li, and D.A. Hopkins. Integration rules and experimental evidences for the stability of time domain integral equations. *International Applied Computational Electromagnetics Society Symposium(ACES)*, 2017.

[21] Jielin Li, Daniel S. Weile, and D.A. Hopkins. Integral accuracy and experimental evidence for the stability of time domain integral equations. *International Applied Computational Electromagnetics Society Symposium (ACES)*, 2018.

[22] X.Wang, R. A.Wildman, D. S.Weile, and P. Monk. A finite difference delay modeling approach to the discretization of the time domain integral equations of electromagnetics. *IEEE Transactions on Antennas and Propagation*, 56:2442–2452, 2008.

[23] A. J. Pray, N. V. Nair, and B. Shanker. Stability properties of the time domain electric field integral equationusing a separable approximation for the convolution with the retarded potential. *IEEE Transactions on Antennas and Propagation*, 60:3772–3781, 2012.

[24] M. J. Bluck and S. P. Walker. Time-domain bie analysis of large three-dimensional electromagnetic scattering problems. *IEEE Transactions on Antennas and Propagation*, 45:894–901, 1997.

[25] S. Dodson, S. P. Walker, and M. J. Bluck. Implicitness and stability of time domain integral equation scattering analysis. *Appl. Comput. Electromagn. Soc. J.*, 13:291–301, 1998.

[26] Ying Zhao, Dazhi Ding, and Rushan Chen. A discontinuous galerkin time-domain integral equation method for electromagnetic scattering from pec objects. *IEEE Transactions on Antennas and Propagation*, 64(6):2410–2415, 2016.

[27] Li Huang, Yi-Bei Hou, Hao-Xuan Zhang, Liang Zhou, and Wen-Yan Yin. A discontinuous galerkin time-domain integral equation method for electromagnetic scattering from pec objects. *IEEE Transactions on Electromagnetic Compatibility*, 2018.

[28] D. N. Pattanayak and E. Wolf. General form and new interpretation of the ewald-oseen extincition theorem. *Optics communications*, 6(3):217–220, 1972.

[29] Yong Zeng, Walter Hoyer, Jinjie Liu, Stephan W. Koch, and Jerome V. Moloney. Classical theory for second-harmonic generation from metallic nanoparticles. *Physical Review B*, 79:235109, 2009.