UiT
THE ARCTIC
UNIVERSITY
OF NORWAY

FACULTY OF SCIENCE AND TECHNOLOGY

Department of Mathematics and Statistics

# Group Cohomology and Extensions

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**Markus Nordvoll Breivik**

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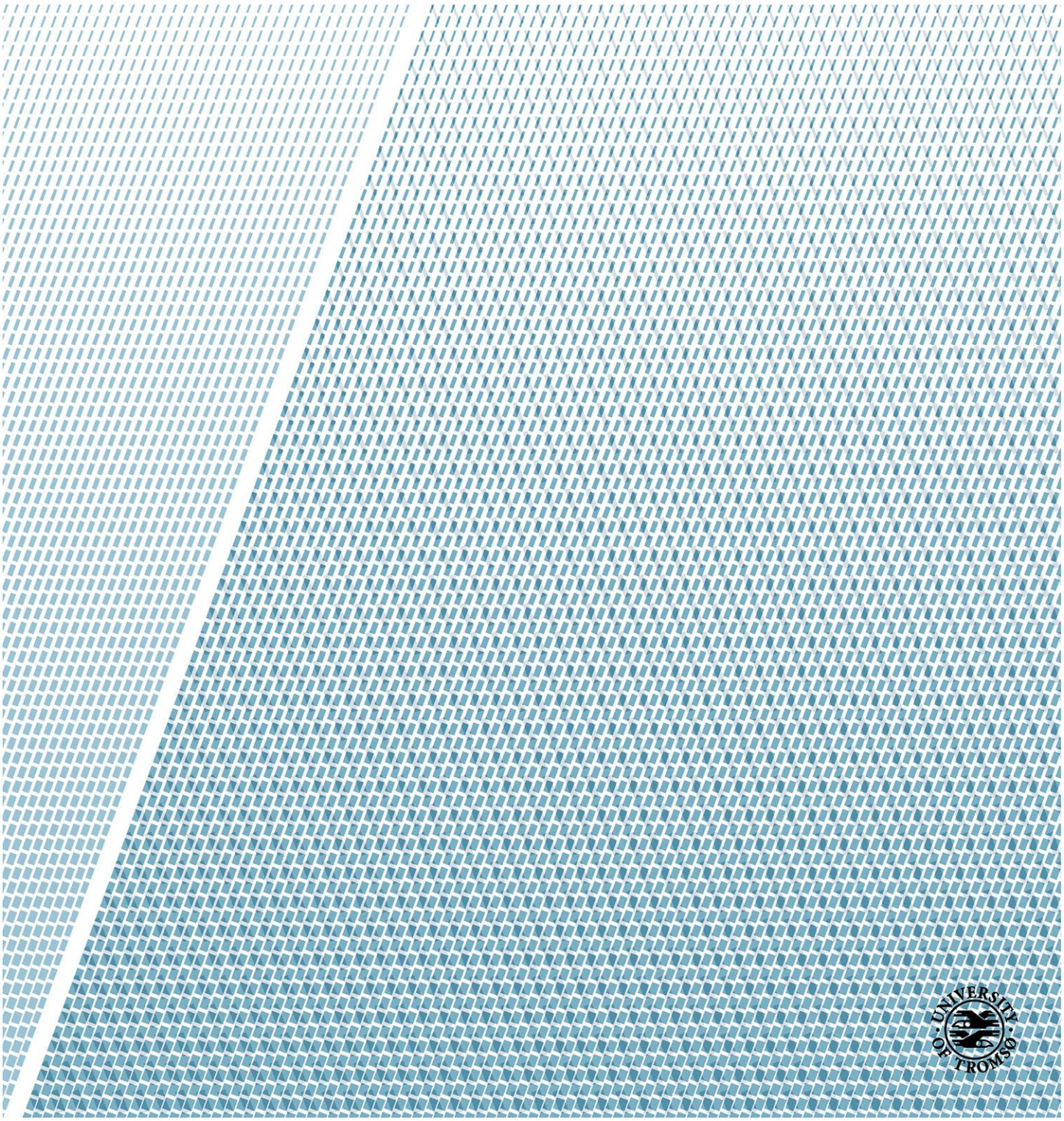

# GROUP COHOMOLOGY AND EXTENSIONS

MARKUS NORDVOLL BREIVIK

Abstract. The goal of this thesis is to classify all extensions where the kernel has order $p^s$ and the cokernel has order $p^t$, $p$ is a prime, and $1 \leq s, t \leq 2$. We determine (up to weak congruence) the different combinations of kernel, cokernel and operators, and for each case, calculate the second cohomology group. By comparing resolutions, we get an explicit correspondence between the second cohomology group and the group of congruence classes of extensions. Using this construction, we determine (up to congruence) the extensions for the different combinations.

## Contents



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## 0. Introduction

An extension of $A$ by $G$ is a short exact sequence

$$\varepsilon = (1 \to A \to E \to G \to 1).$$

Identifying $A$ with its image in $E$, we see that an extension of $A$ by $G$ is a group $E$ in which $A$ is a normal subgroup and $E/A \cong G$. Given groups $A$ and $G$, the extension problem is to determine all extensions of $A$ by $G$.

When $A$ is abelian, an extension of $A$ by $G$ determines a $G$-module structure (Definition 1.5) on $A$ , $\xi : G \to \mathrm{Aut}(A)$ (Proposition 1.25), so we can split the extension problem into sub-problems, namely, to determine the extensions of $A$ by $G$ that realizes the action $\xi : G \to \mathrm{Aut}(A)$. For a pair $A$ and $G$ with action $\xi : G \to \mathrm{Aut}(A)$, we write $A^{\xi}$ and $G$.

Congruent extensions (Definition 1.39) determine the same action, and the set of congruence classes of extensions of $A^{\xi}$ by $G$ is denoted by $E\left(G, A^{\xi}\right)$. By [ML95, Theorem IV.4.2] we have

$$E\left(G, A^{\xi}\right) \cong H^2\left(G, A^{\xi}\right)$$

(see also Theorem 1.45), where

$$H^n\left(G, A^{\xi}\right) := \mathrm{Ext}^n_{\mathbb{Z}G}\left(\mathbb{Z}^{triv}, A^{\xi}\right),$$

and $\mathbb{Z}^{triv}$ is the abelian group $\mathbb{Z}$ considered as a **trivial** $\mathbb{Z}G$-module (${}^g a = a$ for any $a \in \mathbb{Z}$ and $g \in G$). This means that we can determine the elements in $E\left(G, A^{\xi}\right)$ by calculating cohomology groups. Unfortunately, this correspondence can be difficult to use in practice. The map between $E\left(G, A^{\xi}\right)$ and $H^2\left(G, A^{\xi}\right)$ is only made explicit when $H^2\left(G, A^{\xi}\right)$ is calculated using the bar resolution, which has great theoretical applications, but is unsuitable for computation. The way one goes about it practically is to calculate $H^2\left(G, A^{\xi}\right)$ using a projective resolution specific to $\left(G, A^{\xi}\right)$, and then find a chain map between the resolutions which induce isomorphisms between the cohomology groups.

**Remark 0.1.** *We can summarize the above by noting that when $A$ is abelian, we can find all extensions of $A$ by $G$ by:*

1. *Determining the possible actions $\xi : G \to \mathrm{Aut}(A)$.*
2. *For all of the actions found in (1), calculate the groups $H^2\left(G, A^{\xi}\right)$ using a resolution suitable for computation.*
3. *Find a correspondence $E\left(G, A^{\xi}\right) \cong H^2\left(G, A^{\xi}\right)$, for each $H^2\left(G, A^{\xi}\right)$ found in (2), and determine each congruence class corresponding to $s \in H^2\left(G, A^{\xi}\right)$.*

**Remark 0.2.** *When $A$ is non-abelian the situation is trickier as one has to consider abstract kernels and 3-dimensional cohomology groups (See* [ML95, Chapter IV.8]*). This machinery is not needed here.*

The goal of this thesis is to classify all extensions in which the kernel and cokernel have orders $p^s$ and $p^t$ respectively (finite $p$-groups), where $1 \leq s, t \leq 2$. If an extension satisfies these conditions, we say it is of type

$$p^s \to p^{s+t} \to p^t.$$

Most cases $s + t \leq 3$ were done in [EP18].

From the viewpoint of homological algebra, finite $p$-groups are interesting since the cohomology groups are large (i.e. there are many extensions). For $|G| = n$ we have

$$n \cdot H^2(G, A) = 0$$

by [ML95, Proposition IV.5.3]. For $|A| = m$ we have

$$m \cdot H^2(G, A) = 0$$

since by Proposition 1.20

$$H^2(G,A) \cong \frac{\{\text{cocycles}\}}{\{\text{coboundaries}\}}$$

and cocycles are functions with values in $A$. So when $|G| = n$ and $|A| = m$, Bézout's formula gives

$$\gcd(m,n) \cdot H^2(G,A) = 0. \tag{1}$$

Therefore, if we want an interesting $H^2(G,A)$, we need $\gcd(m,n)$ to be large.

**Example 0.3.** *Let $p$, $q$, and $r$ be different primes. By equation (1), it follows that:*

1. *If $|G| = p^s$ and $|A| = q^t$, then $H^2(G,A) = 0$.*
2. *If $|G| = p^s r$ and $|A| = q^t r$, then $H^2(G,A)$ is a direct sum of finitely many copies of $\mathbb{I}_r$.*

Another reason for our interest in $p$-groups is due to a theorem of Sylow, which states that any group $E$ of order $p^s$ is nilpotent. Moreover, there is a tower

$$0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{s-1} \subseteq E_s = E$$

such that $|E_k| = p^k$, $E_k \lhd E$, and

$$\frac{E_{k+1}}{E_k} \subseteq Z\left(\frac{E}{E_k}\right).$$

Hence if we had determined all $p$-groups of order up to $p^i$, $i = \left\lceil \frac{s}{2} \right\rceil$, we could find all groups of order $p^s$ by describing all extensions

$$1 \to E_i \to E \to G \to 1$$

where $|G| = p^{s-i}$. Thus if we were successful in the goal of our thesis, we would survey all groups of order $p, p^2, p^3$, and $p^4$ as a bonus.

**Remark 0.4.** *By* [DF04, 6.1 Theorem 3] *any finite nilpotent group is a product of $p$-groups, so in order to classify all extensions of finite nilpotent groups, then we first need to do so for finite $p$-groups.*

In the thesis we classify up to a weak congruence (Definition 1.35) the different combination of $G$ and $A^\xi$ arising in extensions of type

$$\begin{aligned} p^s &\to p^{s+t} \to p^t, \\ 1 &\leq s,t \leq 2 \end{aligned}$$

(Theorem 2.8). In total, there are 15 combinations to consider (see Table 2.13). The cokernel $G$ can either be cyclic, or a product of cyclic groups (dicyclic). In both cases there are textbook $\mathbb{Z}G$-resolutions (Section 1.4), which work for any $A^\xi$, which we call the **special** resolutions. We use the special resolutions to calculate $H^2_{\text{spec}}\left(G, A^\xi\right)$ for 15 different cases (Theorem 2.14). We construct machinery (Theorems 2.3 and 2.6) that allow us to go from $H^2_{\text{spec}}\left(G, A^\xi\right)$ to $E\left(G, A^\xi\right)$. For each element $s \in H^2_{\text{spec}}\left(G, A^\xi\right)$, we get generators and relations for $E^s$, the middle group of a representative of the congruence class corresponding to $s$. Using the generators and relations we match $E^s$ with a group $E$ from [Bur55] (Appendix B.2), and in so doing determine $[\varepsilon_s]$ (see Section 5.1 for a description of the procedure).

Once we have determined $[\varepsilon_s]$ for all $s \in H^2_{\text{spec}}\left(G, A^\xi\right)$, we will have found every congruence class of extensions of $A^\xi$ by $G$. We have succeeded in solving the extension problem for all pairs with $s + t \leq 3$ (Theorems 2.16, 2.18, and 2.20). In the case $s + t = 4$ (Theorem 2.22), we have solved the majority of cases, where extensions of $\mathbb{I}_p \times \mathbb{I}_p$ by $\mathbb{I}_p \times \mathbb{I}_p$ are unfinished.

In the future, it could be interesting to:

(1) Finish determining the congruence classes of $A = \mathbb{I}_p \times \mathbb{I}_p$ by $G = \mathbb{I}_p \times \mathbb{I}_p$ (trivial and non-trivial action).
   (a) The case $A = A^{\text{triv}}$ is difficult since
   $$H^2\left(\left(\mathbb{I}_p \times \mathbb{I}_p\right)^{\text{triv}}, \mathbb{I}_p \times \mathbb{I}_p\right) \cong \left(\mathbb{I}_p\right)^6$$
   is very large. The abelian extensions are finished, but the rest remain.
   (b) For the case $A = A^{\xi}$, we have
   $$H^2\left(\left(\mathbb{I}_p \times \mathbb{I}_p\right)^{\xi}, \mathbb{I}_p \times \mathbb{I}_p\right) \cong \begin{cases} \mathbb{I}_{2,} & p = 2 \\ \left(\mathbb{I}_p\right)^3 & p \neq 2. \end{cases}$$
   The case $p = 2$ is simple and is done, while $p \neq 2$ is unfinished. The cohomology group is relatively large, and the rules for $E^s$ are complicated (See Appendix C).

(2) After having found all congruence classes for the cases listed in Theorem 2.8, determine the weak congruence classes. This can be done by
   (a) Constructing weak congruences directly. For instance, in the case $G = \mathbb{I}_p = \langle x \rangle$, $A = \mathbb{I}_p = \langle z \rangle$ we have the class of the split extension
   $$\mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p$$
   and the non-split ones, for $s \in \left(\mathbb{I}_p\right)^*$
   $$\begin{aligned} \varepsilon_s \quad : \quad & \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \\ & \iota_s : z \mapsto P^{s'p} \\ & \pi_s : P \mapsto x \\ & s' \equiv s^{-1} \left(\operatorname{mod} p\right). \end{aligned}$$
   Clearly the triple $\Gamma = \left(\alpha, 1_{\mathbb{I}_{p^2}}, 1_{\mathbb{I}_p}\right)$, where
   $$\begin{array}{rccc} \alpha : \mathbb{I}_p & \rightarrow & \mathbb{I}_p \\ z & \mapsto & z^{rs'} \end{array}$$
   defines a weak congruence $\varepsilon_s \cong \varepsilon_r$, for any $s, r \in \left(\mathbb{I}_p\right)^*$. So there are two weak congruence classes for extensions $p \rightarrow p^2 \rightarrow p$,namely
   $$\mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p$$
   and
   $$\begin{aligned} & \mathbb{I}_p \overset{\iota_1}{\rightarrowtail} \left(\mathbb{I}_{p^2} = \langle P \rangle\right) \overset{\pi_1}{\twoheadrightarrow} \mathbb{I}_p \\ & \iota_1 : z \mapsto P^p \\ & \pi_1 : P \mapsto x. \end{aligned}$$
   In general, it is not the case that if two extensions $\varepsilon$ and $\varepsilon'$ of $A$ by $G$, have the same middle group that they are weakly congruent. We conjecture that this the case here, but we have not showed it. A better method for determining weak congruence classes could be:
   (b) Letting a group that we call $\operatorname{Aut}(G, A)$ (not included in the present text!) act on extensions of $A$ by $G$ (and hence on $E(G, A)$), in such a way that the orbits under the action are precisely the weak congruence classes. Then we can, using the isomorphisms
   $$E(G, A) \tilde{\rightarrow} H^2_{\text{special}}(G, A)$$
   from Theorems 2.3 and 2.6 induce an action of $\operatorname{Aut}(G, A)$ on $H^2_{\text{special}}(G, A)$. Then in order to find the weak congruence classes we:
      (i) Find the orbits of $H^2_{\text{special}}(G, A)$.

(ii) Pick a representative of each orbit, and find the extension it classifies. Then we are done.

This procedure could also significantly simplify the procedure for determining the congruence classes. In the thesis, we partitioned $H^2(G,A)$ arbitrarily, and just found the extensions that the partition classified. Instead we could partition $H^2_{\text{special}}(G,A)$ by the orbits of $\text{Aut}(G,A)$, find the extension of a easy cocycle of each orbit, and let $\text{Aut}(G,A)$ act on it to determine the rest in the class.

0.1. **Structure of the thesis.** We start by introducing group cohomology and the bar resolution in Section 1.1. In Section 1.2 we treat group extensions, where we define the notions of (weak) congruence. In Section 1.3 we discuss the connection of group cohomology with group extensions. We deal with the special resolutions in Section 1.4. In Section 1.5 we state and prove the Constructive Lifting Theorem (Theorem 1.53).

In Section 2 we state the main results of the thesis. The abstract machinery is located in Section 2.1, and in it we give the explicit correspondence between $H^2_{\text{spec}}(G,A)$ and $E(G,A)$. The computational results are in Section 2.2.

Section 3 contains proofs of statements from Section 1.

The next sections are dedicated the proving the main results. In Section 4 we prove the statements of Section 2.1, and Theorems 2.8 and 2.14 from Section 2.2. We describe the procedure for determining extensions in Section 5.1, and in Sections 5.2 to 5.5 we find them.

The Appendix contains results from homological algebra, a treatment on group presentations, lists of $p$-groups up to order $p^4$, and some rules that we derived for extensions of $(\mathbb{I}_p \times \mathbb{I}_p)^{\varsigma}$ by $\mathbb{I}_p \times \mathbb{I}_p$.

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## 1. Preliminaries

**Notation 1.1.**

(1) $\mathbb{I}_n := \mathbb{Z}/n\mathbb{Z}$. *If $p$ is a prime, then $\mathbb{I}_p$ is a field, denoted also by $\mathbb{F}_p$.*
(2) *For $a \in (\mathbb{I}_n)^*$, we let $a' \in \mathbb{I}_n$ be such that*

$$a' \equiv a^{-1} \pmod{n}$$

**Notation 1.2.** *For $a, m \in \mathbb{Z}$*

$$[a]_m := a \pmod{m}$$

**Notation 1.3.** *Let $A$ be an $R$-module and $r$ an element of $R$. Then*

(1) $A^{fix} = \{a \in A : ra = a, \forall r \in R\}$.
(2) ${}_{[r]}A = \{a \in A : ra = 0\}$.

**Notation 1.4.** *If a group $G$ is presented by $S \subseteq G$ subject to relations $R$, we write*

$$G = \langle S : R \rangle .$$

*We shorten relations of the type $w = 1$, just writing $w$ instead, for example*

$$\left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle$$

*means the group presented by two generators $P$, $Q$, and relations*

$$\begin{aligned} P^4 &= 1, \\ Q^2 &= 1, \\ Q^{-1}PQ &= P^3. \end{aligned}$$

*See Definition B.1 and Remark B.3.*

### 1.1. Cohomology and the bar resolution.

**Definition 1.5.** *Let $G$ be a group (or a monoid), then a left $G$-module $A$, is an abelian group $A$ together with a homomorphism*

$$\xi : G \to \mathrm{Aut}(A) .$$

*For $x \in G$ and $a \in A$ we write*

$${}^x a := [\xi(x)](a) .$$

**Remark 1.6.** *It is sometimes convenient to use the exponential notation for function values:*

$${}^f x := f(x) .$$

*The above action would look like this:*

$${}^x a := {}^{\xi(x)} a = {}^{\left(\xi\, x\right)} a.$$

**Notation 1.7.** *Given an abelian group $A$, and an action*

$$\xi : G \to \mathrm{Aut}(A) ,$$

*let $A^\xi$ denote the corresponding $G$-module.*

**Definition 1.8.** *Let $G$ be a group, then the* ***integral group ring of*** *$G$, $\mathbb{Z}G$, has as its elements finite sums*

$$\sum_{x \in G} m_x \langle x \rangle , m_x \in \mathbb{Z}, x \in G$$

*with addition*

$$\sum_{x \in G} m_x \langle x \rangle + \sum_{x \in G} n_x \langle x \rangle = \sum_{x \in G} (m_x + n_x) \langle x \rangle ,$$

*and multiplication*

$$\left(\sum_{x\in G} m_x \langle x\rangle\right)\left(\sum_{x\in G} n_x \langle x\rangle\right) = \sum_{x,y\in G} m_x n_y \langle xy\rangle = \sum_{z\in G}\left(\sum_{\substack{x,y\in G\\ xy=z}} m_x n_y\right)\langle z\rangle .$$

**Remark 1.9.** *As an abelian group, $\mathbb{Z}G$ is free, with the set of generators*

$$\{\langle x\rangle : x\in G\}.$$

**Remark 1.10.** *Elements $\langle x\rangle$ belong to the group* ***ring*** *$\mathbb{Z}G$. We will use, however, similar notations $\langle x\rangle$, $\langle x,y\rangle$, $\langle x,y,z\rangle$ for the (sub)groups generated by $\{x\}$, $\{x,y\}$, $\{x,y,z\}$ etc., hoping that that would not lead to a confusion.*

**Remark 1.11.** *We consider only* ***left*** *modules in this thesis.*

**Notation 1.12.** *We use notations $R$-*Mod *or ${}_R$*Mod *for (left) $R$-modules, and $G$-*Mod *or ${}_G$*Mod *for (left) $G$-modules.*

**Proposition 1.13.** *The categories ${}_{\mathbb{Z}G}$*Mod *and ${}_G$*Mod *are equivalent.*

*Proof.* See [ML95, Proposition IV.1.2]. □

**Remark 1.14.** *We will frequently use this fact, switching between the notations*

$$\left(\sum_{x\in G} m_x \langle x\rangle\right) a = \sum_{x\in G} m_x\ {}^x a.$$

**Definition 1.15.** *Let $G$ be a group and let $A\in$ ${}_{\mathbb{Z}G}$*Mod*, then*

$$H^n(G,A) = \operatorname{Ext}^n_{\mathbb{Z}G}\left(\mathbb{Z}^{triv}, A\right),$$

*where*

$$\xi = triv : G\to \operatorname{Aut}(A)$$

*is the trivial action ($^\xi x \equiv 1_A$) is the $n$th* ***cohomology group*** *of $G$ with coefficients in $A$.*

The significance of group cohomology comes from the (normalized) bar resolution, which we will now discuss.

Let $\beta_n$ be the free $G$-module with generators $[x_1,x_2,\dots,x_n]$, $x_i\in G$, which we may also think of as the free abelian group generated by elements of the form $x[x_1,x_2,\dots,x_n]$. Let $D_n$ be the submodule generated by elements of the form

$$[x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n],\ 1\le i\le n$$

(the degenerate elements). Then we define

$$B_n = \beta_n/D_n.$$

The notation for $[x_1,x_2,\dots,x_n]+D_n\in B_n$ is $[x_1|x_2|\dots|x_n]$.

Differentials $\partial_{n-1}:\beta_n\to\beta_{n-1}, n>0$ on generators are given by

$$\begin{aligned}\partial_{n-1}([x_1,x_2,\dots,x_n]) &= x_1[x_2,\dots,x_n] + \sum_{i=1}^{n-1}(-1)^i[x_1,\dots,x_i\cdot x_{i+1},\dots,x_n] + \\ &\quad + (-1)^n[x_1,\dots,x_{n-1}],\end{aligned}$$

which also work for $B_n$ because $\partial_{n-1}(D_n)\subseteq D_{n-1}$.

Define $\mathbb{Z}$-maps $S_n:\beta_n\to\beta_{n+1}$ by

$$S_{-1}(x[\,]) = [x],\ S_n(x[x_1,\dots,x_n]) = [x,x_1,\dots,x_n]$$

which work the same for $B_n$ since $S_n(D_n)\subseteq D_{n+1}$.

**Remark 1.16.** *The $\mathbb{Z}$-maps $S_n : B_n \to B_{n+1}$ above are examples of what are called contractions. See Appendix A for a definition and properties of contractible complexes.*

Observe that $B_0$ is the free $\mathbb{Z}G$-module with generator $[\,]$, and so is isomorphic to $\mathbb{Z}G$ via the map $[\,] \mapsto \langle 1 \rangle$. The map

$$\begin{array}{rcl} \varepsilon : \mathbb{Z}G & \to & \mathbb{Z} \\ \sum\limits_{\substack{g \in G \\ \text{finite}}} m_g \langle g \rangle & \mapsto & \sum\limits_{\substack{g \in G \\ \text{finite}}} m_g \end{array}$$

is called the **augmentation**, and is clearly a surjective $\mathbb{Z}G$-map.

**Theorem 1.17.** *The (normalized) bar resolution $(B_n, \partial_n)$ with augmentation is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}^{triv}$.*

*Proof.* See [ML95, Theorem IV.5.1] □

Let $A \in {}_{\mathbb{Z}G}\,\mathrm{Mod}$, and let

$$0 \leftarrow \mathbb{Z}^{\mathrm{triv}} \xleftarrow{\varepsilon} B_0 \xleftarrow{d_0} B_1 \xleftarrow{d_1} B_2 \leftarrow \cdots$$

be the normalized bar resolution. Apply $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ to the above complex, with $\mathbb{Z}^{\mathrm{triv}}$ deleted to obtain

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(B_0, A) \xrightarrow{\partial_0^*} \mathrm{Hom}_{\mathbb{Z}G}(B_1, A) \xrightarrow{\partial_1^*} \mathrm{Hom}_{\mathbb{Z}G}(B_2, A) \xrightarrow{\partial_2^*} \cdots$$

and recall that since $B_n$ is free with generators $[x_1|\ldots|x_n]$, we know that any homomorphism

$$\Phi : B_n \to A$$

is the unique extension of a map

$$\varphi : G^n \to A,$$

where

$$\Phi([x_1|\ldots|x_n]) = \varphi(x_1, \ldots, x_n).$$

Identifying $\varphi$ with $\Phi$, and labeling

$$B^n := \mathrm{Hom}_{\mathbb{Z}G}(B_n, A), \delta^n := \partial_n^*$$

we get the cochain complex

$$0 \to B^0 \xrightarrow{\delta^0} B^1 \xrightarrow{\delta^1} B^2 \to \cdots$$

where

$$\begin{array}{rcl} (\delta^n \varphi)(x_1, \ldots, x_{n+1}) & = & {}^{x_1}\varphi(x_2, \ldots, x_{n+1}) + \sum\limits_{i=1}^{n} (-1)^i \varphi(x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_{n+1}) + \\ & & + (-1)^{n+1} \varphi(x_1, \ldots, x_n). \end{array}$$

**Definition 1.18.** *Let $A \in {}_{\mathbb{Z}G}\,\mathrm{Mod}$.*

1. *A map $\varphi : G^n \to A$ is called a **cochain**.*
2. *We say that $\varphi$ is **normalized cochain** if $\varphi(x_1, \ldots, x_n) = 0$ whenever any $x_i = 1$.*
3. *A **cocycle** $\varphi$ is a cochain with the property that $\delta^n \varphi = 0$.*
4. *A cochain $\varphi$ is **coboundary** if $\varphi = \delta^{n-1}\psi$, for some cochain $\psi \in B^{n-1}$.*

**Example 1.19.**

(1) *Let $\varphi : G \times G \to A$ be a cochain. Then $\varphi$ is a cocycle if and only if for all $x, y, z \in G$*

$$\left(\delta^2\varphi\right)(x,y,z) = {}^{x}\varphi(y,z) - \varphi(xy,z) + \varphi(x,yz) - \varphi(xy) = 0$$

*i.e.*

$${}^{x}\varphi(y,z) + \varphi(x,yz) = \varphi(xy,z) + \varphi(xy).$$

(2) *Let $\varphi : G^2 \to A$ be a cochain. Then $\varphi$ is a coboundary if for some cochain $\psi \in B^1$,*

$$\varphi(x,y) = \left(\delta^1\psi\right)(x,y) = {}^{x}\psi(y) - \psi(xy) + \psi(x).$$

*The equation*

$$\begin{aligned}
&\left(\delta^2\varphi\right)(x,y,z) = \left[\delta^2\left(\delta^1\psi\right)\right](x,y,z) \\
= \;& {}^{x}\left(\delta^1\psi\right)(y,z) - \left(\delta^1\psi\right)(xy,z) + \left(\delta^1\psi\right)(x,yz) - \left(\delta^1\psi\right)(x,y) \\
= \;& {}^{x}({}^{y}\psi(z) - \psi(yz) + \psi(y)) - ({}^{xy}\psi(z) - \psi(xyz) + \psi(xy)) + \\
& + ({}^{x}\psi(yz) - \psi(xyz) + \psi(x)) - ({}^{x}\psi(y) - \psi(xy) + \psi(x)) \\
= \;& 0
\end{aligned}$$

*verifies that coboundaries are cocycles.*

**Proposition 1.20.** *Let $G$ be a group and $A \in {}_{G}\mathrm{Mod}$, then*

$$\begin{aligned}
H^n(G,A) &= \frac{\ker\left(B^n \overset{\delta^n}{\to} B^{n+1}\right)}{\mathrm{Im}\left(B^{n-1} \overset{\delta^{n-1}}{\to} B^n\right)} \\
&= \frac{\{cocycles\}}{\{coboundaries\}}.
\end{aligned}$$

**Remark 1.21.** *We will label cohomology groups calculated using the bar resolutions as $H^n_{bar}$.*

## 1.2. **Group extensions.**

**Definition 1.22.** *Let $A$ and $G$ be groups. An **extension** $\varepsilon$ of $A$ by $G$ is a short exact sequence*

$$\varepsilon : 1 \to A \overset{\iota}{\to} E \overset{\pi}{\to} G \to 1.$$

*An extension $\varepsilon$ **splits** is $\pi$ has a right inverse, i.e. there is a homomorphism $\nu : G \to E$ such that $\pi \circ \nu = 1_G$.*

**Definition 1.23.** *Let*

$$\varepsilon : 1 \to A \overset{\iota}{\to} E \overset{\pi}{\to} G \to 1$$

*be an extension of $A$ by $G$, then a **section** of $\varepsilon$ is a map (of sets) $\sigma : G \to E$ with $\pi \circ \sigma = 1_G$. We require further that $\sigma(1) = 1$.*

**Proposition 1.24.** *Let $E$ be an extension of $A$ by $G$. Then conjugation in $E$ determines a homomorphism*

$$\begin{aligned}
\theta : E &\to \mathrm{Aut}(\iota A) \\
x &\mapsto \theta(x) : \iota(a) \mapsto x\iota(a)\,x^{-1}.
\end{aligned}$$

**Proposition 1.25.** *Let $A$ be an abelian group. Then an extension $\varepsilon$ of $A$ by $G$ makes $A$ into a $G$-module.*

*Proof.* Since $A$ is abelian we know that $\theta(\iota(A)) = \{1_{\iota A}\}$, so that if elements of $E$ are congruent modulo $\iota A$, their action on $A$ coincides. Thus, let

$$\sigma : G \to E$$

be a section $\pi$. Then any other section $\sigma'$ will be congruent to $\sigma$ modulo $\iota A$ :

$$\begin{aligned} \pi\left(\sigma'(x)\,\sigma(x)^{-1}\right) &= \pi(\sigma'(x))\,\pi(\sigma(x))^{-1} = xx^{-1} \\ &= 1 \in \ker(\pi) = \iota A. \end{aligned}$$

Hence the map

$$\begin{aligned} \xi : G &\to \mathrm{Aut}(\iota A) \\ x &\mapsto \xi(x) : \iota(a) \mapsto \sigma(x)\,\iota(a)\,\sigma(x)^{-1} \end{aligned}$$

is a well defined homomorphism, which gives us an action of $G$ on $\iota A$, and hence on $A$. □

**Remark 1.26.** *Notice that, though the action $\xi$ seems to depend on the section $\sigma$, $\xi$ does not in fact depend on $\sigma$.*

**Remark 1.27.** *An old-fashioned name for the action $\xi$ is* ***operators*** *(the group $G$ acts on $A$ by the operators $\xi$).*

**Definition 1.28.** *Let $(A, \xi)$ be a $G$-module. An extension $\varepsilon$ of $A$ by $G$* ***realizes the action****, if for all $x \in G$*

$$^{x}a = [\xi(x)](a).$$

*By an extension of $A^{\xi}$ by $G$, we mean an extension of $A$ by $G$ that realizes the action $\xi$.*

**Definition 1.29.** *An extension $\varepsilon$ of $A$ by $G$ in which $\iota(A) \subseteq Z(E)$, where $Z(E)$ is the center of $E$, is called central.*

**Proposition 1.30.** *Let $\varepsilon$ be an extension of $A$ by $G$. Then $\varepsilon$ is central if and only if the action of $G$ on $A$ is trivial.*

**Definition 1.31.** *Let $(A, \xi)$ be a $G$-module. The* ***semidirect product*** *of $A$ and $G$, $A \rtimes_{\xi} G$ is a group with elements are of the form*

$$(a, x),\ a \in A, x \in G,$$

*and with multiplication*

$$(a, x)(b, y) = (a\ {}^{x}b, xy).$$

*Define maps*

$$\begin{aligned} \iota : A &\to A \rtimes_{\xi} G \\ a &\mapsto (a, 1) \end{aligned}$$

*and*

$$\begin{aligned} \pi : A \rtimes_{\xi} G &\to G \\ (a, x) &\mapsto x. \end{aligned}$$

**Remark 1.32.** *It is convenient sometimes to write $ax$ instead of $(a, x)$.*

**Proposition 1.33.** *The semidirect product with the maps $\iota$ and $\pi$*

$$1 \to A \xrightarrow{\iota} A \rtimes_{\xi} G \xrightarrow{\pi} G \to 1$$

*is an extension of $A$ by $G$ realizing the action $\xi$. Furthermore, the extension splits.*

*Proof.* The maps $\iota$ and $\pi$ are clearly homomorphisms, injective and surjective respectively. The equality

$$xax^{-1} = (1,x)\left(a,x^{-1}\right) = \left({}^{x}a, xx^{-1}\right) = \left({}^{x}a, 1\right)$$

shows that $A \rtimes_\xi G$ realizes the action. A splitting $\gamma : G \to A \rtimes_\xi G$ is given by

$$\gamma : x \mapsto (1,x)\,.$$

This is obviously a homomorphism with $\pi \circ \gamma = 1_G$. □

**Example 1.34.** *If the action of $G$ on $A$ is trivial, then $A \rtimes_\xi G = A \times G$.*

**Definition 1.35.** *If $\varepsilon$ and $\varepsilon'$ are extensions, then a **morphism** $\Gamma : \varepsilon \to \varepsilon'$ is a triple $(\alpha, \beta, \gamma)$ of morphisms such that*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
1 & \longrightarrow & A' & \xrightarrow{i'} & E' & \xrightarrow{p'} & G' & \longrightarrow & 1
\end{array}$$

*commutes.*

*The morphism $\Gamma : \varepsilon \to \varepsilon'$ is an **isomorphism** if each of the components are isomorphisms, and we write $\varepsilon \cong \varepsilon'$.*

**Remark 1.36.** *Clearly the relation $\cong$ on extensions is an equivalence relation.*

**Remark 1.37.** *We will also call an isomorphism of extensions a **weak congruence** (compare with a **congruence** defined below).*

**Proposition 1.38.** *If $\Gamma : \varepsilon \to \varepsilon'$ is a weak congruence of extensions, then the action of $\varepsilon'$ is determined by $\varepsilon$:*

$$\left[\xi'(\gamma x)\right](\alpha c) = \alpha\left(\left[\xi(x)\right](c)\right),$$

*or equivalently, in the exponential notation:*

$${}^{\xi'({}^{\gamma}x)}({}^{\alpha}c) = {}^{\alpha}\left({}^{(\xi x)}c\right),$$

*$x \in G$, $c \in A$.*

*Proof.* It is clear that

$$\sigma' = \beta\sigma\gamma^{-1}$$

is a section of $\pi'$. It follows from the commutativity of the diagram that

$$\begin{aligned}
\iota' &= \beta\iota\alpha^{-1}, \\
(\iota')^{-1} &= \alpha\iota^{-1}\beta^{-1}.
\end{aligned}$$

Then

$$\begin{aligned}
{}^{\xi'({}^{\gamma}x)}({}^{\alpha}c) &= (i')^{-1}\left[\sigma'({}^{\gamma}x)\cdot {}^{i'}({}^{\alpha}c)\cdot\left(\sigma'({}^{\gamma}x)\right)^{-1}\right] = \\
&= {}^{\alpha\iota^{-1}\beta^{-1}}\left[{}^{\beta\sigma\gamma^{-1}}({}^{\gamma}x)\cdot {}^{\beta\iota\alpha^{-1}}({}^{\alpha}c)\cdot\left({}^{\beta\sigma\gamma^{-1}}({}^{\gamma}x)\right)^{-1}\right] = \\
&= {}^{\alpha\iota^{-1}\beta^{-1}}\left[{}^{\beta\sigma}x\cdot {}^{\beta\iota\alpha^{-1}}({}^{\alpha}c)\cdot\left({}^{\beta\sigma}x\right)^{-1}\right] = \\
&= {}^{\alpha\iota^{-1}}\left[{}^{\sigma}x\cdot {}^{\iota\alpha^{-1}}({}^{\alpha}c)\cdot({}^{\sigma}x)^{-1}\right] = \\
&= {}^{\alpha\iota^{-1}}\left[{}^{\sigma}x\cdot {}^{\iota}c\cdot({}^{\sigma}x)^{-1}\right] = {}^{\alpha}\left({}^{(\xi x)}c\right).
\end{aligned}$$

□

**Definition 1.39.** *We say that two extensions $\varepsilon$ and $\varepsilon'$ of $A$ by $G$ are* ***congruent*** *(equivalent) ($\varepsilon \sim \varepsilon'$) if there is a group homomorphism $\gamma : E \to E'$ so that $(1_A, \gamma, 1_G)$ is a morphism between $\varepsilon$ and $\varepsilon'$, i.e.,*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_G} & & \\ 1 & \longrightarrow & A' & \xrightarrow{i'} & E' & \xrightarrow{p'} & G' & \longrightarrow & 1 \end{array} \tag{2}$$

*commutes.*

Clearly congruent extensions are weakly congruent, but the converse needs not hold.

**Remark 1.40.** *We can and will now assume without loss of generality that the maps $A \xrightarrow{\iota} E$ and $E \xrightarrow{\pi} G$ are the inclusion and projection, respectively. Furthermore, when we have $A \in {}_{\mathbb{Z}G}\mathrm{Mod}$, and say that $\varepsilon$ is an extension of $A$ (as a $G$-module) by $G$, we mean that $\varepsilon$ realizes the action.*

**Notation 1.41.** *The set of classes of congruent extensions is denoted $E(G, A)$ (or $E\left(G, A^{\xi}\right)$, when $\xi$ is a given action).*

**Remark 1.42.** *In fact, $E(G, A)$ has a natural structure of an abelian group. The group operation can be defined* ***internally****, using extensions. However, we will only consider the group structure on $E(G, A)$ inherited from the isomorphism*

$$E(G, A) \simeq H^2(G, A),$$

*proved below.*

**Proposition 1.43.** *Any extension of $A$ by $G$ that splits is congruent to the semidirect product $A \rtimes_{\xi} G$.*

*Proof.* See [ML95, Section IV.4.3]. □

**Definition 1.44.** *Let*

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

*be an extension of $A$ by $G$.*

(1) *If $G = \mathbb{I}_m = \langle t \rangle$, then a section $\sigma : G \to E$ of $\pi$ is called* ***simple*** *if*

$$\sigma\left(t^i\right) = \{t\}^i, 0 \le i < m$$

*for $\{t\} \in E$ such that $\pi(\{t\}) = 1$.*

(2) *If $G = \mathbb{I}_m \times \mathbb{I}_n = \langle x \rangle \times \langle y \rangle$, then a section $\sigma : G \to E$ of $\pi$ is called* ***simple*** *if*

$$\sigma\left(x^i y^j\right) = \{x\}^i \{y\}^j, 0 \le i \le m, 0 \le j \le n$$

*for some $\{x\}, \{y\} \in E$ with $\pi(\{x\}) = x$ and $\pi(\{y\}) = y$.*

1.3. **The isomorphism $H^2_{\mathbf{bar}} \cong E(G, A)$.** Let

$$1 \to A \to E \xrightarrow{\pi} G \to 1$$

be an extension of $A$ by $G$, with $x : G \to E$ being a section of $\pi$, that is,

$$\begin{aligned} \pi(x_g) &= g \text{ for } \forall g \in G, \\ x_1 &= 1. \end{aligned}$$

Then every element $e \in E$ can be written uniquely in the form

$$e = a \cdot x_g, \text{ some } a \in A, g \in G.$$

Hence as a set, we may think of $E$ as

$$E = A \times G = \{(a, g) : a \in A, g \in G\}.$$

The following equality

$$\pi(x_{gh}) = gh = \pi(x_g)\pi(x_h) = \pi(x_g x_h)$$

shows that $x_g x_h \equiv x_{gh} \pmod{A}$, i.e.

$$x_g x_h = \varphi(g, h)\, x_{gh},\ \forall g, h \in G.$$

This gives a multiplication compatible with the description of $E$ as a set of tuples:

$$(a, g)(b, h) \mapsto (ax_g)(bx_h) = a\left(x_g b x_g^{-1}\right) x_g x_h = a \cdot {}^g b \cdot \varphi(g, h)\, x_{gh} \mapsto (a \cdot {}^g b \cdot \varphi(g, h), gh)$$

so

$$(a, g)(b, h) = (a \cdot {}^g b \cdot \varphi(g, h), gh).$$

**Theorem 1.45.**

(1) *The function $\varphi : G \times G \to A$ is a normalized cocycle (Definition 1.18).*
(2) *The element $(1, 1)$ is the identity element of $E$, and the inverse of $(a, g)$ is $\left({}^{g^{-1}}\left(a^{-1}\right) \cdot \varphi\left(g^{-1}, g\right)^{-1}, g^{-1}\right)$.*
(3) *Consider the commutative diagram is*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_G} & & \\ 1 & \longrightarrow & A' & \longrightarrow & E' & \xrightarrow{\pi'} & G' & \longrightarrow & 1 \end{array}$$

*Let $x : G \to E$ be a section of $\pi$ ($\pi \circ x = 1_G$) and $y : G \to E'$ be a section of $\pi'$ ($\pi' \circ y = 1_G$). Then for $\forall g \in G$*

$$\pi'(\beta(x_g)) = \pi(x_g) = g$$

*shows that $y_g \equiv \beta(x_g) \pmod{A}$ $\forall g \in G$, i.e.*

$$y_g = \xi(g)\, \beta(x_g),\ \xi : G \to A.$$

*Let $\varphi, \psi : G \times G \to A$ be the cocycles of $x$ and $y$ respectively. Then the cocycles $\varphi$ and $\psi$ are congruent modulo coboundaries.*
(4) *Finally: let $G$ be a group, $A$ be a $G$-module, and $\omega$ be the function which assigns to each extension of $A$ by $G$ realizing the action, the congruence class of one of its cocycles. Then $\omega$ induces a bijection*

$$\omega : E(G, A) \leftrightarrow H^2(G, A)$$

*where the class of the semidirect product $A \rtimes G$ corresponds to $0 \in H^2(G, A)$.*

*Proof.* See Section 3.2. □

**Proposition 1.46.** *Let $G = \mathbb{I}_m = \langle t \rangle$,*

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

*be an extension of $A$ by $G$, and let $\sigma$ be a simple section of $\pi$, with $\sigma(t) = \{t\}$. Then the corresponding cocycle $\varphi_\sigma : G \times G \to A$ is given by*

$$\varphi_\sigma\left(t^i, t^j\right) = \{t\}^{i+j-[i+j]_m} = \begin{cases} \{t\}^m & \text{if} \quad i + j \geq m \\ 1 & \text{if} \quad i + j < m \end{cases}.$$

*Proof.* Let $\varphi$ be the cocycle of $\sigma$. Then

$$\sigma\left(t^i\right)\sigma\left(t^j\right)=\varphi\left(t^i,t^j\right)\sigma\left(t^{[i+j]_m}\right)$$

for all $i,j$. Using the fact that $\sigma$ is simple we get the equation

$$\{t\}^{i+j}=\varphi\left(t^i,t^j\right)\{t\}^{[i+j]_m},$$

which is equivalent to

$$\varphi\left(t^i,t^j\right)=\{t\}^{i+j-[i+j]_m}.$$

□

**Proposition 1.47.** *Let $G=\mathbb{I}_m\times\mathbb{I}_n=\langle x\rangle\times\langle y\rangle$,*

$$1\to A\xrightarrow{\iota}E\xrightarrow{\pi}G\to 1$$

*be an extension of $A$ by $G$, and let $\sigma$ be a simple section of $\pi$, with $\sigma\left(x^iy^j\right)=\{x\}^i\{y\}^j$. Then the corresponding cocycle $\varphi_\sigma: G\times G\to A$ is given by*

$$\left(x^iy^j,x^ky^l\right)\longmapsto\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^{i+r}y^d}V\cdot\begin{cases}1 & , & i+k<m,j+l<n\\ W & , & i+k\geq m,j+l<n\\ {}^{x^{i+k}}U & , & i+k<m,j+l\geq n\\ W\,{}^{x^{i+k-m}}U & , & i+k\geq m,j+l\geq n\end{cases},$$

*where*

$$\begin{bmatrix}U\\V\\W\end{bmatrix}=\begin{bmatrix}\{y\}^n\\ \{y\}\{x\}\{y\}^{-1}\{x\}^{-1}\\ \{x\}^m\end{bmatrix}\in A^3.$$

*Proof.* See Section 3.1. □

1.4. **Special Resolutions.** The bar resolution tells us what cohomology of groups means, but it is not suitable for computation. Fortunately in the case when $G=\mathbb{I}_m$ is cyclic, there is a textbook resolution which we will call the **special resolution**. In the case when $G=\mathbb{I}_m\times\mathbb{I}_n$ is dicyclic, the total complex of the tensor product of the special resolutions for $\mathbb{I}_m$ and $\mathbb{I}_n$, is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}^{\text{triv}}$, which we will also call the special resolution.

1.4.1. *The case $G=\mathbb{I}_m$.* Consider the sequence

$$0\leftarrow\mathbb{Z}^{\text{triv}}\xleftarrow{d_{-1}}P_0\xleftarrow{d_0}P_1\xleftarrow{d_1}P_2\xleftarrow{d_2}P_3\leftarrow\dots$$

where for all $n\geq 0$:

$$\begin{aligned}P_n &= \mathbb{Z}G=\left\{\sum_{i=0}^{m-1}a_i\left\langle x^i\right\rangle : a_i\in\mathbb{Z}\right\};\\ d_{-1} &= \varepsilon:\langle 1\rangle\mapsto 1,\\ d_{2n} &: \langle 1\rangle\mapsto D{:=}\langle x\rangle-\langle 1\rangle,\\ d_{2n+1} &: \langle 1\rangle\mapsto N{:=}\sum_{i=0}^{m-1}\left\langle x^i\right\rangle.\end{aligned}$$

**Remark 1.48.** *Since the $P_n$'s are free $\mathbb{Z}G$-modules, it is enough to define the homomorphisms on the generator $\langle 1\rangle$, and extend by $\mathbb{Z}G$-linearity.*

**Proposition 1.49.** *Let $G = \mathbb{I}_m$. Then complex $0 \leftarrow \mathbb{Z}^{triv} \leftarrow P_\bullet$ is a free $\mathbb{Z}G$-module resolution of $\mathbb{Z}^{triv}$, which we call the* ***special resolution****. A contraction for the special resolution is given by*

$$\begin{aligned} S_{-1}(1) &= \langle 1 \rangle, \\ S_{2n}\left(\langle x^i \rangle\right) &= \begin{cases} \sum_{j=0}^{i-1} \langle x^j \rangle & , i > 0 \\ 0 & , i = 0 \end{cases}, \\ S_{2n+1}\left(\langle x^i \rangle\right) &= \begin{cases} 0 & , i < m-1 \\ \langle 1 \rangle & , i = m-1. \end{cases} \end{aligned}$$

*Proof.* For the proof, see Section 3.3. □

Let us calculate the cohomology groups. We have

$$0 \leftarrow \mathbb{Z}^{\text{triv}} \overset{d_{-1}}{\leftarrow} P_0 \overset{d_0}{\leftarrow} P_1 \overset{d_1}{\leftarrow} P_2 \overset{d_2}{\leftarrow} P_3 \leftarrow \dots$$

which when we delete $\mathbb{Z}^{\text{triv}}$ gives

$$0 \leftarrow P_0 \overset{d_0}{\leftarrow} P_1 \overset{d_1}{\leftarrow} P_2 \overset{d_2}{\leftarrow} P_3 \leftarrow \dots$$

Applying $\text{Hom}_{\mathbb{Z}G}(-, A)$:

$$0 \to \text{Hom}_{\mathbb{Z}G}(P_0, A) \overset{d_0^*}{\to} \text{Hom}_{\mathbb{Z}G}(P_1, A) \overset{d_1^*}{\to} \text{Hom}_{\mathbb{Z}G}(P_2, A) \to \cdots$$

where for $\varphi : \mathbb{Z}G \to A$ we have

$$d_n^* : \varphi \mapsto \varphi \circ d_n.$$

Note that

$$\begin{aligned} \text{Hom}_{\mathbb{Z}G}(P_n, A) &\to A \\ \varphi &\mapsto \varphi(\langle 1 \rangle) \end{aligned}$$

is a natural isomorphism, with inverse

$$\begin{aligned} A &\to \text{Hom}_{\mathbb{Z}G}(P_n, A) \\ a &\mapsto \varphi : \langle 1 \rangle \mapsto a \end{aligned}$$

Thus the above cochain complex is isomorphic to

$$0 \to A \overset{d^0}{\to} A \overset{d^1}{\to} A \to \cdots$$

where

$$\begin{aligned} d^{2k} &: a \mapsto Da = (\langle x \rangle - \langle 1 \rangle) a = {}^x a - a, \\ d^{2k+1} &: a \mapsto Na = \left( \sum_{j=0}^{m-1} \langle x^j \rangle \right) a = \sum_{j=0}^{m-1} {}^{x^j} a \end{aligned}$$

**Theorem 1.50.** [ML95, Theorem 7.1] *Let $G = \mathbb{I}_m$ and $A \in {}_G\text{Mod}$, then for any integer $k \geq 0$ :*

$$\begin{aligned} H^0_{special}(G, A) &= A^{fix}, \\ H^{2k}_{special}(G, A) &= \frac{A^{fix}}{NA}, \\ H^{2k+1}_{special}(G, A) &= \frac{\{a \in A : N \cdot a = 0\}}{DA}. \end{aligned}$$

*Proof.* Going through the different cases:

(1) $H^0_{\text{special}}(G, A) = \frac{\ker\left(A \overset{d^0}{\to} A\right)}{\text{Im}(0 \to A)} = \{a \in A : {}^x a - a = 0\} = A^{\text{fix}}$.

(2) $H^{2k}_{\text{special}}(G,A) = \frac{\ker\left(A \overset{d^{2k}}{\to} A\right)}{\operatorname{Im}\left(A \overset{d^{2k-1}}{\to} A\right)} = \frac{\{a\in A:\ {}^x a - a = 0\}}{\{Na : a\in A\}} = \frac{A^{\text{fix}}}{NA}$.

(3) $H^{2k+1}_{\text{special}}(G,A) = \frac{\ker\left(A \overset{d^{2k+1}}{\to} A\right)}{\operatorname{Im}\left(A \overset{d^{2k}}{\to} A\right)} = \frac{\{a\in A : Na = 0\}}{\{Da : a\in A\}}$.

□

1.4.2. *The case $G = \mathbb{I}_m \times \mathbb{I}_n$.* We also need a resolution of $\mathbb{Z}^{\text{triv}}$ in the case when $G = \mathbb{I}_m \times \mathbb{I}_n$. We follow the procedure outlined in [HS97, Chapter VI, Section 15]. Let

$$\begin{aligned} G_1 &= \mathbb{I}_m = \langle x \rangle, \\ G_2 &= \mathbb{I}_n = \langle y \rangle, \end{aligned}$$

and

$$G = G_1 \times G_2 = \left\{ x^i y^j : 0 \le i \le m-1, 0 \le j \le n-1 \right\}.$$

By Proposition 1.49 we have free $\mathbb{Z}G_i$-resolutions

$$0 \leftarrow \mathbb{Z}^{\text{triv}} \leftarrow P^i_\bullet.$$

As described in Appendix A.0.1, the part $\operatorname{Tot}_s\left(P^1_\bullet \otimes_{\mathbb{Z}} P^2_\bullet\right)$, $s \le 3$, of the total complex looks like this:

$$\begin{array}{ccccccc}
\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & & & & & & \\
\downarrow{\scriptstyle 1\otimes D_y} & & & & & & \\
\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & \xleftarrow{D_x \otimes 1} & \mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & & & & \\
\downarrow{\scriptstyle 1\otimes N_y} & & \downarrow{\scriptstyle -(1\otimes N_y)} & & & & \\
\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & \xleftarrow{D_x \otimes 1} & \mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & \xleftarrow{N_x \otimes 1} & \mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & & \\
\downarrow{\scriptstyle 1\otimes D_y} & & \downarrow{\scriptstyle -(1\otimes D_y)} & & \downarrow{\scriptstyle 1\otimes D_y} & & \\
\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & \xleftarrow{D_x \otimes 1} & \mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & \xleftarrow{N_x \otimes 1} & \mathbb{Z}G_1 \otimes \mathbb{Z}G_2 & \xleftarrow{D_x \otimes 1} & \mathbb{Z}G_1 \otimes \mathbb{Z}G_2
\end{array}$$

We note that $\mathbb{Z}$ is a PID and that both $P^1_\bullet, P^2_\bullet$ are projective and hence flat as complexes over $\mathbb{Z}$, so by the Künneth formula (Theorem A.13), there is a short exact sequence

$$\bigoplus_{p+q=n} H_p\left(P^1_\bullet\right) \otimes_{\mathbb{Z}} H_q\left(P^2_\bullet\right) \rightarrowtail H_n\left(\operatorname{Tot}\left(P^1_\bullet \otimes_{\mathbb{Z}} P^2_\bullet\right)\right) \twoheadrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}^{\mathbb{Z}}_1\left(H_p\left(P^1_\bullet\right), H_q\left(P^2_\bullet\right)\right).$$

Since the $P^i_\bullet$'s are exact, it follows that $H_p\left(P^i_\bullet\right) = 0$ and thus the SES reduces to

$$0 \rightarrowtail H_n\left(\operatorname{Tot}\left(P^1_\bullet \otimes_{\mathbb{Z}} P^2_\bullet\right)\right) \twoheadrightarrow 0$$

i.e.

$$H_n\left(\operatorname{Tot}\left(P^1_\bullet \otimes_{\mathbb{Z}} P^2_\bullet\right)\right) \cong 0$$

which shows that $\operatorname{Tot}\left(P^1_\bullet \otimes_{\mathbb{Z}} P^2_\bullet\right)$ is exact. Hence we have a $\mathbb{Z}$-resolution

$$0 \leftarrow \mathbb{Z}^{\text{triv}} \leftarrow \operatorname{Tot}\left(P^1_\bullet \otimes_{\mathbb{Z}} P^2_\bullet\right)$$

with the obvious augmentation.

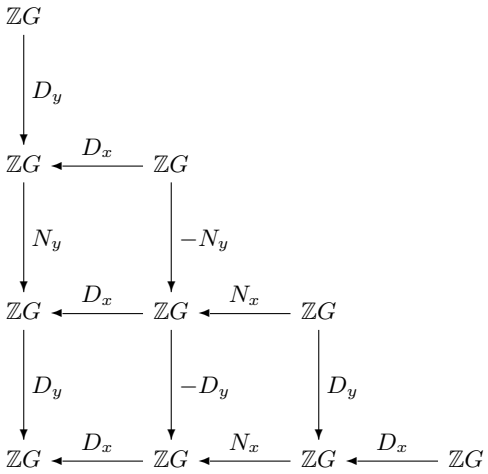
We will now show that this $\mathbb{Z}$-resolution is in fact a $\mathbb{Z}G$-resolution. We make $\mathbb{Z}G_1 \otimes \mathbb{Z}G_2$ into a $\mathbb{Z}G$-module by

$$^{x^i y^j}(a \otimes b) := \left(\left\langle x^i \right\rangle a\right) \otimes \left(\left\langle y^j \right\rangle b\right)$$

which is easily seen to be compatible with the differentials. Applying the isomorphism

$$\begin{aligned} \mathbb{Z}G_1 \otimes \mathbb{Z}G_2 &\to \mathbb{Z}G \\ \langle x \rangle \otimes \langle y \rangle &\mapsto \langle xy \rangle \end{aligned}$$

to our bicomplex, yields a bicomplex (over $\mathbb{Z}G$)




Hence the complex

$$0 \leftarrow \mathbb{Z}^{triv} \overset{d_{-1}}{\leftarrow} \mathbb{Z}G \overset{d_0}{\leftarrow} \mathbb{Z}G \bigoplus \mathbb{Z}G \overset{d_1}{\leftarrow} \mathbb{Z}G \bigoplus \mathbb{Z}G \bigoplus \mathbb{Z}G \leftarrow \cdots$$

where the first few differentials are given by

$$\begin{aligned} d_{-1} &= \varepsilon, \\ d_0 \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} D_y & D_x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = D_y a + D_x b, \\ d_1 \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} N_y & D_x & 0 \\ 0 & -D_y & N_x \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} N_y a + D_x b \\ -D_y b + N_x c \end{bmatrix} \\ d_2 \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &= \begin{bmatrix} D_y & D_x & 0 & 0 \\ 0 & -N_y & N_x & 0 \\ 0 & 0 & D_y & D_x \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} D_y a + D_x b \\ -N_y b + N_x c \\ D_y c + D_x d \end{bmatrix}. \end{aligned}$$

is a $\mathbb{Z}G$-resolution, which we call the **special resolution**.

Applying $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ to

$$0 \leftarrow \mathbb{Z}G \overset{d_0}{\leftarrow} \mathbb{Z}G \bigoplus \mathbb{Z}G \overset{d_1}{\leftarrow} \mathbb{Z}G \bigoplus \mathbb{Z}G \bigoplus \mathbb{Z}G \leftarrow \cdots$$

gives

$$A \overset{d_0^*}{\to} A^2 \overset{d_1^*}{\to} A^3 \to \dots$$

via the natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{i=1}^{m} \mathbb{Z}G, A\right) &\to \prod_{i=1}^{m} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \to A^m \\ \varphi &\mapsto (\varphi \circ \iota_i) \mapsto (\varphi \circ \iota_i(\langle 1 \rangle)) \end{aligned}$$

So

$$H^2(G,A) \cong \frac{\ker\left(A^3 \overset{d_2^*}{\to} A^4\right)}{\operatorname{Im}\left(A^2 \overset{d_1^*}{\to} A^3\right)}$$

where

$$d_2^*\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} D_y & 0 & 0 \\ D_x & -N_y & 0 \\ 0 & N_x & D_y \\ 0 & 0 & D_x \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} D_y a \\ D_x a - N_y b \\ N_x b + D_y c \\ D_x c \end{bmatrix}$$

and

$$d_1^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} N_y & 0 \\ D_x & -D_y \\ 0 & N_x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} N_y a \\ D_x a - D_y b \\ N_x b \end{bmatrix}$$

We have in fact proved the following theorem:

**Theorem 1.51.** *Let $G = \mathbb{I}_m \times \mathbb{I}_n = \langle x \rangle \times \langle y \rangle$, and let $A$ be a $G$-module. Then*

$$H^2(G,A) \cong \frac{\ker\left(A^3 \overset{d_2^*}{\to} A^4\right)}{\operatorname{Im}\left(A^2 \overset{d_1^*}{\to} A^3\right)},$$

*where*

$$\begin{aligned} d_2^*\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= \begin{bmatrix} D_y a \\ D_x a - N_y b \\ N_x b + D_y c \\ D_x c \end{bmatrix}, \\ d_1^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= \begin{bmatrix} N_y a \\ D_x a - D_y b \\ N_x b \end{bmatrix}. \end{aligned}$$

**Proposition 1.52.** *Let $G = \mathbb{I}_m \times \mathbb{I}_n = \langle x \rangle \times \langle y \rangle$, and let $A$ be a $G$-module. Then the special resolution*

$$0 \leftarrow \mathbb{Z}^{triv} \overset{\varepsilon}{\leftarrow} \mathbb{Z}G \overset{d_0}{\leftarrow} \mathbb{Z}G \oplus \mathbb{Z}G \overset{d_1}{\leftarrow} \mathbb{Z}G \oplus \mathbb{Z}G \oplus \mathbb{Z}G \leftarrow \cdots$$

*is contractible as a complex of $\mathbb{Z}$-modules, and (up to homotopy) the first few maps are given by*

$$\begin{aligned}
S_{-1}(1) &= \langle 1 \rangle, \\
S_0\left(\left\langle x^i y^j \right\rangle\right) &= \begin{bmatrix} \sum_{k=0}^{j-1} \left\langle x^i y^k \right\rangle \\ \sum_{k=0}^{i-1} \left\langle x^k \right\rangle \end{bmatrix}, \\
S_1\left(\begin{bmatrix} \left\langle x^i y^j \right\rangle \\ 0 \end{bmatrix}\right) &= \begin{cases} \begin{bmatrix} \left\langle x^i \right\rangle \\ 0 \\ 0 \end{bmatrix} & j = n-1 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & j < n-1 \end{cases}, \\
S_1\left(\begin{bmatrix} 0 \\ \left\langle x^i y^j \right\rangle \end{bmatrix}\right) &= \begin{cases} \begin{bmatrix} 0 \\ -\sum_{k=0}^{j-1} \left\langle x^i y^k \right\rangle \\ \langle 1 \rangle \end{bmatrix} & , i = m-1 \\ \begin{bmatrix} 0 \\ -\sum_{k=0}^{j-1} \left\langle x^i y^k \right\rangle \\ 0 \end{bmatrix} & , i < m-1 \end{cases}
\end{aligned}$$

*Proof.* For the proof, see Section 3.4. □

1.5. **Comparisons of Resolutions.** The Comparison Theorem (Theorem A.6) tells us that $H^n_{\text{bar}}$ and $H^n_{\text{special}}$ are isomorphic, but it does not specify the isomorphism. We will need an explicit description of this isomorphism, and so we introduce the constructive lifting theorem. It is inspired by the field of relative homological algebra, in particular [ML95, Chapter IX. Theorem 6.2], and it uses the condition that our resolutions are **free** (not just projective).

**Theorem 1.53.** *(Constructive Lifting Theorem) Let $L$ be a ring and $K \subseteq L$ a subring, and suppose that we have:*

(1) *A complex $0 \leftarrow A \leftarrow P_\bullet$ in ${}_L\text{Mod}$, where $P_n$ are free, i.e.*

$$P_n = \bigoplus_{X_n} L = \left\{ \sum_{\substack{x \in X_n \\ finite}} l_x [x] : l_x \in L \right\}$$

*for some index set $X_n$.*

(2) *A complex $0 \leftarrow B \leftarrow Q_\bullet$ in ${}_L\text{Mod}$ is contractible in ${}_K\text{Mod}$ with contraction $S$ (e.g. a projective resolution of $B$).*

*Then for any $L$-map $f : A \to B$, the family of $L$-maps $(f_n)$ defined recursively on generators by*

$$f_n[x] := S_{n-1} f_{n-1} d_{n-1}[x], \ f_0[x] := S_{-1} f d_{-1}[x].$$

*is a lifting of $f$ (in ${}_L\text{Mod}$).*

*Proof.* (By induction) When $n = 0$, we have $d_{-1} f_0 = d_{-1} f$ by definition of $f_0$. Let $n \geq 0$ and assume that

$$f_{n-1} d_{n-1} = d_{n-1} f_n.$$

Then

$$\begin{aligned} f_n d_n [x] &= 1_{Q_n} f_n d_n [x] = (S_{n-1} d_{n-1} + d_n S_n) f_n d_n [x] \\ &= S_{n-1} d_{n-1} f_n d_n [x] + d_n S_n f_n d_n [x] \\ &= S_{n-1} (d_{n-1} f_n) d_n [x] + (d_n S_n f_n) d_n [x] \\ &= S_{n-1} f_{n-1} \underbrace{d_{n-1} d_n}_{0} [x] + f_{n+1} d_n [x] \\ &= f_{n+1} d_n [x] . \end{aligned}$$

Since $P_n$, $n \geq 0$ are free modules, we know that $f_\bullet$ extends uniquely to $\mathbb{Z}G$-maps. □

The following diagram illustrates the situation in Theorem 1.53:

$$\begin{array}{ccccccccccc} A & \xleftarrow{d_{-1}} & P_0 & \xleftarrow{d_0} & P_1 & \xleftarrow{d_1} & P_2 & \longleftarrow & \cdots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ B & \xrightarrow{S_{-1}} & Q_0 & \xrightarrow{S_0} & Q_1 & \xrightarrow{S_1} & Q_2 & \longrightarrow & \cdots \end{array}$$

Now we have the tools to construct liftings (of $1_{\mathbb{Z}}$)

$$P_\bullet^{\text{special}} \xrightarrow{f} B_\bullet^{\text{bar}}, \text{ and}$$
$$B_\bullet^{\text{bar}} \xrightarrow{g} P_\bullet^{\text{special}}$$

to get isomorphisms

$$H^n_{\text{bar}} (G, A) \xrightarrow{f^*} H^n_{\text{special}} (G, A) , \text{ and}$$
$$H^n_{\text{special}} (G, A) \xrightarrow{g^*} H^n_{\text{bar}} (G, A) .$$

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## 2. Main Results

### 2.1. **Machinery.**

2.1.1. *Extensions by a cyclic group.*

**Definition 2.1.** *Let $G = \mathbb{I}_m = \langle x \rangle$ and let $A$ be a $G$-module. Then the **special** cohomology group is*

$$H^2_{spec}(G, A) = \frac{A^{fix}}{NA},$$

*where $N = \sum_{i=0}^{m-1} \langle x^i \rangle$.*

**Remark 2.2.** *In fact, it is possible (and natural) to define $H^n_{spec}$ for all $n$ (see Theorem 1.50), but at this stage we are interested only in $H^2_{spec}$.*

**Theorem 2.3.**

(1) *Let $G = \mathbb{I}_m$ and $A$ be a $G$-module. Then the map*

$$\begin{array}{rcl} H^2_{spec}(G, A) = \dfrac{A^{fix}}{NA} & \to & H^2_{bar}(G, A) \\ a + NA & \mapsto & \left( \left(x^i, x^j\right) \mapsto \left\{ \begin{array}{ll} 0 & , i + j < m \\ a & , i + j \geq m \end{array} \right. \right) + \partial B^1, \end{array}$$

*is an isomorphism.*

(2) *Let $G = \mathbb{I}_m$ and $A$ be a $G$-module. Then the map*

$$\begin{array}{rcl} H^2_{bar}(G, A) & \to & H^2_{spec}(G, A) \\ \varphi + \delta B^1 & \mapsto & \displaystyle\sum_{j=1}^{m-1} \varphi\left(x^j, x\right) + NA \end{array}$$

*is an isomorphism, which is inverse to the previous one.*

(3) *Let $G = \mathbb{I}_m$, $A$ a $G$-module, and $\omega$ the map which sends an extension*

$$\varepsilon : 1 \to A \overset{\iota}{\to} E \overset{\pi}{\to} G \to 1$$

*of $A$ by $G$ realizing the action, to the element*

$$\{x\}^m + NA \in H^2_{special}(G, A),$$

*where $\{x\} \in E^s$ is a representative of $x$. Then $\omega$ induces a bijection*

$$\begin{array}{rcl} \omega : E(G, A) & \leftrightarrow & H^2_{special}(G, A) \\ [\varepsilon] & \mapsto & \{x\}^m + NA. \end{array}$$

*Proof.* For the proof, see Section 4.1. □

**Remark 2.4.** *To see how we apply the above Theorem, see Section 5.1.*

2.1.2. *Extensions by a dicyclic group.*

**Definition 2.5.** *Let $G = \mathbb{I}_m \times \mathbb{I}_n = \langle x, y \rangle$ and let $A$ be a $G$-module. Then the **special** cohomology group is*

$$H^2_{spec}(G, A) = \frac{\ker\left(A^3 \overset{d_2^*}{\to} A^4\right)}{\mathrm{Im}\left(A^2 \overset{d_1^*}{\to} A^3\right)},$$

*where*

$$d_2^*\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} D_y a \\ D_x a - N_y b \\ N_x b + D_y c \\ D_x c \end{bmatrix},$$

*and*

$$d_1^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} N_y a \\ D_x a - D_y b \\ N_x b \end{bmatrix}.$$

**Theorem 2.6.**

(1) *Let $G = \mathbb{I}_m \times \mathbb{I}_n$ and $A$ be a $G$-module. Then the map*

$$\begin{array}{rcl} H^2_{spec}(G,A) = \dfrac{\ker\left(A^3 \overset{d_2^*}{\to} A^4\right)}{\operatorname{Im}\left(A^2 \overset{d_1^*}{\to} A^3\right)} & \to & H^2_{bar}(G,A) \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} + d_1^* A^1 & \mapsto & \varphi + \delta B^1 \end{array}$$

*where*

$$\begin{array}{rcll} \varphi & : & G^2 \to A & \\ & & \left(x^i y^j, x^k y^l\right) \mapsto & -\sum_{d=0}^{k-1}\sum_{\alpha=0}^{j-1} {}^{x^{i+d}y^\alpha} b \\ & & & + {}^{x^{[i+k]_m}} a \qquad \textit{(if } j+l \geq n\textit{)} \\ & & & + c \qquad\qquad\quad \textit{(if } i+k \geq m\textit{)} \end{array}$$

*is an isomorphism.*

(2) *Let $G = \mathbb{I}_m \times \mathbb{I}_n$ and $A$ be a $G$-module. Then the map*

$$\begin{array}{rcl} H^2_{bar}(G,A) & \to & H^2_{spec}(G,A) \\ \varphi + \delta B^1 & \mapsto & \begin{bmatrix} \sum_{k=0}^{n-1} \varphi\left(y^k, y\right) \\ \varphi(x,y) - \varphi(y,x) \\ \sum_{k=0}^{m-1} \varphi\left(x^k, x\right) \end{bmatrix} + d_1^* A^2 \end{array}$$

*is an isomorphism, which is inverse to the previous one.*

(3) *Let $G = \mathbb{I}_m \times \mathbb{I}_n$, $A$ a $G$-module, and $\omega$ the map which sends an extension*

$$\varepsilon : 1 \to A \overset{\iota}{\to} E \overset{\pi}{\to} G \to 1$$

*of $A$ by $G$ realizing the action, to the element*

$$\begin{bmatrix} U \\ -V \\ W \end{bmatrix} + d_1^* A^2 \in H^2_{special}(G,A),$$

*where $\{x\}, \{y\} \in E$ are representatives of $x$ and $y$, and*

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \{y\}^n \\ \{y\}\{x\}\{y\}^{-1}\{x\}^{-1} \\ \{x\}^m \end{bmatrix}.$$

*Then $\omega$ induces a bijection*

$$\begin{array}{rcl} \omega : E(G,A) & \leftrightarrow & H^2_{special}(G,A) \\ [\varepsilon] & \mapsto & \begin{bmatrix} U \\ -V \\ W \end{bmatrix} + d_1^* A^2. \end{array}$$

*Proof.* For the proof, see Section 4.2. □

**Remark 2.7.** *To see how we apply the above Theorem, see Section 5.1.*

2.2. **Computations.** As mentioned previously, the goal of this thesis is to describe all extensions (up to a weak congruence)

$$1 \to A \to ? \to G \to 1$$

where:

$$\begin{aligned} |G| &= p^t, \\ |A| &= p^s, \\ 1 &\leq s, t \leq 2, \end{aligned}$$

without using extra requirements on $A$, or on the action of $G$ on $A$. The following theorem reduces all cases to an essentially smaller number of those.

**Theorem 2.8.** *Up to weak congruence, List 2.9 to List 2.12 below give all the combinations of $A, G$ and $\xi$ arising in extensions*

$$\begin{aligned} p^s &\to p^{s+t} \to p^t, \\ 1 &\leq s, t \leq 2. \end{aligned}$$

*Proof.* See Section 4.3. □

**List 2.9.** *Extensions* $1 \to p \to p^2 \to p \to 1$

(1) $A = \mathbb{I}_p$
  (a) $G = \mathbb{I}_p = \langle x \rangle$ :
    (i) *Trivial action*

**List 2.10.** *Extensions* $1 \to p \to p^3 \to p^2 \to 1$

(1) $A = \mathbb{I}_p$
  (a) $G = \mathbb{I}_{p^2} = \langle x \rangle$ :
    (i) *Trivial action.*
  (b) $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x, y \rangle$ :
    (i) *Trivial action.*

**List 2.11.** *Extensions* $1 \to p^2 \to p^3 \to p \to 1$

(1) $A = \mathbb{I}_{p^2}$.
  (a) $G = \mathbb{I}_p = \langle x \rangle$ :
    (i) *Trivial action.*
    (ii) *Non-trivial action, given by*

$$^{x}a = (1+p)\, a.$$

(2) $A = \mathbb{I}_p \times \mathbb{I}_p$.
  (a) $G = \mathbb{I}_p = \langle x \rangle$ :
    (i) *Trivial action.*
    (ii) *Non-trivial action, given by*

$$^{x}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}.$$

**List 2.12.** *Extensions* $1 \to p^2 \to p^4 \to p^2 \to 1$

(1) $A = \mathbb{I}_{p^2}$.
  (a) $G = \mathbb{I}_{p^2} = \langle x \rangle$ :
    (i) *Trivial action.*
    (ii) *Non-trivial action, given by*

$$^{x^i}a = a\,(1+ip)\,.$$

  (b) $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x, y \rangle$ :
    (i) *Trivial action.*

(ii) *Non-trivial action, given by*

$$x^i y^j a = a\,(1+ip)\,.$$

(2) $A = \mathbb{I}_p \times \mathbb{I}_p$.
  (a) $G = \mathbb{I}_{p^2} = \langle x \rangle$ :
    (i) *Trivial action.*
    (ii) *Non-trivial action, given by*

$$x \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}.$$

  (b) $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x, y \rangle$ :
    (i) *Trivial action.*
    (ii) *Non-trivial action, given by*

$$x^i y^j \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+ib \\ b \end{bmatrix}.$$

If we let $\xi$ to denote the non-trivial action, and 1 the trivial, then Theorem 2.8 states that the following table gives (up to weak congruence) all combinations $G, A^\xi$ arising in extensions

$$\begin{array}{ccc} p^s & \to & p^{s+t} \to p^t, \\ 1 & \leq & s,t \leq 2 \end{array}$$

**Table 2.13.** *Table of possible combinations of $G, A$, and actions.*

| $G$ | $A$ | *Action* |
|---|---|---|
| $\mathbb{I}_p$ | $\mathbb{I}_p$ | 1 |
| $\mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | 1 |
| $\mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | 1 |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p$ | 1 |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_{p^2}$ | 1 |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p \times \mathbb{I}_p$ | 1 |
| $\mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | $\xi$ |
| $\mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\xi$ |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_{p^2}$ | $\xi$ |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\xi$ |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p$ | 1 |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | 1 |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | 1 |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | $\xi$ |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\xi$ |

Below is the list of groups $H^2(G, A)$ for various pairs $(G, A^\eta)$ where $\eta$ is either the trivial action, or the only (up to a weak congruence) non-trivial action.

**Theorem 2.14.** *The table below give the complete list of the groups $H^2(G, A)$ arising in connection with extensions*

$$\begin{array}{ccc} p^s & \longrightarrow & p^{s+t} \longrightarrow p^t, \\ 1 & \leq & s,t \leq 2. \end{array}$$

*Proof.* For the proof, see Section 4.4. □

**Table 2.15.** *Table of cohomologies.*

| | | | $H^2(G,A)$ | |
|---|---|---|---|---|
| $G$ | $A$ | *Action* | $p \neq 2$ | $p = 2$ |
| $\mathbb{I}_p$ | $\mathbb{I}_p$ | 1 | $\mathbb{I}_p$ | |
| $\mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | 1 | $\mathbb{I}_p$ | |
| $\mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | 1 | $\mathbb{I}_p \times \mathbb{I}_p$ | |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p$ | 1 | $\mathbb{I}_p$ | |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_{p^2}$ | 1 | $\mathbb{I}_{p^2}$ | |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p \times \mathbb{I}_p$ | 1 | $\mathbb{I}_p \times \mathbb{I}_p$ | |
| $\mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | $\xi$ | $\{0\}$ | $\mathbb{I}_2$ |
| $\mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\xi$ | $\mathbb{I}_p$ | $\{0\}$ |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_{p^2}$ | $\xi$ | $\mathbb{I}_p$ | |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\xi$ | $\mathbb{I}_p$ | |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p$ | 1 | $\mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$ | |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | 1 | $\mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$ | |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | 1 | $(\mathbb{I}_p \times \mathbb{I}_p)^3$ | |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | $\xi$ | $\mathbb{I}_p$ | |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\xi$ | $(\mathbb{I}_p)^3$ | $\mathbb{I}_2$ |

*A blank entry in the column $p = 2$ means that the cases $p = 2$ and $p \neq 2$* ***do not differ****.*

**Theorem 2.16.** *Up to a weak congruence, the extensions in List 2.17 below are all the congruence classes for $p \to p^2 \to p$.*

*Proof.* For the proof, see Section 5.2. □

**List 2.17.** *Extensions $p \to p^2 \to p$*

(1) $G = \mathbb{I}_p = \langle x \rangle$, $A = (\mathbb{I}_p)^{triv} = \langle z \rangle$
  (a) $s \in H^2(G,A) \cong \mathbb{I}_p$ :
    (i) $s = 0$*:*

$$\mathbb{I}_p \overset{\iota}{\rightarrowtail} \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p$$

    (ii) $s \neq 0$*:*

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{s'p}$$
$$\pi_s : P \mapsto x$$
$$s' \equiv s^{-1} \pmod{p}$$

**Theorem 2.18.** *Up to a weak congruence, the extensions in List 2.19 below are all the congruence classes for $p^2 \to p^3 \to p$.*

*Proof.* For the proof, see Section 5.3. □

**List 2.19.** *Extensions $p^2 \to p^3 \to p$*

(1) $G = \mathbb{I}_p = \langle x \rangle$, $A = \left(\mathbb{I}_{p^2}\right)^{triv} = \langle z \rangle$,
  $s \in H^2(G,A) \cong \mathbb{I}_p$
  (a) $s = 0$*:*

$$\mathbb{I}_p \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p$$

(b) $s \neq 0$:

$$\begin{aligned}
&\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \\
&\iota_s : z \mapsto P^{ps'} \\
&\pi_s : P \mapsto x \\
&s' \equiv s^{-1} \pmod{p}
\end{aligned}$$

(2) $G = \mathbb{I}_p = \langle x \rangle$, $A = \left(\mathbb{I}_{p^2}\right)^{\xi} = \langle z \rangle$,
$s \in H^2(G, A) \cong \begin{cases} \mathbb{I}_2 & \text{if } p = 2 \\ 0 & \text{if } p \neq 0 \end{cases}$

(a) $s = 0$:

$$\begin{aligned}
&\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \\
&\iota : z \mapsto P \\
&\pi : P^i Q^j \mapsto Q^{-j}
\end{aligned}$$

(b) $p = 2, s = 1$:

$$\begin{aligned}
&\left(\mathbb{I}_{p^2} = \langle z \rangle\right) \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^2 \right\rangle \overset{\pi}{\twoheadrightarrow} \left(\mathbb{I}_p = \langle x \rangle\right) \\
&\iota : z \mapsto P \\
&\pi : P^i Q^j \mapsto x^j
\end{aligned}$$

(3) $G = \mathbb{I}_p = \langle x \rangle, A = \left(\mathbb{I}_p \times \mathbb{I}_p\right)^{triv} = \langle y \rangle \times \langle z \rangle$,
$s = \begin{bmatrix} u \\ v \end{bmatrix} \in H^2(G, A) \cong \mathbb{I}_p \times \mathbb{I}_p$

(a) $s = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p$$

(b) $s = \begin{bmatrix} u \\ v \end{bmatrix}$:

(i) $u \neq 0$:

$$\begin{aligned}
&\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \\
&\iota : \begin{array}{l} y \mapsto P^{u'p} Q^{-u'v} \\ z \mapsto Q \end{array} \\
&\pi : P^i Q^j \mapsto x^i \\
&u' \equiv u^{-1} \pmod{p}
\end{aligned}$$

(ii) $u = 0, v \neq 0$:

$$\begin{aligned}
&\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \\
&\iota : \begin{array}{l} y \mapsto Q \\ z \mapsto P^{v'p} \end{array} \\
&\pi : P^i Q^j \mapsto x^i \\
&v' \equiv v^{-1} \pmod{p}
\end{aligned}$$

(4) $G = \mathbb{I}_p = \langle x \rangle, A = \left(\mathbb{I}_p \times \mathbb{I}_p\right)^{\xi} = \langle y \rangle \times \langle z \rangle$,
$s \in H^2(G, A) \cong \begin{cases} \{0\} & \text{if } p = 2 \\ \mathbb{I}_p & \text{if } p \neq 2 \end{cases}$

(a) $p = 2$:

(i) $s = 0$:

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2$$
$$\iota : \begin{array}{l} y \mapsto P^2 \\ z \mapsto Q \end{array}$$
$$\pi : \quad P^i Q^j \mapsto x^i$$

(b) $p \neq 2$:

(i) $s = 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle \begin{array}{ll} P, Q, R : & P^p, Q^p, R^p, R^{-1}QR = QP, \\ & R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p$$
$$\iota : y^i z^j \mapsto P^i R^j,$$
$$\pi : P^i Q^j R^k \mapsto x^j$$

(ii) $s \neq 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p$$
$$\iota_s : y^i z^j \mapsto P^{is'p} Q^{js'}$$
$$\pi_s : P^i Q^j \mapsto x^i$$
$$s' \equiv s^{-1} \pmod{p}$$

**Theorem 2.20.** *Up to a weak congruence, the extensions in List 2.21 below are all the congruence classes for $p \to p^3 \to p^2$.*

*Proof.* For the proof, see Section 5.4. □

**List 2.21.** *Extensions $p \to p^3 \to p^2$*

(1) $G = \mathbb{I}_{p^2} = \langle x \rangle, A = (\mathbb{I}_p)^{triv} = \langle z \rangle$,
$s \in H^2(G, A) \cong \mathbb{I}_p$

(a) $s = 0$:

$$\mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

(b) $s \neq 0$:

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2}$$
$$\iota_s : z \mapsto P^{s'p^2}$$
$$\pi_s : P \mapsto x$$
$$s' \equiv s^{-1} \pmod{p}$$

(2) $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x, y \rangle, A = (\mathbb{I}_p)^{triv} = \langle z \rangle$,
$s = \begin{bmatrix} u \\ -v \\ w \end{bmatrix} \in H^2(G, A) \cong (\mathbb{I}_p)^3$

(a) $s = 0$: *The extension is split*

$$\mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p \times \mathbb{I}_p$$

(b) $v = 0, u \neq 0$:

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{u'p}$$
$$\pi_s : P^i Q^j \mapsto x^j y^{i - wu'j}$$
$$u' \equiv u^{-1} \pmod{p}$$

(c) $v = u = 0, w \neq 0$:

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{w'p}$$
$$\pi_s : P^i Q^j \mapsto x^i y^j$$
$$w' \equiv w^{-1} \pmod{p}$$

(d) $v \neq 0, p = 2$

(i) $u = w = 0$:

$$\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota : z \mapsto P^2$$
$$\pi : P^i Q^j \mapsto x^{i+j} y^j$$

(ii) $u = 1, w = 0$:

$$\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota : z \mapsto P^2$$
$$\pi : P^i Q^j \mapsto x^j y^i$$

(iii) $u = 0, w = 1$:

$$\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota : z \mapsto P^2$$
$$\pi : P^i Q^j \mapsto x^i y^j$$

(iv) $u = w = 1$:

$$\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^2 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota : z \mapsto P^2$$
$$\pi : P^i Q^j \mapsto x^i y^j$$

(e) $v \neq 0, p \neq 2$

(i) $u = w = 0$:

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{l} P^p, Q^p, R^p, R^{-1}QR = QP, \\ R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{v'}$$
$$\pi_s : P^i Q^j R^k \mapsto x^j y^k$$
$$v' \equiv v^{-1} \pmod{p}$$

(ii) $u \neq 0$:

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{u'p}$$
$$\pi_s : P^i Q^j \mapsto x^{-jv'u} y^{i+jv'w}$$
$$u' \equiv u^{-1} \pmod{p}, v' \equiv v^{-1} \pmod{p}$$

(iii) $u = 0, w \neq 0$:

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{w'p}$$
$$\pi_s : P^i Q^j \mapsto x^i y^{jv'w}$$
$$w' \equiv w^{-1} \pmod{p}, v' \equiv v^{-1} \pmod{p}$$

**Theorem 2.22.** *Up to a weak congruence, the extensions in List 2.23 to List 2.26 below are all the congruence classes for $p^2 \to p^4 \to p^2$.*

*Proof.* See Section 5.5. □

**List 2.23.** *Extensions of $A = \mathbb{I}_{p^2} = \langle z \rangle$ by $G = \mathbb{I}_{p^2} = \langle x \rangle$*

(1) *Trivial action,*
$s \in H^2(G, A) \cong \mathbb{I}_{p^2}$*:*
(a) $s = 0$*: The extension is split*

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

(b) $s \in \left(\mathbb{I}_{p^2}\right)^*$*:*

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^4} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota_s : z \mapsto P^{p^2} \\ &\pi_s : P^i \mapsto x^{is'} \\ &s' \equiv s^{-1} \left(\mathrm{mod}\, p^2\right) \end{aligned}$$

(c) $s \in {}_{[p]}\mathbb{I}_{p^2} = p\mathbb{I}_{p^2}$*, i.e.* $s = rp, 1 \leq r < p$*:*

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota_s : z \mapsto P^{r'p} Q \\ &\pi_s : P^i Q^j \mapsto x^{i - jr'p} \\ &r' \equiv r^{-1} \left(\mathrm{mod}\, p\right) \end{aligned}$$

(2) *Non-trivial action,*
$s \in H^2(G, A) \cong \mathbb{I}_p$
(a) $s = 0$*:*

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^{p^2}, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota : z \mapsto P \\ &P^i Q^j \mapsto x^{-j} \end{aligned}$$

(b) $s \neq 0$*:*

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota_s : z \mapsto P^{s'p} Q^{s'} \\ &\pi_s : P^i Q^j \mapsto x^i \end{aligned}$$

**List 2.24.** *Extensions of $A = \mathbb{I}_p \times \mathbb{I}_p = \langle z, Z \rangle$ by $G = \mathbb{I}_{p^2} = \langle x \rangle$*

(1) *Trivial action*
$s = \begin{bmatrix} u \\ v \end{bmatrix} \in H^2(G, A) \cong \mathbb{I}_p \times \mathbb{I}_p$
(a) $s = 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}.$$

(b) $u \neq 0$*:*

$$\begin{aligned} &\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P, Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota_s : \begin{array}{l} z \mapsto P^{u'p^2} Q^{-u'v} \\ Z \mapsto Q \end{array} \\ &\pi_s : P^i Q^j \mapsto x^i \end{aligned}$$

(c) $u = 0, v \neq 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P, Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2}$$
$$\iota_s : \begin{array}{l} z \mapsto Q \\ Z \mapsto P^{v'p^2} \end{array}$$
$$\pi_s : P^i Q^j \mapsto x^i$$

(2) *Non-trivial action*
$s \in H^2(G, A) \cong \mathbb{I}_p$
(a) $s = 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}PR = PQ, \\ Q^{-1}PQ = P, R^{-1}QR = Q \end{array} \right\rangle \twoheadrightarrow \mathbb{I}_{p^2}$$
$$\iota : \begin{array}{l} z \mapsto Q \\ Z \mapsto R \end{array}$$
$$\pi : P^i Q^j R^k \mapsto x^i$$

(b) $s \neq 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \twoheadrightarrow \mathbb{I}_{p^2}$$
$$\iota_s : \begin{array}{l} z \mapsto P^{s'p^2} \\ Z \mapsto P^{-s'p^2} Q^{s'} \end{array}$$
$$\pi_s : P^i Q^j \mapsto x^i$$

**List 2.25.** *Extensions of* $A = \mathbb{I}_{p^2} = \langle z \rangle$ *by* $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x, y \rangle$

(1) *Trivial action*
$s = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in H^2(G, A) \cong \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$
(a) $s = 0$:

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p \times \mathbb{I}_p.$$

(b) $v = 0$:
(i) $u \neq 0$:

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{u'p}$$
$$\pi_s : P^i Q^j \mapsto x^j y^{i - ju'w}$$

(ii) $u = 0, w \neq 0$:

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{w'p}$$
$$\pi_s : P^i Q^j \mapsto x^i y^j$$

(c) $v \neq 0$:
(i) $u = w = 0$:

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{v'}$$
$$\pi_s : P^i Q^j R^k \mapsto x^{-k} y^j$$

(ii) $u \neq 0$:

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : z \mapsto P^{u'p}$$

$$\pi_s : P^i Q^j \mapsto x^{-juv'} y^{i+jv'wp}$$

(iii) $u = 0, w \neq 0$:

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : z \mapsto P^{w'p}$$

$$\pi_s : P^i Q^j \mapsto x^i y^{jv'w}$$

(2) *Non-trivial action*

$$s \in H^2(G, A) \cong \begin{cases} \frac{(\mathbb{I}_p)^2}{\langle (1,1) \rangle} & p \geq 3 \\ {}_{[p]}\mathbb{I}_{p^2} & p = 2 \end{cases} \cong \mathbb{I}_p.$$

(a) $s = 0$:

$$\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}PR = P^{1+p}, \\ P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota : z \mapsto P$$

$$\pi : P^i Q^j R^k \mapsto x^{-k} y^j.$$

(b) $s \neq 0, p = 2$:

$$\mathbb{I}_4 \overset{\iota_s}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{c} P^4, Q^4, R^2, Q^{-1}PQ = P^{-1}, Q^2 = P^2, \\ R^{-1}QR = Q, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$$\iota_s : z \mapsto P$$

$$\pi_s : P^i Q^j R^k \mapsto x^j y^k$$

(c) $s \neq 0, p \neq 2$:

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : z \mapsto PQ$$

$$\pi_s : P^i Q^j R^k \mapsto x^{-k} y^{(i-j)s'}$$

The list below is unfinished.

**List 2.26.** *Extensions of* $A = \mathbb{I}_p \times \mathbb{I}_p = \langle z, Z \rangle$ *by* $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x, y \rangle$

(1) *Trivial Action:*

$$s = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} (u_1, u_2) \\ (v_1, v_2) \\ (w_1, w_2) \end{bmatrix} \in H^2(G, A)$$

(a) $s = 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail (\mathbb{I}_p \times \mathbb{I}_p) \times (\mathbb{I}_p \times \mathbb{I}_p) \twoheadrightarrow \mathbb{I}_p \times \mathbb{I}_p$$

(b) $v = 0, u_1 \neq 0$:

(i) $u_1 w_2 \not\equiv u_2 w_1 \pmod{p}$:

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left( \mathbb{I}_{p^2} \times \mathbb{I}_{p^2} = \langle P \rangle \times \langle Q \rangle \right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto P^{u_1' p} Q^{-u_2(u_1 w_2 - u_2 w_1)' p} \\ Z \mapsto Q^{u_1(u_1 w_2 - u_2 w_1)' p} \end{array}$$

$$\pi_s : P^i Q^j \mapsto x^j y^{i - j u_1' w_1}$$

(ii) $u_2w_1 \equiv u_1w_2 \pmod p$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P\rangle \times \langle Q\rangle \times \langle R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : \begin{array}{l} z \mapsto P^{u_1'p}R^{-u_1'u_2} \\ Z \mapsto R \end{array}$$
$$\pi_s : P^iQ^j \mapsto x^j y^{i-ju_1'w_1}$$

(c) $v = 0, u_1 = 0, u_2 \neq 0$*:*

(i) $w_1 \neq 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_{p^2} = \langle P, Q\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : \begin{array}{l} z \mapsto Q^{w_1'p} \\ Z \mapsto P^{u_2'p} \end{array}$$
$$\pi_s : P^iQ^j \mapsto x^j y^{i-ju_2'w_2}$$

(ii) $w_1 = 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P\rangle \times \langle Q\rangle \times \langle R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : \begin{array}{l} z \mapsto R \\ Z \mapsto P^{u_2'p} \end{array}$$
$$\pi_s : P^iQ^jR^k \mapsto x^j y^{i-ju_2'w_2}$$

(d) $v = 0, u = 0, w_1 \neq 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P, Q, R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : \begin{array}{l} z \mapsto P^{w_1'p}R^{-w_1'w_2} \\ Z \mapsto Q \end{array}$$
$$\pi_s : P^iQ^jR^k \mapsto x^i y^j$$

(e) $v = 0, u = 0, w_1 = 0, w_2 \neq 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P, Q, R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : \begin{array}{l} z \mapsto R \\ Z \mapsto P^{w_2'p} \end{array}$$
$$\pi_s : P^iQ^jR^k \mapsto x^i y^j$$

(f) $v \neq 0$*: Unfinished.*

(2) *Non-trivial action:*

$$s \in H^2_{spec}\left(\mathbb{I}_p \times \mathbb{I}_p, (\mathbb{I}_p \times \mathbb{I}_p)^\xi\right) \cong \begin{cases} (\mathbb{I}_p)^3, & p \geq 3 \\ \mathbb{I}_2, & p = 2 \end{cases}$$

(a) $p = 2, s = 0$ :

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{l} P^4, Q^2, R^2, R^{-1}PR = P^3, \\ P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$
$$\iota : \begin{array}{l} z \mapsto P^2 \\ Z \mapsto PR \end{array}$$
$$\pi : P^iQ^jR^k \mapsto x^{i+k}y^j$$

(b) $p = 2, s = 1$*:*

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{c} P^4, Q^2, R^2, R^{-1}QR = QP^2, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$$\iota : \begin{array}{c} z \mapsto P^2 \\ Z \mapsto Q \end{array}$$

$$\pi : P^i Q^j R^k \mapsto x^k y^i$$

(c) $p \neq 2$*: Unfinished.*

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## 3. Proofs from Preliminaries

### 3.1. **Proof of Proposition 1.47.**

*Proof.* We do this by finding a general multiplication formula for $E$. We know that every element of $E$ is of the form $a\{x\}^i\{y\}^j$, $a \in A$ with multiplication

$$\begin{aligned}\left(a\{x\}^i\{y\}^j\right)\left(b\{x\}^k\{y\}^l\right) &= a\left(\{x\}^i\{y\}^j b\{y\}^{-j}\{x\}^{-i}\right)\{x\}^i\{y\}^j\{x\}^k\{y\}^l \\ &= a\left({}^{x^iy^j}b\right)\{x\}^i\{y\}^j\{x\}^k\{y\}^l.\end{aligned}$$

(1) Consider $\{y\}^j\{x\}^k\{y\}^l$:

$$\begin{aligned}\{y\}^j\{x\}^k\{y\}^l &= \left(\{y\}^j\{x\}^k\{y\}^{-j}\right)\{y\}^j\{y\}^l = {}^{y^j}\left(\{x\}^k\right)\{y\}^{j+l} \\ &= \left({}^{y^j}\{x\}\right)^k\{y\}^{j+l}\end{aligned}$$

(2)

$$\begin{aligned}{}^{y^j}\{x\} &= {}^{y^{j-1}}\left({}^{\{y\}}\{x\}\right) = {}^{y^{j-1}}(V\{x\}) = {}^{y^{j-1}}V\cdot{}^{y^{j-1}}\{x\} \\ &= {}^{y^{j-1}}V\,{}^{y^{j-2}}V\ldots{}^{y}V\,V\{x\} \\ &\overset{v\in A}{=} \left(V\cdot{}^{y}V\cdot{}^{y^2}V\cdot\ldots\cdot{}^{y^{j-1}}V\right)\{x\} \\ &= \left(\prod_{d=0}^{j-1}{}^{y^d}V\right)\{x\}.\end{aligned}$$

(3) Set $c = \prod_{d=1}^{j-1}{}^{y^d}V$ for convenience, so that

$$\begin{aligned}\left({}^{y^j}\{x\}\right)^k &= (c\{x\})^k = c\underbrace{\{x\}\,c\{x\}\ldots c\{x\}}_{k\text{-times}} \\ &= c\,{}^{x}c\{x\}^2\,c\{x\}\ldots c\{x\} \\ &= c\,{}^{x}c\,{}^{x^2}c\ldots{}^{x^{k-1}}c\{x\}^k \\ &= \left(\prod_{r=0}^{k-1}{}^{x^r}c\right)\{x\}^k \\ &= \left(\prod_{r=0}^{k-1}{}^{x^r}\prod_{d=0}^{j-1}{}^{y^d}V\right)\{x\}^k \\ &= \left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^ry^d}V\right)\{x\}^k.\end{aligned}$$

So $\{y\}^j\{x\}^k\{y\}^l = \left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^ry^d}V\right)\{x\}^k\{y\}^{j+l}$.

(4) Next

$$\begin{aligned}\{x\}^i\left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^ry^d}V\right) &= {}^{x^i}\left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^ry^d}V\right)\{x\}^i \\ &= \left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^{i+r}y^d}V\right)\{x\}^i.\end{aligned}$$

(5) Finally

$$\begin{aligned}
&\left(a\{x\}^i\{y\}^j\right)\left(b\{x\}^k\{y\}^l\right)\\
=\;& a\left({}^{x^iy^j}b\right)\left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^{i+r}y^d}V\right)\{x\}^{i+k}\{y\}^{j+l}\\
=\;& a\left({}^{x^iy^j}b\right)\left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^{i+r}y^d}V\right)W^{\left\lfloor\frac{i+k}{m}\right\rfloor}\{x\}^{(i+k)\bmod m}U^{\left\lfloor\frac{j+l}{n}\right\rfloor}\{y\}^{(j+l)\bmod n}\\
=\;& a\left({}^{x^iy^j}b\right)\left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^{i+r}y^d}V\right)\cdot\begin{cases}\{x\}^{i+k}\{y\}^{j+l} & , \quad i+k<m, j+l<n\\ W\{x\}^{i+k-m}\{y\}^{j+l} & , \quad i+k\geq m, j+l<n\\ \{x\}^{i+k}U\{y\}^{j+l-n} & , \quad i+k<m, j+l\geq n\\ W\{x\}^{i+k-m}U\{y\}^{j+l-n} & , \quad i+k<m, j+l<n\end{cases}\\
=\;& a\left({}^{x^iy^j}b\right)\left(\prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^{i+r}y^d}v\right)\cdot\begin{cases}\{x\}^{i+k}\{y\}^{j+l} & , \quad i+k<m, j+l<n\\ W\{x\}^{i+k-m}\{y\}^{j+l} & , \quad i+k\geq m, j+l<n\\ {}^{x^{i+k}}U\{x\}^{i+k}\{y\}^{j+l-n} & , \quad i+k<m, j+l\geq n\\ W\,{}^{x^{i+k-m}}U\{x\}^{i+k-m}\{y\}^{j+l-n} & , \quad i+k\geq m, j+l\geq n\end{cases}.
\end{aligned}$$

hence the cocycle of $\sigma$ is given by

$$\begin{aligned}
\varphi_\sigma \quad &: \quad G\times G\to A\\
\left(x^iy^j,x^ky^l\right) \quad &\longmapsto \quad \prod_{r=0}^{k-1}\prod_{d=0}^{j-1}{}^{x^{i+r}y^d}V\cdot\begin{cases}1 & , \quad i+k<m, j+l<n\\ W & , \quad i+k\geq m, j+l<n\\ {}^{x^{i+k}}U & , \quad i+k<m, j+l\geq n\\ W\,{}^{x^{i+k-m}}U & , \quad i+k\geq m, j+l\geq n\end{cases}.
\end{aligned}$$

□

3.2. **Proof of Theorem 1.45.**

**Lemma 3.1.** *For $\forall g\in G$,*

$$^{g^{-1}}\varphi\left(g,g^{-1}\right)=\varphi\left(g^{-1},g\right).$$

*Proof.* By the cocycle identity $\left(\text{substituting } g=g^{-1},h=g,k=g^{-1}\right)$ :

$$\begin{aligned}
{}^{g^{-1}}\varphi\left(g,g^{-1}\right)\cdot\varphi\left(g^{-1},gg^{-1}\right) &= \varphi\left(g^{-1},g\right)\cdot\varphi\left(g^{-1}g,g^{-1}\right)\\
&\Updownarrow\\
{}^{g^{-1}}\varphi\left(g,g^{-1}\right)\cdot\underbrace{\varphi\left(g^{-1},1\right)}_{1} &= \varphi\left(g^{-1},g\right)\cdot\underbrace{\varphi\left(1,g^{-1}\right)}_{1}\\
&\Updownarrow\\
{}^{g^{-1}}\varphi\left(g,g^{-1}\right) &= \varphi\left(g^{-1},g\right).
\end{aligned}$$

□

*Proof.* Going through the different points:

(1) We show that $\varphi: G\times G\to A$ is a normalized cocycle.
   (a) The equalities

$$\begin{aligned}
x_g &= x_g\cdot 1=x_gx_1=\varphi(g,1)\,x_{g\cdot 1}=\varphi(g,1)\,x_g,\\
x_h &= 1\cdot x_h=x_1x_h=\varphi(1,h)\,x_{1\cdot h}=\varphi(1,h)\,x_h
\end{aligned}$$

along with right cancellation shows that

$$\varphi(g,1)=1=\varphi(1,h),$$

i.e. $\varphi$ is normalized.

(b) By associativity of $E$ :

$$[(a,g)(b,h)](c,k) = (a,g)[(b,h)(c,k)].$$

Now

$$\begin{aligned} \text{LHS} &= (a\cdot {}^{g}b\cdot\varphi(g,h), gh)(c,k) \\ &= \left(a\cdot {}^{g}b\cdot\varphi(g,h)\cdot {}^{gh}c\cdot\varphi(gh,k), (gh)k\right) \\ &= \left(a\cdot {}^{g}b\cdot {}^{gh}c\cdot\varphi(g,h)\cdot\varphi(gh,k), ghk\right). \end{aligned}$$

While

$$\begin{aligned} \text{RHS} &= (a,g)\left(b\cdot {}^{h}c\cdot\varphi(h,k), hk\right) \\ &= \left(a\cdot {}^{g}\left[b\cdot {}^{h}c\cdot\varphi(h,k)\right]\cdot\varphi(g,hk), g(hk)\right) \\ &= \left(a\cdot {}^{g}b\cdot {}^{gh}c\cdot {}^{g}\varphi(h,k)\cdot\varphi(g,hk), ghk\right). \end{aligned}$$

Thus

$${}^{g}\varphi(h,k)\cdot\varphi(g,hk) = \varphi(g,h)\cdot\varphi(gh,k)$$

or since $A$ is Abelian:

$${}^{g}\varphi(h,k)\cdot\varphi(gh,k)^{-1}\cdot\varphi(g,hk)\cdot\varphi(g,h)^{-1} = 1$$

which is equivalent to $\partial\varphi(g,h,k) = 0$ in additive notation.

(2) Identity and inverse elements:

(a) By normalization we have

$$\begin{aligned} (1,1)(a,g) &= \left(1\cdot {}^{1}a\cdot\varphi(1,g), 1\cdot g\right) \\ &= (a\cdot 1, g) = (a,g) \end{aligned}$$

and

$$\begin{aligned} (a,g)(1,1) &= (a\cdot {}^{g}1\cdot\varphi(g,1), g\cdot 1) \\ &= (a,g). \end{aligned}$$

(b) Since $A$ is Abelian:

$$\begin{aligned} & (a,g)\left({}^{g^{-1}}\left(a^{-1}\right)\cdot\varphi\left(g^{-1},g\right)^{-1}, g^{-1}\right) \\ = & \left(a\cdot {}^{g}\left[{}^{g^{-1}}\left(a^{-1}\right)\cdot\varphi\left(g^{-1},g\right)^{-1}\right]\cdot\varphi\left(g,g^{-1}\right), g\cdot g^{-1}\right) \\ = & \left(a\cdot a^{-1}\cdot\left({}^{g}\varphi\left(g^{-1},g\right)\right)^{-1}\cdot\varphi\left(g,g^{-1}\right), 1\right) \\ = & \left(1\cdot\varphi\left(g,g^{-1}\right)^{-1}\cdot\varphi\left(g,g^{-1}\right), 1\right) \text{ by Lemma 3.1 above} \\ = & (1,1), \end{aligned}$$

and conversely

$$\begin{aligned} & \left({}^{g^{-1}}\left(a^{-1}\right)\cdot\varphi\left(g^{-1},g\right)^{-1}, g^{-1}\right)(a,g) \\ = & \left({}^{g^{-1}}\left(a^{-1}\right)\cdot\varphi\left(g^{-1},g\right)^{-1}\cdot {}^{g^{-1}}a\cdot\varphi\left(g^{-1},g\right), g^{-1}g\right) \\ = & \left({}^{g^{-1}}\left(a^{-1}a\right)\cdot\varphi\left(g^{-1},g\right)^{-1}\cdot\varphi\left(g^{-1},g\right), 1\right) \\ = & (1,1). \end{aligned}$$

(3) We have

$$y_g y_h = \psi(g,h)\, y_{gh} = \psi(g,h)\,\xi(gh)\,\beta(x_{gh})$$

and

$$y_g y_h = [\xi(g)\beta(x_g)][\xi(h)\beta(x_h)] = \xi(g)\cdot {}^{g}\xi(h)\cdot\varphi(g,h)\,\beta(x_{gh})$$

so

$$\psi(g,h)\,\xi(gh)\,\beta(x_{gh}) = \xi(g)\cdot\ {}^{g}\xi(h)\cdot\varphi(g,h)\,\beta(x_{gh})$$

or

$$\psi(g,h) = \xi(g)\cdot\ {}^{g}\xi(h)\cdot\varphi(g,h)\cdot\xi(gh)^{-1}.$$

Since $A$ is Abelian

$$\psi(g,h)\cdot\varphi(g,h)^{-1} = {}^{g}\xi(h)\cdot\xi(gh)^{-1}\cdot\xi(g)$$

or written additively

$$\begin{aligned}\psi(g,h)-\varphi(g,h) &= {}^{g}\xi(h)-\xi(gh)+\xi(g)\\ &= (\partial\xi)(g,h).\end{aligned}$$

(4) The only thing that remains to prove, is that the semidirect product (i.e. the **split** extension) corresponds to the zero element of $H^2(G,A)$. If the extension splits, then the section $\sigma$ is a homomorphism, giving the zero cocycle. Conversely, if the cocycle is zero, then $\sigma$ is a homomorphism, and the extension splits. See also [ML95, Theorem IV.4.1].

□

### 3.3. **Proof of Proposition 1.49.**

*Proof.* We need to show that $0 \leftarrow \mathbb{Z}^{\mathrm{triv}} \leftarrow P_\bullet$ is a chain complex of $\mathbb{Z}G$-modules, and that it is contractible (over $\mathbb{Z}$), with the given contraction. Then by Corollary A.4 it will be exact and hence a $\mathbb{Z}G$-module resolution of $\mathbb{Z}^{\mathrm{triv}}$.

(1) For $0 \leftarrow \mathbb{Z}^{\mathrm{triv}} \leftarrow P_\bullet$ to be a chain complex of $\mathbb{Z}G$-modules, we need for $dd = 0$. Again it is enough to check on generators:

$$\begin{aligned}d_{-1}d_0\langle 1\rangle &= d_{-1}(\langle x\rangle - \langle 1\rangle) = 1-1=0,\\ N(\langle x\rangle-\langle 1\rangle) &= \sum_{i=0}^{m-1}\langle x^{i+1}\rangle - \sum_{i=0}^{m-1}\langle x^i\rangle = 0 = (\langle x\rangle-\langle 1\rangle)N,\end{aligned}$$

which proves that

$$d_{2k}d_{2k+1}=0,\ d_{2k+1}d_{2k+2}=0\ \forall k\geq 0,$$

as was to be shown.

(2) Recall that a contraction (Definition A.2)

$$S_n : P_n \to P_{n+1}$$

where $P_{-1} = \mathbb{Z}^{\mathrm{triv}}$ is a family of $\mathbb{Z}$-maps (NB: not necessarily $\mathbb{Z}G$-maps!) which satisfies

$$d_nS_n + S_{n-1}d_{n-1} = 1_{P_n},$$

which is equivalent to

$$d_nS_n = 1_{P_n} - S_{n-1}d_{n-1}. \tag{3}$$

We use equation (3) to calculate the contraction recursively. As $\mathbb{Z}$-modules, $P_n$ is generated by elements $\langle x^i\rangle$, $0\leq i<m$, and so it is enough to define $S_n$ on the generators $\langle x^i\rangle$.

$$\begin{array}{ccccccccccccc} 0 & \leftarrow & \mathbb{Z}^{\mathrm{triv}} & \overset{d_{-1}}{\leftarrow} & P_0 & \overset{d_0}{\leftarrow} & P_1 & \overset{d_1}{\leftarrow} & P_2 & \overset{d_2}{\leftarrow} & P_3 & \leftarrow & \dots\\ & \underset{0}{\rightarrow} & & \underset{S_{-1}}{\rightarrow} & & \underset{S_0}{\rightarrow} & & \underset{S_1}{\rightarrow} & & \underset{S_2}{\rightarrow} & & \rightarrow & \end{array}$$

$S_{-1}$ : We need

$$1_{\mathbb{Z}^{\mathrm{triv}}} = d_{-1}S_{-1},$$

or equivalently $\varepsilon S_{-1}(1) = 1$. Using the fact that $S_{-1}$ must be an $\mathbb{Z}$-map, it is clear that

$$S_{-1}(1) = \langle 1 \rangle .$$

$S_0$ :

$$\begin{aligned} & (1_{\mathbb{Z}G} - S_{-1}\varepsilon)\langle x^i \rangle = \langle x^i \rangle - \langle 1 \rangle \\ = \; & (\langle x \rangle - \langle 1 \rangle)\left(\langle x^{i-1} \rangle + \langle x^{i-2} \rangle + \cdots + \langle 1 \rangle\right) \\ = \; & d_0 \sum_{j=0}^{i-1} \langle x^j \rangle . \end{aligned}$$

Thus

$$S_0\langle x^i \rangle = \begin{cases} \sum_{j=0}^{i-1} \langle x^j \rangle & , i > 0 \\ 0 & , i = 0 \end{cases} .$$

$S_1$:

$$\begin{aligned} & (1_{\mathbb{Z}G} - S_0 d_0)\langle x^i \rangle = \langle x^i \rangle - S_0\langle x^{i+1} \rangle + S_0\langle x^i \rangle \\ = \; & \begin{cases} \langle 1 \rangle - \langle 1 \rangle + 0 & , i = 0 \\ \langle x^i \rangle - \sum_{j=0}^{i} \langle x^j \rangle + \sum_{j=0}^{i-1} \langle x^j \rangle & , 0 < i < m-1 \\ \langle x^{m-1} \rangle - 0 + \sum_{j=0}^{m-2} \langle x^j \rangle & , i = m-1 \end{cases} \\ = \; & \begin{cases} 0 & , i < m-1 \\ N & , i = m-1 \end{cases} = d_1 \begin{cases} 0 & , i < m-1 \\ \langle 1 \rangle & , i = m-1 \end{cases} . \end{aligned}$$

Hence

$$S_1\langle x^i \rangle = \begin{cases} 0 & , i < m-1 \\ \langle 1 \rangle & , i = m-1 \end{cases} .$$

This completes the base step. Finally we need to show that

$$\begin{aligned} S_{2n}\left(\langle x^i \rangle\right) &= \begin{cases} \sum_{j=0}^{i-1} \langle x^j \rangle & , i > 0 \\ 0 & , i = 0 \end{cases} , \\ S_{2n+1}\left(\langle x^i \rangle\right) &= \begin{cases} 0 & , i < m-1 \\ \langle 1 \rangle & , i = m-1. \end{cases} \end{aligned}$$

using induction. Assume the inductive hypothesis, i.e. that it holds for $n \geq 0$. Then

$$\begin{aligned} & (1_{\mathbb{Z}G} - S_{2n+1}d_{2n+1})\langle x^i \rangle = \langle x^i \rangle - S_{2n+1}\left(N\langle x^i \rangle\right) = \langle x^i \rangle - S_1\left(N\langle x^i \rangle\right) \\ = \; & \langle x^i \rangle - \langle 1 \rangle = d_{2(n+1)}\left(\begin{cases} \sum_{j=0}^{i-1} \langle x^j \rangle & , i > 0 \\ 0 & , i = 0 \end{cases}\right) \end{aligned}$$

showing that

$$S_{2(n+1)}\langle t^i \rangle = \begin{cases} \sum_{j=0}^{i-1} \langle x^j \rangle & , i > 0 \\ 0 & , i = 0 \end{cases} .$$

Next:

$$\begin{aligned} & \left(1_{\mathbb{Z}G} - S_{2(n+1)}d_{2(n+1)}\right)\langle x^i \rangle \\ = \; & \langle x^i \rangle - S_{2(n+1)}\langle x^{i+1} \rangle + S_{2(n+1)}\langle x^i \rangle \\ = \; & \langle x^i \rangle - S_0\langle x^{i+1} \rangle + S_0\langle x^i \rangle \\ = \; & d_{2(n+1)+1}\left(\begin{cases} 0 & , i < m-1 \\ \langle 1 \rangle & , i = m-1 \end{cases}\right) \end{aligned}$$

which shows that

$$S_{2(n+1)+1}\langle x^i \rangle = \begin{cases} 0 & , i < m-1 \\ \langle 1 \rangle & , i = m-1 \end{cases} .$$

□

3.4. **Proof of Proposition 1.52.**

*Proof.* Let $Q_\bullet$ be the positive complex whose entries are given by

$$\begin{aligned} Q_0 &= \mathbb{Z}^{\mathrm{triv}}, \\ Q_i &= P_{i-1}, \end{aligned}$$

and whose differentials $\partial_i : Q_{i+1} \to Q_i$ are given by

$$\begin{aligned} \partial_0 &= \varepsilon, \\ \partial_i &= \partial_{i-1}. \end{aligned}$$

Then as a complex in ${}_{\mathbb{Z}}\mathrm{Mod}$, it is exact and projective, and hence by Corollary A.10 contractible. We calculate the contraction recursively by

$$d_n S_n = 1_{P_n} - S_{n-1} d_{n-1}.$$

$S_{-1}$ : We need $d_{-1}S_{-1} = 1_{\mathbb{Z}}$, clearly $S_{-1}(1) = \langle 1 \rangle$ does the trick, so

$$S_{-1}(1) = \langle 1 \rangle .$$

$S_0$ : We have

$$\begin{aligned} & (1_{\mathbb{Z}G} - S_{-1}d_{-1})\left(\langle x^i y^j \rangle\right) \\ = \; & \langle x^i y^j \rangle - \langle 1 \rangle = \left(\langle x^i y^j \rangle - \langle x^i \rangle\right) + \left(\langle x^i \rangle - \langle 1 \rangle\right) \\ = \; & D_y\left(\langle x^i \rangle + \langle x^i y \rangle + \cdots + \langle x^i y^{j-1} \rangle\right) + D_x\left(\langle 1 \rangle + \langle x \rangle + \ldots \langle x^{i-1} \rangle\right) \\ = \; & \begin{bmatrix} D_y & D_x \end{bmatrix} \begin{bmatrix} \langle x^i \rangle + \langle x^i y \rangle + \cdots + \langle x^i y^{j-1} \rangle \\ \langle 1 \rangle + \langle x \rangle + \ldots \langle x^{i-1} \rangle \end{bmatrix} \\ = \; & d_0 \left( \begin{bmatrix} \langle x^i \rangle + \langle x^i y \rangle + \cdots + \langle x^i y^{j-1} \rangle \\ \langle 1 \rangle + \langle x \rangle + \ldots \langle x^{i-1} \rangle \end{bmatrix} \right) \end{aligned}$$

hence

$$\begin{aligned} S_0\left(\langle x^i y^j \rangle\right) &= \begin{bmatrix} \langle x^i \rangle + \langle x^i y \rangle + \cdots + \langle x^i y^{j-1} \rangle \\ \langle 1 \rangle + \langle x \rangle + \ldots \langle x^{i-1} \rangle \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \sum_{k=0}^{i-1} \langle x^k \rangle \end{bmatrix}. \end{aligned}$$

$S_1$ : We proceed componentwise:

$$\begin{aligned}
&\left(1_{\mathbb{Z}G\bigoplus\mathbb{Z}G}-S_0d_0\right)\left(\begin{bmatrix}\langle x^iy^j\rangle\\0\end{bmatrix}\right)=\begin{bmatrix}\langle x^iy^j\rangle\\0\end{bmatrix}-S_0\left(\begin{bmatrix}D_y & D_x\end{bmatrix}\begin{bmatrix}\langle x^iy^j\rangle\\0\end{bmatrix}\right)\\
=\;&\begin{bmatrix}\langle x^iy^j\rangle\\0\end{bmatrix}-S_0\left(D_y\langle x^iy^j\rangle\right)=\begin{bmatrix}\langle x^iy^j\rangle\\0\end{bmatrix}-S_0\left(\left\langle x^iy^{[j+1]_n}\right\rangle\right)+S_0\left(\langle x^iy^j\rangle\right)\\
=\;&\begin{cases}\begin{bmatrix}\langle x^iy^j\rangle-0+\sum_{k=0}^{j-1}\langle x^iy^k\rangle\\0-\sum_{k=0}^{i-1}\langle x^k\rangle+\sum_{k=0}^{i-1}\langle x^k\rangle\end{bmatrix} & ,j=n-1\\ \begin{bmatrix}\langle x^iy^j\rangle-\sum_{k=0}^{j}\langle x^iy^k\rangle+\sum_{k=0}^{j-1}\langle x^iy^k\rangle\\0-\sum_{k=0}^{i-1}\langle x^k\rangle+\sum_{k=0}^{i-1}\langle x^k\rangle\end{bmatrix} & ,j<n-1\end{cases}\\
=\;&\begin{cases}\begin{bmatrix}\sum_{k=0}^{j}\langle x^iy^k\rangle\\0\end{bmatrix} & ,j=n-1\\ \begin{bmatrix}0\\0\end{bmatrix} & ,j<n-1\end{cases}=\begin{cases}\begin{bmatrix}N_y\langle x^i\rangle\\0\end{bmatrix} & ,j=n-1\\ \begin{bmatrix}N_y0\\0\end{bmatrix} & ,j<n-1\end{cases}\\
=\;&\begin{cases}\begin{bmatrix}N_y & D_x & 0\\0 & -D_y & N_x\end{bmatrix}\begin{bmatrix}\langle x^i\rangle\\0\\0\end{bmatrix} & ,j=n-1\\ \begin{bmatrix}N_y & D_x & 0\\0 & -D_y & N_x\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix} & ,j<n-1\end{cases}=\begin{cases}d_1\begin{bmatrix}\langle x^i\rangle\\0\\0\end{bmatrix} & ,j=n-1\\ d_1\begin{bmatrix}0\\0\\0\end{bmatrix} & ,j<n-1\end{cases}.
\end{aligned}$$

Hence

$$S_1\left(\begin{bmatrix}\langle x^iy^j\rangle\\0\end{bmatrix}\right)=\begin{cases}\begin{bmatrix}\langle x^i\rangle\\0\\0\end{bmatrix} & ,j=n-1\\ \begin{bmatrix}0\\0\\0\end{bmatrix} & ,j<n-1\end{cases}$$

For the other component:

$$
\begin{aligned}
& \left(1_{\mathbb{Z}G \bigoplus \mathbb{Z}G} - S_0 d_0\right) \left( \begin{bmatrix} 0 \\ \langle x^i y^j \rangle \end{bmatrix} \right) \\
= & \begin{bmatrix} 0 \\ \langle x^i y^j \rangle \end{bmatrix} - S_0 \left( \begin{bmatrix} D_y & D_x \end{bmatrix} \begin{bmatrix} 0 \\ \langle x^i y^j \rangle \end{bmatrix} \right) \\
= & \begin{bmatrix} 0 \\ \langle x^i y^j \rangle \end{bmatrix} - S_0 \left( \left\langle x^{[i+1]_m} y^j \right\rangle \right) + S_0 \left( \langle x^i y^j \rangle \right) \\
= & \begin{bmatrix} 0 \\ \langle x^i y^j \rangle \end{bmatrix} + \begin{cases} - \begin{bmatrix} \sum_{k=0}^{j-1} \langle y^k \rangle \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \sum_{k=0}^{i-1} \langle x^k \rangle \end{bmatrix} & , i = m-1 \\ - \begin{bmatrix} \sum_{k=0}^{j-1} \langle x^{i+1} y^k \rangle \\ \sum_{k=0}^{i} \langle x^k \rangle \end{bmatrix} + \begin{bmatrix} \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \sum_{k=0}^{i-1} \langle x^k \rangle \end{bmatrix} & , i < m-1 \end{cases} \\
= & \begin{cases} \begin{bmatrix} \sum_{k=0}^{j-1} \langle x^i y^k \rangle - \sum_{k=0}^{j-1} \langle y^k \rangle \\ \langle x^i y^j \rangle + \sum_{k=0}^{i-1} \langle x^k \rangle \end{bmatrix} & , i = m-1 \\ \begin{bmatrix} - \sum_{k=0}^{j-1} \langle x^{i+1} y^k \rangle + \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \langle x^i y^j \rangle - \sum_{k=0}^{i} \langle x^k \rangle + \sum_{k=0}^{i-1} \langle x^k \rangle \end{bmatrix} & , i < m-1 \end{cases} \\
= & \begin{cases} \begin{bmatrix} \left( \langle x^i \rangle - \langle 1 \rangle \right) \sum_{k=0}^{j-1} \langle y^k \rangle \\ \langle x^i y^j \rangle - \langle x^i \rangle + N_x \langle 1 \rangle \end{bmatrix} & , i = m-1 \\ \begin{bmatrix} (\langle 1 \rangle - \langle x \rangle) \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \langle x^i y^j \rangle - \langle x^i \rangle \end{bmatrix} & , i < m-1 \end{cases} \\
= & \begin{cases} \begin{bmatrix} D_x \left( \langle 1 \rangle + \langle x \rangle + \cdots + \langle x^{i-1} \rangle \right) \sum_{k=0}^{j-1} \langle y^k \rangle \\ D_y \left( \langle x^i \rangle + \langle x^i y \rangle + \cdots + \langle x^i y^{j-1} \rangle \right) + N_x \langle 1 \rangle \end{bmatrix} & , i = m-1 \\ \begin{bmatrix} -D_x \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ D_y \left( \langle x^i \rangle + \langle x^i y \rangle + \cdots + \langle x^i y^{j-1} \rangle \right) \end{bmatrix} & , i < m-1 \end{cases} \\
= & \begin{cases} \begin{bmatrix} -D_x \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ D_y \sum_{k=0}^{j-1} \langle x^i y^k \rangle + N_x \langle 1 \rangle \end{bmatrix} & , i = m-1 \\ \begin{bmatrix} -D_x \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ D_y \sum_{k=0}^{j-1} \langle x^i y^k \rangle \end{bmatrix} & , i < m-1 \end{cases} \\
= & d_1 \begin{cases} \begin{bmatrix} 0 \\ - \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \langle 1 \rangle \end{bmatrix} & i = m-1 \\ \begin{bmatrix} 0 \\ - \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ 0 \end{bmatrix} & i < m-1 \end{cases}
\end{aligned}
$$

where the second to last equation follows from when $i = m-1$, we have

$$
\begin{aligned}
& D_x \left( \langle 1 \rangle + \langle x \rangle + \cdots + \langle x^{i-1} \rangle \right) \\
= & D_x \left( \langle 1 \rangle + \langle x \rangle + \cdots + \langle x^{i-1} \rangle + \langle x^i \rangle \right) - D_x \langle x^i \rangle \\
= & D_x N_x - D_x \langle x^i \rangle = -D_x \langle x^i \rangle ,
\end{aligned}
$$

and hence

$$D_x\left(\langle 1\rangle + \langle x\rangle + \cdots + \left\langle x^{i-1}\right\rangle\right)\sum_{k=0}^{j-1}\left\langle y^k\right\rangle = -D_x\sum_{k=0}^{j-1}\left\langle x^i y^k\right\rangle.$$

Finally we get

$$S_1\left(\begin{bmatrix}0\\ \left\langle x^i y^j\right\rangle\end{bmatrix}\right) = \begin{cases}\begin{bmatrix}0\\ -\sum_{k=0}^{j-1}\left\langle x^i y^k\right\rangle\\ \langle 1\rangle\end{bmatrix} & , i = m-1\\ \begin{bmatrix}0\\ -\sum_{k=0}^{j-1}\left\langle x^i y^k\right\rangle\\ 0\end{bmatrix} & , i < m-1\end{cases}$$

□

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## 4. Proof of Main Results, 1

### 4.1. **Proof of Theorem 2.3.**

*Proof.* We will get (1) and (2) from the constructive lifting theorem (Theorem 1.53), and (3) will follow from (2) when we restrict our attention to cocycles arising from special sections.

(1) We shall construct $(g_i)_{i=0}^2$ in the diagram:

$$\begin{array}{ccccccccc}
\mathbb{Z}^{\text{triv}} & \xleftarrow{\partial_{-1}} & B_0 & \xleftarrow{\partial_0} & B_1 & \xleftarrow{\partial_1} & B_2 & \xleftarrow{} & \cdots \\
\downarrow{\scriptstyle 1_{\mathbb{Z}}} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\
\mathbb{Z}^{\text{triv}} & \xrightarrow{S_{-1}} & P_0 & \xrightarrow{S_0} & P_1 & \xrightarrow{S_1} & P_2 & \xrightarrow{} & \cdots
\end{array}$$

Going as in Theorem 1.53, we have

$$g_n\left[x\right] = S_{n-1}g_{n-1}\partial_{n-1}\left[x\right]$$

on generators $\left[x\right]$.

$g_0$ : Clearly $g_0\left[\,\right] = \left\langle 1\right\rangle$ is the only homomorphism that makes the first square commute. So

$$g_0\left[\,\right] = \left\langle 1\right\rangle.$$

$g_1$ : Recall $S_0\left(\left\langle x^i\right\rangle\right) = \sum_{j=0}^{k-1}\left\langle x^j\right\rangle$, with the understanding that $S_0\left(\left\langle x^i\right\rangle\right) = 0$ if $i = 0$. We have

$$\begin{aligned}
&\left[x^i\right] \overset{\partial_0}{\mapsto} \left\langle x^i\right\rangle\left[\,\right] - \left[\,\right] \\
&\overset{g_0}{\mapsto} \left\langle x^i\right\rangle - \left\langle 1\right\rangle \\
&\overset{S_0}{\mapsto} \sum_{j=0}^{i-1}\left\langle x^j\right\rangle,
\end{aligned}$$

and hence

$$g_1\left[x^i\right] = \sum_{j=0}^{i-1}\left\langle x^j\right\rangle.$$

$g_2$ : We saw that

$$S_1\left(\left\langle x^i\right\rangle\right) = \begin{cases} 0 & ,i < m-1 \\ \left\langle 1\right\rangle & ,i = m-1 \end{cases},$$

so

$$\begin{aligned}
&\left[x^i, x^j\right] \overset{\partial_1}{\mapsto} \left\langle x^i\right\rangle\left[x^j\right] - \left[x^{[i+j]_m}\right] + \left[x^i\right] \\
&\overset{g_1}{\mapsto} \sum_{k=0}^{j-1}\left\langle x^{i+k}\right\rangle - \sum_{k=0}^{[i+j]_m-1}\left\langle x^k\right\rangle + \sum_{k=0}^{i-1}\left\langle x^k\right\rangle \\
&\overset{S_1}{\mapsto} S_1\left(\sum_{k=0}^{j-1}\left\langle x^{i+k}\right\rangle\right) + 0 + 0 \\
&= \begin{cases} 0 & ,i+j < m \\ \left\langle 1\right\rangle & ,i+j \geq m \end{cases}.
\end{aligned}$$

Thus

$$g_2\left[x^i, x^j\right] = \begin{cases} 0 & ,i+j < m \\ \left\langle 1\right\rangle & ,i+j \geq m \end{cases}.$$

The Comparison Theorem (Theorem A.6) guarantees that the induced maps $g_n^* : H^n_{\text{special}}(G, A) \to H^n_{\text{bar}}(G, A)$ are isomorphisms. Applying $\text{Hom}_{\mathbb{Z}G}(-, A)$ to $g_2 : B_2 \to P_2$ gives

$$\begin{array}{rcl} g_2^* : \text{Hom}_{\mathbb{Z}G}(P_2, A) & \to & \text{Hom}_{\mathbb{Z}G}(B_2, A) \\ \varphi & \mapsto & \left( \varphi \circ g_2 : \left(x^i, x^j\right) \mapsto \left\{ \begin{array}{ll} 0 & , i+j < m \\ \langle 1 \rangle & , i+j \geq m \end{array} \right. \right). \end{array}$$

Using the natural isomorphism

$$\begin{array}{rcl} A & \tilde{\to} & \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) = \text{Hom}_{\mathbb{Z}G}(P_2, A) \\ a & \mapsto & (\langle 1 \rangle \mapsto a) \end{array}$$

gives us

$$\begin{array}{rcl} g_2^* : A & \to & \text{Hom}_{\mathbb{Z}G}(B_2, A) \\ a & \mapsto & \left( \left(x^i, x^j\right) \mapsto \left\{ \begin{array}{ll} 0 & , i+j < m \\ a & , i+j \geq m \end{array} \right. \right), \end{array}$$

as desired.

(2) We will construct $(f_i)_{i=0}^2$ in

$$\begin{array}{ccccccccccc} \mathbb{Z}^{\text{triv}} & \xleftarrow{d_{-1}} & P_0 & \xleftarrow{d_0} & P_1 & \xleftarrow{d_1} & P_2 & \longleftarrow & \cdots \\ \downarrow{\scriptstyle 1_{\mathbb{Z}}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ \mathbb{Z}^{\text{triv}} & \xrightarrow{S_{-1}} & B_0 & \xrightarrow{S_0} & B_1 & \xrightarrow{S_1} & B_2 & \longrightarrow & \cdots \end{array}$$

From Theorem 1.53, we define

$$f_n[x] = S_{n-1} f_{n-1} \partial_{n-1}[x]$$

on generators $[x]$.

$f_0$: We know that $\langle 1 \rangle$ generates $\mathbb{Z}G = P_n$, so

$$\langle 1 \rangle \overset{d_{-1}}{\mapsto} 1 \overset{1_{\mathbb{Z}}}{\mapsto} 1 \overset{S_{-1}}{\mapsto} [\,]$$

gives

$$f_0(\langle 1 \rangle) = [\,].$$

$f_1$: Recall that $S_0(x[\,]) = [x]$, so

$$\begin{array}{ll} \langle 1 \rangle & \overset{d_0}{\mapsto} \langle x \rangle - \langle 1 \rangle \overset{f_0}{\mapsto} \langle x \rangle [\,] - [\,] \\ & \overset{S_0}{\mapsto} [x] - [1], \end{array}$$

and hence

$$f_1(\langle 1 \rangle) = [x] - [1]$$

which becomes

$$f_1(\langle 1 \rangle) = [x]$$

in the normalized case.

$f_2$: Recall that $S_1(x[x_1]) = [x, x_1]$ :

$$\begin{aligned}\langle 1\rangle &\overset{d_1}{\mapsto} \sum_{j=0}^{m-1} \langle x^j\rangle \\ &\overset{f_1}{\mapsto} \sum_{j=0}^{m-1} \langle x^j\rangle [x] - \sum_{j=0}^{m-1} \langle x^j\rangle [1] \\ &\overset{S_1}{\mapsto} \sum_{j=0}^{m-1} \left[x^j, x\right] - \sum_{j=0}^{m-1} \left[x^j, 1\right].\end{aligned}$$

Therefore

$$f_2(\langle 1\rangle) = \sum_{j=0}^{m-1} \left[x^j, x\right] - \sum_{j=0}^{m-1} \left[x^j, 1\right],$$

which reduces to

$$f_2(\langle 1\rangle) = \sum_{j=1}^{m-1} \left[x^j | x\right]$$

in the normalized case.

The Comparison Theorem (Theorem A.6) guarantees that the induced maps $f_n^* : H^n_{\text{bar}}(G, A) \to H^n_{\text{special}}(G, A)$ are isomorphisms, hence we only need to verify that the induced map is as claimed. Applying $\text{Hom}_{\mathbb{Z}G}(-, A)$ to $f_2 : P_2 \to B_2$ gives

$$\begin{aligned} f_2^* : \text{Hom}_{\mathbb{Z}G}(B_2, A) = B^2 &\to \text{Hom}_{\mathbb{Z}G}(P_2, A) \\ \varphi &\mapsto \varphi \circ f_2. \end{aligned}$$

But $\text{Hom}_{\mathbb{Z}G}(P_2, A) = \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \tilde{\to} A$ via $\psi \mapsto \psi(\langle 1\rangle)$, so

$$\begin{aligned} f_2^* : B^2 &\to A \\ \varphi &\mapsto (\varphi \circ f_2)(\langle 1\rangle) \end{aligned}$$

and

$$(\varphi \circ f_2)(\langle 1\rangle) = \sum_{j=0}^{m-1} \varphi\left(x^j, x\right),$$

as claimed.

(3) By Theorem 1.45 we know that sending an extension

$$\varepsilon : 1 \to A \overset{\iota}{\to} E \overset{\pi}{\to} G \to 1$$

to the congruence class of a cocycle belonging to a simple section induces an isomorphism

$$E(G, A) \tilde{\to} H^2_{\text{bar}}(G, A).$$

By Proposition 1.46 we know that the cocycle of a simple section will be of the form

$$\varphi_\sigma\left(x^i, x^j\right) = \begin{cases} \{x\}^m & \text{if} \quad i + j \geq m \\ 1 & \text{if} \quad i + j < m \end{cases},$$

for some representative $\{x\} \in E$. Hence we have an isomorphism

$$\begin{aligned} E(G, A) &\tilde{\to} H^2_{\text{bar}}(G, A) \\ [\varepsilon] &\mapsto \varphi_\sigma + \delta B^1. \end{aligned}$$

Composing with the isomorphism

$$\begin{aligned} H^2_{\text{bar}}(G,A) &\tilde{\to} H^2_{\text{special}}(G,A) \\ \varphi + \delta B^1 &\mapsto \sum_{j=1}^{m-1} \varphi\left(x^j, x\right) + NA \end{aligned}$$

from (2), we get

$$\begin{aligned} E(G,A) &\tilde{\to} H^2_{\text{special}}(G,A) \\ [\varepsilon] &\mapsto \sum_{j=1}^{m-1} \varphi_\sigma\left(x^j, x\right) + NA = \{x\}^m + NA. \end{aligned}$$

□

4.2. **Proof of Theorem 2.6.**

*Proof.* We will get (1) and (2) from the Constructive Lifting Theorem (Theorem 1.53). That they are inverses follows directly from The Comparison Theorem (Theorem A.6). Next, (3) will follow from (2) when we restrict our attention to cocycles arising from special sections.

(1) Recall the formulas for the contraction (Proposition 1.52):

$$\begin{aligned} S_{-1}(1) &= \langle 1 \rangle, \\ S_0\left(\langle x^i y^j \rangle\right) &= \begin{bmatrix} \sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \sum_{k=0}^{i-1} \langle x^k \rangle \end{bmatrix} \\ S_1\left(\begin{bmatrix} \langle x^i y^j \rangle \\ 0 \end{bmatrix}\right) &= \begin{cases} \begin{bmatrix} \langle x^i \rangle \\ 0 \\ 0 \end{bmatrix} & , j = n-1 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & , j < n-1 \end{cases} \\ S_1\left(\begin{bmatrix} 0 \\ \langle x^i y^j \rangle \end{bmatrix}\right) &= \begin{cases} \begin{bmatrix} 0 \\ -\sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ \langle 1 \rangle \end{bmatrix} & , i = m-1 \\ \begin{bmatrix} 0 \\ -\sum_{k=0}^{j-1} \langle x^i y^k \rangle \\ 0 \end{bmatrix} & , i < m-1 \end{cases} \end{aligned}$$

$$\begin{array}{ccccccccccc} \mathbb{Z}^{\text{triv}} & \xleftarrow{\partial_{-1}} & B_0 & \xleftarrow{\partial_0} & B_1 & \xleftarrow{\partial_1} & B_2 & \longleftarrow & \cdots \\ \downarrow{\scriptstyle 1_{\mathbb{Z}}} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\ \mathbb{Z}^{\text{triv}} & \xrightarrow{S_{-1}} & P_0 & \xrightarrow{S_0} & P_1 & \xrightarrow{S_1} & P_2 & \longrightarrow & \cdots \end{array}$$

$g_0$: Clearly

$$\begin{aligned} g_0 : \mathbb{Z}G[\,] &\to \mathbb{Z}G \\ [\,] &\mapsto \langle 1 \rangle \end{aligned}$$

does the trick.

$g_1$ :

$$\begin{array}{ll} \left[x^i y^j\right] & \overset{\partial_0}{\mapsto} \left\langle x^i y^j \right\rangle [\,] - [\,] \\ & \overset{g_0}{\mapsto} \left\langle x^i y^j \right\rangle - \langle 1 \rangle \\ & \overset{S_0}{\mapsto} \begin{bmatrix} \left\langle x^i \right\rangle + \left\langle x^i y \right\rangle + \cdots + \left\langle x^i y^{j-1} \right\rangle \\ \langle 1 \rangle + \langle x \rangle + \cdots + \left\langle x^{i-1} \right\rangle \end{bmatrix} \end{array}$$

with the understanding that either component becomes zero if $i = 0$ or $j = 0$ respectively. Hence

$$\begin{array}{rcl} g_1 \left[x^i y^j\right] & = & \begin{bmatrix} \left\langle x^i \right\rangle + \left\langle x^i y \right\rangle + \cdots + \left\langle x^i y^{j-1} \right\rangle \\ \langle 1 \rangle + \langle x \rangle + \cdots + \left\langle x^{i-1} \right\rangle \end{bmatrix} \\ & = & \begin{bmatrix} \sum_{d=0}^{j-1} \left\langle x^i y^d \right\rangle \\ \sum_{d=0}^{i-1} \left\langle x^d \right\rangle \end{bmatrix}. \end{array}$$

$g_2$:

$$\left[x^i y^j, x^k y^l\right] \overset{\partial_1}{\mapsto} \left\langle x^i y^j \right\rangle \left[x^k y^l\right] - \left[x^{[i+k]_m} y^{[j+l]_n}\right] + \left[x^i y^j\right]$$

Now: $1_{\mathbb{Z}G \bigoplus \mathbb{Z}G} = \iota_1\pi_1 + \iota_2\pi_2$, and hence $g_2 = S_1 \left(\iota_1\pi_1 + \iota_2\pi_2\right) g_1\partial_1 = S_1\iota_1\pi_1 g_1\partial_1 + S_1\iota_2\pi_2 g_1\partial_1$. Let us consider the first term:

$$\begin{array}{ll} & \left\langle x^i y^j \right\rangle \left[x^k y^l\right] - \left[x^{[i+k]_m} y^{[j+l]_n}\right] + \left[x^i y^j\right] \\ \downarrow & \pi_1 \circ g_1 \\ & \left\langle x^i y^j \right\rangle \sum_{d=0}^{l-1} \left\langle x^k y^d \right\rangle - \sum_{d=0}^{[j+l]_n - 1} \left\langle x^{[i+k]_m} y^d \right\rangle + \sum_{d=0}^{j-1} \left\langle x^i y^d \right\rangle \\ = & \sum_{d=0}^{l-1} \left\langle x^{[i+k]_m} y^{j+d} \right\rangle - \sum_{d=0}^{[j+l]_n - 1} \left\langle x^{[i+k]_m} y^d \right\rangle + \sum_{d=0}^{j-1} \left\langle x^i y^d \right\rangle \\ \downarrow & S_1 \circ \iota_1 \ \ (\text{since } [j+l]_n - 1, j-1 < n-1) \\ & \begin{cases} \begin{bmatrix} \left\langle x^{[i+k]_m} \right\rangle \\ 0 \\ 0 \end{bmatrix} & , j + l \geq n \\ \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & , j + l < n \end{cases} \end{array}$$

Now for the second term:

$$\begin{array}{rl} & \left\langle x^i y^j \right\rangle \left[x^k y^l\right] - \left[x^{[i+k]_m} y^{[j+l]_n}\right] + \left[x^i y^j\right] \\ \downarrow & \pi_2 \circ g_1 \\ & \left\langle x^i y^j \right\rangle \sum_{d=0}^{k-1} \left\langle x^d \right\rangle - \sum_{d=0}^{[i+k]_m - 1} \left\langle x^d \right\rangle + \sum_{d=0}^{i-1} \left\langle x^d \right\rangle \\ = & \sum_{d=0}^{k-1} \left\langle x^{i+d} y^j \right\rangle - \underbrace{\sum_{d=0}^{[i+k]_m - 1} \left\langle x^d \right\rangle + \sum_{d=0}^{i-1} \left\langle x^d \right\rangle}_{\text{No power of } y} \\ \downarrow & S_1 \circ \iota_2 \\ & \begin{bmatrix} 0 \\ -\sum_{d=0}^{k-1} \sum_{\alpha=0}^{j-1} \left\langle x^{i+d} y^\alpha \right\rangle \\ 0 \end{bmatrix} + \begin{cases} \begin{bmatrix} 0 \\ 0 \\ \langle 1 \rangle \end{bmatrix} & , i+k \geq m \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & , i+k < m \end{cases} \end{array}$$

Hence

$$\begin{array}{rl} & g_2\left(\left[x^i y^j, x^k y^l\right]\right) \\ = & \begin{bmatrix} \left\langle x^{[i+k]_m} \right\rangle & , j+l \geq n \\ -\sum_{d=0}^{k-1} \sum_{\alpha=0}^{j-1} \left\langle x^{i+d} y^\alpha \right\rangle & \\ \langle 1 \rangle & , i+k \geq m \end{bmatrix} \end{array}$$

where the first and third component are zero if the conditions to the right are not met.

Theorem A.6 guarantees that the induced maps $g_n^* : H^n_{\text{special}}(G, A) \to H^n_{\text{bar}}(G, A)$ are isomorphisms. We therefore have only to check that the induced map is as claimed. Applying $\text{Hom}_{\mathbb{Z}G}(-, A)$ to $g_2 : B_2 \to P_2$ gives

$$\begin{array}{rcl} g_2^* : \text{Hom}_{\mathbb{Z}G}(P_2, A) & \to & \text{Hom}_{\mathbb{Z}G}(B_2, A) \\ \varphi & \mapsto & \varphi \circ g_2. \end{array}$$

Using the natural isomorphism

$$\begin{array}{rcl} A^n & \tilde{\to} & \text{Hom}_R(R^n, A) \\ \underline{a} & \mapsto & \left(\varphi_a : \underline{r} \mapsto \sum_{i=1}^n r_i a_i\right) \end{array}$$

we get

$$\begin{array}{rcl} A^3 & \to & \text{Hom}_{\mathbb{Z}G}(B_2, A) \\ \underline{a} & \mapsto & \left(\varphi_{\underline{a}} \circ g_2 : B_2 \to A\right) \end{array}$$

where

$$\begin{array}{rcll} \varphi_{\underline{a}} \circ g_2 & : & B_2 \to A & \\ & & \left[x^i y^j | x^k y^l\right] \mapsto & -\sum_{d=0}^{k-1} \sum_{\alpha=0}^{j-1} {}^{x^{i+d} y^\alpha} a_2 \\ & & & + {}^{x^{[i+k]_m}} a_1 \qquad \text{(if } j+l \geq n\text{)} \\ & & & + a_3 \qquad \text{(if } i+k \geq m\text{)} \end{array} .$$

on generators $\left[x^i y^j | x^k y^l\right]$. Recalling that we identified $\varphi \in \mathrm{Hom}_{\mathbb{Z}G}(B_2, A)$ with $\left(\varphi : G^2 \to A\right) \in B^2$

$$\varphi\left(x^i y^j, x^k y^l\right) = \varphi\left(\left[x^i y^j | x^k y^l\right]\right)$$

we get

$$\begin{array}{rcll} \varphi & = & \varphi_{\underline{a}} \circ g_2 : G^2 \to A & \\ & & \left(x^i y^j, x^k y^l\right) \mapsto & -\sum_{d=0}^{k-1} \sum_{\alpha=0}^{j-1} {}^{x^{i+d} y^{\alpha}} a_2 \\ & & & + {}^{x^{[i+k]_m}} a_1 \qquad (\text{if } j+l \geq n) \\ & & & + a_3 \qquad (\text{if } i+k \geq m) \end{array}$$

as desired.

(2) Using Theorem 1.53, we construct the lifting of $\mathbb{Z}^{\mathrm{triv}} = \mathbb{Z}^{\mathrm{triv}}$ inductively by the formula

$$f_n[x] = S_{n-1} f_{n-1} d_{n-1}[x]$$

on generators $[x]$.

$$\begin{array}{ccccccccccc} \mathbb{Z}^{\mathrm{triv}} & \xleftarrow{d_{-1}} & P_0 & \xleftarrow{d_0} & P_1 & \xleftarrow{d_1} & P_2 & \longleftarrow & \cdots \\ \downarrow{\scriptstyle 1_{\mathbb{Z}}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ \mathbb{Z}^{\mathrm{triv}} & \xrightarrow{S_{-1}} & B_0 & \xrightarrow{S_0} & B_1 & \xrightarrow{S_1} & B_2 & \longrightarrow & \cdots \end{array}$$

Recall the formulas for the contraction (Remark 1.16):

$$\begin{array}{rcl} S_{-1}(1) & = & [\,], \\ S_0(x[\,]) & = & [x], \\ S_1(x[x_1]) & = & [x, x_1]. \end{array}$$

$f_0$ :

$$P_0 = \mathbb{Z}G,$$

which is generated by $\langle 1 \rangle$ as a $\mathbb{Z}G$-module. So

$$\langle 1 \rangle \overset{d_{-1}}{\mapsto} 1 \overset{1_{\mathbb{Z}}}{\mapsto} 1 \overset{S_{-1}}{\mapsto} [\,].$$

Hence

$$f_0(\langle 1 \rangle) = [\,]$$

$f_1$ :

$$P_1 = \mathbb{Z}G \bigoplus \mathbb{Z}G,$$

which is generated by

$$\begin{bmatrix} \langle 1 \rangle \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \langle 1 \rangle \end{bmatrix}$$

as a $\mathbb{Z}G$-module. Now:

$$\begin{bmatrix} \langle 1 \rangle \\ 0 \end{bmatrix} \quad \overset{d_0}{\mapsto} \langle y \rangle - \langle 1 \rangle \overset{f_0}{\mapsto} \langle y \rangle [\,] - [\,]$$

$$\overset{S_0}{\mapsto} [y] - [1]$$

and recall that $[1] = 0$ in the normalized bar resolution. Next

$$\begin{bmatrix} 0 \\ \langle 1 \rangle \end{bmatrix} \quad \overset{d_0}{\mapsto} \langle x \rangle - \langle 1 \rangle \overset{f_0}{\mapsto} \langle x \rangle [\,] - [\,]$$

$$\overset{S_0}{\mapsto} [x] - [1].$$

Hence

$$f_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} [y]-[1] & [x]-[1] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a\,[y] + b\,[x] - (a+b)\,[1]\,.$$

or in the normalized case

$$f_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} [y] & [x] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a\,[y] + b\,[x]\,.$$

$f_2$ : The module $P_2 = \mathbb{Z}G \bigoplus \mathbb{Z}G \bigoplus \mathbb{Z}G$, is generated by

$$\begin{bmatrix} \langle 1\rangle \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \langle 1\rangle \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ \langle 1\rangle \end{bmatrix}.$$

We have

$$\begin{aligned} \begin{bmatrix} \langle 1\rangle \\ 0 \\ 0 \end{bmatrix} & \overset{d_1}{\mapsto} \begin{bmatrix} \langle 1\rangle + \langle y\rangle + \langle y^2\rangle + \cdots + \langle y^{n-1}\rangle \\ 0 \end{bmatrix} \\ & \overset{f_1}{\mapsto} \left(\langle 1\rangle + \langle y\rangle + \langle y^2\rangle + \cdots + \langle y^{n-1}\rangle\right)([y]-[1]) \\ = & \sum_{k=0}^{n-1} \langle y^k\rangle\,[y] - \sum_{k=0}^{n-1} \langle y^k\rangle\,[1] \\ & \overset{S_1}{\mapsto} \sum_{k=0}^{n-1} [y^k, y] - \sum_{k=0}^{n-1} [y^k, 1]\,. \end{aligned}$$

While

$$\begin{aligned} & \begin{bmatrix} 0 \\ \langle 1\rangle \\ 0 \end{bmatrix} \overset{d_1}{\mapsto} \begin{bmatrix} \langle x\rangle - \langle 1\rangle \\ \langle 1\rangle - \langle y\rangle \end{bmatrix} \\ & \overset{f_1}{\mapsto} (\langle x\rangle - \langle 1\rangle)\,[y] + (\langle 1\rangle - \langle y\rangle)\,[x] - (\langle x\rangle - \langle y\rangle)\,[1] \\ = & \langle x\rangle\,[y] - \langle x\rangle\,[1] + \langle y\rangle\,[1] - \langle y\rangle\,[x] + [x] - [y] \\ & \overset{S_1}{\mapsto} [x,y] - [x,1] + [y,1] - [y,x] + [1,x] - [1,y] \\ = & [x,y] - [y,x]\,. \end{aligned}$$

And

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ \langle 1\rangle \end{bmatrix} & \overset{d_1}{\mapsto} \begin{bmatrix} 0 \\ N_x \end{bmatrix} \overset{f_1}{\mapsto} N_x\,([x]-[1]) \\ = & \sum_{k=0}^{m-1} \langle x^k\rangle\,([x]-[1]) \\ & \overset{S_1}{\mapsto} \sum_{k=0}^{m-1} \left(\left[x^k, x\right] - \left[x^k, 1\right]\right) \\ = & \sum_{k=0}^{m-1} \left[x^k, x\right] - \sum_{k=0}^{m-1} \left[x^k, 1\right]. \end{aligned}$$

Thus

$$f_2\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{n-1} [y^k, y] - \sum_{k=0}^{n-1} [y^k, 1] \\ [x,y] - [y,x] \\ \sum_{k=0}^{m-1} \left[x^k, x\right] - \sum_{k=0}^{m-1} \left[x^k, 1\right] \end{bmatrix},$$

or

$$f_2\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{n-1} \left[y^k|y\right] \\ [x|y] - [y|x] \\ \sum_{k=0}^{m-1} \left[x^k|x\right] \end{bmatrix}$$

in the normalized case.

Theorem A.6 guarantees that the induced maps $f_n^* : H_{\text{bar}}^n(G,A) \to H_{\text{special}}^n(G,A)$ are isomorphisms. We therefore have only to check that the induced map is as claimed in Theorem 2.6. Applying $\text{Hom}_{\mathbb{Z}G}(-,A)$ to $f_2 : P_2 \to B_2$ gives

$$\begin{array}{rcl} f_2^* : B^2 & \to & \text{Hom}_{\mathbb{Z}G}(P_2,A) = \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G \oplus \mathbb{Z}G \oplus \mathbb{Z}G, A) \\ \varphi & \mapsto & \varphi \circ f_2, \end{array}$$

and using the natural isomorphism

$$\begin{array}{rcl} \text{Hom}_{\mathbb{Z}G}\left(\bigoplus_{i=1}^{3} \mathbb{Z}G, A\right) & \tilde{\to} & A^3 \\ \psi & \mapsto & \begin{bmatrix} (\psi \circ \iota_1)(\langle 1 \rangle) \\ (\psi \circ \iota_2)(\langle 1 \rangle) \\ (\psi \circ \iota_3)(\langle 1 \rangle) \end{bmatrix} \end{array}$$

where $\iota_i$ are the canonical injections, we get

$$\begin{array}{rcl} f_2^* : B^2 & \to & A^3 \\ \varphi & \mapsto & \begin{bmatrix} \sum_{k=0}^{n-1} \varphi\left(y^k, y\right) \\ \varphi(x,y) - \varphi(y,x) \\ \sum_{k=0}^{m-1} \varphi\left(x^k, x\right) \end{bmatrix}. \end{array}$$

(3) Let $0 \le i,k < m$ and $0 \le j,l < n$, then the defining equation for $\varphi_\sigma$ is:

$$\sigma\left(x^i y^j\right) \sigma\left(x^k y^l\right) = \varphi_\sigma\left(x^i y^j, x^k y^l\right) \sigma\left(x^i y^j x^k y^l\right).$$

Using the definition of $\sigma$ and the fact that $xy = yx$ we get

$$\{x\}^i \{y\}^j \{x\}^k \{y\}^l = \varphi_\sigma\left(x^i y^j, x^k y^l\right) \{x\}^{[i+k]_m} \{y\}^{[j+l]_n},$$

so

$$\varphi_\sigma\left(x^i y^j, x^k y^l\right) = \{x\}^i \{y\}^j \{x\}^k \{y\}^l \left(\{x\}^{[i+k]_m} \{y\}^{[j+l]_n}\right)^{-1}$$

Hence (additive notation)

$$\begin{array}{rcl} \varphi_\sigma\left(y^k, y\right) & = & \{y\}^{k+1} \left(\{y\}^{[k+1]_n}\right)^{-1} = \begin{cases} \{y\}^n = U & \text{if} \quad k \ge n-1 \\ 0 & \text{if} \quad k < n-1 \end{cases}, \\ \varphi_\sigma(x,y) & = & \{x\}\{y\}(\{x\}\{y\})^{-1} = 0, \\ \varphi_\sigma(y,x) & = & \{y\}\{x\}(\{x\}\{y\})^{-1} = V, \\ \varphi_\sigma\left(x^k, x\right) & = & \{x\}^{k+1} \left(\{x\}^{[k+1]_m}\right)^{-1} = \begin{cases} \{x\}^m = W & \text{if} \quad k \ge m-1 \\ 0 & \text{if} \quad k < m-1 \end{cases} \end{array}$$

Applying the isomorphism

$$\begin{array}{rcl} H_{\text{bar}}^2(G,A) & \to & H_{\text{spec}}^2(G,A) \\ \varphi + \delta B^1 & \mapsto & \begin{bmatrix} \sum_{k=0}^{n-1} \varphi\left(y^k, y\right) \\ \varphi(x,y) - \varphi(y,x) \\ \sum_{k=0}^{m-1} \varphi\left(x^k, x\right) \end{bmatrix} + d_1^* A^2 \end{array}$$

from (2) to $\varphi_\sigma + \delta B^1$ gives

$$\begin{bmatrix} \sum_{k=0}^{n-1} \varphi_\sigma\left(y^k, y\right) \\ \varphi_\sigma(x,y) - \varphi_\sigma(y,x) \\ \sum_{k=0}^{m-1} \varphi_\sigma\left(x^k, x\right) \end{bmatrix} + \delta B^1 = \begin{bmatrix} 0+0+\cdots+0+U \\ -V \\ 0+0+\cdots+0+W \end{bmatrix} + \delta B^1 = \begin{bmatrix} U \\ -V \\ W \end{bmatrix} + \delta B^1$$

as desired.

□

4.3. **Proof of Theorem 2.8.**

**Lemma 4.1.** $|\mathrm{GL}_2(\mathbb{I}_p)| = \left(p^2-1\right)\left(p^2-p\right) = p(p-1)^2(p+1)$.

*Proof.* Any invertible matrix

$$X = [\mathbf{c}_1, \mathbf{c}_2] \in \mathrm{GL}_2(\mathbb{I}_p)$$

consists of two columns. The first column $\mathbf{c}_1$, is nonzero so we have $p^2-1$ possible entries. In order for the matrix to be invertible we need that $\mathbf{c}_2$ not be a multiple of $\mathbf{c}_1$, i.e. $\mathbf{c}_2 \notin \mathbb{I}_p\mathbf{c}_1$. There are $p$ possibilities for $\mathbf{c}_2 \in \mathbb{I}_p\mathbf{c}_1$, and hence we have $p^2-p$ choices for $\mathbf{c}_2$. □

*Proof.* (Of the Theorem 2.8.) We know from abstract algebra that (up to isomorphism) $\mathbb{I}_p$ is the only group of order $p$, and that $\mathbb{I}_{p^2}$, $\mathbb{I}_p \times \mathbb{I}_p$ are the only groups of order $p^2$.

If $s = t = 1$, then the only combination is $A = G = \mathbb{I}_p$. We have

$$\mathrm{Aut}(A) = \mathrm{Aut}(\mathbb{I}_p) = (\mathbb{I}_p)^* \cong \mathbb{I}_{p-1}$$

so if $\varphi$ is any action

$$\varphi : G \to \mathrm{Aut}(A)$$

then it must be trivial, since $|\xi(G)|$ must divide $|G| = p$ and $|\mathrm{Aut}(A)| = p-1$.

If $s = 1$, $t = 2$ then the possible combinations of $A$ and $G$ are $\left(\mathbb{I}_p, \mathbb{I}_{p^2}\right)$ and $(\mathbb{I}_p, \mathbb{I}_p \times \mathbb{I}_p)$. Again, in either case $G$ is a $p$-group, so for $|\varphi(G)|$ must divide $p$ and $p-1$, hence the only option is $\varphi(G) = \{1_A\}$.

If $s = 2$, $t = 1$ the we need to check the different cases individually.

(1) $A = \mathbb{I}_{p^2}$, then $\mathrm{Aut}(A) = \left(\mathbb{I}_{p^2}\right)^* \cong \mathbb{I}_p \times \mathbb{I}_{p-1}$by [DF04, Section 9.5, Corollary 20]. Let

$$\varphi : G \to Aut(A)$$

be an action. Since $|G| = p$, there are two possibilities:
   (a) $\varphi(G) = \{1\}$, i.e., the action is trivial.
   (b) $G \cong \varphi(G) = \{1, 1+p, 1+2p, \ldots, 1+(p-1)p\}$. Changing the generator $x$ of $G$, we can assume that

$$\varphi(1 + p\mathbb{Z}) = 1 + p,$$

   and for $a \in A$,

$$^x a = (1+p)\, a.$$

(2) $A = \mathbb{I}_p \times \mathbb{I}_p$, then $\mathrm{Aut}(A) = \mathrm{GL}_2(\mathbb{I}_p) = \mathrm{GL}_2(\mathbb{F}_p)$.
   (a) The action of $\varphi$ is trivial.
   (b) The action of $\varphi$ is non-trivial, and it is given by

$$^x \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix},$$

   which we show in further down.

For $s = t = 2$, then we have the cases

(1) $A = \mathbb{I}_{p^2}$, then $\mathrm{Aut}(A) = \left(\mathbb{I}_{p^2}\right)^* \cong \mathbb{I}_p \times \mathbb{I}_{p-1}$ by [DF04, Section 9.5, Corollary 20]. Let

$$\varphi : G \longrightarrow Aut(A)$$

be action. Since $|G| = p^2$, there are two possibilities:

(a) $\varphi(G) = \{1\}$, i.e., the action is trivial.

(b) $\varphi(G) = \{1, 1+p, 1+2p, \ldots, 1+(p-1)p\}$.

(i) $G = \langle x \rangle = \mathbb{I}_{p^2}$, $\ker \varphi = p\mathbb{I}_{p^2}$. Changing the generator $x$ of $G$, we can assume that

$$\varphi\left(1 + p^2\mathbb{Z}\right) = 1 + p,$$

and for $a \in A$,

$${}^{x^i}a = a(1+ip).$$

(ii) $G = \langle x \rangle \times \langle y \rangle = \mathbb{I}_p \times \mathbb{I}_p$. Changing the generators $x$ and $y$, we can assume that $x$ acts trivially, and $y$ acts like this:

$${}^{y}a = a(1+p).$$

Or vice versa: $y$ acts trivially, and

$${}^{x}a = a(1+p).$$

(2) $A = \mathbb{I}_p \times \mathbb{I}_p, \mathrm{Aut}(A) = \mathrm{GL}_2(\mathbb{I}_p) = \mathrm{GL}_2(\mathbb{F}_p)$.

(a) The action $\varphi$ is trivial.

(b) The action is non-trivial, and since $|G| = p^2$, the only possible order for $\varphi(G)$ is $p$ (Lemma 4.1). So $\varphi(G)$ is a cyclic subgroup $\langle Y \rangle \subseteq \mathrm{GL}_2(\mathbb{I}_p)$ for some matrix $Y$, with $Y^p = 1$. Let $m_Y(t)$ be the minimal polynomial, and $\chi_Y(t)$ be the characteristic polynomial. Then by the Cayley-Hamilton Theorem [DF04, Section 12.2, Proposition 20 (2)] we know that $m_Y(t)$ divides $\chi_Y(t)$, which is of degree 2. Since $Y^p = 1$, we see that $t^p - 1$ is an invariant factor of $Y$, and hence we know that

$$m_Y(t) \,|\, t^p - 1.$$

We therefore know that $m_Y(t)$ is either $t-1$, or $t^2-1$.

(i) $m_Y(t) = t - 1$, then $Y - 1 = 0$ and so $Y = 1$ and hence the action is trivial, contradiction.

(ii) $m_Y(t) = t^2 - 1$, then the modified Frobenius form of $Y$ is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

In fact, by slightly changing the construction of the Frobenius form, we can assume that it is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$Y = P^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P$$

for some $P \in GL_2(\mathbb{F}_p)$. Apply this $P$ (or, may be, $P^{-1}$) to the generators of $A$. This allows us to assume that

$$Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\varphi(G) = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{F}_p \right\}.$$

If $G = \langle x \rangle \times \langle y \rangle = \mathbb{I}_p \times \mathbb{I}_p$, then, changing the generators $x$ and $y$, we can assume that $y$ acts trivially, and $x$ acts as above (or *vice versa*).

□

4.4. **Proof of Theorem 2.14.**

*Proof.* By Theorem 2.8, the different cases for the kernel, cokernel, and actions are (up to weak equivalence) all that arise in connection with extensions

$$\begin{array}{rcl} p^s & \to & p^{s+t} \to p^t, \\ 1 & \leq & s, t \leq 2. \end{array}$$

We consider the case $G$ cyclic and the case $G$ dicyclic separately.

(1) $G = \mathbb{I}_m$, then we recall that

$$H^2_{\text{spec}}(G, A) = \frac{A^{\text{fix}}}{NA}.$$

We treat the cases with trivial action together, and the cases with non-trivial action individually.

(a) Trivial action: Since ${}^x a = a$ for $\forall a \in A$ we have $A^{\text{fix}} = A$. The equation

$$Na = \sum_{i=0}^{m-1} {}^{x^i} a = \sum_{i=0}^{m-1} a = ma$$

shows that $NA = mA$. Thus

$$H^2_{\text{spec}}(G, A) = \frac{A}{mA},$$

which when combined with Lagrange's Theorem gives the sub-table

**Table 4.2.** *For any prime $p$ and and $G$ acting trivially on $A$:*

| $G$ | $A$ | $H^2(G, A)$ |
|---|---|---|
| $\mathbb{I}_p$ | $\mathbb{I}_p$ | $\mathbb{I}_p$ |
| $\mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | $\mathbb{I}_p$ |
| $\mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p$ | $\mathbb{I}_p$ |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_{p^2}$ | $\mathbb{I}_{p^2}$ |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ |

(b) $G = \mathbb{I}_p$ and $A = \mathbb{I}_{p^2}$ where the action of $G$ on $A$ is given by

$${}^x a = (1 + p)\, a.$$

The equation

$$Da = {}^x a - a = (1 + p)\, a - a = pa$$

shows that $a \in A^{\text{fix}}$ if and only if $pa = 0$, so

$$A^{\text{fix}} = {}_{[p]}\mathbb{I}_{p^2} \cong p\mathbb{I}_{p^2}.$$

Next we have

$$\begin{array}{rcl} Na & = & \displaystyle\sum_{i=0}^{p-1} {}^{x^i} a = \sum_{i=0}^{p-1} (1 + ip)\, a = \sum_{i=0}^{p-1} a + p \sum_{i=0}^{p-1} ia \\ & = & \displaystyle pa + pa \frac{p(p-1)}{2} = \begin{cases} 0 & \text{if} \quad p = 2 \\ pa & \text{if} \quad p \neq 2 \end{cases}, \end{array}$$

giving us

$$NA = \begin{cases} 0 & \text{if} \quad p = 2 \\ p\mathbb{I}_{p^2} & \text{if} \quad p \neq 2 \end{cases}.$$

Hence

$$H^2(G, A) \cong \begin{cases} p\mathbb{I}_{p^2} & \text{if} \quad p = 2 \\ 0 & \text{if} \quad p \neq 2 \end{cases} \cong \begin{cases} \mathbb{I}_2 & \text{if} \quad p = 2 \\ 0 & \text{if} \quad p \neq 2 \end{cases}.$$

(c) $G = \mathbb{I}_p$ and $A = \mathbb{I}_p \times \mathbb{I}_p$ with

$$^x \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}.$$

We have

$$\begin{aligned} D \begin{bmatrix} a \\ b \end{bmatrix} &= {}^x \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a+b \\ b \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \end{aligned}$$

so

$$A^{\text{fix}} = \mathbb{I}_p \times \{0\}.$$

Next

$$\begin{aligned} N \begin{bmatrix} a \\ b \end{bmatrix} &= \sum_{i=0}^{p-1} {}^{x^i} \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{i=0}^{p-1} \begin{bmatrix} a+ib \\ b \end{bmatrix} = p \begin{bmatrix} a \\ b \end{bmatrix} + \sum_{i=0}^{p-1} \begin{bmatrix} ib \\ 0 \end{bmatrix} \\ &= 0 + b \sum_{i=0}^{p-1} i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = b \frac{p(p-1)}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{cases} \begin{bmatrix} b \\ 0 \end{bmatrix} & \text{if} \quad p = 2 \\ 0 & \text{if} \quad p \neq 2 \end{cases}. \end{aligned}$$

Therefore

$$A^N = \begin{cases} \mathbb{I}_2 \times \{0\} & \text{if} \quad p = 2 \\ \{0\} & \text{if} \quad p \neq 2 \end{cases},$$

and so

$$\frac{A^{\text{fix}}}{A^N} = \begin{cases} \frac{\mathbb{I}_p \times \{0\}}{\mathbb{I}_2 \times \{0\}} & \text{if} \quad p = 2 \\ \frac{\mathbb{I}_p \times \{0\}}{\{0\}} & \text{if} \quad p \neq 2 \end{cases} \cong \begin{cases} \{0\} & \text{if} \quad p = 2 \\ \mathbb{I}_p & \text{if} \quad p \neq 2 \end{cases}.$$

(d) $G = \mathbb{I}_{p^2}$ and $A = \mathbb{I}_{p^2}$, where

$$^x a = (1+p)\, a = a + pa,$$

so $a \in A^{\text{fix}}$ if and only if $pa = 0$, i.e. $a \in {}_{[p]}A$. Hence $A^{\text{fix}} = {}_{[p]}A$. Next

$$\begin{aligned} Na &= \sum_{i=0}^{p^2-1} \left\langle c^i \right\rangle a = \sum_{i=0}^{p^2-1} (a + ipa) = a \sum_{i=0}^{p^2-1} 1 + ap \sum_{i=0}^{p^2-1} i \\ &= ap^2 + ap \frac{\left(p^2-1\right) p^2}{2} = 0. \end{aligned}$$

Thus $NA \cong \{0\}$ and so

$$H^2_{\text{spec}} \left( \mathbb{I}_{p^2}, \left( \mathbb{I}_{p^2} \right)^{\xi} \right) \cong {}_{[p]}\mathbb{I}_{p^2} \cong \mathbb{I}_p$$

(e) $G = \mathbb{I}_{p^2}$, $A = \mathbb{I}_p \times \mathbb{I}_p$ with the action being given by

$$^{x^i} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+ib \\ b \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} a \\ b \end{bmatrix} \in A^{\text{fix}} \Leftrightarrow b = 0,$$

showing that $A^{\text{fix}} = \mathbb{I}_p \times \{0\}$. Next

$$\begin{aligned} N \cdot \begin{bmatrix} a \\ b \end{bmatrix} &= \sum_{m=0}^{p^2-1} \begin{bmatrix} a+mb \\ b \end{bmatrix} = p^2 \begin{bmatrix} a \\ b \end{bmatrix} + b \begin{bmatrix} \sum_{m=0}^{p^2-1} m \\ p^2 \end{bmatrix} \\ &= 0 + b \begin{bmatrix} \frac{(p^2-1)p^2}{2} \\ 0 \end{bmatrix} = 0, \end{aligned}$$

shows that $NA = \{0\}$. Thus $H^2_{\text{special}}\left(\mathbb{I}_{p^2}, (\mathbb{I}_p \times \mathbb{I}_p)^{\xi}\right) \cong \mathbb{I}_p \times \{0\}$. Finally we get

**Table 4.3.** *With non-trivial action*

| | | $H^2(G,A)$ | |
|---|---|---|---|
| $G$ | $A$ | $p \neq 2$ | $p = 2$ |
| $\mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | 0 | $\mathbb{I}_2$ |
| $\mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p$ | 0 |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_{p^2}$ | $\mathbb{I}_p$ | |
| $\mathbb{I}_{p^2}$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p$ | |

(2) $G = \mathbb{I}_m \times \mathbb{I}_n$, then by Theorem 1.51

$$H^2(G,A) \cong \frac{\ker\left(A^3 \overset{d_2^*}{\to} A^4\right)}{\operatorname{Im}\left(A^2 \overset{d_1^*}{\to} A^3\right)},$$

where

$$\begin{aligned} d_2^*\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= \begin{bmatrix} D_y a \\ D_x a - N_y b \\ N_x b + D_y c \\ D_x c \end{bmatrix}, \\ d_1^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= \begin{bmatrix} N_y a \\ D_x a - D_y b \\ N_x b \end{bmatrix}. \end{aligned}$$

(a) Trivial action, then for any $a \in A$

$$D_x a = \langle x \rangle a - a = a - a = 0 = D_y a$$

and

$$\begin{aligned} N_x a &= \sum_{i=1}^{m} \langle x \rangle^i a = ma, \\ N_y a &= na. \end{aligned}$$

Thus

$$d_2^*\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -nb \\ mb \\ 0 \end{bmatrix}$$

and

$$d_1^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} na \\ 0 \\ mb \end{bmatrix}.$$

So we have

$$\ker\left(A^3 \overset{d_2^*}{\to} A^4\right) = A \times {}_{[\operatorname{lcm}(m,n)]}A \times A$$

while

$$\mathrm{Im}\left(A^2 \overset{d_1^*}{\to} A^3\right) = nA \times 0 \times mA$$

showing that

$$\begin{aligned} H^2(G,A) &\cong \frac{A \times {}_{[\mathrm{lcm}(m,n)]}A \times A}{nA \times 0 \times mA} && (4)\\ &\cong \frac{A}{nA} \times {}_{[\mathrm{lcm}(m,n)]}A \times \frac{A}{mA}. && (5) \end{aligned}$$

In our case $m = n = p$, so (4) becomes

$$H^2(G,A) \cong \frac{A}{pA} \times {}_{[p]}A \times \frac{A}{pA} \qquad (6)$$

which when combined with Lagrange's Theorem gives

**Table 4.4.** *For any prime $p$ and and $G$ acting trivially on $A$:*

| $G$ | $A$ | $H^2(G,A)$ |
|---|---|---|
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$ |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | $\mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$ |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $(\mathbb{I}_p \times \mathbb{I}_p)^3$ |

(b) $G = \mathbb{I}_p \times \mathbb{I}_p$, $A = \mathbb{I}_{p^2}$ with action given by

$${}^{x^i y^j}a = (1+ip)\,a = a + ipa.$$

We have

$$\begin{aligned} D_x a &= {}^{x}a - a = pa, \\ N_x a &= \sum_{i=0}^{p-1} {}^{x^i}a = \sum_{i=0}^{p-1}(a+ipa) \\ &= pa + pa\sum_{i=0}^{p-1} i = pa + \frac{(p-1)\,p^2}{2}a \\ &= \begin{cases} pa+0 & p \geq 3 \\ 2a+2a, & p=2 \end{cases} = \begin{cases} pa & p \geq 3 \\ 0 & p=2 \end{cases}, \\ D_y a &= {}^{y}a - a = a - a = 0, \\ N_y a &= \sum_{i=0}^{p-1} {}^{y^i}a = pa. \end{aligned}$$

Hence

$$\begin{aligned} d_2^*\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= \begin{bmatrix} D_y a \\ D_x a - N_y b \\ N_x b + D_y c \\ D_x c \end{bmatrix} = \begin{bmatrix} 0 \\ pa - pb \\ \begin{cases} pb & p \geq 3 \\ 0 & p=2 \end{cases} + 0 \\ pc \end{bmatrix} \\ &= p\begin{bmatrix} 0 \\ a-b \\ \begin{cases} b & p \geq 3 \\ 0 & p=2 \end{cases} \\ c \end{bmatrix}, \end{aligned}$$

and therefore

$$\ker\left(d_2^* : A^3 \to A^4\right)) = \begin{cases} \left({}_{[p]}\mathbb{I}_{p^2}\right)^3 & p \geq 3 \\ \langle (p,p) \rangle \times \left({}_{[p]}\mathbb{I}_{p^2}\right) \subseteq \left({}_{[p]}\mathbb{I}_{p^2}\right)^3 & p = 2 \end{cases}.$$

Next, we calculate the image:

$$\begin{aligned} d_1^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= \begin{bmatrix} N_y a \\ D_x a - D_y b \\ N_x b \end{bmatrix} = \begin{bmatrix} pa \\ pa - 0 \\ \begin{cases} pb & p \geq 3 \\ 0 & p = 2 \end{cases} \end{bmatrix} \\ &= p \begin{bmatrix} a \\ a \\ \begin{cases} b & p \geq 3 \\ 0 & p = 2 \end{cases} \end{bmatrix}, \end{aligned}$$

which means

$$\operatorname{Im}\left(d_1^* : A^2 \to A^3\right) = \begin{cases} \langle (p,p) \rangle \times \left({}_{[p]}\mathbb{I}_{p^2}\right) \subseteq \left({}_{[p]}\mathbb{I}_{p^2}\right)^3, & p \geq 3 \\ \langle (p,p) \rangle \times \{0\} \subseteq \left({}_{[p]}\mathbb{I}_{p^2}\right)^3, & p = 2 \end{cases}.$$

Thus

$$\begin{aligned} H^2_{\text{spec}}\left(\mathbb{I}_p \times \mathbb{I}_p, \left(\mathbb{I}_{p^2}\right)^{\xi}\right) &= \begin{cases} \frac{\left({}_{[p]}\mathbb{I}_{p^2}\right)^3}{\langle (p,p) \rangle \times \left({}_{[p]}\mathbb{I}_{p^2}\right)} & p \geq 3 \\ \frac{\langle (p,p) \rangle \times \left({}_{[p]}\mathbb{I}_{p^2}\right)}{\langle (p,p) \rangle \times \{0\}} & p = 2 \end{cases} \\ &\cong \begin{cases} \frac{\left({}_{[p]}\mathbb{I}_{p^2}\right)^2}{\langle (p,p) \rangle} \times \{0\} & p \geq 3 \\ \left(\{0\}\right)^2 \times {}_{[p]}\mathbb{I}_{p^2} & p = 2 \end{cases} \\ &\cong \begin{cases} \frac{\left(\mathbb{I}_p\right)^2}{\langle (1,1) \rangle} & p \geq 3 \\ {}_{[p]}\mathbb{I}_{p^2} & p = 2 \end{cases} \cong \mathbb{I}_p. \end{aligned}$$

(c) $G = \mathbb{I}_p \times \mathbb{I}_p$, $A = \mathbb{I}_p \times \mathbb{I}_p$ with action given by

$${}^{x^i y^j}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + ib \\ b \end{bmatrix}.$$

This gives formulas

$$\begin{aligned} D_x\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= \begin{bmatrix} a+b \\ b \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \\ N_x\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= \sum_{i=0}^{p-1} \begin{bmatrix} a+ib \\ b \end{bmatrix} = p \begin{bmatrix} a \\ 0 \end{bmatrix} + b \sum_{i=0}^{p-1} \begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= 0 + b \begin{bmatrix} \frac{(p-1)p}{2} \\ p \end{bmatrix} = \begin{bmatrix} \frac{(p-1)p}{2} b \\ 0 \end{bmatrix} \\ &= \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & p \geq 3 \\ \\ \begin{bmatrix} b \\ 0 \end{bmatrix} & p = 2 \end{cases}, \\ D_y\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ N_y\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) &= p \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Thus, setting $a=(a_1,a_2)\,, b=(b_1,b_2)\,,$ and $c=(c_1,c_2)$ we calculate

$$\begin{aligned} d_2^*\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= \begin{bmatrix} D_y a \\ D_x a - N_y b \\ N_x b + D_y c \\ D_x c \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ (a_2,0) - 0 \\ \begin{cases} 0, & p \geq 3 \\ (b_2,0)\,, & p=2 \end{cases} \\ (c_2,0) \end{bmatrix} + 0 \\ &= \begin{bmatrix} 0 \\ (a_2,0) \\ \begin{cases} 0, & p \geq 3 \\ (b_2,0)\,, & p=2 \end{cases} \\ (c_2,0) \end{bmatrix}, \end{aligned}$$

so

$$\begin{aligned} &\ker\left(d_2^* : (\mathbb{I}_p \times \mathbb{I}_p)^3 \to (\mathbb{I}_p \times \mathbb{I}_p)^4\right) \\ = &\begin{cases} (\mathbb{I}_p \times \{0\}) \times (\mathbb{I}_p \times \mathbb{I}_p) \times (\mathbb{I}_p \times \{0\})\,, & p \geq 3 \\ (\mathbb{I}_p \times \{0\})^3\,, & p=2 \end{cases}. \end{aligned}$$

Next

$$d_1^*\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} N_y a \\ D_x a - D_y b \\ N_x b \end{bmatrix} = \begin{bmatrix} 0 \\ (a_2,0) \\ \begin{cases} 0, & p \geq 3 \\ (b_2,0)\,, & p=2 \end{cases} \end{bmatrix}$$

and hence

$$\begin{aligned} &\operatorname{Im}\left(d_1^* : (\mathbb{I}_p \times \mathbb{I}_p)^2 \to (\mathbb{I}_p \times \mathbb{I}_p)^3\right) \\ = &\begin{cases} (\{0\})^2 \times (\mathbb{I}_p \times \{0\}) \times (\{0\})^2\,, & p \geq 3 \\ (\{0\})^2 \times (\mathbb{I}_p \times \{0\})^2\,, & p=2 \end{cases}. \end{aligned}$$

Finally

$$\begin{aligned} &\frac{\ker\left(d_2^* : (\mathbb{I}_p \times \mathbb{I}_p)^3 \to (\mathbb{I}_p \times \mathbb{I}_p)^4\right)}{\operatorname{Im}\left(d_1^* : (\mathbb{I}_p \times \mathbb{I}_p)^2 \to (\mathbb{I}_p \times \mathbb{I}_p)^3\right)} \\ = &\begin{cases} (\mathbb{I}_p \times \{0\}) \times (\{0\} \times \mathbb{I}_p) \times (\mathbb{I}_p \times \{0\})\,, & p \geq 3 \\ (\mathbb{I}_p \times \{0\}) \times (\{0\})^2 \times (\{0\})^2\,, & p=2 \end{cases} \\ \cong &\begin{cases} (\mathbb{I}_p)^3\,, & p \geq 3 \\ \mathbb{I}_2, & p=2 \end{cases}. \end{aligned}$$

**Table 4.5.** *With non-trivial action*

| | | $H^2(G,A)$ | |
|---|---|---|---|
| $G$ | $A$ | $p \neq 2$ | $p=2$ |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_{p^2}$ | $\mathbb{I}_p$ | |
| $\mathbb{I}_p \times \mathbb{I}_p$ | $\mathbb{I}_p \times \mathbb{I}_p$ | $(\mathbb{I}_p)^3$ | $\mathbb{I}_2$ |

□

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## 5. Proof of Main Results, 2

5.1. **On determining extensions.** Here we explain how we use our main tools, Theorem 2.3 and Theorem 2.6, to determine the extensions which $H^2(G,A)$ classify. They give us generators and relations for the middle $E^s$ of a representative of the congruence class $[\varepsilon_s]$.

(1) If $G=\mathbb{I}_m$, then given an element

$$s\cdot NA\in H^2_{\text{special}}(G,A)=\frac{A^{\text{fix}}}{NA}$$

we know that the congruence class $[\varepsilon_s]$ it corresponds to will have a representative

$$1\to A\overset{\iota}{\to}E^s\overset{\pi}{\to}G\to 1$$

where every element of $E^s$ is of the form

$$a\{x\}^i, 0\leq i<m, a\in A$$

and maps given by

$$\begin{aligned}\iota &: a\mapsto a\\ \pi &: a\{x\}^i\mapsto x^i.\end{aligned}$$

Then if $S$ is a generating set of $A$, i.e. $A=\langle S:R\rangle$ for some $R\subseteq F(S)$, then $E^s$ is generated by $S\cup\{\{x\}\}$ subject to the relations

$$\begin{aligned}&R,\\ &\{x\}^m\in s\cdot NA,\\ &{}^xa=\{x\}a\{x\}^{-1}.\end{aligned}$$

Where the relation ${}^xa=\{x\}a\{x\}^{-1}$ follows from the fact that the extension is compatible with the action of $G$ on $A$.

(2) $G=\mathbb{I}_m\times\mathbb{I}_n$ , then given an element

$$s\cdot\operatorname{Im}\left(A^2\overset{d_1^*}{\to}A^3\right)\in H^2_{\text{special}}(G,A)=\frac{\ker\left(A^3\overset{d_2^*}{\to}A^4\right)}{\operatorname{Im}\left(A^2\overset{d_1^*}{\to}A^3\right)},$$

the class $[\varepsilon_s]$ it corresponds to will have a representative

$$1\to A\overset{\iota}{\to}E^s\overset{\pi}{\to}G\to 1$$

where every element of $E^s$ is of the form

$$a\{x\}^i\{x\}^j$$

for some $a\in A$, $0\leq i,<m, 0\leq j<n$. If $A=\langle S:R\rangle$, then $E^s$ will be generated by $S\cup\{\{x\},\{y\}\}$ subject to the relations

$$\begin{aligned}&R,\\ &\begin{bmatrix}U\\ V^{-1}\\ W\end{bmatrix}=\begin{bmatrix}\{y\}^n\\ \{x\}\{y\}\{x\}^{-1}\{y\}^{-1}\\ \{x\}^m\end{bmatrix}\in s\cdot\operatorname{Im}\left(A^2\overset{d_1^*}{\to}A^3\right),\\ &\{x\}a\{x\}^{-1}={}^xa,\\ &\{y\}a\{y\}^{-1}={}^ya,\forall a\in A,\end{aligned}$$

where of course last two relations follow from the fact that the extension is compatible with the action of $G$ on $A$.

So now, in either case, we have generators and relations for $E^s$. Using these generators and relations, we want to find out which of the groups in section B.2 our group $E^s$ corresponds to. We do this as follows: Suppose we have a candidate

$$E = \langle S : R \rangle, S = \{s_1, \ldots, s_k\}$$

from section B.2. If the candidate is any good, we should have

$$|E| = |E^s| = |A|\,|G|.$$

We find a subset $\{e_1, \ldots e_k\} \subseteq E^s$ that generates $E^s$ and satisfies the relations $R$ of $E$. The assignment

$$e_i \to s_i, i = 1, \ldots, k$$

where $s_i \in S$ is the element in $E$ that satisfies the same relations, induces an epimorphism

$$\psi : E^s \to E,$$

and hence

$$E \cong E^s / \ker(\psi).$$

Since $|E^s| = |E|$, and $\ker(\psi) \leq E^s$, it follows that $\ker \psi = \{1\}$, and so $\psi$ is an isomorphism.

**Remark 5.1.** *When $E^s$ is abelian, finding a candidate is usually easy because of the Fundamental Theorem of Finitely Abelian Groups* [DF04]. *Expression $E^s$ in terms of relation matrix one can algorithmically find the candidate $E$, see* [Vin03, Chapter 9.1].

After we find an isomorphism

$$\psi : E^s \to E$$

with our candidate $E$, we get a new representative of $[\varepsilon_s]$,

$$1 \to A \xrightarrow{\iota_s} E \xrightarrow{\pi_s} G \to 1$$

where the maps $\iota_s$ and $\pi_s$ are constructed so that the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\iota} & E^s & \xrightarrow{\pi} & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 1_G} & & \\
1 & \longrightarrow & A & \xrightarrow{\iota_s} & E & \xrightarrow{\pi_s} & G & \longrightarrow & 1
\end{array}$$

commutes. That is, we let

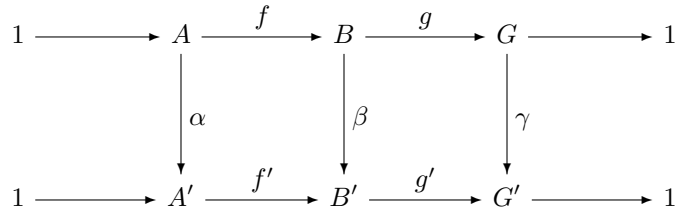
$$\begin{aligned} \iota_s &:= \psi \circ \iota, \\ \pi_s &:= \pi \circ \psi^{-1} \end{aligned}$$

for then $\iota_s = \psi \circ \iota$ by definition and for any $e \in E^s$

$$\begin{aligned} \pi(e) &= \pi\left(\left(\psi^{-1} \circ \psi\right)(e)\right) = \left(\pi \circ \psi^{-1}\right)(\psi(e)) \\ &= \pi_s(\psi(e)) = (\pi_s \circ \psi)(e) \end{aligned}$$

showing that $\pi = \pi_s \circ \psi$. Obviously the maps are homomorphisms, and exactness follows by Lemma 5.2 below.

**Lemma 5.2.** *Assume that the following diagram commutes, and the vertical arrows are isomorphisms.*




*Then the bottom row is exact if and only if the top row is exact.*

*Proof.* Diagram chase. □

That is the general procedure. The following rules for $E^s$ will be useful:

**Lemma 5.3.** *Let $G = \mathbb{I}_m = \langle x \rangle$, and $a \in A$.*

$$\left(a \{x\}^j\right)^k = \left(\prod_{d=0}^{k-1} {}^{x^{dj}}a\right) \{x\}^{kj}$$

*Proof.* (By Induction) Obviously holds in the cases $k = 0, 1$. Let $k > 0$ and assume it holds for $k-1$. Then

$$\left(a \{x\}^j\right)^k = \left(a \{x\}^j\right)^{k-1} \left(a \{x\}^j\right) = \left(\left(\prod_{d=0}^{k-2} {}^{x^{dj}}a\right) \{x\}^{(k-1)j}\right) \left(a \{x\}^j\right)$$

$$= \left(\prod_{d=0}^{k-2} {}^{x^{dj}}a\right) \left(\{x\}^{(k-1)j} a \{x\}^{-(k-1)j}\right) \{x\}^j \{x\}^{(k-1)j}$$

$$= \left(\prod_{d=0}^{k-2} {}^{x^{dj}}a\right) {}^{x^{(k-1)j}}a \{x\}^{kj} = \left(\prod_{d=0}^{k-1} {}^{x^{dj}}a\right) \{x\}^{kj}.$$

□

**Lemma 5.4.** *(Pascal's identity.) For $1 \leq k \leq n$*

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

**Lemma 5.5.** *For Let $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x \rangle \times \langle y \rangle$ act trivially on $A$. Let*

$$s = \begin{bmatrix} U \\ V^{-1} \\ W \end{bmatrix} \cdot \left[\mathrm{Im}\left(A^2 \longrightarrow A^3\right)\right] \in H^2(G, A)$$

*and*

$$1 \longrightarrow A \longrightarrow E^s \longrightarrow G \longrightarrow 1$$

*be the corresponding extension. Then for $m, n \in \mathbb{N}$, we have*

1. $\{x\}^{-1} \{y\}^m \{x\} = V^m \{y\}^m$,
2. $\{x\}^{-n} \{y\} \{x\}^n = V^n \{y\}$,
3. $\{x\}^{-n} \{y\}^m \{x\}^n = V^{mn} \{y\}^m$,
4. *For any $k \in \mathbb{N}_0$*

$$\left(\{x\}^n \{y\}^m\right)^k = V^{\binom{k}{2}mn} \{x\}^{kn} \{y\}^{km}$$

*so long as we define*

$$\binom{a}{b} = 0, a < b.$$

*The same formulas hold when the roles of $\{x\}$ and $\{y\}$ are interchanged, with $V$ becoming $V^{-1}$.*

*Proof.* Recall that

$$V = \{y\}\{x\}\{y\}^{-1}\{x\}^{-1} \in A,$$

and since the action of $G$ on $A$ is trivial

$$\begin{aligned} \{x\}^{-1}\{y\}\{x\} &= V\{y\} \\ \{y\}^{-1}\{x\}\{y\} &= V^{-1}\{x\}. \end{aligned}$$

Going through the list:

(1) $\{x\}^{-1}\{y\}^m\{x\} = \left(\{x\}^{-1}\{y\}\{x\}\right)^m = (V\{y\})^m = V^m\{y\}^m$.

(2) $\{x\}^{-n}\{y\}\{x\}^n = \{x\}^{-n+1}\left(\{x\}^{-1}\{y\}\{x\}\right)\{x\}^{n-1} = \{x\}^{-n+1}(V\{y\})\{x\}^{n-1}$
$= V\{x\}^{-n+1}\{y\}\{x\}^{n-1} = \cdots = V^n\{y\}$.

(3) $\{x\}^{-n}\{y\}^m\{x\}^n = \left(\{x\}^{-n}\{y\}\{x\}^n\right)^m = (V^n\{y\})^m = V^{mn}\{y\}^m$.

(4) (By induction) For $k = 1$

$$(\{x\}^n\{y\}^m)^1 = \{x\}^n\{y\}^m = V^{0\cdot mn}\{x\}^n\{y\}^m = V^{\binom{1}{2}mn}\{x\}^n\{y\}^m.$$

Inductive step: Let $k > 1$ and assume that the hypothesis holds for $k-1$, then

$$\begin{aligned} & (\{x\}^n\{y\}^m)^k = (\{x\}^n\{y\}^m)^{k-1}(\{x\}^n\{y\}^m) \\ = \ & \left(V^{\binom{k-1}{2}mn}\{x\}^{(k-1)n}\{y\}^{(k-1)m}\right)(\{x\}^n\{y\}^m) \\ = \ & V^{\binom{k-1}{2}mn}\{x\}^{kn}\left(\{x\}^{-n}\{y\}^{(k-1)m}\{x\}^n\right)\{y\}^m \\ = \ & V^{\binom{k-1}{2}mn}\{x\}^{kn}V^{(k-1)mn}\{y\}^{(k-1)m}\{y\}^m \\ = \ & V^{\left(\binom{k-1}{2}+\binom{k-1}{1}\right)mn}\{x\}^{kn}\{y\}^{km} \\ = \ & V^{\binom{k}{2}mn}\{x\}^{kn}\{y\}^{km}, \end{aligned}$$

where the final equality follows from Lemma 5.4 above.

□

5.2. **Proof of Theorem 2.16.**

*Proof.* By Theorem 2.8, the only case is $G = \mathbb{I}_p = \langle x\rangle$, and $A = \mathbb{I}_p = \langle z\rangle$ with trivial action. From Theorem 2.14 we have $H^2(G, A) \cong \mathbb{I}_p$. Let $s \in H^2_{\text{special}}(G, A) = \frac{A}{A^N} = \frac{A}{\{0\}} \cong \mathbb{I}_p$. We use the construction from Theorem 2.3 and follow the procedure described in Section 5.1 in order to determine the extensions.

(1) $s = 0$, the extension is split

$$\mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p.$$

(2) $s \neq 0$: A representative for $[\varepsilon_s]$ is given by

$$(\mathbb{I}_p = \langle z\rangle) \overset{\iota_s}{\rightarrowtail} E \overset{\pi_s}{\twoheadrightarrow} (\mathbb{I}_p = \langle x\rangle)$$

where $E$ is an abelian group generated by $z$ and $\{x\}$, subject to the relations $z^p = 1, \{x\}^p = z^s$. Since $z = \{x\}^{s'p}$, we see that $E = \langle\{x\}\rangle \cong \mathbb{I}_{p^2} = \langle P\rangle$. Hence our representative is congruent to

$$\begin{aligned} & \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} = \langle P\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \\ & \iota_s : z \mapsto P^{s'p} \\ & \pi_s : P \mapsto x \end{aligned}$$

□

5.3. **Proof of Theorem 2.18.**

*Proof.* By Theorem 2.8, the different combinations of $G$ and $A^{\xi}$ in List 2.19 are (up to weak equivalence) all that arise in connection with extensions

$$1 \to p^2 \to p^3 \to p \to 1.$$

For further explanation on how we determine the extensions, see Section 5.1.

(1) $G = \mathbb{I}_p, A = \left(\mathbb{I}_{p^2}\right)^{\text{triv}}$:
Let $s \in H^2(G, A) = A/pA = \mathbb{I}_{p^2}/p\mathbb{I}_{p^2} \cong \mathbb{I}_p$, where an isomorphism $\mathbb{I}_{p^2}/p\mathbb{I}_{p^2} \cong \mathbb{I}_p$ is given by

$$a + p\mathbb{I}_{p^2} \mapsto a \,(\mathrm{mod}\, p)\,.$$

Let $G$ be generated by $x$, and $A$ generated by $z$.
(a) $s = 0$: The extension is split

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p.$$

(b) $s \neq 0$: Then a representative of $[\varepsilon_s]$ is given by

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} E^s \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p$$

where

$$E^s = \left\langle z, \{x\} : z^{p^2} = 1, \{x\}^p = z^s, z\,\{x\} = \{x\}\,z \right\rangle.$$

Since $s \neq 0$, it has an inverse $s'$ modulo $p$. Hence

$$z = \{x\}^{ps'}$$

which shows that $E \cong \mathbb{I}_{p^3} = \langle P \rangle$, and so

$$\begin{array}{rcl} \gamma : E^s & \to & \left(\mathbb{I}_{p^3} = \langle P \rangle\right) \\ z & \mapsto & P^{ps'} \\ \{x\} & \mapsto & P \end{array}$$

is an isomorphism. Thus the extension is congruent to

$$\begin{array}{l} \mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \\ \iota_s : z \mapsto P^{ps'} \\ \pi_s : P \mapsto x \end{array}$$

(2) $G = \mathbb{I}_p, A = \left(\mathbb{I}_{p^2}\right)^{\xi}$:
Recall that the action $\xi$ is given by

$${}^{x}z = z^{1+p},$$

and that

$$H^2(G, A) = \left\{ \begin{array}{cccc} 2\mathbb{I}_4/\{0\} & \text{if} & p = 2 \\ p\mathbb{I}_{p^2}/p\mathbb{I}_{p^2} & \text{if} & p \neq 0 \end{array} \right. \cong \left\{ \begin{array}{cccc} \mathbb{I}_2 & \text{if} & p = 2 \\ \{0\} & \text{if} & p \neq 0 \end{array} \right. .$$

Let $s \in H^2(G, A)$:
(a) $s = 0$, $p$ is any prime: The extension is split

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \rtimes_{\xi} \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p$$

where $\mathbb{I}_{p^2} \rtimes_{\xi} \mathbb{I}_p$ is generated by $z, \{x\}$ subject to the relations

$$z^{p^2} = 1, \{x\}^p = 1, \{x\}\,z\,\{x\}^{-1} = z^{1+p}.$$

If we change generator $\{x\}$ to $\{x\}^{-1}$ we get

$$\left(\{x\}^{-1}\right)^{-1} z \{x\}^{-1} = \{x\} z \{x\}^{-1} = z^{1+p}$$

so $\mathbb{I}_{p^2} \rtimes_\xi \mathbb{I}_p \cong \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle$ via the map

$$\begin{aligned} z &\mapsto P, \\ \{x\} &\mapsto Q^{-1}. \end{aligned}$$

Thus, the extension is congruent to

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \\ &\iota : z \mapsto P \\ &\pi : P^i Q^j \mapsto Q^{-j} \end{aligned}$$

(b) $p = 2, s = 1$: A representative of $[\varepsilon_s]$ is given by

$$\left(\mathbb{I}_{p^2} = \langle z \rangle\right) \overset{\iota}{\rightarrowtail} E^s \overset{\pi}{\twoheadrightarrow} (\mathbb{I}_p = \langle x \rangle)$$

where $E^s$ has generators $z, \{x\}$ with relations

$$z^4 = 1, \{x\}^2 = z^2, \{x\}^{-1} z \{x\} = z^3,$$

which clearly is isomorphic to the group $\left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^2 \right\rangle$ via the map

$$\begin{aligned} z &\mapsto P, \\ \{x\} &\mapsto Q. \end{aligned}$$

Hence the extension is congruent to

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^2 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \\ &\iota : z \mapsto P \\ &\pi : P^i Q^j \mapsto x^j \end{aligned}$$

(3) $G = \mathbb{I}_p, A = (\mathbb{I}_p \times \mathbb{I}_p)^{\text{triv}}$:
From Table 2.15, an element $s \in H^2(G, A)$ is of the form

$$s = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{I}_p \times \mathbb{I}_p.$$

(a) $s = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: The extension is split

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p$$

(b) $u \neq 0$, then there is $u' \equiv u^{-1} \pmod{p}$. A representative for $[\varepsilon_s]$ is given by

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} E^s \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p$$

where $E^s$ is abelian, generated by $y, z, \{x\}$ with relations

$$\begin{aligned} y^p &= z^p = 1, \\ \{x\}^p &= y^u z^v. \end{aligned}$$

Taking the relation $\{x\}^p = y^u z^v$ and rasing it to the power $u'$ gives

$$\begin{aligned} \{x\}^{u'p} &= y z^{u'v} \\ y &= \{x\}^{u'p} z^{-u'v} \end{aligned}$$

showing that $\{x\}, z$ generate $E$. Hence $E \cong \mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle$ via the map

$$\begin{array}{rcl} \{x\} & \mapsto & P \\ z & \mapsto & Q, \end{array}$$

and the extension is congruent to

$$\begin{array}{l} \mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \\ \iota: \quad y \mapsto P^{u'p} Q^{-u'v} \\ \qquad z \mapsto Q \\ \pi: P^i Q^j \mapsto x^i \end{array}$$

(c) $u = 0, v \neq 0$, then there is $v' \equiv v^{-1} \pmod{p}$. A representative for $[\varepsilon_s]$ is given by

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p$$

where $E$ is abelian, generated by $y, z, \{x\}$ with relations

$$\begin{array}{rcl} y^p & = & z^p = 1, \\ \{x\}^p & = & z^v, \end{array}$$

showing that $z = \{x\}^{v'p}$. Thus $E = \langle \{x\}, y \rangle \cong \mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle$ via the map

$$\begin{array}{rcl} \{x\} & \mapsto & P, \\ y & \mapsto & Q, \end{array}$$

and the extension is

$$\begin{array}{l} \mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \\ \iota: \quad y \mapsto Q \\ \qquad z \mapsto P^{v'p} \\ \pi: P^i Q^j \mapsto x^i \end{array}$$

(4) $G = \mathbb{I}_p, A = (\mathbb{I}_p \times \mathbb{I}_p)^{\xi}$:
From Theorem 2.8 (in list 2.11) the action of $G$ on $A$ (in additive notation) is given by

$${}^x \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}$$

which corresponds to

$${}^x \left(y^i z^j\right) = y^{i+j} z^j$$

in multiplicative notation. From Table 2.15, we have

$$H^2_{\text{special}}(G, A) = \begin{cases} \frac{\mathbb{I}_2 \times \{0\}}{\mathbb{I}_2 \times \{0\}} & \text{if} \quad p = 2 \\ \frac{\mathbb{I}_p \times \{0\}}{\{0\}} & \text{if} \quad p \neq 2 \end{cases} \cong \begin{cases} \{0\} & \text{if} \quad p = 2 \\ \mathbb{I}_p & \text{if} \quad p \neq 2 \end{cases}.$$

Let $s \in H^2(G, A)$
(a) $p = 2$ : The only case is $s = 0$, so the extension is split

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} (\mathbb{I}_2 \times \mathbb{I}_2) \rtimes_{\xi} \mathbb{I}_2 \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2.$$

Let us determine $(\mathbb{I}_2 \times \mathbb{I}_2) \rtimes_\xi \mathbb{I}_2$. It has generators $y, z\{x\}$ with relations

$$\begin{aligned} y^2 &= z^2 = \{x\}^2 = 1, \\ \{x\} y \{x\}^{-1} &= y, \{x\} z \{x\}^{-1} = yz, \\ yz &= zy. \end{aligned}$$

The equality $\{x\} z \{x\}^{-1} = yz$ is equivalent to

$$\{x\}^{-1} z \{x\} = yz$$

since $\{x\}^{-1} = \{x\}$. Observe that $z\{x\}$ and $z$ generate $(\mathbb{I}_2 \times \mathbb{I}_2) \rtimes_\xi \mathbb{I}_2$ since

$$\begin{aligned} (z\{x\})^2 &= (z\{x\})(z\{x\}) = z^{\,x} z \{x\}^2 \\ &= z^{\,x} z = zyz = y \end{aligned}$$

which also shows that

$$|z\{x\}| = 4.$$

Next, the equality

$$z^{-1}(z\{x\}) z = \{x\} z = (z\{x\})^{-1} = (z\{x\})^3$$

shows that

$$\begin{aligned} P &\mapsto z\{x\} \\ Q &\mapsto z \end{aligned}$$

induces an isomorphism $\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \rangle \cong (\mathbb{I}_2 \times \mathbb{I}_2) \rtimes_\xi \mathbb{I}_2$. In terms of our original generators, this means

$$\begin{aligned} y &\mapsto P^2 \\ z &\mapsto Q \\ \{x\} &\mapsto P^3 Q. \end{aligned}$$

since

$$y = (z\{x\})^2 \mapsto P^2$$

and

$$P^3 Q = QP \mapsto z(z\{x\}) = \{x\}.$$

Hence the extension is congruent to

$$\begin{aligned} &\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2 \\ &\iota : \quad y \mapsto P^2 \\ &\qquad z \mapsto Q \\ &\pi : P^i Q^j \mapsto x^i \end{aligned}$$

(b) $p \neq 0, s = 0$ :The extension is split

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} (\mathbb{I}_p \times \mathbb{I}_p) \rtimes_\xi \mathbb{I}_p \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \to 1,$$

where $(\mathbb{I}_p \times \mathbb{I}_p) \rtimes_\xi \mathbb{I}_p$ is generated by $y, z, \{x\}$ with relations

$$\begin{aligned} y^p &= z^p = \{x\}^p = 1, \\ yz &= zy \\ \{x\} y \{x\}^{-1} &= y, \\ \{x\} z \{x\}^{-1} &= yz. \end{aligned}$$

Rewriting the relations to

$$\begin{aligned} y^p &= z^p = \{x\}^p = 1, z^{-1}\{x\}z = \{x\}y, \\ z^{-1}yz &= y, \{x\}^{-1}y\{x\} = y \end{aligned}$$

we see that

$$\begin{aligned} y &\mapsto P \\ z &\mapsto R \\ \{x\} &\mapsto Q \end{aligned}$$

induces an isomorphism

$$E^s \cong \left\langle P,Q,R: \begin{array}{c} P^p, Q^p, R^p, R^{-1}QR = QP, \\ R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle.$$

Hence our extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P,Q,R: \begin{array}{c} P^p, Q^p, R^p, R^{-1}QR = QP, \\ R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p$$

$\iota : y^i z^j \mapsto P^i R^j,$
$\pi : P^i Q^j R^k \mapsto x^j$

(c) $p \neq 0, s \neq 0$ : A representative for $[\varepsilon_s]$ is

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p,$$

where $E$ is generated by $y, z, \{x\}$ with relations

$$\begin{aligned} y^p &= z^p = 1, \\ \{x\}^p &= y^s, \\ yz &= zy \\ \{x\}y\{x\}^{-1} &= y, \\ \{x\}z\{x\}^{-1} &= yz. \end{aligned}$$

We claim that this group is isomorphic to

$$\left\langle P,Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle.$$

To see this, note that

$$\{x\}^p = y^s,$$

which means that $\{x\}$ has order $p^2$. We will let $\{x\}$ act as $P$ and $z^s$ act as $Q$. Then

$$\begin{aligned} z^{-s}\{x\}z^s &= z^{-s}\left(\{x\}z^s\{x\}^{-1}\right)\{x\} = z^{-s}\left(\{x\}z\{x\}^{-1}\right)^s\{x\} \\ &= z^{-s}(yz)^s\{x\} = y^s\{x\} = \{x\}^p\{x\} = \{x\}^{1+p}. \end{aligned}$$

Thus assigning $\{x\}$ to $P$ and $z^s$ to $Q$ does indeed yield an isomorphism. Hence our extension is congruent to

$$(\mathbb{I}_p \times \mathbb{I}_p = \langle y,z\rangle) \overset{\iota_s}{\rightarrowtail} \left\langle P,Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi_s}{\twoheadrightarrow} (\mathbb{I}_p = \langle x\rangle)$$

$\iota_s : y^i z^j \mapsto P^{is'p} Q^{js'}$
$\pi_s : P^i Q^j \mapsto x^i$

□

5.4. **Proof of Theorem 2.20.**

*Proof.* By Theorem 2.8 our combinations of $G$ and $A^\xi$ are all (up to weak equivalence) that arise in connection with extensions

$$1 \to p \to p^3 \to p^2 \to 1.$$

Throughout this proof we will use Table 2.15 and the approach to determining extensions from Section 5.1.

(1) $G = \mathbb{I}_{p^2}$, $A = (\mathbb{I}_p)^{\text{triv}}$:
Write $G = \langle x \rangle = \mathbb{I}_{p^2}$, and $A = \langle z \rangle = \mathbb{I}_p$, and let $s \in H^2(G, A) \cong \mathbb{I}_p$.
(a) $s = 0$ : The extension is split

$$\mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

(b) $s \neq 0$ : Then a representative of $[\varepsilon_s]$ is given by

$$\mathbb{I}_p \overset{\iota}{\rightarrowtail} E^s \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2}$$

where $E^s$ consists of elements $z, \{x\}$, with relations

$$z^p = 1, \{x\}^{p^2} = z^s, z\{x\} = \{x\} z,$$

and maps

$$\begin{aligned} \iota &: z \mapsto z, \\ \pi &: z^i \{x\}^j \mapsto x^j. \end{aligned}$$

Since $s \neq 0$ we know that there exists $s' \equiv s^{-1} \pmod{p}$, and hence

$$z = \{x\}^{s'p^2}$$

which shows that the assignment

$$\{x\} \mapsto P$$

(under which $z = \{x\}^{s'p^2} \mapsto P^{s'p^2}$) induces an isomorphism

$$E^s \cong \left(\mathbb{I}_{p^3} = \langle P \rangle\right).$$

Hence the extension is given by

$$\begin{aligned} &\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota_s : z \mapsto P^{s'p^2} \\ &\pi_s : P \mapsto x \end{aligned}$$

(2) $G = \mathbb{I}_p \times \mathbb{I}_p$, $A = (\mathbb{I}_p)^{\text{triv}}$:
Write $G = \langle x, y \rangle$, $A = \langle z \rangle$, and let $s = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \left[\text{Im}\left(A^2 \longrightarrow A^3\right)\right] \in H^2(G, A) \cong (\mathbb{I}_p)^3$.
(a) $s = 0$: The extension is split

$$\mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p \times \mathbb{I}_p$$

(b) $v = 0$: Then $\{x\}\{y\} = \{y\}\{x\}$, and so $E^s$ is abelian.
(i) $u \neq 0$, then

$$\begin{aligned} z &= \{y\}^{u'p} \\ \{x\}^p &= \{y\}^{wu'p} \end{aligned}$$

where $u' \equiv u^{-1} \pmod{p}$. Hence we see that $E = \left\langle \{x\} \{y\}^{-wu'}, \{y\} \right\rangle$, where

$$\begin{aligned} \left(\{x\} \{y\}^{-wu'}\right)^p &= \{x\}^p \{y\}^{-wu'p} \\ &= \{y\}^{wu'p} \{y\}^{-wu'p} = 1 \\ \{y\}^{p^2} &= 1, \end{aligned}$$

and hence

$$E^s \cong \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right)$$

via the assignment

$$\begin{aligned} \{y\} &\mapsto P \\ \{x\} \{y\}^{-wu'} &\mapsto Q. \end{aligned}$$

Since

$$\begin{aligned} \iota_s(z) &= \{y\}^{u'p}, \\ \pi_s\left(\{y\}^i \left(\{x\} \{y\}^{-wu'}\right)^j\right) &= x^j y^{i-wu'j} \end{aligned}$$

the extension is given by

$$\begin{aligned} &\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota_s : z \mapsto P^{u'p} \\ &\pi_s : P^i Q^j \mapsto x^j y^{i-wu'j} \end{aligned}$$

(ii) $u = 0, w \neq 0$, then we get relations

$$z^p, \{x\}^p = z^w, \{y\}^p.$$

Thus $E = \langle \{x\}, \{y\} \rangle \cong \mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle$, and the extension is given by

$$\begin{aligned} &\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota_s : z \mapsto P^{w'p} \\ &\pi_s : P^i Q^j \mapsto x^i y^j \end{aligned}$$

(c) $v \neq 0, p = 2$:
Then $v = 1$ is the only possible value.

(i) $u = w = 0$: We have relations

$$\begin{aligned} &z^2, \{x\}^2, \{y\}^2, \\ &\{x\} \{y\} \{x\}^{-1} \{y\}^{-1} = z \\ &\{x\} z = z \{x\}, z \{y\} = \{y\} z, \end{aligned}$$

from which we see

$$\begin{aligned} (\{x\} \{y\})^2 &= (\{x\} \{y\}) (\{x\} \{y\}) \\ &= (z \{y\} \{x\}) \{x\} \{y\} = z \\ (\{x\} \{y\})^3 &= z \{x\} \{y\} = \{y\} \{x\} \\ (\{x\} \{y\})^4 &= 1 \end{aligned}$$

and

$$\begin{aligned} \{x\}^{-1} (\{x\} \{y\}) \{x\} &= \{x\} (\{x\} \{y\}) \{x\} = \{x\}^2 \{y\} \{x\} \\ &= \{y\} \{x\} = (\{x\} \{y\})^3. \end{aligned}$$

Hence we have an isomorphism

$$\begin{array}{rcl} E^s & \to & \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \\ \{x\}\{y\} & \mapsto & P \\ \{x\} & \mapsto & Q, \end{array}$$

and the extension is congruent to

$$\begin{aligned} &\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota : z \mapsto P^2 \\ &\pi : P^i Q^j \mapsto x^{i+j} y^j \end{aligned}$$

(ii) $u = 1, w = 0$ : We have relations

$$\begin{aligned} &z^2, \{x\}^2, \{y\}^2 = z, \\ &\{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z \\ &\{x\} z = z\{x\}, z\{y\} = \{y\} z. \end{aligned}$$

Since $\{y\}^4 = z^2 = 1$, and

$$\begin{array}{rcl} \{x\}^{-1}\{y\}\{x\} & = & \{x\}\{y\}\{x\}^{-1} = \{y\} z \\ & = & \{y\}\{y\}^2 = \{y\}^3 \end{array}$$

we have an isomorphism

$$\begin{array}{rcl} E & \to & \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \\ \{y\} & \mapsto & P \\ \{x\} & \mapsto & Q. \end{array}$$

Hence our extension is congruent to

$$\begin{aligned} &\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota : z \mapsto P^2 \\ &\pi : P^i Q^j \mapsto x^j y^i \end{aligned}$$

(iii) $u = 0, w = 1$ : We have relations

$$\begin{aligned} &z^2, \{x\}^2 = z, \{y\}^2, \\ &\{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z \\ &\{x\} z = z\{x\}, z\{y\} = \{y\} z. \end{aligned}$$

so $\{x\}^4 = 1$,and

$$\{y\}^{-1}\{x\}\{y\} = \{x\} z = \{x\}^3.$$

Thus

$$\begin{array}{rcl} E & \to & \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \\ \{x\} & \mapsto & P \\ \{y\} & \mapsto & Q, \end{array}$$

is an isomorphism, and

$$\begin{aligned} &\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota : z \mapsto P^2 \\ &\pi : P^i Q^j \mapsto x^i y^j \end{aligned}$$

(iv) $u = v = 1$ : We have relations

$$\begin{aligned} &z^2, \{x\}^2 = z, \{y\}^2 = z, \\ &\{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z \\ &\{x\} z = z\{x\}, z\{y\} = \{y\} z. \end{aligned}$$

The middle equation is equivalent to

$$\{y\}^{-1}\{x\}\{y\} = \{y\}^{-1} z \{y\}\{x\} = z\{x\} = \{x\}^3 = \{x\}^{-1}$$

and hence the assignment

$$\begin{aligned} \gamma : E &\to \left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^2 \right\rangle \\ \{x\} &\mapsto P \\ \{y\} &\mapsto Q \end{aligned}$$

is an isomorphism, and our extension is congruent to

$$\begin{aligned} &\mathbb{I}_p \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^2 \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota : z \mapsto P^2 \\ &\pi : P^i Q^j \mapsto x^i y^j \end{aligned}$$

(d) $v \neq 0, p \neq 2$:

(i) If $u = w = 0$, then we get relations

$$\begin{aligned} &z^p, \{x\}^p, \{y\}^p, \\ &\{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z^v \\ &\{x\} z = z\{x\}, z\{y\} = \{y\} z. \end{aligned}$$

Since $E$ is central (Definition 1.29), the middle equation yields

$$\begin{aligned} \{x\}^{-1}\{y\}\{x\} &= z^{-v}\{y\}, \\ \{y\}^{-1}\{x\}\{y\} &= z^{v}\{x\}. \end{aligned}$$

From this, we see that

$$\begin{aligned} \gamma : E &\to \left\langle \begin{array}{ll} P, Q, R : & P^p, Q^p, R^p, R^{-1}QR = QP, \\ & R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle \\ z &\mapsto P^{v'} \\ \{x\} &\mapsto Q \\ \{y\} &\mapsto R, \end{aligned}$$

where $v' \equiv v^{-1} \pmod{p}$, is an isomorphism. Hence the extension is given by

$$\begin{aligned} &\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left\langle \begin{array}{ll} P, Q, R : & P^p, Q^p, R^p, R^{-1}QR = QP, \\ & R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota_s : z \mapsto P^{v'} \\ &\pi_s : P^i Q^j R^k \mapsto x^j y^k \end{aligned}$$

(ii) $u \neq 0$ : We get the relations

$$\begin{aligned} &z^p, \{x\}^p = z^w, \{y\}^p = z^u, \\ &\{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z^v \\ &\{x\} z = z\{x\}, z\{y\} = \{y\} z. \end{aligned}$$

From which we see that $\{y\}^p = z^u$ generate $\langle z \rangle$, since $z = \{y\}^{u'p}$. So we have

$$\{y\}^{p^2} = 1, \{x\}^p = z^w,$$

and we claim that

$$E_s \cong \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle.$$

Observe that

$$\{x\}\{y\}\{x\}^{-1} = z^v\{y\} = \{y\}^{vu'p}\{y\},$$

i.e.

$$\left(\{x\}^{-1}\right)^{-1}\{y\}\left(\{x\}^{-1}\right) = \{y\}^{vu'p}\{y\},$$

and

$$\{x\}^{v'u}\{y\}\{x\}^{-v'u} = (z^v)^{v'u}\{y\} = \{y\}^{1+p},$$

so

$$\left(\{x\}^{-v'u}\right)^{-1}\{y\}\left(\{x\}^{-v'u}\right) = \{y\}^{1+p}.$$

Thus

$$\left(\{x\}^{-v'u}\{y\}^m\right)^{-1}\{y\}\left(\{x\}^{-v'u}\{y\}^m\right) = \{y\}^{1+p}.$$

We want $\{x\}^{-v'u}\{y\}^m$ to have the role of $Q$, so we need it to have order $p$. By Lemma 5.5

$$\begin{aligned}\left(\{x\}^{-v'u}\{y\}^m\right)^p &= z^{-\binom{p}{2}m(-v'u)v}\{x\}^{-v'up}\{y\}^{mp} = z^{\binom{p}{2}mu}\{x\}^{-v'up}\{y\}^{mp} \\ &= \{x\}^{-v'up}\{y\}^{mp} = z^{-v'uw}z^{mu} = z^{u(-v'w+m)}\end{aligned}$$

since $\binom{p}{2} \equiv 0 \pmod p$ when $p$ is odd. So if we set $m = v'w$, then we get

$$\left(\{x\}^{v'u}\{y\}^m\right)^p = 1$$

as desired. Hence the assignment

$$\begin{aligned}P &\mapsto \{y\} \\ Q &\mapsto \{x\}^{-v'u}\{y\}^{v'w}\end{aligned}$$

defines an isomorphism. Finally the extension is congruent to

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : z \mapsto P^{u'p}$$

$$\pi_s : P^iQ^j \mapsto x^{-jv'u}y^{i+jv'w}$$

(iii) $u = 0, w \neq 0$, then we get the relations

$$\begin{aligned}&z^p, \{x\}^p = z^w, \{y\}^p, \\ &\{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z^v \\ &\{x\}z = z\{x\}, z\{y\} = \{y\}z.\end{aligned}$$

We observe that $\{x\}$ generate $\langle z \rangle$ as

$$\{x\}^{pw'} = z,$$

where $w' \equiv w^{-1} \pmod p$. Again we claim that

$$E_s \cong \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle,$$

and this time we shall let $\{x\}$ take on the role of $P$. Observe that

$$\{y\}^{-1}\{x\}\{y\} = z^v\{x\} = \{x\}^{vw'p}\{x\}$$

and hence

$$\{y\}^{-v'w}\{x\}\{y\}^{v'w} = (z^v)^{v'w}\{x\} = z^w\{x\} = \{x\}^{1+p}.$$

Therefore the isomorphism is given by

$$\begin{aligned} P &\mapsto \{x\} \\ Q &\mapsto \{y\}^{v'w}, \end{aligned}$$

and so the extension is congruent to

$$\mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : z \mapsto P^{w'p}$$

$$\pi_s : P^iQ^j \mapsto x^i y^{jv'w}$$

□

5.5. **Proof of Theorem 2.22.**

*Proof.* By Theorem 2.8, the combination of $A$, $G$, and action $\varphi$ in the lists are (up to weak equivalence) all that arise in connection with extensions $p^2 \to p^4 \to p^2$.

(1) The contents of List 2.23 are extensions of $\mathbb{I}_{p^2}$ by $\mathbb{I}_{p^2}$. The case when the action is trivial follow from Lemma 5.7. The case with non-trivial action follow from Lemma 5.10.
(2) In List 2.24, extensions of $(\mathbb{I}_p \times \mathbb{I}_p)^{\text{triv}}$ and $(\mathbb{I}_p \times \mathbb{I}_p)^{\xi}$ by $\mathbb{I}_{p^2}$ are those in Lemma 5.12 and Lemma 5.14, respectively.
(3) Extensions of $\mathbb{I}_{p^2}$ by $\mathbb{I}_p \times \mathbb{I}_p$ in which the action is trivial come from Lemma 5.16, and those in which the action is non-trivial come from Lemma 5.19.
(4) The extensions of $\mathbb{I}_p \times \mathbb{I}_p$ of $\mathbb{I}_p \times \mathbb{I}_p$ are covered in Lemma 5.21 and Lemma 5.22.

□

5.5.1. *Extensions of $\mathbb{I}_{p^2}$ by $\mathbb{I}_{p^2}$.*

**Remark 5.6.** *Write $G = \langle x \rangle$ and $A = \langle z \rangle$.*

Trivial action.

**Lemma 5.7.** *Below are all the congruence classes of extensions $A = \mathbb{I}_{p^2}$ by $G = \mathbb{I}_{p^2}$, where $G$ acts trivially on $A$. Let $s \in H^2(G, A) \cong \mathbb{I}_{p^2}$*

(1) $s = 0$ :

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

(2) $s \in \left(\mathbb{I}_{p^2}\right)^*$ :

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^4} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2}$$

$$\iota_s : z \mapsto P^{p^2}$$

$$\pi_s : P^i \mapsto x^{is'}$$

(3) $s = rp, 1 \le r < p$ :

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2}$$

$$\iota_s : z \mapsto P^{r'p}Q$$

$$\pi_s : P^iQ^j \mapsto x^{i-jr'p}$$

*Proof.* From Theorem 2.14 we know that $H^2(G,A) \cong \mathbb{I}_{p^2}$. Let $s \in \mathbb{I}_{p^2}$ and note that by Theorem 2.3 a representative $E^s$ of the equivalence class $[\varepsilon_s]$ has generators $z$ and $\{x\}$ subject to the relations

$$\begin{aligned} z^{p^2} = 1, \{x\}^{p^2} = z^s, \\ \{x\} z \{x\}^{-1} = z. \end{aligned}$$

(1) $s = 0$: The extension is split

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

(2) $s \in \left(\mathbb{I}_{p^2}\right)^*$: By assumption $s$ has an inverse $s' \pmod p$, so

$$z = \left(\{x\}^{p^2}\right)^{s'} = \{x\}^{s'p^2}$$

which means that $\{x\}$ generates $E^s$. The equation

$$\left(\{x\}^{s'}\right)^{p^4} = \left(\{x\}^{s'p^2}\right)^{p^2} = z^{p^2} = 1$$

shows that the assignment

$$\{x\}^{s'} \mapsto P$$

induces an isomorphism

$$E^s = \left\langle \{x\}^{s'} \right\rangle \cong \mathbb{I}_{p^4} = \langle P \rangle .$$

Since

$$z = \{x\}^{s'p^2} \mapsto P^{p^2}$$

we see that our extension is congruent to

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^4} = \langle P \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota_s : z \mapsto P^{p^2} \\ &\pi_s : P^i \mapsto x^{is'} \end{aligned}$$

(3) $s \in p\mathbb{I}_{p^2} \setminus \{0\}$: Then $s = rp$ for some $1 \leq r < p$, and we have

$$\{x\}^{p^2} = z^{rp}$$

which implies that

$$\{x\}^{p^3} = \left(\{x\}^{p^2}\right)^p = (z^{rp})^p = 1$$

and

$$z^p = \{x\}^{r'p^2} .$$

We see that $\{x\}$ and $z\{x\}^{-r'p}$ generate $E^s$, and that

$$\left(z\{x\}^{-r'p}\right)^p = z^p \{x\}^{-r'p^2} = \left(\{x\}^{r'p^2}\right) \{x\}^{-r'p^2} = 1.$$

Hence the assignment

$$\begin{aligned} \{x\} &\mapsto P \\ z\{x\}^{-r'p} &\mapsto Q \end{aligned}$$

induces an isomorphism

$$E^s = \left\langle \{x\}, z\{x\}^{-r'p} \right\rangle \cong \mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle .$$

Since

$$z = \left(\{x\}^{r'p} \{x\}^{-r'p}\right) z = \{x\}^{r'p} \left(z\{x\}^{-r'p}\right) \mapsto P^{r'p} Q$$

and

$$\{x\}^i \left(z \{x\}^{-r'p}\right)^j = z^j \{x\}^{i-jr'p}$$

we see that our extension is congruent to

$$\begin{aligned}
&\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_{p^2} \\
&\iota_s : z \mapsto P^{r'p} Q \\
&\pi_s : P^i Q^j \mapsto x^{i-jr'p}
\end{aligned}$$

□

Non-trivial action.

**Remark 5.8.** *When $A$ is written additively, action on $G$ on $A$ is given by*

$$^{x^i} a = (1 + ip)\, a.$$

*In multiplicative notation this corresponds to*

$$^{x^i} a = a^{1+ip}.$$

*Since both are cyclic, it is enough to specify what the generator $x$ of $G$ does to $z$ of $A$:*

$$^{x} z = z^{1+p}.$$

**Lemma 5.9.** *Let $a \in A$, then*

$$\left(a \{x\}^j\right)^k = a^{k + \frac{k(k-1)}{2} jp} \{x\}^{kj}, k \geq 0.$$

*Proof.* By Lemma 5.3

$$\begin{aligned}
&\left(a \{x\}^j\right)^k = \left(\prod_{d=0}^{k-1} {}^{x^{dj}} a\right) \{x\}^{kj} = \left(\prod_{d=0}^{k-1} a^{1+djp}\right) \{x\}^{kj} \\
= \;& a^{\sum_{d=0}^{k-1}(1+djp)} \{x\}^{kj} = a^{k+jp\sum_{d=0}^{k-1} d} \{x\}^{kj} = a^{k+jp\frac{k(k-1)}{2}} \\
= \;& a^{k\left(1+\frac{k-1}{2}jp\right)} \{x\}^{kj}.
\end{aligned}$$

□

**Lemma 5.10.** *Below are all the congruence classes of extensions $A = \mathbb{I}_{p^2}$ by $G = \mathbb{I}_{p^2}$, where $G$ acts non-trivially on $A$.*
*Let $s \in H^2(G, A) \cong \mathbb{I}_p$*

(1) *$s = 0$:*

$$\begin{aligned}
&\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^{p^2}, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\
&\iota : z \mapsto P \\
&P^i Q^j \mapsto x^{-j}
\end{aligned}$$

(2) *$s \neq 0$:*

$$\begin{aligned}
&\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^{p}, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\
&\iota_s : z \mapsto P^{r'p} Q^{r'} \\
&\pi_s : P^i Q^j \mapsto x^i
\end{aligned}$$

*Proof.* From Theorem 2.14 we know that $H^2(G, A) \cong {}_{[p]}\mathbb{I}_{p^2} \cong \mathbb{I}_p$. Let $s \in \mathbb{I}_p$ and note that by Theorem 2.3 a representative $E^s$ of the equivalence class $[\varepsilon_s]$ has generators $z$ and $\{x\}$ subject to the relations

$$\begin{aligned}
&z^{p^2} = 1, \{x\}^{p^2} = z^{sp}, \\
&^{x} z = \{x\} z \{x\}^{-1} = z^{1+p}.
\end{aligned}$$

(1) $s = 0$: This is the split extension

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \rtimes_\xi \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

where

$$\mathbb{I}_{p^2} \rtimes_\xi \mathbb{I}_{p^2} = \left\langle z, \{x\} : z^{p^2}, \{x\}^{p^2}, \{x\}\, z\, \{x\}^{-1} = z^{1+p} \right\rangle.$$

Clearly

$$\mathbb{I}_{p^2} \rtimes_\xi \mathbb{I}_{p^2} \cong \left\langle P, Q : P^{p^2}, Q^{p^2}, Q^{-1}PQ = P^{1+p} \right\rangle$$

via the assignment

$$\begin{aligned} z &\mapsto P \\ \{x\}^{-1} &\mapsto Q. \end{aligned}$$

Since

$$\{x\} = \left(\{x\}^{-1}\right)^{-1} \mapsto Q^{-1}$$

we see that the extension is congruent to

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P, Q : P^{p^2}, Q^{p^2}, Q^{-1}PQ = P^{1+p} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota : z \mapsto P \\ &P^i Q^j \mapsto x^{-j} \end{aligned}$$

(2) $s \neq 0$: Then

$$E^s = \left\langle z, \{x\} : z^{p^2}, \{x\}^{p^2} = z^{sp}, \{x\}\, z\, \{x\}^{-1} = z^{1+p} \right\rangle.$$

We claim that the assignment

$$\begin{aligned} \{x\} &\mapsto P \\ z^s \{x\}^{-p} &\mapsto Q \end{aligned}$$

induces an isomorphism

$$E^s \cong \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle.$$

The equation

$$\{x\}^{p^3} = \left(\{x\}^{p^2}\right)^p = (z^{sp})^p = \left(z^{p^2}\right)^s = 1$$

shows that $|\{x\}| = p^3$. By Lemma 5.9

$$\begin{aligned} \left(z^s \{x\}^{-p}\right)^p &= (z^s)^{p + \frac{p(p-1)}{2} p(-p)} \{x\}^{p(-p)} \\ &= z^{sp} \{x\}^{-p^2} = \{x\}^{p^2} \{x\}^{-p^2} = 1, \end{aligned}$$

so $\left|z^s \{x\}^{-p}\right| = p$, as desired. All that remains is to check $Q^{-1}PQ = P^{1+p^2}$, and to do that we need the following:

$$\begin{aligned} z^{-1} \{x\} z &= z^{-1} \left(\{x\}\, z\, \{x\}^{-1}\right) \{x\} = z^{-1} \left(z^{1+p}\right) \{x\} = z^p \{x\} \\ z^{-2} \{x\} z^2 &= z^{-1} \left(z^p \{x\}\right) z = z^p \left(z^{-1} \{x\} z\right) = z^{2p} \{x\} \\ &\vdots \\ z^{-n} \{x\} z^n &= z^{np} \{x\}. \end{aligned}$$

Hence

$$\begin{aligned}
&\left(z^s\{x\}^{-p}\right)^{-1}\{x\}\left(z^s\{x\}^{-p}\right) = \{x\}^p\left(z^{-s}\{x\}z^s\right)\{x\}^{-p}\\
=\ &\{x\}^p\left(z^{sp}\{x\}\right)\{x\}^{-p} = \{x\}^p z^{sp}\{x\}^{1-p}\\
=\ &\{x\}^p\{x\}^{p^2}\{x\}^{1-p} = \{x\}^{1+p^2}
\end{aligned}$$

as was to be shown. We have

$$z = (z^s)^{s'} = \left(z^s\left(\{x\}^{-p}\{x\}^p\right)\right)^{s'}$$

$$= \left(\left(z^r\{x\}^{-p}\right)\{x\}^p\right)^{r'} \mapsto (QP^p)^{r'}.$$

We want to write the image of $z$; $(QP^p)^{s'}$ in the form $P^iQ^j$. To do that we need some formulas for $\left\langle P,Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2}\right\rangle$. We have

$$Q^{-1}PQ = P^{1+p^2} \Rightarrow QPQ^{-1} = P^{1-p^2}$$

so

$$QP = P^{1-p^2}Q$$

which by induction gives

$$QP^p = P^{p(1-p^2)}Q = P^pQ.$$

Thus

$$z \mapsto (QP^p)^{s'} = (QP^p)^{s'} = P^{s'p}Q^{s'}$$

and since

$$\pi_s\left(\{x\}^i\left(z^s\{x\}^{-p}\right)^j\right) = x^{i-jp}$$

the extension is congruent to

$$\begin{aligned}
&\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P,Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2}\right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2}\\
&\iota_s : z \mapsto P^{s'p}Q^{s'}\\
&\pi_s : P^iQ^j \mapsto x^{i-jp}
\end{aligned}$$

□

5.5.2. *Extensions of $\mathbb{I}_p \times \mathbb{I}_p$ by $\mathbb{I}_{p^2}$.*

**Remark 5.11.** *Write $G = \mathbb{I}_{p^2} = \langle x\rangle$ and $A = \mathbb{I}_p \times \mathbb{I}_p = \langle z, Z\rangle$.*

Trivial action.

**Lemma 5.12.** *Below are all the congruence classes of extensions $A = \mathbb{I}_p \times \mathbb{I}_p$ by $G = \mathbb{I}_{p^2}$, where $G$ acts trivially on $A$. Let $s = \begin{bmatrix} u \\ v \end{bmatrix} \in H^2(G, A) \cong \mathbb{I}_p \times \mathbb{I}_p$*

(1) *$s = 0$:*

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

(2) *$u \neq 0$:*

$$\begin{aligned}
&\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P,Q\rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2}\\
&\iota_s : \begin{array}{l} z \mapsto P^{u'p^2}Q^{-u'v}\\ Z \mapsto Q \end{array}\\
&\pi_s : P^iQ^j \mapsto x^i
\end{aligned}$$

(3) $u = 0, v \neq 0$:

$$\begin{aligned}
&\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P, Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\
&\iota_s : \begin{array}{l} z \mapsto Q \\ Z \mapsto P^{v' p^2} \end{array} \\
&\pi_s : P^i Q^j \mapsto x^i
\end{aligned}$$

*Proof.* From Theorem 2.14 we know that $H^2(G, A) \cong \mathbb{I}_p \times \mathbb{I}_p$. Let

$$s = \begin{bmatrix} u \\ v \end{bmatrix} \in H^2(G, A) \cong \mathbb{I}_p \times \mathbb{I}_p$$

and note that by Theorem 2.3 a representative $E^s$ of the equivalence class $[\varepsilon_s]$ has generators $z$,$Z$ and $\{x\}$ subject to the relations

$$\begin{aligned}
&z^p, Z^p, \{x\}^{p^2} = z^u Z^v, \\
&{}^x z = \{x\} z \{x\}^{-1} = z, \\
&{}^x Z = \{x\} Z \{x\}^{-1} = Z, \\
&z Z z^{-1} = Z.
\end{aligned}$$

From the relations we see that for any $s$, $E^s$ will be abelian.

(1) $s = 0$: The extension is split

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

(2) $u \neq 0$: Then

$$\{x\}^{p^2} = z^u Z^v$$

implies that

$$z = \left(Z^{-v} \{x\}^{p^2}\right)^{u'} = Z^{-u'v} \{x\}^{u' p^2},$$

so $Z$ and $\{x\}$ generate $E^s$. We claim that

$$\begin{aligned}
\{x\} &\mapsto P \\
Z &\mapsto Q
\end{aligned}$$

induces an isomorphism

$$E^s = \langle \{x\}, Z \rangle \cong \mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P, Q \rangle.$$

The equations

$$\begin{aligned}
\{x\}^{p^3} &= \left(\{x\}^{p^2}\right)^p = (z^u Z^v)^p = 1, \\
Z^p &= 1, \\
\{x\} Z &= Z \{x\}
\end{aligned}$$

shows that this is indeed the case. Since

$$z = Z^{-u'v} \{x\}^{u' p^2} = \{x\}^{u' p^2} Z^{-u'v} \mapsto P^{u' p^2} Q^{-u'v}$$

and

$$\pi_s\left(\{x\}^i Z^j\right) = x^i$$

the extension is congruent to

$$\begin{aligned}
&\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P, Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\
&\iota_s : \begin{array}{l} z \mapsto P^{u' p^2} Q^{-u'v} \\ Z \mapsto Q \end{array} \\
&\pi_s : P^i Q^j \mapsto x^i
\end{aligned}$$

(3) $u = 0, v \neq 0$: Then

$$\{x\}^{p^2} = Z^v$$

implies that

$$Z = \{x\}^{v'p^2},$$

so $\{x\}$ and $z$ generate $E^s$. We claim that

$$\begin{aligned} \{x\} &\mapsto P \\ z &\mapsto Q \end{aligned}$$

induces an isomorphism

$$E^s = \langle \{x\}, z \rangle \cong \mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P, Q \rangle .$$

Indeed,

$$\begin{aligned} \{x\}^{p^3} &= \left(\{x\}^{p^2}\right)^p = (Z^v)^p = 1, \\ z^p &= 1, \\ \{x\} z &= z \{x\} \end{aligned}$$

shows this statement to be true. Hence the extension is congruent to

$$\begin{aligned} &\mathbb{I}_p \times \mathbb{I}_p \overset{\iota}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P, Q \rangle\right) \overset{\pi}{\twoheadrightarrow} \mathbb{I}_{p^2} \\ &\iota_s : \begin{array}{l} z \mapsto Q \\ Z \mapsto P^{v'p^2} \end{array} \\ &\pi_s : P^i Q^j \mapsto x^i \end{aligned}$$

□

Non-trivial action.

**Remark 5.13.** *When $A$ is written additively, action on $G$ on $A$ is given by*

$$^x \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}.$$

*In multiplicative notation this becomes*

$$^x \left(z^i Z^j\right) = z^{i+j} Z^j$$

*which is equivalent to*

$$\begin{aligned} {}^x z &= z, \\ {}^x Z &= zZ. \end{aligned}$$

**Lemma 5.14.** *Below are all the congruence classes of extensions $A = \mathbb{I}_p \times \mathbb{I}_p$ by $G = \mathbb{I}_{p^2}$, where $G$ acts non-trivially on $A$.*
*Let $s \in H^2(G, A) \cong \mathbb{I}_p$*

(1) $s = 0$*:*

$$\begin{aligned} &\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}PR = PQ, \\ Q^{-1}PQ = P, R^{-1}QR = Q \end{array} \right\rangle \twoheadrightarrow \mathbb{I}_{p^2} \\ &\iota : \begin{array}{l} z \mapsto Q \\ Z \mapsto R \end{array} \\ &\pi : P^i Q^j R^k \mapsto x^i \end{aligned}$$

(2) $s \neq 0$:

$$\begin{aligned}
&\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \twoheadrightarrow \mathbb{I}_{p^2} \\
&\iota_s : \begin{array}{l} z \mapsto P^{s'p^2} \\ Z \mapsto P^{-s'p^2}Q^{s'} \end{array} \\
&\pi_s : P^iQ^j \mapsto x^i
\end{aligned}$$

*Proof.* From Theorem 2.14 we know that

$$H^2(G, A) \cong \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \in \mathbb{I}_p \times \mathbb{I}_p \right\} = \mathbb{I}_p \times \{0\} \cong \mathbb{I}_p.$$

Let $s \in \mathbb{I}_p$, and note that by Theorem 2.3 a representative $E^s$ of the equivalence class $[\varepsilon_s]$ has generators $z$,$Z$ and $\{x\}$ subject to the relations

$$\begin{aligned}
&z^p, Z^p, \{x\}^{p^2} = z^s, \\
&{}^x z = \{x\} z \{x\}^{-1} = z \\
&{}^x Z = \{x\} Z \{x\}^{-1} = zZ, \\
&zZz^{-1} = Z.
\end{aligned}$$

Since $\{x\} Z \{x\}^{-1} = zZ$, we see that none of the representatives are going to be abelian.

(1) $s = 0$: The extension is split

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail (\mathbb{I}_p \times \mathbb{I}_p) \rtimes_\xi \mathbb{I}_{p^2} \twoheadrightarrow \mathbb{I}_{p^2}$$

We claim that

$$\begin{aligned}
z &\mapsto Q \\
Z &\mapsto R \\
\{x\} &\mapsto P
\end{aligned}$$

induces an isomorphism

$$(\mathbb{I}_p \times \mathbb{I}_p) \rtimes_\xi \mathbb{I}_{p^2} \cong \left\langle \begin{array}{ll} P, Q, R : & P^{p^2}, Q^p, R^p, R^{-1}PR = PQ, \\ & Q^{-1}PQ = P, R^{-1}QR = Q \end{array} \right\rangle.$$

Since

$$\begin{aligned}
\{x\}^{p^2} &= z^0 = 1, \\
z^p &= 1, \\
Z^p &= 1
\end{aligned}$$

we see that the orders are correct. Next, the equations

$$\begin{aligned}
R^{-1}PR &= Z^{-1}\{x\}Z = Z^{-1}\{x\}Z\left(\{x\}^{-1}\{x\}\right) \\
&= Z^{-1}\left(\{x\}Z\{x\}^{-1}\right)\{x\} = Z^{-1}(zZ)\{x\} \\
&= z\{x\} = \{x\}z = PQ, \\
Q^{-1}PQ &= z^{-1}\{x\}z = z^{-1}\{x\}z\left(\{x\}^{-1}\{x\}\right) \\
&= z^{-1}\left(\{x\}z\{x\}^{-1}\right)\{x\} = z^{-1}z\{x\} \\
&= \{x\} = P, \\
R^{-1}QR &= Z^{-1}zZ = Z^{-1}zZ\left(z^{-1}z\right) = Z^{-1}\left(zZz^{-1}\right)z \\
&= Z^{-1}Zz = z = Q,
\end{aligned}$$

verifies the remaining relations. Hence the extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \left\langle P,Q,R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}PR = PQ, \\ Q^{-1}PQ = P, R^{-1}QR = Q \end{array} \right\rangle \twoheadrightarrow \mathbb{I}_{p^2}$$

$$\iota : \begin{array}{l} z \mapsto Q \\ Z \mapsto R \end{array}$$

$$\pi : P^i Q^j R^k \mapsto x^i$$

(2) $s \neq 0$: Then the equation

$$\{x\}^{p^2} = z^s$$

implies that

$$z = \{x\}^{s'p^2}.$$

Hence $\{x\}$ and $Z$ generate $E^s$, with order $p^3$ and $p$ respectively. The assignment

$$\begin{array}{rcl} \{x\} & \mapsto & P \\ Z^s \{x\}^{p^2} & \mapsto & Q \end{array}$$

induces an isomorphism

$$E^s \cong \left\langle P,Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle.$$

Indeed, the order of $\{x\}$ is $p^3$ as noted above and

$$\left(Z^s \{x\}^{p^2}\right)^p = (Z^s z^s)^p = 1$$

shows that $Z^s \{x\}^{p^2}$ has order $p$. Next

$$\begin{array}{rcl} Q^{-1}PQ & = & \left(Z^s \{x\}^{p^2}\right)^1 \{x\} \left(Z^s \{x\}^{p^2}\right) = \left(\{x\}^{-p^2} Z^{-s}\right) \{x\} \left(Z^s \{x\}^{p^2}\right) \\ & = & \left(z^{-s} Z^{-s}\right) \{x\} \left(Z^s z^s\right) = Z^{-s} \{x\} Z^s = z^s \{x\} = \{x\}^{p^2} \{x\} \\ & = & \{x\}^{1+p^2} = P^{1+p^2} \end{array}$$

where the equation

$$Z^{-s} \{x\} Z^s = z^s \{x\}$$

follows from

$$\begin{array}{rcl} Z^{-1} \{x\} Z & = & Z^{-1} \{x\} Z \left(\{x\}^{-1} \{x\}\right) = Z^{-1} \left(\{x\} Z \{x\}^{-1}\right) \{x\} \\ & = & Z^{-1} (zZ) \{x\} = z \{x\} \end{array}$$

and induction. Since

$$\begin{array}{l} z = \{x\}^{s'p^2} \mapsto P^{s'p^2}, \\ \pi_s \left(\{x\}^i \left(Z^s \{x\}^{p^2}\right)^j\right) = x^{i-jp^2} = x^i, \\ Z^s = \left(Z^s \{x\}^{p^2}\right) \{x\}^{-p^2} \mapsto QP^{-p^2}, \end{array}$$

and because $z^{-s} \mapsto P^{-p^2}$ is in the center

$$Z \mapsto \left(QP^{-p^2}\right)^{s'} = P^{-s'p^2} Q^{s'}$$

the extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \twoheadrightarrow \mathbb{I}_{p^2}$$
$$\iota_s : \begin{array}{l} z \mapsto P^{s'p^2} \\ Z \mapsto P^{-s'p^2}Q^{s'} \end{array}$$
$$\pi_s : P^iQ^j \mapsto x^i$$

□

5.5.3. *Extensions of $\mathbb{I}_{p^2}$ by $\mathbb{I}_p \times \mathbb{I}_p$.*

**Remark 5.15.** *Write $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x, y \rangle$, and $A = \mathbb{I}_{p^2} = \langle z \rangle$.*

Trivial action.

**Lemma 5.16.** *Below are all the congruence classes of extensions $A = \mathbb{I}_{p^2}$ by $G = \mathbb{I}_p \times \mathbb{I}_p$, where $G$ acts trivially on $A$. Let $s = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in H^2(G, A) \cong \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$*

(1) *$s = 0$:*
$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p \times \mathbb{I}_p.$$
(2) *$v = 0$:*
   (a) *$u \neq 0$:*
   $$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
   $$\iota_s : z \mapsto P^{u'p}$$
   $$\pi_s : P^iQ^j \mapsto x^j y^{i-ju'w}$$
   (b) *$u = 0, w \neq 0$:*
   $$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
   $$\iota_s : z \mapsto P^{w'p}$$
   $$\pi_s : P^iQ^j \mapsto x^i y^j$$
(3) *$v \neq 0$:*
   (a) *$u = w = 0$:*
   $$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
   $$\iota_s : z \mapsto P^{v'}$$
   $$\pi_s : P^iQ^jR^k \mapsto x^{-k}y^j$$
   (b) *$u \neq 0$:*
   $$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
   $$\iota_s : z \mapsto P^{u'p}$$
   $$\pi_s : P^iQ^j \mapsto x^{-juv'}y^{i+jv'wp}$$
   (c) *$u = 0, w \neq 0$:*
   $$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
   $$\iota_s : z \mapsto P^{w'p}$$
   $$\pi_s : P^iQ^j \mapsto x^i y^{jv'w}$$

*Proof.* From Theorem 2.14 we know that

$$H^2(G,A) \cong \mathbb{I}_p \times {}_{[p]}\mathbb{I}_{p^2} \times \mathbb{I}_p \cong \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p.$$

Let

$$s = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in H^2(G,A) \cong \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$$

and note that by Theorem 2.6 a representative $E^s$ of the equivalence class $[\varepsilon_s]$ has generators $z$,$Z$ and $\{x\}$ subject to the relations

$$\begin{aligned} &z^{p^2}, \{y\}^p = z^u, \{x\}^p = z^w, \\ &V^{-1} = \{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z^{vp}, \\ &{}^x z = \{x\} z \{x\}^{-1} = z, \\ &{}^y z = \{y\} z \{y\}^{-1} = z. \end{aligned}$$

(1) $s = 0$: The extension is split

$$\mathbb{I}_{p^2} \rightarrowtail \mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p \twoheadrightarrow \mathbb{I}_p \times \mathbb{I}_p$$

(2) $v = 0$: Then $E^s$ will be abelian since the generators all commute.

(a) $u \neq 0$: Then

$$\{y\}^p = z^u$$

implies that

$$\begin{aligned} z &= \{y\}^{u'p}, \\ \{x\}^p &= z^w = \{y\}^{u'wp} \end{aligned}$$

and hence $E^s = \langle \{x\}, \{y\} \rangle$. We claim that

$$\begin{aligned} \{y\} &\mapsto P \\ \{x\}\{y\}^{-u'w} &\mapsto Q \end{aligned}$$

gives an isomorphism

$$E^s \cong \mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle.$$

It is clear that $\{y\}$ and $\{x\}\{y\}^{-u'w}$ generate $E^s$, the order of $\{y\}$ is $p^3$, and $\left|\{x\}\{y\}^{-u'w}\right| = p$ since

$$\left(\{x\}\{y\}^{-u'w}\right)^p = \{x\}^p \{y\}^{-u'wp} = \{x\}^p \{x\}^{-p} = 1.$$

Since

$$\pi_s\left(\{y\}^i \left(\{x\}\{y\}^{-u'w}\right)^j\right) = x^j y^{i-ju'w}$$

the extension is congruent to

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota_s : z \mapsto P^{u'p} \\ &\pi_s : P^i Q^j \mapsto x^j y^{i-ju'w} \end{aligned}$$

(b) $u = 0, w \neq 0$: Then

$$\begin{aligned} \{y\}^p &= 1, \\ \{x\}^p &= z^w \end{aligned}$$

implies that

$$z = \{x\}^{w'p}$$

and hence $E^s = \langle \{x\}, \{y\} \rangle$, where the orders of $\{x\}$ and $\{y\}$ are $p^3$ and $p$, respectively. Hence

$$\begin{aligned} \{x\} &\mapsto P \\ \{y\} &\mapsto Q \end{aligned}$$

gives an isomorphism $E^s \cong \mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle$, and our extension is congruent to

$$\begin{aligned} &\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left( \mathbb{I}_{p^3} \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle \right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota_s : z \mapsto P^{w'p} \\ &\pi_s : P^i Q^j \mapsto x^i y^j \end{aligned}$$

(3) $v \neq 0$: Then the group $E^s$ will not be abelian.
   (a) $u = w = 0$: We claim that

$$\begin{aligned} z^v &\mapsto P \\ \{y\} &\mapsto Q \\ \{x\}^{-1} &\mapsto R \end{aligned}$$

gives an isomorphism

$$E^s \cong \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle.$$

Indeed, the orders are all correct, $z^v$ commutes with everything, and

$$\begin{aligned} R^{-1}QR &= \left( \{x\}^{-1} \right)^{-1} \{y\} \{x\}^{-1} = \{x\} \{y\} \{x\}^{-1} \\ &= z^{vp} \{y\} = \{y\} z^{vp} = QP^p. \end{aligned}$$

Since

$$\pi_s \left( \left( z^v \right)^i \{y\}^j \left( \{x\}^{-1} \right)^k \right) = x^{-k} y^j$$

the extension is congruent to

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q, R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\begin{aligned} &\iota_s : z \mapsto P^{v'} \\ &\pi_s : P^i Q^j R^k \mapsto x^{-k} y^j \end{aligned}$$

   (b) $u \neq 0$: By Lemma 4

$$\begin{aligned} \left( \{x\}^n \{y\}^m \right)^k &= V^{\binom{k}{2} mn} \{x\}^{kn} \{y\}^{km} \\ &= z^{-pv\binom{k}{2} mn} \{x\}^{kn} \{y\}^{km} \end{aligned} \tag{7}$$

since $V^{-1} = z^{pv}$ implies that $V = z^{-pv}$. Then

$$z = \{y\}^{u'p}$$

and hence

$$\{x\}^p = \{y\}^{u'wp}.$$

Also

$$\{x\} \{y\} \{x\}^{-1} = z^{vp} \{y\} = \left( \{y\}^{u'p} \right)^{vp} \{y\} = \{y\}^{1+u'vp^2},$$

or more generally

$$\{x\}^m \{y\} \{x\}^{-m} = \{y\}^{1+mu'vp^2}. \tag{8}$$

We claim that

$$\begin{aligned} \{y\} &\mapsto P \\ \{x\}^{-uv'}\{y\}^{v'wp} &\mapsto Q \end{aligned}$$

gives an isomorphism

$$E^s \cong \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle.$$

Indeed, the order of $\{y\}$ is $p^3$,

$$\begin{aligned} & \left(\{x\}^{-uv'}\{y\}^{v'wp}\right)^{-1}\{y\}\left(\{x\}^{-uv'}\{y\}^{v'wp}\right) \\ = \;& \{y\}^{-v'wp}\left(\{x\}^{uv'}\{y\}\{x\}^{-uv'}\right)\{y\}^{v'wp} \\ = \;& \{y\}^{-v'wp}\left(\{y\}^{1+p^2}\right)\{y\}^{v'wp} = \{y\}^{1+p^2} \end{aligned}$$

by equation (8), and

$$\left(\{x\}^{-uv'}\{y\}^{v'wp}\right)^p = z^{-pv\binom{p}{2}(-uv')(v'wp)}\{x\}^{-uv'p}\{y\}^{v'wp^2}$$

$$= \left(z^{(uv'w)\binom{p}{2}}\right)^{p^2}(\{x\}^p)^{-uv'}\{y\}^{v'wp^2} = 1\cdot\left(\{y\}^{u'wp}\right)^{-uv'}\{y\}^{v'wp^2}$$

$$= 1$$

by equation (7). Since

$$\pi_s\left(\{y\}^i\left(\{x\}^{-uv'}\{y\}^{v'wp}\right)^j\right) = x^{-juv'}y^{i+jv'wp}$$

and

$$z = \{y\}^{u'p} \mapsto P^{u'p}$$

we see that our extension is congruent to

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : z \mapsto P^{u'p}$$

$$\pi_s : P^iQ^j \mapsto x^{-juv'}y^{i+jv'wp}.$$

(c) $u = 0, w \neq 0$: Then

$$z = \{x\}^{w'p}$$

and

$$\{x\}\{y\}\{x\}^{-1} = z^{vp}\{y\} = \left(\{x\}^{w'p}\right)^{vp}\{y\} = \{x\}^{vw'p^2}\{y\}$$

which implies

$$\{y\}^{-1}\{x\}\{y\} = \{x\}^{1+vw'p^2}$$

or that

$$\{y\}^{-v'w}\{x\}\{y\}^{v'w} = \{x\}^{1+\left(vw'p^2\right)v'w} = \{x\}^{1+p^2}.$$

Hence assignment

$$\begin{aligned} \{x\} &\mapsto P \\ \{y\}^{v'w} &\mapsto Q \end{aligned}$$

is an isomorphism

$$E^s \cong \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle,$$

and our extension is congruent to

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : z \mapsto P^{w'p}$$
$$\pi_s : P^i Q^j \mapsto x^i y^{jv'w}.$$

□

Non-trivial action.

**Remark 5.17.** *When $A$ is written additively, action on $G$ on $A$ is given by*

$$^{x^i y^j} a = (1 + ip)\, a$$

*In multiplicative notation this becomes*

$$^{x^i y^j} a = a^{(1+ip)}$$

*which is equivalent to*

$$\begin{aligned} {}^{x}z &= z^{1+p}, \\ {}^{y}z &= z. \end{aligned}$$

The following map will be useful

**Lemma 5.18.** *The map*

$$\begin{aligned} \varphi : \frac{(\mathbb{I}_p)^2}{\langle (1,1) \rangle} &\to \mathbb{I}_p \\ (a,b) + \langle (1,1) \rangle &\mapsto a - b \end{aligned}$$

*is an isomorphism with inverse*

$$\begin{aligned} \psi : \mathbb{I}_p &\to \frac{(\mathbb{I}_p)^2}{\langle (1,1) \rangle} \\ a &\mapsto (a,0) + \langle (1,1) \rangle \end{aligned}$$

*Proof.* We need to check that the maps are well defined, homomorphisms, and inverses of each other. Obviously $\psi$ is well defined, and a homomorphism.

(1) Suppose

$$(a', b') \equiv (a, b) \left(\mathrm{mod} \left\langle (1,1) \right\rangle\right).$$

Then by definition

$$(a', b') - (a, b) = (a' - a, b' - b) \in \langle (1,1) \rangle$$

i.e.

$$(a', b') = (a, b) + (c, c) = (a + c, b + c)$$

for some $c \in \mathbb{I}_p$. Then

$$\begin{aligned} \varphi\left((a', b') + \langle (1,1) \rangle\right) &= a' - b' = (a + c) - (b + c) \\ &= a - b = \varphi\left((a, b) + \langle (1,1) \rangle\right) \end{aligned}$$

which shows that $\varphi$ is well defined.

(2) Let $(a, b), (c, d) \in (\mathbb{I}_p)^2$. Then

$$\varphi\left(((a, b) + \langle (1,1) \rangle) + ((c, d) + \langle (1,1) \rangle)\right) = \varphi\left((a + c, b + d) + \langle (1,1) \rangle\right)$$
$$= (a + c) - (b + d) = (a - b) + (c - d)$$
$$= \varphi\left((a, b) + \langle (1,1) \rangle\right) + \varphi\left((c, d) + \langle (1,1) \rangle\right)$$

which shows that $\varphi$ is a homomorphism.

(3) Let $a \in \mathbb{I}_p$, then the equation

$$\varphi(\psi(a)) = \varphi((a,0) + \langle(1,1)\rangle) = a - 0 = a$$

shows that $\varphi \circ \psi = 1_{\mathbb{I}_p}$. Conversely, let $(a,b) \in (\mathbb{I}_p)^2$, then

$$\begin{aligned} & \psi(\varphi((a,b) + \langle(1,1)\rangle)) = \psi(a-b) \\ = \;& (a-b,0) + \langle(1,1)\rangle \\ = \;& ((a-b,0) + (b,b))\langle(1,1)\rangle \\ = \;& (a,b) + \langle(1,1)\rangle \end{aligned}$$

which shows that $\psi \circ \varphi = 1_{\frac{(\mathbb{I}_p)^2}{\langle(1,1)\rangle}}$.

□

**Lemma 5.19.** *Below are all the congruence classes of extensions $A = \mathbb{I}_{p^2}$ by $G = \mathbb{I}_p \times \mathbb{I}_p$, where $G$ acts non-trivially on $A$. Let*

$$s \in H^2(G,A) \cong \begin{cases} \frac{(\mathbb{I}_p)^2}{\langle(1,1)\rangle} & p \geq 3 \\ {}_{[p]}\mathbb{I}_{p^2} & p = 2 \end{cases} \cong \mathbb{I}_p.$$

(1) $s = 0$*:*

$$\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle P,Q,R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}PR = P^{1+p}, \\ P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$\iota : z \mapsto P$

$\pi : P^i Q^j R^k \mapsto x^{-k} y^j$.

(2) $s \neq 0, p = 2$*:*

$$\mathbb{I}_4 \overset{\iota_s}{\rightarrowtail} \left\langle P,Q,R : \begin{array}{c} P^4, Q^4, R^2, Q^{-1}PQ = P^{-1}, Q^2 = P^2, \\ R^{-1}QR = Q, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$\iota_s : z \mapsto P$

$\pi_s : P^i Q^j R^k \mapsto x^j y^k$.

(3) $s \neq 0, p \neq 2$*:*

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P,Q,R : \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$\iota_s : z \mapsto PQ$

$\pi_s : P^i Q^j R^k \mapsto x^{-k} y^{(i-j)a'}$.

*Proof.* From Theorem 2.14 we know that

$$\begin{aligned} H^2(G,A) &\cong \begin{cases} \frac{\left({}_{[p]}\mathbb{I}_{p^2}\right)^3}{\langle(p,p)\rangle \times \left({}_{[p]}\mathbb{I}_{p^2}\right)} & p \geq 3 \\ \frac{\langle(p,p)\rangle \times \left({}_{[p]}\mathbb{I}_{p^2}\right)}{\langle(p,p)\rangle \times \{0\}} & p = 2 \end{cases} \cong \begin{cases} \frac{\left({}_{[p]}\mathbb{I}_{p^2}\right)^2}{\langle(p,p)\rangle} \times \{0\} & p \geq 3 \\ (\{0\})^2 \times {}_{[p]}\mathbb{I}_{p^2} & p = 2 \end{cases} \\ &\cong \begin{cases} \frac{(\mathbb{I}_p)^2}{\langle(1,1)\rangle} & p \geq 3 \\ {}_{[p]}\mathbb{I}_{p^2} & p = 2 \end{cases} \cong \mathbb{I}_p. \end{aligned}$$

Let $s \in \mathbb{I}_p$

(1) $s = 0$: The extension is split

$$1 \to \mathbb{I}_{p^2} \to \mathbb{I}_{p^2} \rtimes ({}_{\xi}\mathbb{I}_p \times \mathbb{I}_p) \to \mathbb{I}_p \times \mathbb{I}_p \to 1.$$

The group $\mathbb{I}_{p^2} \rtimes_\xi (\mathbb{I}_p \times \mathbb{I}_p)$ has generators $z, \{x\}, \{y\}$ with relations

$$\begin{aligned} &z^{p^2} = \{x\}^p = \{y\}^p = 1, \\ &\{x\}^{-1} \{y\} \{x\} = \{y\}, \\ &\{x\}^{-1} z \{x\} = z^{1-p}, \\ &\{y\}^{-1} z \{y\} = z. \end{aligned}$$

We see that $\mathbb{I}_{p^2} \rtimes_\xi (\mathbb{I}_p \times \mathbb{I}_p)$ is isomorphic to the group

$$\left\langle \begin{array}{ll} P, Q, R: & P^{p^2}, Q^p, R^p, R^{-1}PR = P^{1+p}, \\ & P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle$$

under the map

$$\begin{aligned} z &\mapsto P \\ \{x\} &\mapsto R^{-1} \\ \{y\} &\mapsto Q \end{aligned}$$

defined on generators. Thus the extension is congruent to

$$\mathbb{I}_{p^2} \overset{\iota}{\rightarrowtail} \left\langle \begin{array}{ll} P, Q, R: & P^{p^2}, Q^p, R^p, R^{-1}PR = P^{1+p}, \\ & P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota : z \mapsto P$$

$$\pi : P^i Q^j R^k \mapsto x^{-k} y^j.$$

We note that we made no assumption on $p$ being odd, and

$$\left\langle \begin{array}{ll} P, Q, R: & P^{p^2}, Q^p, R^p, R^{-1}PR = P^{1+p}, \\ & P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle$$

also works for $p = 2$.

(2) $s \neq 0, p = 2$: Then

$$H^2(G, A) \cong (\{0\})^2 \times {}_{[2]}\mathbb{I}_4$$

and since ${}_{[2]}\mathbb{I}_4 = \{0, 2\}$ we see that

$$s = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Thus $E^s$ will have relations

$$\begin{aligned} &z^4 = 1, \{x\}^2 = z^2, \{y\}^2 = 1, \\ &\{x\}^{-1} z \{x\} = z^3, \\ &\{y\}^{-1} z \{y\} = z, \\ &\{x\}^{-1} \{y\} \{x\} = \{y\}. \end{aligned}$$

and we see that $E^s$ is isomorphic to the group

$$\left\langle \begin{array}{ll} P, Q, R: & P^4, Q^4, R^2, Q^{-1}PQ = P^{-1}, Q^2 = P^2, \\ & R^{-1}QR = Q, R^{-1}PR = P \end{array} \right\rangle$$

via the assignment

$$\begin{aligned} z &\mapsto P, \\ \{x\} &\mapsto Q, \\ \{y\} &\mapsto R. \end{aligned}$$

Hence the extension is congruent to

$$\mathbb{I}_4 \overset{\iota_s}{\rightarrowtail} \left\langle \begin{array}{cc} P,Q,R: & P^4,Q^4,R^2,Q^{-1}PQ=P^{-1},Q^2=P^2, \\ & R^{-1}QR=Q,R^{-1}PR=P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$\iota_s : z \mapsto P$

$\pi_s : P^i Q^j R^k \mapsto x^j y^k$.

(3) $s \neq 0, p \neq 2$: We have the isomorphism

$$\begin{array}{rcl} \varphi : \dfrac{(\mathbb{I}_p)^2}{\langle (1,1) \rangle} & \to & \mathbb{I}_p \\ (a,b) + \langle (1,1) \rangle & \mapsto & a-b \end{array}$$

from Lemma 5.18. So if $s \neq 0$, then

$$s = (a,0) + \langle (1,1) \rangle$$

for some $a \in (\mathbb{I}_p)^*$. Choosing the representative $(a,0)$ we get relations

$$\begin{array}{l} z^{p^2} = \{x\}^p = 1, \\ \{y\}^p = z^{pa}, \\ \{x\}^{-1} \{y\} \{x\} = \{y\}, \\ \{x\}^{-1} z \{x\} = z^{1-p}, \\ \{y\}^{-1} z \{y\} = z. \end{array}$$

We identify $s$ with $a$, so the second relation becomes

$$\{y\}^p = z^{ps}.$$

Claim that

$$E^s \cong \left\langle \begin{array}{cc} P,Q,R: & P^{p^2},Q^p,R^p,R^{-1}QR=QP^p, \\ & Q^{-1}PQ=P,R^{-1}PR=P \end{array} \right\rangle$$

via the assignment

$$\begin{array}{rcl} \{y\}^{s'} & \mapsto & P \\ \{x\}^{-1} & \mapsto & R \\ z\{y\}^{-s'} & \mapsto & Q, \end{array}$$

where $s' \equiv s^{-1} \pmod{p}$. Then

$$\left(z\{y\}^{-s'}\right)^p = z^p \{y\}^{-s'p} = z^p z^{-p} = 1$$

and

$$\begin{array}{rcl} R^{-1}QR & = & \{x\}\left(z\{y\}^{-s'}\right)\{x\}^{-1} = \left(\{x\}z\{x\}^{-1}\right)\{y\}^{-s'} \\ & = & z^{1+p}\{y\}^{-s'} = z\{y\}^{-s'}\{y\}^{ps'} = QP^p, \\ Q^{-1}PQ & = & z^{-1}\{y\}^{s'} z = \{y\}^{s'} = P, \\ R^{-1}PR & = & \{x\}\{y\}^{s'}\{x\}^{-1} = \left(\{x\}\{y\}\{x\}^{-1}\right)^{s'} \\ & = & \{y\}^{s'} = P, \end{array}$$

as desired. Since

$$z = z\left(\{y\}^{-s'}\{y\}^{s'}\right) = \left(z\{y\}^{-s'}\right)\{y\}^{s'} \mapsto QP = PQ$$

and

$$\pi_s\left(\left(\{y\}^{s'}\right)^i \left(z\{y\}^{-s'}\right)^j \left(\{x\}^{-1}\right)^k\right) = x^{-k} y^{s'(i-j)}$$

we see that our extension is congruent to

$$\mathbb{I}_{p^2} \overset{\iota_s}{\rightarrowtail} \left\langle P,Q,R: \begin{array}{c} P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : z \mapsto PQ$$

$$\pi_s : P^iQ^jR^k \mapsto x^{-k}y^{(i-j)s'}$$

□

5.5.4. *Extensions of $\mathbb{I}_p \times \mathbb{I}_p$ by $\mathbb{I}_p \times \mathbb{I}_p$.*

**Remark 5.20.** *Write $G = \mathbb{I}_p \times \mathbb{I}_p = \langle x \rangle \times \langle y \rangle$, and $A = \mathbb{I}_p \times \mathbb{I}_p = \langle z \rangle \times \langle Z \rangle$.*

Trivial action.

**Lemma 5.21.** *Below are all the congruence classes of abelian extensions of $A = \mathbb{I}_p \times \mathbb{I}_p$ by $G = \mathbb{I}_p \times \mathbb{I}_p$. Let*

$$s = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} (u_1, u_2) \\ (v_1, v_2) \\ (w_1, w_2) \end{bmatrix} \in H^2(G, A)$$

(1) $s = 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \rightarrowtail (\mathbb{I}_p \times \mathbb{I}_p) \times (\mathbb{I}_p \times \mathbb{I}_p) \twoheadrightarrow \mathbb{I}_p \times \mathbb{I}_p$$

(2) $v = 0$*:*

(a) $u_1 \neq 0$*:*

(i) $u_1w_2 \not\equiv u_2w_1 \pmod{p}$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_{p^2} = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto P^{u_1'p}Q^{-u_2(u_1w_2-u_2w_1)'p} \\ Z \mapsto Q^{u_1(u_1w_2-u_2w_1)'p} \end{array}$$

$$\pi_s : P^iQ^j \mapsto x^jy^{i-ju_1'w_1}$$

(ii) $u_2w_1 \equiv u_1w_2 \pmod{p}$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle \times \langle R \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto P^{u_1'p}R^{-u_1'u_2} \\ Z \mapsto R \end{array}$$

$$\pi_s : P^iQ^j \mapsto x^jy^{i-ju_1'w_1}$$

(b) $u_1 = 0, u_2 \neq 0$*:*

(i) $w_1 \neq 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_{p^2} = \left\langle \{y\}, \{x\}\{y\}^{-u_2'w_2} \right\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto Q^{w_1'p} \\ Z \mapsto P^{u_2'p} \end{array}$$

$$\pi_s : P^iQ^j \mapsto x^jy^{i-ju_2'w_2}$$

(ii) $w_1 = 0$*:*

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle \times \langle R \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto R \\ Z \mapsto P^{u_2'p} \end{array}$$

$$\pi_s : P^iQ^jR^k \mapsto x^jy^{i-ju_2'w_2}$$

(c) $u = 0, w_1 \neq 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P, Q, R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : \begin{array}{l} z \mapsto P^{w_1' p} R^{-w_1' w_2} \\ Z \mapsto Q \end{array}$$
$$\pi_s : P^i Q^j R^k \mapsto x^i y^j$$

(d) $u = 0, w_1 = 0, w_2 \neq 0$:

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P, Q, R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$
$$\iota_s : \begin{array}{l} z \mapsto R \\ Z \mapsto P^{w_2' p} \end{array}$$
$$\pi_s : P^i Q^j R^k \mapsto x^i y^j$$

*Proof.* By Theorem 2.14

$$H^2\left(\mathbb{I}_p \times \mathbb{I}_p, \mathbb{I}_p \times \mathbb{I}_p\right) \cong \left(\mathbb{I}_p \times \mathbb{I}_p\right)^3.$$

Write an element $s$ of $H^2\left(\mathbb{I}_p \times \mathbb{I}_p, \mathbb{I}_p \times \mathbb{I}_p\right)$ as

$$s = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} (u_1, u_2) \\ (v_1, v_2) \\ (w_1, w_2) \end{bmatrix} \in H^2(G, A).$$

Then by Theorem 2.6 $E^s$ has generators $z, Z, \{x\}, \{y\}$ with relations

$$\{y\}^p = z^{u_1} Z^{u_2}, \{x\}^p = z^{w_1} Z^{w_2}, z^p = Z^p = 1,$$
$$\{x\}\{y\}\{x\}^{-1}\{y\}^{-1} = z^{v_1} Z^{v_2}, \{x\} z \{x\}^{-1} = z,$$
$$\{y\} z \{y\}^{-1} = z, \{x\} Z \{x\}^{-1} = Z, \{y\} Z \{y\}^{-1} = Z,$$
$$z Z z^{-1} = Z.$$

We proceed as described in Section 5.1.

(1) $s = 0$: Then the extension is split

$$\left(\mathbb{I}_p \times \mathbb{I}_p = \langle z\rangle \times \langle Z\rangle\right) \rightarrowtail \left(\mathbb{I}_p\right)^4 \twoheadrightarrow \left(\mathbb{I}_p \times \mathbb{I}_p = \langle x\rangle \times \langle y\rangle\right)$$

(2) $v = 0$: Then our group $E^s$ is abelian since all of the generators commute with one another.

(a) $u_1 \neq 0$: Then

$$\{y\}^p = z^{u_1} Z^{u_2} \Rightarrow z = Z^{-u_1' u_2} \{y\}^{u_1' p}$$

so $\{y\}, \{x\}$, and $Z$ generate $E^s$, and $|\{y\}| = p^2$. Furthermore, observe that

$$\begin{aligned} \{x\}^p &= z^{w_1} Z^{w_2} = \left(Z^{-u_1' u_2} \{y\}^{u_1' p}\right)^{w_1} Z^{w_2} \\ &= \{y\}^{u_1' w_1 p} Z^{w_2 - u_1' u_2 w_1}, \end{aligned}$$

and

$$\begin{aligned} \left(\{x\}\{y\}^{-u_1' w_1}\right)^p &= \{x\}^p \{y\}^{-u_1' w_1 p} = Z^{w_2 - u_1' u_2 w_1} \\ &= \left(Z^{u_1 w_2 - u_2 w_1}\right)^{u_1'} \end{aligned}$$

so

$$\left|\{x\}\{y\}^{-u_1' w_1}\right| = \begin{cases} p, & u_1 w_2 \equiv u_2 w_1 \pmod p \\ p^2, & \text{otherwise} \end{cases}$$

(i) If $u_1w_2 \not\equiv u_2w_1 \pmod p$,then $u_1w_2 - u_2w_1$ is invertible $(\mathrm{mod}\, p)$, so

$$\left(Z^{u_1w_2-u_2w_1}\right)^{u_1'} = \{x\}^p \{y\}^{-u_1'w_1p}$$

implies that

$$\begin{aligned} Z &= \left(\{x\}^p \{y\}^{-u_1'w_1p}\right)^{u_1(u_1w_2-u_2w_1)'} \\ &= \left(\{x\} \{y\}^{-u_1'w_1}\right)^{u_1(u_1w_2-u_2w_1)'p}, \end{aligned}$$

and thus $E = \left\langle \{y\}, \{x\} \{y\}^{-u_1'w_1} \right\rangle \cong \mathbb{I}_{p^2} \times \mathbb{I}_{p^2} = \langle P \rangle \times \langle Q \rangle$. Since

$$\begin{aligned} Z &= \left(\{x\} \{y\}^{-u_1'w_1}\right)^{u_1(u_1w_2-u_2w_1)'p} \\ &\mapsto Q^{u_1(u_1w_2-u_2w_1)'p}, \\ z &= Z^{-u_1'u_2} \{y\}^{u_1'p} = \{y\}^{u_1'p} Z^{-u_1'u_2} \\ &\mapsto P^{u_1'p} \left(Q^{u_1(u_1w_2-u_2w_1)'p}\right)^{-u_1'u_2} \\ &= P^{u_1'p} Q^{-u_2(u_1w_2-u_2w_1)'p} \end{aligned}$$

and

$$\pi\left(\{y\}^i \left(\{x\} \{y\}^{-u_1'w_1}\right)^j\right) = x^j y^{i-u_1'w_1j}$$

we see that the extension is congruent to

$$\begin{aligned} &\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_{p^2} = \langle P \rangle \times \langle Q \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p \\ &\iota_s : \begin{array}{l} z \mapsto P^{u_1'p} Q^{-u_2(u_1w_2-u_2w_1)'p} \\ Z \mapsto Q^{u_1(u_1w_2-u_2w_1)'p} \end{array} \\ &\pi_s : P^i Q^j \mapsto x^j y^{i-ju_1'w_1} \end{aligned}$$

(ii) If $u_2w_1 \equiv u_1w_2$ then

$$\left(\{x\} \{y\}^{-u_1'w_1}\right)^p = 1$$

so the elements $\{y\}$ and $\{x\} \{y\}^{-u_1'w_1}$ is not enough to generate $E^s$. So we add $Z$ to our generating set. Then the assignment

$$\begin{aligned} \{y\} &\mapsto P \\ \{x\} \{y\}^{-u_1'w_1} &\mapsto Q \\ Z &\mapsto R \end{aligned}$$

induces an isomorphism $E^s \cong \mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P \rangle \times \langle Q \rangle \times \langle R \rangle$. Since

$$\begin{aligned} Z &\mapsto R \\ z &= Z^{-u_1'u_2} \{y\}^{u_1'p} = \{y\}^{u_1'p} Z^{-u_1'u_2} \\ &\mapsto P^{u_1'p} R^{-u_1'u_2} \end{aligned}$$

we see that the extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P\rangle \times \langle Q\rangle \times \langle R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto P^{u_1' p} R^{-u_1' u_2} \\ Z \mapsto R \end{array}$$

$$\pi_s : P^i Q^j \mapsto x^j y^{i - j u_1' w_1}$$

(b) $u_1 = 0, u_2 \neq 0$: Then

$$\{y\}^p = Z^{u_2} \Rightarrow Z = \{y\}^{u_2' p},$$

and hence

$$\{x\}^p = z^{w_1} Z^{w_2} = z^{w_1} \left(\{y\}^{u_2' p}\right)^{w_2} = z^{w_1} \{y\}^{u_2' w_2 p}.$$

We also have

$$\left(\{x\}\{y\}^{-u_2' w_2}\right)^p = \{x\}^p \{y\}^{-u_2' w_2 p} = z^{w_1}$$

so $\{y\}$,$\{x\}\{y\}^{-u_2' w_2}$, and $z$ form a generating set for $E^s$.

(i) So if $w_1 = 0$ we see that $\left(\{x\}\{y\}^{-u_2' w_2}\right)^p = 1$, and we need to keep $z$ in the generating set. Thus the assignment

$$\begin{array}{rcl} \{y\} & \mapsto & P \\ \{x\}\{y\}^{-u_2' w_2} & \mapsto & Q \\ z & \mapsto & R \end{array}$$

induces an isomorphism

$$E^s \cong \mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P\rangle \times \langle Q\rangle \times \langle R\rangle.$$

Since

$$Z = \{y\}^{u_2' p} \mapsto P^{u_2' p}$$

and

$$\pi\left(\{y\}^i \left(\{x\}\{y\}^{-u_2' w_2}\right)^j z^k\right) = x^j y^{i - j u_2' w_2},$$

we see that the extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P\rangle \times \langle Q\rangle \times \langle R\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto R \\ Z \mapsto P^{u_2' p} \end{array}$$

$$\pi_s : P^i Q^j R^k \mapsto x^j y^{i - j u_2' w_2}$$

(ii) if $w_1 \neq 0$ then

$$\left(\{x\}\{y\}^{-u_2' w_2}\right)^p = z^{w_1}$$

implies that

$$z = \left(\{x\}\{y\}^{-u_2' w_2}\right)^{w_1' p}$$

and hence $\{y\}$,$\{x\}\{y\}^{-u_2' w_2}$ generates $E^s$, and both have orders $p^2$. Thus $E^s \cong \mathbb{I}_{p^2} \times \mathbb{I}_{p^2}$ via the assignment

$$\begin{array}{rcl} \{y\} & \mapsto & P \\ \{x\}\{y\}^{-u_2' w_2} & \mapsto & Q, \end{array}$$

and the extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_{p^2} = \left\langle \{y\}, \{x\}\{y\}^{-u_2' w_2} \right\rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto Q^{w_1' p} \\ Z \mapsto P^{u_2' p} \end{array}$$

$$\pi_s : P^i Q^j \mapsto x^j y^{i - j u_2' w_2}$$

(c) $u = 0, w_1 \neq 0$: Then

$$\{x\}^p = z^{w_1} Z^{w_2} \Rightarrow z = Z^{-w_1' w_2} \{x\}^{w_1' p},$$

and hence $E^s = \langle \{x\}, \{y\}, Z \rangle$ is isomorphic to $\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p$ via the assignment

$$\begin{array}{rcl} \{x\} & \mapsto & P \\ \{y\} & \mapsto & Q \\ Z & \mapsto & R. \end{array}$$

Thus our extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P, Q, R \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto P^{w_1' p} R^{-w_1' w_2} \\ Z \mapsto Q \end{array}$$

$$\pi_s : P^i Q^j R^k \mapsto x^i y^j.$$

(d) $u = 0, w_1 = 0, w_2 \neq 0$: Then

$$\{x\}^p = Z^{w_2} \Rightarrow Z = \{x\}^{w_2' p}$$

so

$$E^s = \langle \{x\}, \{y\}, z \rangle \cong \mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p,$$

and the extension is congruent to

$$\mathbb{I}_p \times \mathbb{I}_p \overset{\iota_s}{\rightarrowtail} \left(\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p = \langle P, Q, R \rangle\right) \overset{\pi_s}{\twoheadrightarrow} \mathbb{I}_p \times \mathbb{I}_p$$

$$\iota_s : \begin{array}{l} z \mapsto R \\ Z \mapsto P^{w_2' p} \end{array}$$

$$\pi_s : P^i Q^j R^k \mapsto x^i y^j.$$

□

Non-trivial action. In our multiplicative notation, the action is given by

$${}^{x^i y^j}\left(z^a Z^b\right) = z^{a+ib} Z^b$$

which we can summarize by saying that everything is trivial except for

$${}^{x} Z = zZ.$$

**Lemma 5.22.** *The congruence classes for $G = \mathbb{I}_2 \times \mathbb{I}_2$ by $A = \mathbb{I}_2 \times \mathbb{I}_2$, with non-trivial is given below. Let*
$s \in H^2_{spec}\left(\mathbb{I}_2 \times \mathbb{I}_2, (\mathbb{I}_2 \times \mathbb{I}_2)^{\xi}\right) \cong \mathbb{I}_2$

(1) $s = 0$*:*

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \left\langle \begin{array}{ll} P, Q, R : & P^4, Q^2, R^2, R^{-1} P R = P^3, \\ & P^{-1} Q P = Q, R^{-1} Q R = Q \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$$\iota : \begin{array}{l} z \mapsto P^2 \\ Z \mapsto PR \end{array}$$

$$\pi : P^i Q^j R^k \mapsto x^{i+k} y^j$$

(2) $s = 1$:

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \left\langle \begin{array}{ll} P, Q, R : & P^4, Q^2, R^2, R^{-1}QR = QP^2, \\ & Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$$\iota : \begin{array}{l} z \mapsto P^2 \\ Z \mapsto Q \end{array}$$

$$\pi : P^i Q^j R^k \mapsto x^k y^i$$

*Proof.* By Theorem 2.14:

$$\begin{aligned} H^2_{\text{spec}}\left(\mathbb{I}_2 \times \mathbb{I}_2, (\mathbb{I}_2 \times \mathbb{I}_2)^{\xi}\right) &\cong (\mathbb{I}_2 \times \{0\}) \times (\{0\})^2 \times (\{0\})^2 \\ &\cong \mathbb{I}_2. \end{aligned}$$

□

*Proof.* Let $s \in \mathbb{I}_2$

(1) $s = 0$: The extension is split

$$\mathbb{I}_2 \times \mathbb{I}_2 \rightarrowtail (\mathbb{I}_2 \times \mathbb{I}_2) \rtimes_{\xi} (\mathbb{I}_2 \times \mathbb{I}_2) \twoheadrightarrow \mathbb{I}_2 \times \mathbb{I}_2$$

where $(\mathbb{I}_2 \times \mathbb{I}_2) \rtimes_{\xi} (\mathbb{I}_2 \times \mathbb{I}_2)$ has generators $z, Z, \{x\}$, and $\{y\}$ with relations

$$\begin{array}{l} z^2, Z^2, \{x\}^2, \{y\}^2, \\ z^{-1}Zz = Z, \{x\}^{-1}\{y\}\{x\} = \{y\}, \\ \{x\}\, z\, \{x\}^{-1} = Z, \{y\}\, z\, \{y\}^{-1} = z \\ \{x\}\, Z\, \{x\}^{-1} = zZ, \{y\}\, Z\, \{y\}^{-1} = Z. \end{array}$$

We claim that

$$(\mathbb{I}_2)^2 \rtimes_{\xi} (\mathbb{I}_2)^2 \cong \left\langle \begin{array}{ll} P, Q, R : & P^4, Q^2, R^2, R^{-1}PR = P^3, \\ & P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle$$

under the under the assignment

$$\begin{array}{rcl} Z\{x\} & \mapsto & P \\ \{y\} & \mapsto & Q \\ \{x\} & \mapsto & R. \end{array}$$

Indeed, the equation

$$\begin{aligned} (Z\{x\})^2 &= (Z\{x\})(Z\{x\}) = Z(\{x\}\, Z\{x\}) \\ &= Z\left(\{x\}\, Z\{x\}^{-1}\right) = Z(zZ) = z \end{aligned}$$

show that $(Z\{x\})^4 = z^2 = 1$ and that the elements $Z\{x\}, \{y\}$, and $\{x\}$ generate $(\mathbb{I}_2)^2 \rtimes_{\xi} (\mathbb{I}_2)^2$. Below we verify the remaining relations

$$\begin{aligned} R^{-1}PR &= \{x\}^{-1}(Z\{x\})\{x\} = \left(\{x\}^{-1} Z\{x\}\right)\{x\} \\ &= \left(\{x\}\, Z\{x\}^{-1}\right)\{x\} = (zZ)\{x\} = z(Z\{x\}) \\ &= (Z\{x\})^3 = P^3, \\ P^{-1}QP &= (Z\{x\})^{-1}\{y\}(Z\{x\}) = \{x\}^{-1} Z^{-1}\{y\}\, Z\{x\} \\ &= \{x\}^{-1}\{y\}\{x\} = \{y\} = Q, \\ R^{-1}QR &= \{x\}^{-1}\{y\}\{x\} = \{y\} = Q. \end{aligned}$$

Thus our extension is congruent to

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \left\langle \begin{array}{ll} P,Q,R: & P^4, Q^2, R^2, R^{-1}PR = P^3, \\ & P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$$\iota : \begin{array}{l} z \mapsto P^2 \\ Z \mapsto PR \end{array}$$

$$\pi : P^i Q^j R^k \mapsto x^{i+k} y^j$$

(2) $s = 1$: Then $E^s$ has relations

$$\begin{array}{l} z^2, Z^2, \{x\}^2, \{y\}^2 = z, \\ z^{-1} Z z = Z, \{x\}^{-1} \{y\} \{x\} = \{y\}, \\ \{x\} z \{x\}^{-1} = z, \{y\} z \{y\}^{-1} = z \\ \{x\} Z \{x\}^{-1} = zZ, \{y\} Z \{y\}^{-1} = Z. \end{array}$$

The equation that $\{y\}^4 = z^2 = 1$ shows that $|\{y\}| = 4$, and that the elements $\{y\}, \{x\}$, and $Z$ generate $E^s$. We claim that

$$E^s \cong \left\langle \begin{array}{ll} P,Q,R: & P^4, Q^2, R^2, R^{-1}QR = QP^2, \\ & Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle$$

and that the following assignment is an isomorphism:

$$\begin{array}{rcl} \{y\} & \mapsto & P \\ \{x\} & \mapsto & R \\ Z & \mapsto & Q. \end{array}$$

Obviously the order relations are satisfied, below is the verification of remaining other relations

$$\begin{array}{rcl} R^{-1}QR & = & \{x\}^{-1} Z \{x\} = \{x\} Z \{y\}^{-1} \\ & = & zZ = \{y\}^2 Z = QP^2 \\ Q^{-1}PQ & = & Z^{-1} \{y\} Z = \{y\} = P \\ R^{-1}QR & = & \{x\}^{-1} Z \{x\} = Z = Q. \end{array}$$

Thus the extension is congruent to

$$\mathbb{I}_2 \times \mathbb{I}_2 \overset{\iota}{\rightarrowtail} \left\langle \begin{array}{ll} P,Q,R: & P^4, Q^2, R^2, R^{-1}QR = QP^2, \\ & Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle \overset{\pi}{\twoheadrightarrow} \mathbb{I}_2 \times \mathbb{I}_2$$

$$\iota : \begin{array}{l} z \mapsto P^2 \\ Z \mapsto Q \end{array}$$

$$\pi : P^i Q^j R^k \mapsto x^k y^i$$

□

**Remark 5.23.** *See Appendix C for some of the rules for $E^s$, which will likely be useful when $p \neq 2$.*

## Appendix A. Elements of Homological Algebra

**Definition A.1.** [Rot09, 6.1 Homology Functors, p.337] *Chain maps $f, g : C_\bullet \to C'_\bullet$ are **homotopic**, denoted by $f \simeq g$, if , for all $n$, there is a map $s = (s_n) : C_\bullet \to C'_\bullet$ of degree $+1$ with*

$$f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n.$$

*A map $f : C_\bullet \to C'_\bullet$ is **null-homotopic** if $f \simeq 0$.*

**Definition A.2.** [Rot09, 6.1 Homology Functors, p.337] *A complex $C_\bullet$ in a category $\mathbf{K}$ is **contractible** if its identity $1 = 1_{C_\bullet}$ is null-homotopic; that is, there is $s : C_\bullet \to C_\bullet$ of degree $+1$ with $1 = sd + ds$. Such a map $s$ is called a **contracting homotopy**.*

**Proposition A.3.** [Rot09, Proposition 6.15, p.337] *A contractible complex $C_\bullet$ in a category $\mathbf{K}$ is exact.*

A complex in $R$-Mod can also be considered as a complex in $\mathbb{Z}$-Mod, and any $R$-map is also a $\mathbb{Z}$-map. It is well-known that a complex is exact in $R$-Mod if an only if it is exact in $\mathbb{Z}$-Mod .

**Corollary A.4.** *A complex $C_\bullet$ in $R$-Mod that is $\mathbb{Z}$-Mod contractible is exact.*

**Remark A.5.** *When we want to show that a complex $(C_\bullet, d_\bullet)$ in $R$-Mod is exact, it is enough to find a family of $\mathbb{Z}$-maps $(s_n : C_{n+1} \to C_n)_{n\in\mathbb{Z}}$ with the property that $1_{C_n} = s_{n-1}d_n + d_{n+1}s_n$, for all $n \in \mathbb{Z}$.*

**Theorem A.6.** *(Comparison Theorem.) If $\varphi : A \to B$ is a module homomorphism, while $\varepsilon : P_\bullet \to A$ is a projective complex over $A$, and $\epsilon : Q_\bullet \to B$ is a resolution of $B$, then there is a chain transformation $f : P_\bullet \to Q_\bullet$ with*

$$\epsilon f = \varphi\varepsilon$$

*and any two such chain transformations are homotopic.*

*Proof.* [ML95, Chapter III, Theorem 6.1] □

**Definition A.7.** *(Lifting.) A chain map $f : P_\bullet \to Q_\bullet$ with the properties in Theorem A.6 is called a **lifting** of $\varphi$.*

**Lemma A.8.** *Under the hypotheses of Theorem A.6, let $f : P_\bullet \to Q_\bullet$ be a lifting of $\varphi : A \to B$, and suppose there is homomorphism*

$$g : A \to Q_0$$

*such that*

$$\epsilon \circ g = \varphi.$$

*Then, $f : P_\bullet \to Q_\bullet$ is null homotopic.*

**Corollary A.9.** *If $P_\bullet$ and $Q_\bullet$ are two projective resolutions of $A$, while $B$ is any module, then*

$$H^n(P_\bullet, B) \cong H^n(Q_\bullet, B)$$

*depends only on $A$ and $B$.*

*Proof.* [ML95, Chapter III, Corollary 6.3] □

Hence, we are guaranteed that

$$H^n_{\text{bar}}(G, A) \cong H^n_{\text{special}}(G, A).$$

**Corollary A.10.** *A projective complex is exact if and only if it is contractible.*

*Proof.* Proposition A.4 shows that contractible implies exact. Conversely, suppose that

$$0 \leftarrow P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} P_2 \leftarrow \cdots$$

is an exact projective complex. Then since $1_{P_\bullet} : P_\bullet \to P_\bullet$ lifts $1_{P_o} : P_o \to P_o$, Lemma A.8 guarantees that $1_{P_\bullet}$ is null homotopic, and hence $P_\bullet$ is contractible. □

A.0.1. *Bicomplexes.*

**Definition A.11.** *A bicomplex over $R$ is a family of $R$-modules $(C_{s,t})_{(s,t)\in\mathbb{Z}\times\mathbb{Z}}$ and two families of $R$-maps*

$$\begin{aligned} d_{s-1,t} &: C_{s,t} \to C_{s-1,t}, \\ \delta_{s,t-1} &: C_{s,t} \to C_{s,t-1} \end{aligned}$$

*such that*

$$dd = 0, \delta\delta = 0, \text{ and } d\delta + \delta d = 0.$$

Given a bicomplex, we form a chain complex $\mathrm{Tot}_\bullet(C_{\bullet\bullet})$ as follows: Let family of modules be given by

$$\mathrm{Tot}_n(C_{\bullet\bullet}) := \bigoplus_{s+t=n} C_{s,t},$$

and the differential

$$D_n : \mathrm{Tot}_{n+1}(C_{\bullet\bullet}) \to \mathrm{Tot}_n(C_{\bullet\bullet})$$

be the unique $R$-map satisfying

$$D_n \iota_{s,t} = d_{s-1,t} + \delta_{s,t-1}$$

where $\iota_{s,t} : C_{s,t} \to \bigoplus_{s+t=n} C_{s,t} = \mathrm{Tot}_n(C_{\bullet\bullet})$ is the canonical injection.

$$\begin{array}{ccc} \mathrm{Tot}_{n+1}(C_{\bullet\bullet}) = \bigoplus_{s+t=n+1} C_{s,t} & \xrightarrow{D_n} & \bigoplus_{s+t=n} C_{s,t} = \mathrm{Tot}_n(C_{\bullet\bullet}) \\ \uparrow{\scriptstyle \iota_{C_{s,t}}} & & \uparrow{\scriptstyle \iota_{C_{s-1,t}} + \iota_{C_{s,t-1}}} \\ C_{s,t} & \xrightarrow{d_{s-1,t} + \delta_{s,t-1}} & C_{s-1,t} \bigoplus C_{s,t-1} \end{array}$$

We need to verify that $\mathrm{Tot}_\bullet(C_{\bullet\bullet})$ is indeed a complex, namely, that $DD = 0$. For any $s, t \in \mathbb{Z}\times\mathbb{Z}$ with $s + t = n$, we have

$$\begin{aligned} D_{n-1}D_n\iota_{C_{s,t}} &= D_{n-1}\left(\iota_{C_{s-1,t}} + \iota_{C_{s,t-1}}\right) \circ (d_{s-1,t} + \delta_{s,t-1}) \\ &= \left(D_{n-1} \circ \iota_{C_{s-1,t}}\right) \circ d_{s-1,t} + \left(D_{n-1} \circ \iota_{C_{s,t-1}}\right) \circ \delta_{s,t-1} \\ &= (d_{s-2,t} + \delta_{s-1,t-1}) \circ d_{s-1,t} + (d_{s-1,t-1} + \delta_{s,t-2}) \circ \delta_{s,t-1} \\ &= d_{s-2,t} \circ d_{s-1,t} + \delta_{s-1,t-1} \circ d_{s-1,t} + d_{s-1,t-1} \circ \delta_{s,t-1} + \delta_{s,t-2} \circ \delta_{s,t-1} \\ &= 0 + \delta_{s-1,t-1} \circ d_{s-1,t} + d_{s-1,t-1} \circ \delta_{s,t-1} + 0 = 0, \end{aligned}$$

where the final equality follows from the condition $d\delta + \delta d = 0$. Hence

$$D_{n-1}D_n : \mathrm{Tot}_{n+1}(C_{\bullet\bullet}) \to \mathrm{Tot}_{n-1}(C_{\bullet\bullet})$$

is the zero map.

**Remark A.12.** *We will restrict ourselves to first quadrant bicomplexes ($C_{s,t} = 0$, if $s < 0$ or $t < 0$) and positive complexes ($D_i = 0$, if $i < 0$).*

**Theorem A.13.** *(Künneth formula) Let $C_\bullet, D_\bullet$ be chain complexes over the PID $R$, and suppose that one of $C_\bullet, D_\bullet$ is flat. Then there is a natural short exact sequence*

$$\bigoplus_{p+q=n} H_p(C_\bullet) \otimes_R H_q(D_\bullet) \rightarrowtail H_n(\mathrm{Tot}(C_\bullet \otimes_R D_\bullet)) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet)).$$

*Proof.* See [HS97, Chapter 5 Theorem 2.1] □

## Appendix B. Groups

B.1. **Presentations of Groups.** Let $S$ be a set and let $F(S)$ be the free group on $S$. Elements of $F(S)$ are of words

$$s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_k^{\alpha_k}, s_i \in S, \alpha_i \in \mathbb{Z},$$

and the operation in $F(S)$ is concatenation of words. If $G = \langle S \rangle$ then we have a unique surjective group homomorphism

$$\pi : F(S) \to G$$

which restricts to the identity on $S$.

**Definition B.1.** [DF04, Sec. 6.3]*Let $G$ be a group and $S \subseteq G$ be a subset such that $G = \langle S \rangle$. A **presentation** of $G$ is a pair $(S, R)$, where $R$ is a set of words in $F(S)$ such that*

$$F(S) \supseteq \overline{\langle R \rangle} = \ker(\pi : F(S) \to G)$$

*where $\overline{\langle R \rangle}$ is the normal closure of $\langle R \rangle \leq F(S)$. The elements of $S$ are called **generators** and those of $R$ are called **relations**.*

**Remark B.2.** *It is clear that every group admits presentations.*

**Remark B.3.** *If $(S, R)$ is a presentation of $G$, it is typical to denote the presentation as*

$$\langle S | R \rangle,$$

*however in this thesis we will the notation*

$$\langle S : R \rangle.$$

B.2. **Groups of order $p^2, p^3$ and $p^4$.** When we are going to determine the extensions, we will check them against the following lists, which come from [Bur55, Chapter V.].

**List B.4.** *Groups of order $p^2$:*

(1) $\mathbb{I}_{p^2}$;
(2) $\mathbb{I}_p \times \mathbb{I}_p$.

**List B.5.** *Groups of order $p^3$, $p$ is an odd prime:*

(1) $\mathbb{I}_{p^3}$;
(2) $\mathbb{I}_{p^2} \times \mathbb{I}_p$;
(3) $\mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$;
(4) $\left\langle P, Q : P^{p^2}, Q^p, Q^{-1}PQ = P^{1+p} \right\rangle$;
(5) $\left\langle \begin{array}{ll} P, Q, R : & P^p, Q^p, R^p, R^{-1}QR = QP, \\ & R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle$.

**List B.6.** *Groups of order $p^3$, $p = 2$ :*

(1) $\mathbb{I}_8$;
(2) $\mathbb{I}_4 \times \mathbb{I}_2$;
(3) $\mathbb{I}_2 \times \mathbb{I}_2 \times \mathbb{I}_2$;

(4) $\left\langle P, Q : P^4, Q^2, Q^{-1}PQ = P^3 \right\rangle$;
(5) $\left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^2 \right\rangle$.

**Remark B.7.** *Note that groups* (1) *to* (4) *in List (B.6) are just those of List (B.5) with* $p = 2$*. Moreover, the groups* (4) *and* (5) *in List (B.5) become isomorphic when* $p = 2$*.*

**List B.8.** *Groups of order* $p^4$*,* $p$ *is an odd prime:*

(1) $\mathbb{I}_{p^4}$;
(2) $\mathbb{I}_{p^3} \times \mathbb{I}_p$;
(3) $\mathbb{I}_{p^2} \times \mathbb{I}_{p^2}$;
(4) $\mathbb{I}_{p^2} \times \mathbb{I}_p \times \mathbb{I}_p$;
(5) $\mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p \times \mathbb{I}_p$;
(6) $\left\langle P, Q : P^{p^3}, Q^p, Q^{-1}PQ = P^{1+p^2} \right\rangle$;
(7) $\left\langle \begin{array}{ll} P, Q, R : & P^{p^2}, Q^p, R^p, R^{-1}QR = QP^p, \\ & Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle$;
(8) $\left\langle P, Q : P^{p^2}, Q^{p^2}, Q^{-1}PQ = P^{1+p} \right\rangle$;
(9) $\left\langle \begin{array}{ll} P, Q, R : & P^{p^2}, Q^p, R^p, R^{-1}PR = P^{1+p}, \\ & P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle$;
(10) $\left\langle \begin{array}{ll} P, Q, R : & P^{p^2}, Q^p, R^p, R^{-1}PR = PQ, \\ & Q^{-1}PQ = P, R^{-1}QR = Q \end{array} \right\rangle$;
(11) $\left\langle \begin{array}{ll} P, Q, R : & P^{p^2}, Q^p, R^p = P^{\alpha p}, Q^{-1}PQ = P^{1+p}, \\ & R^{-1}PR = PQ, R^{-1}QR = Q \end{array} \right\rangle\Big|_{\alpha=0}$;
(12) *As in* (11) *where* $\alpha = 1$;
(13) *As in* (11) *where* $\alpha$ *is any non-quadratic residue* $(\mathrm{mod}\, p)$;
(14) $\left\langle \begin{array}{ll} P, Q, R, S : & P^p, Q^p, R^p, S^p, S^{-1}RS = RP, \\ & S^{-1}QS = Q, S^{-1}PS = P, R^{-1}QR = Q \\ & R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle$;
(15) $p > 3$ :
$\left\langle \begin{array}{ll} P, Q, R, S : & Q^p, R^p, S^p, S^{-1}RS = RQ, S^{-1}QS = QP, \\ & S^{-1}PS = P, R^{-1}QR = Q, \\ & R^{-1}PR = P, Q^{-1}PQ = P \end{array} \right\rangle$,
$p = 3$ :
$\left\langle \begin{array}{ll} P, Q, R : & P^9, Q^3, R^3, Q^{-1}PQ = P, \\ & R^{-1}PR = PQ, R^{-1}QR = P^{-3}Q \end{array} \right\rangle$.

**List B.9.** *Groups of order* $p^4$*,* $p = 2$*:*

(1) $\mathbb{I}_{16}$;
(2) $\mathbb{I}_8 \times \mathbb{I}_2$;
(3) $\mathbb{I}_4 \times \mathbb{I}_4$;
(4) $\mathbb{I}_4 \times \mathbb{I}_2$;
(5) $\mathbb{I}_2 \times \mathbb{I}_2 \times \mathbb{I}_2 \times \mathbb{I}_2$;
(6) $\left\langle P, Q : P^8, Q^2, Q^{-1}PQ = P^5 \right\rangle$;
(7) $\left\langle \begin{array}{ll} P, Q, R : & P^4, Q^2, R^2, R^{-1}QR = QP^2, \\ & Q^{-1}PQ = P, R^{-1}PR = P \end{array} \right\rangle$;
(8) $\left\langle P, Q : P^4, Q^4, Q^{-1}PQ = P^3 \right\rangle$;
(9) $\left\langle \begin{array}{ll} P, Q, R : & P^4, Q^2, R^2, R^{-1}PR = P^3, \\ & P^{-1}QP = Q, R^{-1}QR = Q \end{array} \right\rangle$;
(10) $\left\langle \begin{array}{ll} P, Q, R : & P^4, Q^2, R^2, R^{-1}PR = PQ, \\ & Q^{-1}PQ = P, R^{-1}QR = Q \end{array} \right\rangle$;
(11) $\left\langle \begin{array}{ll} P, Q, R : & P^4, Q^4, R^2, Q^{-1}PQ = P^{-1}, Q^2 = P^2, \\ & R^{-1}QR = Q, R^{-1}PR = P \end{array} \right\rangle$;

(12) $\left\langle P,Q : P^8, Q^2, Q^{-1}PQ = P^{-1} \right\rangle$;
(13) $\left\langle P,Q : P^8, Q^2, Q^{-1}PQ = P^{3} \right\rangle$;
(14) $\left\langle P,Q : P^8, Q^4, Q^{-1}PQ = P^{-1}, Q^2 = P^4 \right\rangle$.

**Remark B.10.** *Note that for groups* (1) *to* (10) *in List* (*B*.9) *are just the corresponding groups in List* (*B*.8) *with* $p = 2$.

## Appendix C. Rules for extensions of $(\mathbb{I}_p \times \mathbb{I}_p)^\xi$ by $\mathbb{I}_p \times \mathbb{I}_p$

We have the following rules for $E^s$:

$$\begin{aligned} z^p, Z^p, \{y\}^p &= z^u, \{x\}^p = z^w, \\ \{x\}\{y\}\{x\}^{-1} &= Z^v\{y\}, z^{-1}Zz = Z, \\ \{x\} z \{x\}^{-1} &= Z, \{y\} z \{y\}^{-1} = z, \\ \{x\} Z \{x\}^{-1} &= zZ, \{y\} Z \{y\}^{-1} = Z \end{aligned}$$

The equation

$$\{x\}\{y\}\{x\}^{-1} = Z^v\{y\}$$

is equivalent to

$$\begin{aligned} \{x\}^{-1}\{y\}\{x\} &= Z^{-v}\{y\}, \\ \{y\}^{-1}\{x\}\{y\} &= Z^{v}\{x\}, \\ \{y\}\{x\}\{y\}^{-1} &= Z^{-v}\{x\}. \end{aligned}$$

The equation

$$\{x\} Z \{x\}^{-1} = zZ$$

is equivalent to

$$\{x\}^{-1} Z \{x\} = z^{-1}Z$$

**Remark C.1.** *As we have done previously, we define* $\binom{m}{n} = 0, m < n$.

**Proposition C.2.** *We have*

$$\{x\}^{-m}\{y\}\{x\}^m = z^{\binom{m}{2}v}Z^{-mv}\{y\}$$

*Proof.* (By induction) Base step holds since

$$\{x\}^{-1}\{y\}\{x\} = Z^{-v}\{y\} = z^{\binom{1}{2}v}Z^{-v}\{y\}.$$

Let $m > 1$ and assume that the hypothesis holds for $m-1$, then

$$\begin{aligned} \{x\}^{-m}\{y\}\{x\}^m &= \{x\}^{-1}\left(\{x\}^{-(m-1)}\{y\}\{x\}^{m-1}\right)\{x\} = \{x\}^{-1}\left(z^{\binom{m-1}{2}v}Z^{-(m-1)v}\{y\}\right)\{x\} \\ &= z^{\binom{m-1}{2}v}\left[\left(\{x\}^{-1}Z^{-(m-1)v}\{x\}\right)\left(\{x\}^{-1}\{y\}\{x\}\right)\right] \\ &= z^{\binom{m-1}{2}v}\left[\left(\{x\}^{-1}Z\{x\}\right)^{-(m-1)v}\left(Z^{-v}\{y\}\right)\right] \\ &= z^{\binom{m-1}{2}v}\left[z^{(m-1)v}Z^{-(m-1)v}Z^{-v}\{y\}\right] \\ &= z^{\binom{m-1}{2}v}z^{(m-1)v}Z^{-mv}\{y\} = z^{\binom{m-1}{2}v}z^{\binom{m-1}{1}v}Z^{-mv}\{y\} \\ &= z^{\binom{m}{2}v}Z^{-mv}\{y\} \end{aligned}$$

by Pascal's identity (Lemma 5.4). □

**Proposition C.3.** *We have*

$$\left(Z^\alpha\{x\}^m\{y\}^n\right)^k = z^{\binom{k}{2}\alpha m + nv\left[\binom{k}{2}\binom{m}{2} - 2\binom{k+1}{3}m^2\right]}Z^{k\alpha - \binom{k}{2}mnv}\{x\}^{km}\{y\}^{kn}$$

*Proof.* (By induction). Base case was shown above. Let $k > 1$ and assume the statement is true for $k-1$, then

$$\begin{aligned}
&(Z^\alpha \{x\}^m \{y\}^n)^k = (Z^\alpha \{x\}^m \{y\}^n)^{k-1} (Z^\alpha \{x\}^m \{y\}^n) \\
= &\left(z^{\binom{k-1}{2}\alpha m + nv\left[\binom{k-1}{2}\binom{m}{2} - 2\binom{(k-1)+1}{3}m^2\right]} Z^{(k-1)\alpha - \binom{k-1}{2}mnv} \{x\}^{(k-1)m} \{y\}^{(k-1)n}\right) \\
&\cdot (Z^\alpha \{x\}^m \{y\}^n) \\
= &\, z^{\binom{k-1}{2}\alpha m + nv\left[\binom{k-1}{2}\binom{m}{2} - 2\binom{k}{3}m^2\right]} Z^{(k-1)\alpha - \binom{k-1}{2}mnv} \{x\}^{(k-1)m} Z^\alpha \{y\}^{(k-1)n} \{x\}^m \{y\}^n \\
= &\, z^{\binom{k-1}{2}\alpha m + nv\left[\binom{k-1}{2}\binom{m}{2} - 2\binom{k}{3}m^2\right]} Z^{(k-1)\alpha - \binom{k-1}{2}mnv} \left(\{x\}^{(k-1)m} Z^\alpha \{x\}^{-(k-1)m}\right) \\
&\cdot \{x\}^{(k-1)m} \{y\}^{(k-1)n} \{x\}^m \{y\}^n \\
= &\, z^{\binom{k-1}{2}\alpha m + nv\left[\binom{k-1}{2}\binom{m}{2} - 2\binom{k}{3}m^2\right]} Z^{(k-1)\alpha - \binom{k-1}{2}mnv} \left(z^{(k-1)\alpha m} Z^\alpha\right) \{x\}^{km} \\
&\cdot \left(\{x\}^{-m} \{y\}^{(k-1)n} \{x\}^m\right) \{y\}^n \\
= &\, z^{\binom{k}{2}\alpha m + nv\left[\binom{k-1}{2}\binom{m}{2} - 2\binom{k}{3}m^2\right]} Z^{k\alpha - \binom{k-1}{2}mnv} \{x\}^{km} \left(z^{\binom{m}{2}v} Z^{-mv} \{y\}\right)^{(k-1)n} \{y\}^n \\
= &\, z^{\binom{k}{2}\alpha m + nv\left[\binom{k-1}{2}\binom{m}{2} - 2\binom{k}{3}m^2\right]} Z^{k\alpha - \binom{k-1}{2}mnv} \{x\}^{km} z^{(k-1)nv\binom{m}{2}} Z^{-(k-1)mnv} \{y\}^{kn} \\
= &\, z^{\binom{k}{2}\alpha m + nv\left[\binom{k}{2}\binom{m}{2} - 2\binom{k}{3}m^2\right]} Z^{k\alpha - \binom{k-1}{2}mnv} \left(\{x\}^{km} Z^{-(k-1)mnv} \{x\}^{-km}\right) \{x\}^{km} \{y\}^{kn} \\
= &\, z^{\binom{k}{2}\alpha m + nv\left[\binom{k}{2}\binom{m}{2} - 2\binom{k}{3}m^2\right]} Z^{k\alpha - \binom{k-1}{2}mnv} \left(z^{-(k-1)km^2nv} Z^{-(k-1)mnv}\right) \{x\}^{km} \{y\}^{kn} \\
= &\, z^{\binom{k}{2}\alpha m + nv\left[\binom{k}{2}\binom{m}{2} - 2\binom{k}{3}m^2 - 2\binom{k}{2}m^2\right]} Z^{k\alpha - \binom{k-1}{2}mnv - \binom{k-1}{1}mnv} \{x\}^{km} \{y\}^{kn} \\
= &\, z^{\binom{k}{2}\alpha m + nv\left[\binom{k}{2}\binom{m}{2} - 2\binom{k+1}{3}m^2\right]} Z^{k\alpha - \binom{k}{2}mnv} \{x\}^{km} \{y\}^{kn}
\end{aligned}$$

where we used Lemma 5.4 several times. □

**Remark C.4.** *Since $\langle z \rangle = Z(E^s)$, Proposition C.3 gives the powers for the most general elements of $E^s$.*

For $p \neq 2$ we have

$$\binom{p}{2} \equiv 0 \pmod{p},$$

and when $p \neq 3$

$$\binom{p+1}{3} \equiv 0 \pmod{p}.$$

So for $p \neq 3$

$$\begin{aligned}
(Z^\alpha \{x\}^m \{y\}^n)^p &= z^{\binom{p}{2}\alpha m + nv\left[\binom{p}{2}\binom{m}{2} - 2\binom{p+1}{3}m^2\right]} Z^{p\alpha - \binom{p}{2}mnv} \{x\}^{pm} \{y\}^{pn} \\
&= (\{x\}^p)^m (\{y\}^p)^n = (z^w)^m (z^u)^n \\
&= z^{wm+un}.
\end{aligned}$$

For $p = 3$

$$\binom{3+1}{3} \equiv 1 \pmod{p}$$

so

$$\begin{aligned}
(Z^\alpha \{x\}^m \{y\}^n)^3 &= z^{\binom{3}{2}\alpha m + nv\left[\binom{3}{2}\binom{m}{2} - 2\binom{4}{3}m^2\right]} Z^{3\alpha - \binom{3}{2}mnv} \{x\}^{3m} \{y\}^{3n} \\
&= z^{-2nm^2v} \{x\}^{3m} \{y\}^{3n} = z^{nm^2v + wm + un}.
\end{aligned}$$

For instance, when $m = n = 0$, $E^s$, $p \neq 3$ has no elements of order greater than $p$, but when $p = 3$ the element $\{x\}\{y\}$ has order $p^2 = 9$ whenever $v \neq 0$. So the case $p = 3$ is different from the case $p > 3$.

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