Homogenization of biased convolution type operators

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Abstract

This paper deals with homogenization of parabolic problems for integral convolution type operators with a non-symmetric jump kernel in a periodic elliptic medium. It is shown that the homogenization result holds in moving coordinates. We determine the corresponding effective velocity and prove that the limit operator is a second order parabolic operator with constant coefficients. We also consider the behaviour of the effective velocity in the case of small antisymmetric perturbations of a symmetric kernel, in particular we show that the Einstein relation holds for the studied periodic environment.

Keywords: homogenization in moving coordinates, periodic medium, non-local operator, non-symmetric convolution kernel

1 Introduction

The paper deals with homogenization of parabolic problems for an integral convolution type operator of the form

$$(Lu)(x) \ = \int\limits_{\mathbb{R}^d} a(x-y)\mu(x,y)(u(y)-u(x))dy \tag{1}$$

with a non-symmetric jump kernel $a(z)$ and a periodic positive function $\mu(x,y)$.
In our previous work [6] we considered an integral convolution type operator defined by

$$(Lu)(x) \ = \ \lambda(x)\int\limits_{\mathbb{R}^d} a(x-y)\mu(y)(u(y)-u(x))dy \tag{2}$$

under the assumption that $\lambda(x)$ and $\mu(y)$ are bounded positive periodic functions characterizing the properties of the medium, and $a(z)$ is the jump kernel being a positive integrable function such that $a(-z)=a(z)$. We then made a diffusive scaling of this operator

$$(L^\varepsilon u)(x) \ = \ \varepsilon^{-d-2}\lambda\Big(\frac{x}{\varepsilon}\Big)\int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\mu\Big(\frac{y}{\varepsilon}\Big)(u(y)-u(x))dy, \tag{3}$$

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where ε is a positive scaling factor, $\varepsilon \ll 1$. Then we proved the homogenization result for the operators L^ε. More precisely, we proved that the family L^ε converges, as $\varepsilon \to 0$, to a second order divergence form elliptic operator with constant coefficient in the so-called G-topology that is for any $m > 0$ the family of operators $(-L^\varepsilon + m)^{-1}$ converges strongly in $L^2(\mathbb{R}^d)$ to the operator $(-L^0 + m)^{-1}$ where $L^0 = \Theta^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$ with a positive definite constant matrix Θ.

In this work we consider homogenization problems for convolution type operators L with a kernel of the form $a(x-y)\mu(x,y)$, where the function $a(z)$ is not assumed to be even. More precisely, we assume that $a(z)$ is the generic non-negative integrable function in $\mathbb{R}^d$ that has finite second moments. Concerning the coefficient $\mu(x,y)$ we assume that this function is periodic both in x and y and satisfies the estimates $0 < \alpha_1 \le \mu(x,y) \le \alpha_2$ for some positive constants α_1 and α_2.

In this framework it is natural to study the evolution version of the corresponding homogenization problem. Namely, we are going to investigate the limit behaviour of a solution to the following parabolic equation:

$$\partial_t u(x,t) - (L^\varepsilon u)(x,t) \;=\; 0, \qquad u(x,0) = u_0(x). \tag{4}$$

Clearly, under the above conditions on a and μ the effective velocity need not be zero. This raises the following two natural problems: to determine the effective velocity, and to obtain homogenization results in the corresponding moving coordinates. In the paper we address both this questions. The main homogenization results are formulated in Theorem 2.1 below.

We also consider a small antisymmetric perturbation of a symmetric kernel and study how the effective velocity and other effective characteristics react on this small perturbation. These results are summarized in Lemma 8.1. In particular, we prove that the Einstein relation holds for the perturbation of special structure.

It is interesting to compare the effective behaviour of parabolic equations for nonlocal non-symmetric convolution type operators and for differential operators of convection-diffusion type. Homogenization problems for non-stationary convection-diffusion equations in periodic media have been investigated in the works [1], [2]. It was shown in [1], [2], that the homogenization takes place in the moving coordinates $X(t) = x - \frac{b}{\varepsilon}\, t$ with an appropriate constant vector b. For an elliptic diffusion in a periodic environment and in a random ergodic environment with a finite range of dependence the Einstein relation was proved in [3], for a random walk with i.i.d. conductances it was justified in [4].

2 Problem setup and main results

In this section we provide all the conditions on the coefficients of operator L and then formulate our main results.

Regarding the function $a(\cdot)$ we assume that

$$a(z) \in L^1(\mathbb{R}^d), \;\; a(z) \ge 0, \;\; \hat a(\eta) \in L^2(\mathbb{T}^d), \tag{5}$$

and

$$\|a\|_{L^1(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} a(z)\, dz = a_1 > 0; \qquad \int\limits_{\mathbb{R}^d} |z|^2 a(z)\, dz < \infty. \tag{6}$$

The function $\mu(x,y)$ is periodic in both variables and bounded from above and from below:

$$0 < \alpha_1 \le \mu(x,y) \le \alpha_2 < \infty. \tag{7}$$

From now on we identify periodic functions in $\mathbb{R}^d$ with functions defined on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The operator L is a bounded not necessary symmetric operator in $L^2(\mathbb{R}^d)$, see [6].

In what follows we also use the function

$$\hat{a}(\eta) = \sum_{k \in Z^d} a(\eta + k), \ \eta \in \mathbb{T}^d.$$

Notice that $\hat{a}$ is non-negative, and $\|\hat{a}\|_{L^1(\mathbb{T}^d)} = \|a\|_{L^1(\mathbb{R}^d)}$.

Let us consider the following evolution operator

$$H = \frac{\partial}{\partial t} - L,$$

with L defined in (1). Then, performing the change of variables $x \to \varepsilon x, \ t \to \varepsilon^2 t$, we obtain the family of rescaled operators

$$H^\varepsilon u = \frac{\partial u}{\partial t} - L^\varepsilon u, \quad \text{where } (L^\varepsilon u)(x,t) \ = \ \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big) \mu\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)(u(y,t) - u(x,t)) dy. \tag{8}$$

The main result of this paper is the following homogenization theorem.

Theorem 2.1. *Assume that the functions $a(z)$ and $\mu(x,y)$ satisfy conditions* (5) - (7).

Let $u^\varepsilon(x,t)$ be the solution of the evolution problem

$$\frac{\partial u^\varepsilon}{\partial t} = L^\varepsilon u^\varepsilon, \quad u^\varepsilon(x,0) = \varphi(x), \quad \varphi \in L^2(\mathbb{R}^d), \tag{9}$$

and $u^0(x,t)$ be the solution of a parabolic problem

$$\frac{\partial u^0}{\partial t} = \Theta \cdot \nabla\nabla u^0, \quad u^0(x,0) = \varphi(x), \quad \varphi \in L^2(\mathbb{R}^d). \tag{10}$$

Then there exist a vector $b \in \mathbb{R}^d$ and a positive definite constant matrix Θ such that for any $T > 0$:

$$\|u^\varepsilon\big(x + \frac{b}{\varepsilon}\ t, \ t\big) - u^0(x,t)\|_{L^\infty((0,T), \ L^2(\mathbb{R}^d))} \to 0 \qquad as \ \ \varepsilon \to 0. \tag{11}$$

Observe that

$$\|u^\varepsilon\big(x + \frac{b}{\varepsilon}\ t, \ t\big) - u^0(x,t)\|_{L^\infty((0,T), \ L^2(\mathbb{R}^d))} = \|u^\varepsilon\big(x, \ t\big) - u^0(x - \frac{b}{\varepsilon}\ t, t)\|_{L^\infty((0,T), \ L^2(\mathbb{R}^d))} \tag{12}$$

3 Correctors and auxiliary cell problems

In this section we approximate a solution u^ε of problem (9) using an ansatz constructed in terms of a solution u^0 of the limit problem (10) with the same initial condition φ. To this end we consider auxiliary periodic problems, whose solutions (the so-called correctors) are used in the construction of this ansatz and define the coefficients Θ of effective operator in (10). We first deal with functions from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ that are smooth in t on any interval $t \in (0,T)$.

For a given $u \in C^\infty((0,T), \mathcal{S}(\mathbb{R}^d))$ we introduce the following ansatz:

$$w^\varepsilon(x,t) \ = \ u(x - \frac{b}{\varepsilon} t, t) + \varepsilon \varkappa_1(\frac{x}{\varepsilon}) \cdot \nabla u(x - \frac{b}{\varepsilon} t, t) + \varepsilon^2 \varkappa_2(\frac{x}{\varepsilon}) \cdot \nabla\nabla u(x - \frac{b}{\varepsilon} t, t), \tag{13}$$

where the vector $b \in \mathbb{R}^d$ and correctors $\varkappa_1 \in (L^2(\mathbb{T}^d))^d$ and $\varkappa_2 \in (L^2(\mathbb{T}^d))^{d^2}$ (a vector function $\varkappa_1$ and a matrix function $\varkappa_2$) will be defined below.

Lemma 3.1. *Assume that $u \in C^\infty((0,T),\mathcal{S}(\mathbb{R}^d))$. Then there exist functions $\varkappa_1 \in (L^2(\mathbb{T}^d))^d$ and $\varkappa_2 \in (L^2(\mathbb{T}^d))^{d^2}$, a vector $b \in \mathbb{R}^d$ and a positive definite matrix Θ such that for the function w^ε defined by* (13) *we obtain*

$$H^\varepsilon w^\varepsilon(x,t) \;:=\; \frac{\partial w^\varepsilon}{\partial t} - L^\varepsilon w^\varepsilon \;=\; \Big(\frac{\partial u}{\partial t}(x^\varepsilon,t) - \Theta\cdot\nabla\nabla u(x^\varepsilon,t) \;+\; \phi^\varepsilon(x^\varepsilon,t)\Big)\big|_{x^\varepsilon = x-\frac{b}{\varepsilon}t}, \tag{14}$$

where

$$\lim_{\varepsilon\to 0}\|\phi^\varepsilon\|_{L^\infty((0,T),\ L^2(\mathbb{R}^d))} = 0. \tag{15}$$

Proof. Substituting the expression on the right-hand side of (13) for u in (8) and using the notation $x^\varepsilon = x - \frac{b}{\varepsilon}t$ we get

$$\begin{aligned} H^\varepsilon w^\varepsilon(x,t) \;&=\; \frac{\partial w^\varepsilon(x,t)}{\partial t} - \frac{1}{\varepsilon^{d+2}}\int\limits_{\mathbb{R}^d} a\big(\frac{x-y}{\varepsilon}\big)\mu\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)(w^\varepsilon(y,t)-w^\varepsilon(x,t))dy \\ &=\; \big(-\frac{b}{\varepsilon}\big)\cdot\nabla u(x^\varepsilon,t) + \frac{\partial u}{\partial t}(x^\varepsilon,t) + \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}\big)\otimes\big(-\frac{b}{\varepsilon}\big)\cdot\nabla\nabla u(x^\varepsilon,t) \\ +\, \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}\big)\cdot\nabla\frac{\partial u}{\partial t}(x^\varepsilon,t) &+ \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}\big)\otimes\big(-\frac{b}{\varepsilon}\big)\cdot\nabla\nabla\nabla u(x^\varepsilon,t) + \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}\big)\cdot\nabla\nabla\frac{\partial u}{\partial t}(x^\varepsilon,t) \\ &- \frac{1}{\varepsilon^{d+2}}\int\limits_{\mathbb{R}^d} a\big(\frac{x-y}{\varepsilon}\big)\mu\big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\big)\Big\{u(y^\varepsilon,t) + \varepsilon\varkappa_1\big(\frac{y}{\varepsilon}\big)\cdot\nabla u(y^\varepsilon,t) + \\ +\, \varepsilon^2\varkappa_2\big(\frac{y}{\varepsilon}\big)\cdot\nabla\nabla u(y^\varepsilon,t) &- u(x^\varepsilon,t) - \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}\big)\cdot\nabla u(x^\varepsilon,t) - \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}\big)\cdot\nabla\nabla u(x^\varepsilon,t)\Big\}dy, \end{aligned} \tag{16}$$

where the symbol $\otimes$ stands for tensor product, in particular

$$\varkappa_2\big(\frac{x}{\varepsilon}\big)\otimes\big(-\frac{b}{\varepsilon}\big)\cdot\nabla\nabla\nabla u = \varkappa_2^{ij}\big(\frac{x}{\varepsilon}\big)\big(-\frac{b^k}{\varepsilon}\big)\,\partial_{x^i}\partial_{x^j}\partial_{x^k}u.$$

Here and in the sequel we assume summation over repeated indices.

We collect the terms in (16) that give the main contribution on the right hand side of equality (14); the higher order terms form the remainder ϕ^ε. We do this separately for $\frac{\partial w^\varepsilon}{\partial t}$ and for $L^\varepsilon w^\varepsilon$. For $\frac{\partial w^\varepsilon}{\partial t}$ we obtain

$$\frac{\partial w^\varepsilon(x,t)}{\partial t} = \big(-\frac{b}{\varepsilon}\big)\cdot\nabla u(x^\varepsilon,t) + \frac{\partial u}{\partial t}(x^\varepsilon,t) + \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}\big)\otimes\big(-\frac{b}{\varepsilon}\big)\cdot\nabla\nabla u(x^\varepsilon,t) + \phi_\varepsilon^{(0)}(x,t), \tag{17}$$

with

$$\phi_\varepsilon^{(0)}(x,t) = \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}\big)\cdot\nabla\frac{\partial u}{\partial t}(x^\varepsilon,t) + \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}\big)\otimes\big(-\frac{b}{\varepsilon}\big)\cdot\nabla\nabla\nabla u(x^\varepsilon,t) + \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}\big)\cdot\nabla\nabla\frac{\partial u}{\partial t}(x^\varepsilon,t). \tag{18}$$

After change of variables $z = \frac{x-y}{\varepsilon} = \frac{x^\varepsilon - y^\varepsilon}{\varepsilon}$ we get

$$\begin{aligned} (L^\varepsilon w^\varepsilon)(x,t) \;&=\; \frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}} dz\; a(z)\mu\big(\frac{x}{\varepsilon},\frac{x}{\varepsilon}-z\big)\Big\{u(x^\varepsilon-\varepsilon z,t) + \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}-z\big)\cdot\nabla u(x^\varepsilon-\varepsilon z,t) \\ +\,\varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}-z\big)&\cdot\nabla\nabla u(x^\varepsilon-\varepsilon z,t) - u(x^\varepsilon,t) - \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}\big)\cdot\nabla u(x^\varepsilon,t) - \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}\big)\cdot\nabla\nabla u(x^\varepsilon,t)\Big\}. \end{aligned} \tag{19}$$

Using the following relations

$$\begin{aligned} u(y) \;&=\; u(x) + \int_0^1 \frac{\partial}{\partial q}\,u(x+(y-x)q)\;dq \;=\; u(x) + \int_0^1 \nabla u(x+(y-x)q)\cdot(y-x)\;dq, \\ u(y) \;&=\; u(x) + \nabla u(x)\cdot(y-x) + \int_0^1 \nabla\nabla u(x+(y-x)q)(y-x)\cdot(y-x)(1-q)\;dq \end{aligned}$$

based on the integral form of a remainder in the Taylor expansion and being valid for any $x, y \in \mathbb{R}^d$, we rearrange (19) as follows

$$
\begin{aligned}
&(L^\varepsilon w^\varepsilon)(x,t) = \\
&\frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d} dz\, a(z)\mu\big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} - z\big)\Big\{ (u(x^\varepsilon,t) - \varepsilon z\cdot \nabla u(x^\varepsilon,t) + \varepsilon^2 \int\limits_0^1 \nabla\nabla u(x^\varepsilon - \varepsilon z q, t)\cdot z\otimes z\,(1-q)\,dq \\
&+ \varepsilon\varkappa_1\big(\frac{x}{\varepsilon} - z\big)\cdot\Big(\nabla u(x^\varepsilon,t) - \varepsilon\nabla\nabla u(x^\varepsilon,t)\,z + \varepsilon^2\int_0^1 \nabla\nabla\nabla u(x^\varepsilon - \varepsilon z q,t) z\otimes z (1-q)\,dq\Big) \\
&+ \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon} - z\big)\cdot \nabla\nabla u(x^\varepsilon - \varepsilon z, t) \;-\; u(x^\varepsilon,t) - \varepsilon\varkappa_1\big(\frac{x}{\varepsilon}\big)\cdot\nabla u(x^\varepsilon,t) - \varepsilon^2\varkappa_2\big(\frac{x}{\varepsilon}\big)\cdot\nabla\nabla u(x^\varepsilon,t)\Big\},
\end{aligned}
$$

where

$$
\big\{\nabla\nabla u(\cdot) z\big\}^i = \frac{\partial^2 u}{\partial x^i \partial x^j}(\cdot) z^j \quad \text{and} \quad \big\{\nabla\nabla\nabla u(\cdot) z\otimes z\big\}^i = \frac{\partial^3 u}{\partial x^i\partial x^j\partial x^k}(\cdot) z^j z^k.
$$

Collecting power-like terms in the last relation we obtain

$$
\begin{aligned}
&(L^\varepsilon w^\varepsilon)(x,t) \\
&= \frac{1}{\varepsilon}\nabla u(x^\varepsilon,t)\cdot \int\limits_{\mathbb{R}^d} \Big\{ -z + \varkappa_1\big(\frac{x}{\varepsilon} - z\big) - \varkappa_1\big(\frac{x}{\varepsilon}\big)\Big\} a(z)\mu\big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} - z\big)\,dz \\
&+ \nabla\nabla u(x^\varepsilon,t)\cdot \int\limits_{\mathbb{R}^d}\Big\{\frac{1}{2} z\otimes z - z\otimes\varkappa_1\big(\frac{x}{\varepsilon} - z\big) + \varkappa_2\big(\frac{x}{\varepsilon} - z\big) - \varkappa_2\big(\frac{x}{\varepsilon}\big)\Big\}\, a(z)\mu\big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} - z\big)\,dz \\
&+ \phi_\varepsilon^{(L)}(x,t)
\end{aligned}
\tag{20}
$$

with

$$
\begin{aligned}
\phi_\varepsilon^{(L)}(x,t) \;=\; &\frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d} dz\, a(z)\mu\big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon} - z\big)\Big\{\varepsilon^2\int\limits_0^1 \nabla\nabla u(x^\varepsilon - \varepsilon z q, t)\cdot z\otimes z\,(1-q)\,dq \\
&- \frac{\varepsilon^2}{2}\nabla\nabla u(x^\varepsilon,t)\cdot z\otimes z \;+\; \varepsilon^3\varkappa_1\big(\frac{x}{\varepsilon} - z\big)\cdot\int\limits_0^1 \nabla\nabla\nabla u(x^\varepsilon - \varepsilon z q, t) z\otimes z (1-q)\,dq \\
&- \varepsilon^3\varkappa_2\big(\frac{x}{\varepsilon} - z\big)\cdot\int\limits_0^1 \nabla\nabla\nabla u(x^\varepsilon - \varepsilon z q, t) z\,dq\Big\}.
\end{aligned}
\tag{21}
$$

Thus the remainder term ϕ^ε is the sum

$$
\phi^\varepsilon \;=\; \phi_\varepsilon^{(0)} \;+\; \phi_\varepsilon^{(L)}. \tag{22}
$$

Proposition 3.1. *Let $u \in C^\infty\big((0,T), \mathcal{S}(\mathbb{R}^d)\big)$ Then for the functions $\phi_\varepsilon^{(0)}$ and $\phi_\varepsilon^{(L)}$ given by* (18) *and* (21) *we have*

$$
\|\phi_\varepsilon^{(L)}\|_\infty \;\to\; 0 \qquad \textit{and} \qquad \|\phi_\varepsilon^{(0)}\|_\infty \;\to\; 0 \qquad \textit{as}\ \ \varepsilon \to 0, \tag{23}
$$

where $\|\cdot\|_\infty$ is the norm in $L^\infty\big((0,T), L^2(\mathbb{R}^d)\big)$.

Proof. The convergence (23) for $\phi_\varepsilon^{(0)}$ immediately follows from the representation (18) for this function. For the function $\phi_\varepsilon^{(L)}$, the proof is completely analogous to the proof of Proposition 5 in [6]. $\square$

4 First corrector $\varkappa_1$ and drift b

Our next step of the proof deals with constructing the correctors $\varkappa_1$ and $\varkappa_2$. Denote $\xi = \frac{x}{\varepsilon}$ a variable on the period: $\xi \in \mathbb{T}^d = [0,1]^d$, then $\mu(\xi,\eta), \varkappa_1^i(\xi), \varkappa_2^{ij}(\xi),\ i,j = 1,\dots,d$, are functions on $\mathbb{T}^d$. We collect all the terms of the order ε^{-1} in (17) and (20), and then equate them to 0. This yields the following equation for the vector function $\varkappa_1(\xi) = \{\varkappa_1^i(\xi)\},\ \xi \in \mathbb{T}^d,\ i = 1,\dots,d$, as unknown function and for the unknown vector $b = \{b^i\} \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \Big(- z^i + \varkappa_1^i(\xi - z) - \varkappa_1^i(\xi) \Big) \, a(z)\mu(\xi, \xi - z) \, dz + b^i \ = \ 0 \quad \forall\, i = 1,\dots,d. \tag{24}$$

Here and in what follows $\varkappa_1(q),\ q \in \mathbb{R}^d$, is the periodic extension of $\varkappa_1(\xi),\ \xi \in \mathbb{T}^d$. Notice that (24) is a system of uncoupled equations. After change of variables $q = \xi - z \in \mathbb{R}^d$ equation (24) can be written in the vector form as follows

$$\int_{\mathbb{R}^d} a(\xi - q)\mu(\xi, q)(\varkappa_1(q) - \varkappa_1(\xi)) \, dq \ = \int_{\mathbb{R}^d} a(\xi - q)\mu(\xi, q)(\xi - q) \, dq \ - \ b, \tag{25}$$

or

$$A\varkappa_1 \ = \ h \ = \ f \ - \ b \tag{26}$$

with the operator A in $(L^2(\mathbb{T}^d))^d$ defined by

$$(A\bar\varphi)(\xi) \ = \int_{\mathbb{R}^d} a(\xi - q)\mu(\xi, q)(\bar\varphi(q) - \bar\varphi(\xi)) \, dq \ = \int_{\mathbb{T}^d} \hat a(\xi - \eta)\mu(\xi, \eta)(\bar\varphi(\eta) - \bar\varphi(\xi)) \, d\eta, \tag{27}$$

where

$$\hat a(\eta) \ = \ \sum_{k \in \mathbb{Z}^d} a(\eta + k), \quad \eta \in \mathbb{T}^d, \tag{28}$$

and

$$f \ = \int_{\mathbb{R}^d} a(\xi - q)\mu(\xi, q)(\xi - q) \, dq. \tag{29}$$

Observe that the vector function

$$h(\xi) \ = \int_{\mathbb{R}^d} a(\xi - q)\mu(\xi, q)(\xi - q) \, dq \ - \ b \ \in \ (L^2(\mathbb{T}^d))^d, \tag{30}$$

because it is bounded for all $\xi \in \mathbb{T}^d$:

$$\left| \int_{\mathbb{R}^d} a(\xi - q)(\xi - q)\mu(\xi, q) \, dq \right| \ \le \ \alpha_2 \int_{\mathbb{R}^d} a(z)|z| \, dz \ < \ \infty.$$

In (26) operator A applies component-wise. In what follows, abusing slightly the notation, we use the same notation A for the scalar operator in $L^2(\mathbb{T}^d)$ acting on each component in (26).

Let us denote

$$K\varphi(\xi) = \int_{\mathbb{R}^d} a(\xi - q)\mu(\xi, q)\varphi(q) \, dq, \qquad \varphi \in L^2(\mathbb{T}^d).$$

Proposition 4.1 ([6]). *The operator*

$$K\varphi(\xi) \;=\; \int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\varphi(q)\;dq \;=\; \int\limits_{\mathbb{T}^d} \hat{a}(\xi-\eta)\mu(\xi,\eta)\varphi(\eta)\;d\eta, \quad \varphi\in L^2(\mathbb{T}^d), \tag{31}$$

is a compact operator in $L^2(\mathbb{T}^d)$.

The proof see in [6].

The operator

$$G\varphi(\xi) \;=\; \varphi(\xi)\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\;dq \;=\; \varphi(\xi)\int\limits_{\mathbb{T}^d} \hat{a}(\xi-\eta)\mu(\xi,\eta)\;d\eta, \quad \varphi\in L^2(\mathbb{T}^d), \tag{32}$$

is the operator of multiplication by the function $G(\xi)=\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\;dq$. Observe that

$$0<g_1\le G(\xi)\le g_2<\infty. \tag{33}$$

Thus, the operator A in (27) can be written as $A=K-G$, where G and K were defined in (32) and (31). Therefore $-A$ is the sum of a positive invertible operator G and a compact operator $-K$, and the Fredholm theorem applies to (26).

It will be shown in the next section that $\lambda=1$ is a simple eigenvalue of the operator $(G^{-1}K)^*$ in $L^2(\mathbb{T}^d)$. Denote the corresponding eigenfunction by ψ_0. It is easy to see that the kernel of $(G-K)^*$ has dimension one and that

$$\mathrm{Ker}\,(G-K)^* = G^{-1}(\xi)\psi_0(\xi) =: v_0(\xi). \tag{34}$$

Indeed,

$$(G-K)^*v_0 = \big(G(E-G^{-1}K)\big)^*G^{-1}\psi_0 = \big(E-(G^{-1}K)^*\big)\psi_0 = 0.$$

Then the solvability condition for the equation in (26) reads:

$$\int\limits_{\mathbb{T}^d} h(\xi)v_0(\xi)\;d\xi \;=\; \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} a(\xi-q)\mu(\xi,q)(\xi-q)\;dq\,v_0(\xi)d\xi \;-\; b\int\limits_{\mathbb{T}^d} v_0(\xi)d\xi \;=\; 0. \tag{35}$$

Thus taking the normalized v_0 with $\int_{\mathbb{T}^d} v_0(\xi)d\xi=1$ and choosing b in the following way

$$b \;=\; \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} a(\xi-q)\mu(\xi,q)(\xi-q)\;dq\,v_0(\xi)d\xi, \tag{36}$$

we conclude that the equation in (25) has a unique (up to a constant vector) solution $\varkappa_1\in(L^2(\mathbb{T}^d))^d$.

The properties of the functions ψ_0 and v_0 are discussed in the next section.

5 Ground state

Lemma 5.1. *The operator* $(G^{-1}K)^*$ *is compact in* $L^2(\mathbb{T}^d)$ *and has a simple eigenvalue at* $\lambda=1$. *The corresponding eigenfunction* $\psi_0\in L^2(\mathbb{T}^d)$ *satisfies the equation*

$$(G^{-1}K)^*\psi_0 \;=\; \psi_0 \tag{37}$$

and admits the following estimates:

$$0<\gamma_1\le\psi_0(\xi)\le\gamma_2<\infty \quad \text{for all} \;\; \xi\in\mathbb{T}^d, \tag{38}$$

here $\gamma_1>0$ *and* γ_2 *are positive constants.*

Proof. The compactness of $G^{-1}K$ is an immediate consequence of Proposition 4.1 and estimate (33). The operator $A = K - G$ has the eigenfunction $\varphi_0(\xi) \equiv 1$ with the eigenvalue equal to 0. Thus $\varphi_0(\xi) \equiv 1$ is also the eigenfunction of the operator $G^{-1}K$ that corresponds to the eigenvalue $\lambda = 1$. Moreover, $\lambda = 1$ is the maximal eigenvalue, since the operator $G^{-1}K$ is a stochastic operator. It is clear that $G^{-1}K$ is a positive operator, that is it maps the set of non-negative $L^2(\mathbb{T}^d)$ functions into itself. Moreover, we will now prove that $G^{-1}K$ is a positivity improving operator, and furthermore there exists $N \in \mathbb{N}$ such that

$$f \in L^2(\mathbb{T}^d) \setminus \{0\},\ f(\xi) \geq 0 \ \text{ implies } \ (G^{-1}K)^N f(\xi) \geq \nu_0(f) > 0 \ \text{ for all } \ \xi \in \mathbb{T}^d. \tag{39}$$

Due to representation (31) of the operator K property (39) is a straightforward consequence of the following lemma.

Lemma 5.2. *There exist $N \in \mathbb{N}$ and $\gamma_0 > 0$ such that*

$$\hat{a}^{*N}(\xi) \geq \gamma_0 \quad \forall \xi \in \mathbb{T}^d, \tag{40}$$

where the symbol $*$ stands for the convolution on the torus $\mathbb{T}^d$.

Proof. For proving (40) it is sufficient to show that for any non-negative $a \in L^1(\mathbb{R}^d)$:

$$a(z) \geq 0, \quad \int_{\mathbb{R}^d} a(z)\, dz = 1, \tag{41}$$

there exist $\gamma > 0$ and a ball $B_\delta \in \mathbb{R}^d$ of a radius $\delta > 0$ such that

$$(a * a)(z) > \gamma \quad \forall\, x \in B_\delta. \tag{42}$$

The Lebesgue differentiation theorem states that, given any $f \in L^1(\mathbb{R}^d)$, almost every x is a Lebesgue point of f, i.e.

$$\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y) - f(x)|\, dy = 0, \tag{43}$$

where $B_r(x)$ is a ball centered at x with radius $r > 0$, $|B_r(x)|$ is its Lebesgue measure. Condition (41) implies that there exists a Lebesgue point x_0 such that $a(x_0) = \alpha > 0$. We assume without loss of generality that $x_0 = 0$.

Proposition 5.1. *For any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$:*

$$\mu\{y \in B_\delta(0) :\ a(y) > \frac{\alpha}{2}\} \geq (1 - \varepsilon)\, |B_\delta(0)|. \tag{44}$$

Proof. Using inclusion

$$\{y \in B_\delta(0) :\ a(y) < \frac{\alpha}{2}\} \subset \{y \in B_\delta(0) :\ |a(y) - a(0)| > \frac{\alpha}{2}\}, \quad \text{with } \ a(0) = \alpha,$$

the Chebyshev inequality

$$\mu\{y \in B_\delta(0) :\ |a(y) - a(0)| > \frac{\alpha}{2}\} \leq \frac{2}{\alpha} \int\limits_{B_\delta(0)} |a(y) - a(0)|\, dy,$$

and definition (43) of a Lebesgue point, we get that for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$:

$$\mu\{y \in B_\delta(0) :\ a(y) < \frac{\alpha}{2}\} \leq \mu\{y \in B_\delta(0) :\ |a(y) - a(0)| > \frac{\alpha}{2}\} \leq \varepsilon |B_\delta(0)|. \tag{45}$$

Consequently, inequality (44) holds. □

Notice that $x - y \in B_\delta(0)$, if $x, y \in B_{\frac{\delta}{2}}(0)$. Then it follows from (45) that for any $x \in B_{\frac{\delta}{2}}(0)$ we obtain

$$\mu\{y \in B_{\frac{\delta}{2}}(0) : \ a(y) > \frac{\alpha}{2}, \ a(x-y) > \frac{\alpha}{2}\} \ge |B_{\frac{\delta}{2}}| - 2\mu\{y \in B_\delta(0) : \ a(y) < \frac{\alpha}{2}\} \ge |B_{\frac{\delta}{2}}| - 2\varepsilon |B_\delta|. \quad (46)$$

Choosing $\varepsilon = 2^{-(d+2)}$ and the corresponding $\delta = \delta(\varepsilon)$ we get from (46) the following estimate which is valid for all $x \in B_{\frac{\delta}{2}}(0)$ with $\delta = \delta(\varepsilon)$:

$$\mu\{y \in B_{\frac{\delta}{2}}(0) : \ a(y) > \frac{\alpha}{2}, \ a(x-y) > \frac{\alpha}{2}\} \ge \frac{1}{2}|B_{\frac{\delta}{2}}|. \quad (47)$$

Finally we have for all $x \in B_{\frac{\delta}{2}}(0)$:

$$(a * a)(x) = \int\limits_{\mathbb{R}^d} a(x-y)a(y)dy \ge \int\limits_{|y|<\frac{\delta}{2}, \ a(y)>\frac{\alpha}{2}, \ a(x-y)>\frac{\alpha}{2}} a(x-y)a(y)dy \ge \frac{\alpha^2}{8}|B_{\frac{\delta}{2}}|,$$

which implies (42). Since $\widehat{a^{*N}}(\cdot) = \hat{a}^{\star N}$, the inequality (40) follows, and the proof of Lemma 5.2 is completed. $\square$

As was already explained in the beginning of the proof of Lemma 5.1 the maximal eigenvalue of the operator $G^{-1}K$ is equal to 1. Consequently, the Krein-Rutman theorem ([5], Theorem 6.2) implies that the operator $(G^{-1}K)^*$ has the maximal eigenvalue equal to 1, and from Lemma 5.2 it follows that the corresponding eigenfunction ψ_0 is positive: $\psi_0 > 0$ (the ground state). The fact that $\lambda = 1$ is a simple eigenvalue of the operator $(G^{-1}K)^*$ in the space $L^2(\mathbb{T}^d)$ follows from the positivity improving property (39), see e.g. [5], Section 6.

Thus we have proved the existence and uniqueness of $\psi_0 > 0, \ \psi_0 \in L^2(\mathbb{T}^d)$ that satisfies (37). In particular,

$$\|\psi_0\|_{L^1(\mathbb{T}^d)} = \int\limits_{\mathbb{T}^d} \psi_0(\xi) d\xi > 0.$$

Next we turn to the bounds in (38). Estimates (33) and (40) imply the bound from below:

$$\psi_0(\xi) = \left((G^{-1}K)^*\right)^N \psi_0(\xi) \ge (g_2^{-1}\alpha_1)^N \gamma_0 \int\limits_{\mathbb{T}^d} \psi_0(\eta) d\eta = (g_2^{-1}\alpha_1)^N \gamma_0 \|\psi_0\|_{L^1(\mathbb{T}^d)} \quad \forall \, \xi \in \mathbb{T}^d, \quad (48)$$

where $0 < \|\psi_0\|_{L^1(\mathbb{T}^d)} \le \|\psi_0\|_{L^2(\mathbb{T}^d)}$. The upper bound follows from (5) and (37):

$$\max_\xi \psi_0(\xi) \le \max_\xi \Big| \int\limits_{\mathbb{T}^d} \hat{a}(\eta - \xi)\mu(\eta, \xi) G^{-1}(\eta)\psi_0(\eta) d\eta \Big| \le \alpha_2 g_1^{-1} \|\psi_0\|_{L^2(\mathbb{T}^d)} \|\hat{a}\|_{L^2(\mathbb{T}^d)}. \quad (49)$$

The proof of Lemma 5.1 is completed. $\square$

Corollary 1. *There exists a unique (up to an additive constant) function $v_0 \in L^2(\mathbb{T}^d)$ satisfying*

$$\int\limits_{\mathbb{R}^d} a(q-\xi)\mu(q,\xi)v_0(q) \, dq \ = \ v_0(\xi) \int\limits_{\mathbb{R}^d} a(\xi - q)\mu(\xi, q) dq, \quad (50)$$

i.e. $\mathrm{span}(v_0) = \mathrm{Ker}\,(G-K)^*$. *This function obeys the following lower and upper bounds:*

$$0 < \tilde{\gamma}_1 \le v_0(\xi) \le \tilde{\gamma}_2 < \infty \quad \textit{for all } \ \xi \in \mathbb{T}^d. \quad (51)$$

6 Second corrector $\varkappa_2$ and effective matrix Θ.

We collect now all the terms of the order ε^0 in (17) and (20), and then equate them to the main term on the right-hand side of (14):

$$\frac{\partial u}{\partial t}(x^\varepsilon, t) - \Theta \cdot \nabla\nabla u(x^\varepsilon, t).$$

Notice that time derivatives $\frac{\partial u}{\partial t}(x^\varepsilon, t)$ are mutually cancelled on both sides of this relation, and we obtain an equation for the unknown matrix function $\varkappa_2(\xi) = \{\varkappa_2^{ij}(\xi)\},\ \xi \in \mathbb{T}^d,\ i,j = 1,\dots,d$, and the constant matrix $\Theta = \{\Theta^{ij}\}$. This equation reads

$$\int_{\mathbb{R}^d} a(z)\mu(\xi,\xi-z)(\varkappa_2^{ij}(\xi-z)-\varkappa_2^{ij}(\xi))dz+b^i\varkappa_1^j(\xi)+\int_{\mathbb{R}^d} a(z)\mu(\xi,\xi-z)\big(\frac{1}{2}z^iz^j-z^i\varkappa_1^j(\xi-z)\big)dz \ = \ \Theta^{ij}. \quad (52)$$

Notice that (52) is again a system of uncoupled equations. After change of variables $q = \xi - z \in \mathbb{R}^d$ equation (52) can be written in the vector form as follows

$$\begin{aligned} &-\int_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)(\varkappa_2(q)-\varkappa_2(\xi))\ dq \\ &= \ b\otimes\varkappa_1(\xi) \ + \int_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\Big(\frac{1}{2}(\xi-q)\otimes(\xi-q)-(\xi-q)\otimes\varkappa_1(q)\Big)dq \ - \ \Theta, \end{aligned} \quad (53)$$

or

$$-A\varkappa_2(\xi) \ = \ F(\xi) - \Theta \quad (54)$$

with the operator A defined above in (27) and the following matrix function on the right-hand side:

$$F(\xi) = \ b\otimes\varkappa_1(\xi) + \int_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\Big(\frac{1}{2}(\xi-q)\otimes(\xi-q)-(\xi-q)\otimes\varkappa_1(q)\Big)dq.$$

The equation (54) on $\varkappa_2$ has the same form as equation (26) on $\varkappa_1$. Consequently, using the same reasoning as above we conclude that the solvability condition for (54) leads after simple rearrangements to the following formula for the matrix Θ:

$$\begin{aligned} \Theta^{ij} \ &= \int_{\mathbb{T}^d} F^{ij}(\xi)v_0(\xi)\ d\xi \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\Big(\frac{1}{2}(\xi-q)^i(\xi-q)^j-(\xi-q)^i\varkappa_1^j(q)\Big)v_0(\xi)\ dq\ d\xi \ + \ b^i\int_{\mathbb{T}^d}\varkappa_1^j(\xi)v_0(\xi)\ d\xi \end{aligned} \quad (55)$$

for any i,j, where $v_0 \in L^2(\mathbb{T}^d)$ is the normalized function from $\mathrm{Ker}(-A^*)$, see (34).

Proposition 6.1. *The integrals on the right-hand side of (55) converge. Moreover, the symmetric part of the matrix $\Theta = \{\Theta^{ij}\}$ defined in (55) is positive definite.*

Proof. The first statement of the Proposition immediately follows from the existence of the second moment of the function $a(z)$. Since function $v_0(\xi) > 0$ and satisfies two-sided bounds (51), it is sufficient to prove that the symmetric part of the right-hand side of (55) is positive definite. To prove that Θ is a positive definite matrix we consider the following integrals:

$$I^{ij} = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\big((\xi-q)+(\varkappa_1(\xi)-\varkappa_1(q))\big)^i\big((\xi-q)+(\varkappa_1(\xi)-\varkappa_1(q))\big)^j v_0(\xi)dqd\xi. \quad (56)$$

Our aim is to show that the symmetric part of the right-hand side of (55) is equal to I:

$$I^{ij} \;=\; \Theta^{ij} + \Theta^{ji}. \tag{57}$$

We have

$$\begin{aligned}\Theta^{ij} \;+\; \Theta^{ji} \;=\; & \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)(\xi-q)^i(\xi-q)^j v_0(\xi)d\xi dq \\ - & \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\big((\xi-q)^i\varkappa_1^j(q)+(\xi-q)^j\varkappa_1^i(q)\big)v_0(\xi)\;dq\;d\xi \\ + & \; b^i\int\limits_{\mathbb{T}^d}\varkappa_1^j(\xi)v_0(\xi)\;d\xi \;+\; b^j\int\limits_{\mathbb{T}^d}\varkappa_1^i(\xi)v_0(\xi)\;d\xi.\end{aligned} \tag{58}$$

Let us rewrite I^{ij} as the sum

$$I^{ij} = I_1^{ij} + I_2^{ij} + I_3^{ij},$$

where

$$I_1^{ij} \;=\; \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)(\xi-q)^i(\xi-q)^j v_0(\xi)dqd\xi, \tag{59}$$

$$I_2^{ij} \;=\; \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\big((\xi-q)^i(\varkappa_1(\xi)-\varkappa_1(q))^j+(\varkappa_1(\xi)-\varkappa_1(q))^i(\xi-q)^j\big)v_0(\xi)dqd\xi, \tag{60}$$

$$I_3^{ij} \;=\; \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)(\varkappa_1(\xi)-\varkappa_1(q))^i(\varkappa_1(\xi)-\varkappa_1(q))^j v_0(\xi)dqd\xi. \tag{61}$$

Then I_1^{ij} coincides with the first integral in (58). Let us rewrite the integral in I_2^{ij} as follows:

$$\begin{aligned}I_2^{ij} \;=\; & \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\big((\xi-q)^i\varkappa_1^j(\xi)+(\xi-q)^j\varkappa_1^i(\xi)\big)v_0(\xi)dqd\xi \\ - & \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi-q)\mu(\xi,q)\big((\xi-q)^i\varkappa_1^j(q)+(\xi-q)^j\varkappa_1^i(q)\big)v_0(\xi)dqd\xi \;=\; \tilde{J}_2^{ij} \;+\; J_2^{ij}.\end{aligned} \tag{62}$$

Then J_2^{ij} coincides with the second integral in (58). Further we rearrange the integral $\tilde{J}_2^{ij}$ using (25) and (26) and recalling the definition of the function f in (29):

$$\begin{aligned}\tilde{J}_2^{ij} \;=\; & \int\limits_{\mathbb{T}^d} f^i(\xi)\varkappa_1^j(\xi)v_0(\xi)d\xi \;+\; \int\limits_{\mathbb{T}^d} f^j(\xi)\varkappa_1^i(\xi)v_0(\xi)d\xi \\ = & \int\limits_{\mathbb{T}^d}\varkappa_1^j(\xi)v_0(\xi)\big(b^i + A\varkappa_1^i(\xi)\big)d\xi \;+\; \int\limits_{\mathbb{T}^d}\varkappa_1^i(\xi)v_0(\xi)\big(b^j + A\varkappa_1^j(\xi)\big)d\xi \\ = & \; b^i\int\limits_{\mathbb{T}^d}\varkappa_1^j(\xi)v_0(\xi)d\xi \;+\; b^j\int\limits_{\mathbb{T}^d}\varkappa_1^i(\xi)v_0(\xi)d\xi \\ + & \int\limits_{\mathbb{T}^d}\varkappa_1^j(\xi)v_0(\xi)A\varkappa_1^i(\xi)d\xi \;+\; \int\limits_{\mathbb{T}^d}\varkappa_1^i(\xi)v_0(\xi)A\varkappa_1^j(\xi)d\xi.\end{aligned} \tag{63}$$

Denote

$$D_2^{ij} = \; b^i\int\limits_{\mathbb{T}^d}\varkappa_1^j(\xi)v_0(\xi)d\xi \;+\; b^j\int\limits_{\mathbb{T}^d}\varkappa_1^i(\xi)v_0(\xi)d\xi, \tag{64}$$

$$\tilde{D}_2^{ij} = \int\limits_{\mathbb{T}^d} \varkappa_1^j(\xi) v_0(\xi) A \varkappa_1^i(\xi) d\xi \ + \int\limits_{\mathbb{T}^d} \varkappa_1^i(\xi) v_0(\xi) A \varkappa_1^j(\xi) d\xi. \tag{65}$$

Then D_2^{ij} coincides with the third integral in (58).

We have to show that $I_3^{ij} = -\tilde{D}_2^{ij}$. We have

$$\begin{aligned} I_3^{ij} \ &= \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi - q)\mu(\xi,q)(\varkappa_1(\xi) - \varkappa_1(q))^i \varkappa_1^j(\xi) v_0(\xi) dq d\xi \\ &- \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi - q)\mu(\xi,q)(\varkappa_1(\xi) - \varkappa_1(q))^i \varkappa_1^j(q) v_0(\xi) dq d\xi \\ &= -\int\limits_{\mathbb{T}^d} A \varkappa_1^i(\xi) \varkappa_1^j(\xi) v_0(\xi) d\xi \ + \ J_3^{ij}. \end{aligned} \tag{66}$$

We rearrange J_3^{ij} using (50):

$$\begin{aligned} J_3^{ij} \ &= \ -\int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{R}^d} a(\xi - q)\mu(\xi,q)(\varkappa_1(\xi) - \varkappa_1(q))^i \varkappa_1^j(q) v_0(\xi) dq d\xi \\ &= \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \hat{a}(\xi - q)\mu(\xi,q)(\varkappa_1(q) - \varkappa_1(\xi))^i \varkappa_1^j(q) v_0(\xi) dq d\xi \\ &= \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \hat{a}(q - \xi)\mu(q,\xi)(\varkappa_1(\xi) - \varkappa_1(q))^i \varkappa_1^j(\xi) v_0(q) dq d\xi \\ &= \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \hat{a}(q - \xi)\mu(q,\xi) v_0(q) dq \ \varkappa_1^i(\xi)\varkappa_1^j(\xi) d\xi - \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} a(q - \xi)\mu(q,\xi) v_0(q) \varkappa_1(q)^i \varkappa_1^j(\xi) \ dq d\xi \\ &= \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \hat{a}(\xi - q)\mu(\xi,q)\varkappa_1^j(\xi) dq \ v_0(\xi)\varkappa_1^i(\xi) d\xi - \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \hat{a}(\xi - q)\mu(\xi,q)\varkappa_1^j(q) dq \ v_0(\xi)\varkappa_1^i(\xi) d\xi \\ &= -\int\limits_{\mathbb{T}^d} A \varkappa_1^j(\xi) v_0(\xi) \varkappa_1^i(\xi) d\xi. \end{aligned} \tag{67}$$

Thus $I_3^{ij} = -\tilde{D}_2^{ij}$ and this relation complete the proof of equality (57).

The structure of (56) implies that $(Ir, r) \ge 0, \ \forall\, r \in \mathbb{R}^d$, and moreover $(Ir, r) > 0$ since $v_0 > 0$ and $\varkappa_1(q)$ is the periodic function while q is the linear function, consequently $\big[\big((\xi - q) + (\varkappa_1(\xi) - \varkappa_1(q))\big)\cdot r\big]^2$ can not be identically 0 if $r \ne 0$. □

Thus, Lemma 3.1 is now completely proved. □

7 A priori estimates

Let $u^0(x,t)$ be a solution of (10) with $u^0(x,0) = \varphi \in \mathcal{S}(\mathbb{R}^d)$. Then $u^0(x,t) \in C^\infty((0,T), \mathcal{S}(\mathbb{R}^d))$ for any T and we can define approximation w^ε of u^0 substituting $u^0(\cdot)$ for $u(\cdot)$ in (13). It follows from Lemma 3.1 that w^ε satisfies the following equation

$$\frac{\partial w^\varepsilon}{\partial t} - L^\varepsilon w^\varepsilon = \frac{\partial u^0}{\partial t}(x^\varepsilon, t) - \Theta \cdot \nabla\nabla u^0(x^\varepsilon, t) \ + \ \phi^\varepsilon(x^\varepsilon, t) = \phi^\varepsilon(x^\varepsilon, t), \quad w^\varepsilon(x,0) = \varphi(x) + \psi^\varepsilon(x) \tag{68}$$

where $x^\varepsilon = x - \frac{b}{\varepsilon}\,t$, and

$$\psi^\varepsilon(x) \;=\; \varepsilon\varkappa_1(\frac{x}{\varepsilon})\cdot\nabla\varphi(x) + \varepsilon^2\varkappa_2(\frac{x}{\varepsilon})\cdot\nabla\nabla\varphi(x) \;\in\; L^2(\mathbb{R}^d).$$

Consequently, the difference $v^\varepsilon(x,t) = w^\varepsilon(x,t) - u^\varepsilon(x,t)$, where u^ε is the solution of (9), satisfies the following problem:

$$\frac{\partial v^\varepsilon(x,t)}{\partial t} - L^\varepsilon v^\varepsilon(x,t) = \phi^\varepsilon(x^\varepsilon,t), \quad v^\varepsilon(x,0) = \psi^\varepsilon(x). \tag{69}$$

Notice that by (22) and Proposition 3.1 we have $\|\psi^\varepsilon\|_{L^2(\mathbb{R}^d)} = O(\varepsilon)$ and $\|\phi^\varepsilon\|_\infty = o(1)$, where $\|\cdot\|_\infty$ is the norm in $L^\infty((0,T),L^2(\mathbb{R}^d))$. We are going to show now that the solution v^ε of (69) tends to zero in $L^\infty((0,T),L^2(\mathbb{R}^d))$ as $\varepsilon\to 0$.

Proposition 7.1. *Let v^ε be the solution of (69) with small ψ^ε and ϕ^ε:*

$$\|\phi^\varepsilon\|_\infty = o(1), \quad \|\psi^\varepsilon\|_{L^2(\mathbb{R}^d)} = O(\varepsilon) \quad as\;\; \varepsilon\to 0.$$

Then

$$\|v^\varepsilon\|_\infty \;\to\; 0 \quad as\;\; \varepsilon\to 0. \tag{70}$$

Proof. Since problem (69) is linear, we consider separately two problems:

$$\frac{\partial v_\psi}{\partial t} - L^\varepsilon v_\psi = 0, \quad v_\psi(x,0) = \psi(x), \tag{71}$$

$$\frac{\partial v_\phi}{\partial t} - L^\varepsilon v_\phi = \phi, \quad v_\phi(x,0) = 0, \tag{72}$$

and prove that $\|v_\psi\|_\infty \le C_1\|\psi\|_{L^2(\mathbb{R}^d)}$ and $\|v_\phi\|_\infty^2 \le C_2\|\phi\|_\infty$ with some constants C_1, C_2 that do not depend on ε, however might depend on T. This immediately implies the required relation in (70).

Denote $v_0^\varepsilon(x) = \tilde v_0(\frac{x}{\varepsilon})$, where $\tilde v_0$ is the periodic extension of the function $v_0 \in L^2(\mathbb{T}^d)$ defined in (34), see also Corollary 1. Multiplying equation (71) by $v_\psi(x,t)\,\tilde v_0(\frac{x}{\varepsilon})$ and integrating the resulting relation over $t\in(0,s)$ and $x\in\mathbb{R}^d$ we have

$$\begin{aligned}\int\limits_{\mathbb{R}^d}\int\limits_0^s \frac{\partial v_\psi(x,t)}{\partial t}\, v_\psi(x,t)\,dt\, v_0^\varepsilon(x)\,dx &= \frac12\int\limits_{\mathbb{R}^d} v_\psi^2(x,s)\,v_0^\varepsilon(x)\,dx - \frac12\int\limits_{\mathbb{R}^d}\psi^2(x)\;v_0^\varepsilon(x)\,dx \\ &= \int\limits_0^s\int\limits_{\mathbb{R}^d} v_\psi(x,t)\,v_0^\varepsilon(x)\,L^\varepsilon v_\psi(x,t)\,dx\,dt.\end{aligned} \tag{73}$$

All integrals in (73) exist since v_0 is uniformly bounded, see (51). The last integral in (73) can be analysed in the same way as the term I_3 in the proof of Proposition 6.1, see (66) - (67). This yields

$$\int\limits_{\mathbb{R}^d} L^\varepsilon v_\psi(x,t)\,v_\psi(x,t)\,v_0^\varepsilon(x)\,dx \;=\; (L^\varepsilon v_\psi, v_\psi)_{v_0} \;\le\; 0 \quad \text{for all}\;\; t\in[0,T],$$

and consequently,

$$\int\limits_{\mathbb{R}^d} v_\psi^2(x,s)\,v_0^\varepsilon(x)\,dx \;\le\; \int\limits_{\mathbb{R}^d}\psi^2(x)\;v_0^\varepsilon(x)\,dx \qquad \text{for all}\;\; s\in(0,T).$$

Using the estimates in (51) for v_0 we conclude that

$$\|v_\psi(\cdot,s)\|_{L^2(\mathbb{R}^d)} \;\le\; C_1\|\psi\|_{L^2(\mathbb{R}^d)} \tag{74}$$

with a constant C_1 which does not depend on $s \in (0,T)$. Thus

$$\|v_\psi\|_\infty \ \le\ C_1 \|\psi\|_{L^2(\mathbb{R}^d)}. \tag{75}$$

Using the same reasoning for the second equation (72) we obtain

$$\frac{1}{2}\int\limits_{\mathbb{R}^d} v_\phi^2(x,s)\, v_0^\varepsilon(x)\, dx \ -\ \int\limits_0^s \int\limits_{\mathbb{R}^d} \phi(x,t)\, v_\phi(x,t)\, v_0^\varepsilon(x)\, dx\, dt \ =\ \int\limits_0^s (L^\varepsilon v_\phi, v_\phi)_{v_0}\, dt \ \le\ 0. \tag{76}$$

Recalling the bounds in (51), by the Schwartz inequality we derive from (76) that

$$\frac{\tilde\gamma_1}{2}\|v_\phi(\cdot,s)\|^2_{L^2(\mathbb{R}^d)} \ \le\ \frac{\tilde\gamma_2}{2}\int\limits_0^s \|\phi(\cdot,t)\|_{L^2(\mathbb{R}^d)}\, \|v_\phi(\cdot,t)\|_{L^2(\mathbb{R}^d)}\, dt \ \le\ \frac{\tilde\gamma_2}{2}\, s\, \|\phi\|_\infty\, \|v_\phi\|_\infty \tag{77}$$

for any $s \in (0,T)$. Consequently,

$$\|v_\phi\|_\infty \le C_2(T)\, \|\phi\|_\infty.$$

□

Since $\|w^\varepsilon(x,t) - u^0(x - \frac{b}{\varepsilon}\, t, t)\|_\infty \to 0$ by (13), then (70) immediately yields

$$\|u^\varepsilon(x,t) \ -\ u^0(x - \frac{b}{\varepsilon}\, t, t)\|_\infty \ \to\ 0 \quad \text{or} \quad \|u^\varepsilon(x + \frac{b}{\varepsilon}\, t, t) \ -\ u^0(x,t)\|_\infty \ \to\ 0 \quad \text{as} \ \varepsilon \to 0. \tag{78}$$

Thus we proved (11) for a dense in $L^2(\mathbb{R}^d)$ set of initial data, when $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

We can complete now the proof of Theorem 2.1. For any $\varphi \in L^2(\mathbb{R}^d)$ and for any $\delta > 0$ there exists $\varphi_\delta \in \mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi - \varphi_\delta\|_{L^2(\mathbb{R}^d)} < \delta$. We denote by u_δ^ε and u_δ^0 the solution of (9) and (10) with initial data φ_δ. Since (10) is the standard Cauchy problem for a parabolic operator with constant coefficients, its solution admits the classical upper bound

$$\|u^0(x,t) \ -\ u_\delta^0(x,t)\|_\infty \ \le \|\varphi - \varphi_\delta\|_{L^2(\mathbb{R}^d)} \ < \delta \tag{79}$$

for any $T > 0$. By the estimate in (74) we obtain

$$\|u_\delta^\varepsilon(x,t) \ -\ u^\varepsilon(x,t)\|_\infty \ \le\ C_1\, \delta. \tag{80}$$

Since the upper bounds in (79) - (80) are valid with an arbitrary small $\delta > 0$, then (78) - (80) imply that

$$\|u^\varepsilon(x + \frac{b}{\varepsilon}\, t, t) - u^0(x,t)\|_\infty \to 0, \quad \text{as} \ \varepsilon \to 0.$$

This completes the proof of Theorem 2.1.

8 Small perturbations of symmetric kernels. Einstein relation.

Let us assume in this section that $\mu(\xi,\eta) = \mu(\eta,\xi)$ and consider a kernel $a(z)$ satisfying (5) - (6) of a special form:

$$a(z) \ =\ a_{\rm sym}(z) \ +\ \boldsymbol{\ell}\cdot c(z), \tag{81}$$

where $a_{\rm sym}(-z) = a_{\rm sym}(z)$ is a symmetric function that also satisfies (5) - (6), $c(z)$ is an antisymmetric vector function, that is $\boldsymbol{\ell}\cdot c(z) = \ell^i c^i(z)$, $c^i(-z) = -c^i(z)$, $i = 1,\dots,d$; $c(z)$ satisfies condition (6), and $\boldsymbol{\ell} \in \mathbb{R}^d$ is a constant vector of a small norm. We assume here and in the sequel summation over repeated indices. We also consider in this section a special case of antisymmetric perturbation of the form

$$c_{\boldsymbol{\ell}}(z) = z a_{\rm sym}(z) \omega_{\boldsymbol{\ell}}(z),$$

where $\omega_{\boldsymbol{\ell}}(z) = \omega(|\boldsymbol{\ell}|\,|z|)$, and $\omega(s)$ is a $C_0^\infty(\mathbb{R})$ function such that $0 \le \omega(\cdot) \le 1$, $\omega(s) = 1$ for $s \in [0, \frac14]$, and $\omega(s) = 0$ for $s \ge \frac12$.

Lemma 8.1. *Let $b(\boldsymbol{\ell}) \in \mathbb{R}^d$ be the effective drift vector corresponding to the problem* (24) *with $a(z)$ given by* (81)*. Then, for small $\boldsymbol{\ell}$,*

$$b^i(\boldsymbol{\ell}) \;=\; \ell^j \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\, c^j(z)\, \mu(\xi, \xi - z) dz d\xi \;+\; \ell^j \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\, a_{\rm sym}(z) \mu(\xi, \xi - z)\, \tilde\varphi_0^j(\xi) dz d\xi + O(|\boldsymbol{\ell}|^2), \quad (82)$$

where $\tilde\varphi_0 = \{\tilde\varphi_0^i\} \in (L^2(\mathbb{T}^d))^d$ is the solution of the problem

$$\int\limits_{\mathbb{R}^d} a_{\rm sym}(\xi - q) \mu(\xi, q) \big(\tilde\varphi_0^i(q) - \tilde\varphi_0^i(\xi)\big)\, dq \;=\; 2 \int\limits_{\mathbb{R}^d} c^i(\xi - q) \mu(\xi, q)\, dq \quad (83)$$

with $\int\limits_{\mathbb{T}^d} \tilde\varphi_0^i(\eta) d\eta = 0$.

In the special case, when $c_{\boldsymbol{\ell}}(z) = z\, a_{\rm sym}(z)\, \omega_{\boldsymbol{\ell}}(z)$ and $b(\boldsymbol{\ell})$ is defined by (82) - (83) *with $c(z) = c_{\boldsymbol{\ell}}(z)$, we obtain the so-called Einstein relation:*

$$\frac{\partial b^i(\boldsymbol{\ell})}{\partial \ell^j}\Big|_{\boldsymbol{\ell}=0} \;=\; 2\Theta^{ij}_{\rm sym}, \quad (84)$$

where $\Theta_{\rm sym}$ is the effective matrix of problem (9) *corresponding to the symmetric kernel $a_{\rm sym}(x - y)\mu(x, y)$.*

Remark 8.1. *Notice that the symmetric part of $2\Theta^{ij}_{\rm sym}$ coincides with $I^{ij}_{\rm sym} = \Theta^{ij}_{\rm sym} + \Theta^{ji}_{\rm sym}$.*

Proof. Since the operator K and the function G defined in (31) and (32), respectively, depend on a vector parameter $\boldsymbol{\ell}$ smoothly, and $\lambda = 1$ is a simple eigenvalue of the operator $((G(\cdot))^{-1}K)^*$, then the corresponding eigenfunction $\psi_0 = \psi_0^{\boldsymbol{\ell}} \in L^2(\mathbb{T}^d)$ is also a smooth function of a parameter $\boldsymbol{\ell}$. So is $v_0 = v_0^{\boldsymbol{\ell}}$. Using the perturbation theory arguments we conclude that for small $\boldsymbol{\ell}$ the function $v_0^{\boldsymbol{\ell}} \in L^2(\mathbb{T}^d)$ defined by (34) admits the following representation

$$v_0^{\boldsymbol{\ell}}(\xi) \;=\; \mathbf{1} \;+\; \boldsymbol{\ell}\, \tilde\varphi_0(\xi) \;+\; O(|\boldsymbol{\ell}|^2), \quad \tilde\varphi_0 \in (L^2(\mathbb{T}^d))^d, \quad (85)$$

where $\mathbf{1}$ stands for the function identically equal to 1 on $\mathbb{T}^d$. We used here the fact that

$$\mathrm{span}(\mathbf{1}) = \mathrm{Ker}\,(G_{\rm sym} - K_{\rm sym})^* = \mathrm{Ker}\,(G_{\rm sym} - K_{\rm sym}), \quad (86)$$

where operators K, G are defined by (31) and (32) respectively, and we denote by $K_{\rm sym}$, $G_{\rm sym}$ the operators related to the symmetric kernel $a_{\rm sym}(x - y)\mu(x, y)$.

Substituting (85) in the relation $K^* v_0^{\boldsymbol{\ell}} \;=\; G v_0^{\boldsymbol{\ell}}$ we obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}^d} a_{\rm sym}(q - \xi) \mu(q, \xi) \big(\mathbf{1} + \ell_i \tilde\varphi_0^i(q)\big)\, dq \;+\; \ell^j \int\limits_{\mathbb{R}^d} c^j(q - \xi) \mu(q, \xi) \big(\mathbf{1} + \ell^i \tilde\varphi_0^i(q)\big)\, dq \;+\; O(|\boldsymbol{\ell}|^2) \\ &=\; \big(\mathbf{1} + \ell^i \tilde\varphi_0^i(\xi)\big) \Big[\int\limits_{\mathbb{R}^d} a_{\rm sym}(\xi - q) \mu(\xi, q)\, dq \;+\; \ell^j \int\limits_{\mathbb{R}^d} c^j(\xi - q) \mu(\xi, q)\, dq \Big] \;+\; O(|\boldsymbol{\ell}|^2). \end{aligned} \quad (87)$$

Relations (86) - (87) yield

$$
\begin{aligned}
&\ell^i \int\limits_{\mathbb{R}^d} a_{\rm sym}(\xi-q)\mu(\xi,q)\tilde\varphi_0^i(q)\,dq \;-\; \ell^i \int\limits_{\mathbb{R}^d} c^i(\xi-q)\mu(\xi,q)\,dq \;+\; O(|\boldsymbol{\ell}|^2)\\
&= \;\ell^i\,\tilde\varphi_0^i(\xi) \int\limits_{\mathbb{R}^d} a_{\rm sym}(\xi-q)\mu(\xi,q)\,dq \;+\; \ell^i \int\limits_{\mathbb{R}^d} c^i(\xi-q)\mu(\xi,q)\,dq \;+\; O(|\boldsymbol{\ell}|^2).
\end{aligned} \tag{88}
$$

Collecting the terms of the order $|\boldsymbol{\ell}|$ in (88) we deduce the equation for $\tilde\varphi_0^i$:

$$
\int\limits_{\mathbb{R}^d} a_{\rm sym}(\xi-q)\mu(\xi,q)\big(\tilde\varphi_0^i(q)-\tilde\varphi_0^i(\xi)\big)\,dq \;=\; 2\int\limits_{\mathbb{R}^d} c^i(\xi-q)\mu(\xi,q)\,dq. \tag{89}
$$

Our subsequent reasoning relies on the following statement.

Proposition 8.1. *If $\alpha(-z)=\alpha(z)$ for all $z\in\mathbb{R}^d$ and $\alpha\in L^1(\mathbb{R}^d)$, then*

$$
\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} \alpha(\xi-q)\mu(\xi,q)\,d\xi dq = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} \alpha(\xi-q)\mu(q,\xi)\,d\xi dq; \tag{90}
$$

if $\beta(-z)=-\beta(z)$ for all $z\in\mathbb{R}^d$ and $\beta\in L^1(\mathbb{R}^d)$, then

$$
\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} \beta(\xi-q)\mu(\xi,q)\,d\xi dq = -\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} \beta(\xi-q)\mu(q,\xi)\;d\xi\;dq. \tag{91}
$$

The proof is the same as that of Proposition 7 in [6]. It is straightforward to check that the arguments used in the proof given in [6] also apply to the operators considered here. We leave the details to the reader. □

Since $c^i(-z)=-c^i(z)$ by our assumption, then Proposition 8.1 yields

$$
\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} c^i(\xi-q)\mu(\xi,q)\,dqd\xi \;=\; 0,
$$

and consequently, there exists a unique (up to an additive constant) solution $\tilde\varphi_0\in(L^2(\mathbb{T}^d))^d$ of (89). We choose the additive constant in such a way that $\int\limits_{\mathbb{T}^d}\tilde\varphi_0^i(\xi)d\xi=0$ for any component of $\tilde\varphi_0$. Then (85) implies that $\int\limits_{\mathbb{T}^d} v_0^{\boldsymbol{\ell}}(\xi)d\xi = 1+O(|\boldsymbol{\ell}|^2)$, and from (36) and (85) we obtain that

$$
\begin{aligned}
b^i(\boldsymbol{\ell}) \;&=\; \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\left(a_{\rm sym}(z)+\ell^j c^j(z)\right)\mu(\xi,\xi-z)\,dz\,\big(\mathbf{1}+\ell^j\tilde\varphi_0^j(\xi)\big)\,d\xi \;+\; O(|\boldsymbol{\ell}|^2)\\
&=\;\ell^j \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\,c^j(z)\,\mu(\xi,\xi-z)\,dzd\xi \;+\; \ell^j \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\,a_{\rm sym}(z)\,\mu(\xi,\xi-z)\,\tilde\varphi_0^j(\xi)\,dzd\xi + O(|\boldsymbol{\ell}|^2).
\end{aligned} \tag{92}
$$

In the case when $c_{\boldsymbol{\ell}}(z) \;=\; z\,a_{\rm sym}(z)\,\omega_{\boldsymbol{\ell}}(z)$, it follows from equation (89) that

$$
\tilde\varphi_0^{\boldsymbol{\ell}} = 2\varkappa_{\rm sym} + r_{\boldsymbol{\ell}}, \quad \tilde\varphi_0^{\boldsymbol{\ell}}\in(L^2(\mathbb{T}^d))^d, \tag{93}
$$

where $\varkappa_{\rm sym}$ is the first corrector of the symmetric problem (9) that satisfies the equation

$$\int\limits_{\mathbb{R}^d} a_{\rm sym}(\xi-q)\,\mu(\xi,q)\,\big(\varkappa_{\rm sym}(q)-\varkappa_{\rm sym}(\xi)\big)\,dq \;=\; \int\limits_{\mathbb{R}^d} a_{\rm sym}(\xi-q)\,(\xi-q)\,\mu(\xi,q)\,dq, \tag{94}$$

see also [6], and

$$\|r_{\boldsymbol\ell}\|_{(L^2(\mathbb{T}^d))^d} \;\to\; 0 \quad \text{ as } \boldsymbol\ell\to 0. \tag{95}$$

Indeed, denoting

$$g_{\boldsymbol\ell}(\xi)=\int\limits_{\mathbb{R}^d} z\,a_{\rm sym}(z)\,\big(1-\omega_{\boldsymbol\ell}(z)\big)\,\mu(\xi,\xi-z)\,dz$$

and using (6) we get that

$$\max_{i=1,\dots,d}\,\max_{\xi\in\mathbb{T}^d}|g^i_{\boldsymbol\ell}(\xi)| \;\to\; 0 \text{ as } \boldsymbol\ell\to 0, \quad \text{and consequently } \;\|g_{\boldsymbol\ell}\|_{(L^2(\mathbb{T}^d))^d} \;\to\; 0 \text{ as } \boldsymbol\ell\to 0. \tag{96}$$

After substitution (93) in (89), considering equation (94), we come to a conclusion that $r_{\boldsymbol\ell}$ satisfies the following equation:

$$\int\limits_{\mathbb{T}^d} \hat a_{\rm sym}(\xi-\eta)\,\mu(\xi,\eta)\,(r_{\boldsymbol\ell}(\eta)-r_{\boldsymbol\ell}(\xi))d\eta \;=\; 2g_{\boldsymbol\ell}(\xi), \quad r_{\boldsymbol\ell}\in(L^2(\mathbb{T}^d))^d. \tag{97}$$

We can rewrite (97) as

$$\Big(K_{\rm sym} \;-\; G_{\rm sym}\Big)r_{\boldsymbol\ell} \;=\; 2g_{\boldsymbol\ell}, \tag{98}$$

where the operators

$$K_{\rm sym}f(\xi)=\int\limits_{\mathbb{T}^d}\hat a_{\rm sym}(\xi-\eta)\,\mu(\xi,\eta)\,f(\eta)d\eta,\; G_{\rm sym}f(\xi)=G(\xi)f(\xi),\; G(\xi)=\int\limits_{\mathbb{T}^d}\hat a_{\rm sym}(\xi-\eta)\,\mu(\xi,\eta)\,d\eta$$

apply component-wise. Considering each component of $r_{\boldsymbol\ell}=\{r^i_{\boldsymbol\ell}\}$ separately and applying the Krein-Rutman theorem to the compact positivity improving operator $K_{\rm sym}$ (see [5], Section 6, or [7], Theorem XIII.44), we conclude that the operator $\Big(G^{-1}_{\rm sym}K_{\rm sym}-\mathbb{E}\Big)$ is invertible on $L^2(\mathbb{T}^d)\ominus\{\mathbf{1}\}$. Consequently, equation (97) has a unique solution $r^i_{\boldsymbol\ell}\in L^2(\mathbb{T}^d)\ominus\{\mathbf{1}\}$. Moreover, (96) and (98) imply that

$$\|r^i_{\boldsymbol\ell}\|_{L^2(\mathbb{T}^d)} \;\to\; 0 \quad \text{ as } \boldsymbol\ell\to 0, \tag{99}$$

and thus (95) holds.

Next we substitute the right-hand side of (93) for $\tilde\varphi_0$ in (92), and transform the resulting relation with the help of Proposition 8.1 and (99). This yields

$$\begin{aligned}
b^i(\boldsymbol\ell)&=\ell^j\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\,z^j\,a_{\rm sym}(z)\,\omega_{\boldsymbol\ell}(z)\,\mu(\xi,\xi-z)\,dzd\xi\\
&-\,2\ell^j\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\,\big(\varkappa^j_{\rm sym}(\xi-z)+\frac12\,r^j_{\boldsymbol\ell}(\xi-z)\big)\,a_{\rm sym}(z)\,\omega_{\boldsymbol\ell}(z)\,\mu(\xi,\xi-z)\,dzd\xi+O(|\boldsymbol\ell|^2)\\
&=\ell^j\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\,z^j\,a_{\rm sym}(z)\mu(\xi,\xi-z)dzd\xi\\
&-\,2\ell^j\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{T}^d} z^i\,\varkappa^j_{\rm sym}(\xi-z)\,a_{\rm sym}(z)\mu(\xi,\xi-z)dzd\xi+o(|\boldsymbol\ell|).
\end{aligned}\tag{100}$$

Since in the symmetric case $b(0) = 0$ and $v_0 = \mathbf{1}$, then comparing (100) with (55) and using one more time the statement of Proposition 8.1 we come to (84). □

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