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Preprint submitted to Nonlinear Studies / MESA

Dual quaternion control: a review of recent results within motion control

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Dedicated to the 75th anniversary of Professor Lars-Erik Persson

Abstract. This paper presents a review on recent results in the field of dual quaternion based motion control. In addition, we derive two control laws for trajectory tracking control of a fully actuated rigid-body based on the resemblance of dual quaternion kinematics and dynamics to the quaternion based rotational kinematics and dynamics. A velocity error sliding surface and an integrator backstepping controller is derived and uniform asymptotic stability is shown, and the former is subsequently extended to the problem of trajectory tracking of the underactuated quadrotor platform using the hand-position technique. Numerical simulations demonstrate the theoretical results.

2010 **Mathematics Subject Classification**: 11R52; 15A99
Keywords: Dual Quaternions, Clifford Algebra, Rigid-Body Modelling, Dynamics, Control.

1 Introduction

1.1 Background

Control of rigid-bodies in three-dimensional space is an important challenge with broad impact to a number of mechanical systems including, but not limited to; unmanned aerial vehicles, satellites, autonomous underwater vehicles and robot manipulators. The Newton-Euler equations completely describe the motion of a rigid-body having six degrees of freedom (6-DOF), however the rotational and translational movement is often considered separately resulting in control laws designed to deal with 3+3-DOF motion. The reformulation of the equations of motion using dual quaternions combines translation and rotation into a unified framework allowing efficient and compact notation. Moreover, the common framework facilitates concurrent control law design for full 6-DOF motion. The advantages realizing 6-DOF control, as supposed to 3+3 DOF, are greatest in systems where translational and rotational motion is highly coupled [1]; e.g. underactuated systems such as fixed-wing aerial vehicles and multirotors.

The relevance is especially great for applications such as formation flying, aerial towing, near-earth environment inspection and spacecraft rendezvous and docking. It is imperative for these scenarios that the full 6-DOF coupled motion of a rigid body is taken into account in control design [2], as it is noted in [3] "*..., the stability of the overall 6-DOF system may not be directly implied by the individually stable translation and rotation systems and must be further addressed*". Further, several authors state that dual quaternions is the most compact and efficient way to express motion in 3-D space [4, 5, 6, 7], e.g. in [8] the author notes that dual quaternion algebra, which is isomorphic to the even sub-algebra of Euclidean projective geometric algebra of order three, *is the smallest known algebra that can model Euclidean transformations in a structure preserving manner*. This compactness makes equations of motion derived using dual quaternions well behaved numerically as the co-dimension of the solution space within the integration space is small compared to matrix formulation [9]. Moreover, the straightforward process of normalizing brings the integrated dual quaternion back to the solution space. The main disadvantage of using dual quaternions is the fact that the unit dual quaternion group is endowed with a double representation of every pose in the configuration space, a fact that may lead to unwinding if not special care is taken during control design.

1.2 Previous work

1.2.1 Dual quaternions

When William K. Clifford in 1878 introduced a new multiplication rule into Hermann G. Grassmann's exterior algebra by means of an orthonormal basis he created Clifford's geometric algebras [10]. Grassmann originally intended that his "Extension theory" was to transform geometry into algebra, in short his work describes coordinate free algebraic methods for computing with a space and all its subspaces [11]. In Clifford's seminal paper [12] he combines ideas from the work of Grassmann with ideas from William R. Hamilton, the inventor of the quaternion, by the introduction of a new product. This allowed for the measurement of length and angle and thus allowed him to describe an algebra for metric geometries, not just projective geometry. Notably, two years after Clifford's first publication on Clifford's algebras, Rudolf Lipschitz reinvented it, and was the first to apply it to geometry in his exploration of rotation [13]. A large number of Clifford's algebras and subalgebras have been studied, often independently and under different names; *complex numbers, hyperbolic numbers, dual numbers, quaternions, biquaternions, bicomplex numbers and dual quaternions* to name a few examples.

Dual quaternions were first studied by Clifford in [14] under the name *biquaternions*, however, an early death prevented Clifford from completing the development of the framework [15]. Further work was carried out by McAulay in [16], where the author tries to coin the term *octonions* for Clifford's *biquaternions*. At the same time the German mathematician Eduard Study applied the work of Clifford to kinematics of rigid-bodies in [17] and [18]. Study's work developed the kinematic motion of the rigid body as a point in a six-dimensional manifold in eight-dimensional space, and although Study avoided the term quaternion his work became known as what we now call dual quaternions [15]. The group of unit dual quaternions is a covering group of the Lie group of proper rigid motion, and the associated Lie algebra is captured by Screw theory, developed by Sir Robert Ball [19]. By using pure dual quaternions one is able to represent elements in Screw theory as dual quaternions. Blasche [20] and Dimentberg [21] continued the use of dual quaternions in the study of mechanisms after Study. The authors Yang and Freudenstein were the first to apply line transformation operator mechanisms by using the dual quaternion as the transformation operator in [22]. The work of Study was further developed by Ravani and Roth in [23] to represent Euclidean displacements using four coordinates in a dual-space. This work was later used by Dooley and McCarthy to solve the general dynamics problem using dual quaternions [24]. Despite the numerous results on kinematics, problems arise when one tries to uphold the principle of transference to dynamics. The principle states that [25]; *when dual numbers replace real ones all laws of vector algebra, which describe the kinematics of rigid body with one point fixed, are also valid for motor algebra, which describes a free rigid body*. Part of the problem is caused by the fact that the time derivative of a screw does not follow the dual vector transformation rule, i.e. it's not a proper dual vector but rather a pseudo dual vector as pointed out by Yang in [26]. The same author later extended the use of dual quaternions in one of the earliest application towards rigid-body dynamics in [27].

From Screw theory we have that linear and angular momenta are quantities known as co-screws, as they are intended to be dual to the screws [28]. Co-screws can also be used to represent wrenches, i.e. combinations of force and torque. In order to complete the framework one needs to include an operator that convert velocity into momenta, i.e. screws into co-screws, namely inertia [28]. Roughly speaking, the operator needs to swap the order of the angular and linear parts when mapping from screws to co-screws, to be consistent with Screw theory and the dual vector transformation rule. The first to work on dual number representation of dynamics, Kotelnikov, introduced the concept of the inertia binor. The binor is a combination of two dual matrices, and the binor applied to a dual vector produces a new dual vector. However, Kotelnikovs binor is not analytic [21], and in an attempt to obtain a general formulation [25] introduced an approach for including inertias by using the *dual inertia operator*. The dual-inertia operator can be taught of as the inverse of the dual operator ε, as when it is applied to a dual number returns it's dual part as a real part. Brodsky and Shoham [29] used dual-numbers to express the six-dimensional motion of a rigid-body in a three-dimensional dual-space using this dual-inertia operator in a dual matrix. This keeps the dynamics compact, however the inverse of the dual matrix must be explicitly introduced as it is not well defined. It can be shown by using the dual Moore-Penrose pseudoinverse that the inverse dual matrix with this construction is not associative.

In an effort to improve on this Filipe and Tsiotras introduced the invertible diagonal 8-by-8 dual inertia matrix in [30]. This construction has to be applied with an artificial swap-operator, defined on dual quaternions, that flips the order of angular and linear velocity in the dual vector screws, i.e. making it into a pseudo-dual vector. An improved approach was presented in [31] where an invertible 8-by-8 block anti-diagonal matrix was proposed that works without requiring a swap operator for the dual vector screws and the result is consistent with Screw theory. Dual quaternions have since then been applied to kinematic and dynamic analysis in many research areas such as robotics [32], estimation [33], vision [34], navigation [35], inverse kinematics [36], computer graphics [7] and neuroscience [37].

1.2.2 Dual Quaternion Control

The earliest applications of dual quaternions in control were within the field of robotics; one of the earliest contributions was made by Dooley and McCarthy in [38] where the authors used dual quaternion coordinates for modelling and controlling cooperating robotic arms. In Pham et al. [39] the authors presented developments of the forward kinematic model and Jacobian matrix for a robot manipulator in dual quaternion space for position and orientation control, including a proof of asymptotic stability. A more extensive review of contributions of dual quaternion control in robotics can be found in [40]. The last decade has seen numerous applications of dual quaternions to 6-DOF motion control, especially control of fully-actuated spacecrafts and spacecraft formations. In Han et al. [41] the framework is explored for control of oriented mechanical systems, where the dual quaternion logarithm is defined and subsequently used in the development of feedback control laws for the regulation and tracking. Wang et al. derive in [42] the 6-DOF relative motion model of spacecraft leader-follower formations using unit dual quaternions, where the authors propose two terminal sliding-mode (TSM) control laws that ensures convergence of the dynamical synchronization error to the desired trajectory in finite time, the latter of which handle the dual equilibrium problem associated with quaternion attitude representation. The same year, Wang and Sun extended their TSM controller to a robust adaptive tracking controller that ensures finite time convergence of the relative tracking errors with incomplete information on mass and inertia properties of the spacecraft [43], albeit requiring a priori knowledge of the upper bounds on the mass, maximum eigenvalue of the inertia matrix, bounds on unknown disturbance forces and torques and system states. A related result was presented by Filipe and Tsiotras in [44]; specifically, they present an adaptive tracking controller for satellite proximity operations that ensures almost global asymptotic stability of the linear and angular position and velocity errors in the presence of constant unknown disturbance forces and torques. No a priori information is required on upper bounds for system parameters and states, and in addition sufficient conditions for mass and inertia matrix identification is presented. Wang et al. presented in [45] a PD-like controller for coordinated control of spacecraft formation, while in [46] finite-time stability is shown for a nonlinear adaptive feedback control law which is shown to be fault-tolerant.

Gui and Vukovich apply dual quaternions for satellite pose control in [2], providing proof of almost global asymptotic convergence of the tracking error as well as development of an adaptive controller which provides estimation of unknown parameters and disturbances. In the more general sense the same authors show in [3] that uniform almost global finite-time stability for the pose control of a rigid body can be achieved without velocity feedback, and similar results can be found in [30]. Lee and Mesbahi applied dual quaternions to the challenging task of spacecraft powered descent guidance and control in [47, 31, 48]. In [47] the authors present a Lyapunov-based general framework for analysis of both unconstrained and constrained coupled rotational and translational control problems using unit dual quaternions, while in [31, 48] a model predictive control approach is used to account for line-of-sight and glide slope constraints, as well as ensuring that a fuel optimal trajectory is tracked.

In [1] Price presents results on variable structure nonlinear control; a sliding mode controller is developed for use in dual quaternion systems with unknown control direction, where the creation of multiple sliding surfaces for the system in extended state space mitigates the control problem. Recent results by Dong et al. [49] presents a smooth 6-DOF observer that is combined with an independently designed PD-like state-feedback control law. The separation property between observer and controller is ensured by way of Lyapunov strictification, and shows almost global asymptotic stability of the closed-loop tracking error dynamics. Further, the authors propose a map from unit dual quaternions to the ad-joint representation of the homogeneous transformation matrix, which they call dual transformation matrix.

In Andersen et al. [50] the authors provide a simplified methodology for deriving a backstepping controller by the introduction of an anti-diagonal matrix in the augmented Lyapunov function, thus avoiding the use of swap operators. This subtle addition was also utilized by Gui and de Ruiter at roughly the same time in [51], in where the authors present an adaptive fault-tolerant hybrid integral sliding mode control law, providing proof of global finite-time convergence of the pose tracking errors. Additional results on fault tolerant spacecraft control can be found in Dong et al. [46]. The compactness of dual quaternions makes them well suited for composite systems, a fact that is utilized by Valverde and Tsiotras [52, 40] with their work on spacecraft-mounted robotic systems. Mello et al. present in [53] an application of dual quaternions for modeling and kinematic control of an unmanned aerial manipulator consisting of a quadrotor serially coupled with a three-link manipulator. Despite the now large number of applications of dual quaternions to control of fully-actuated systems, applications to under-actuated systems are sparse; recent result in [54] propose a PD+ trajectory tracking controller for an underactuated quadrotor platform. Roughly speaking, the approach uses two virtual frames to map the underactuated control problem into a fully actuated one thus allowing a plethora of control design techniques to be applied, at the cost of only achieving practical stability.

1.3 Contribution

The contribution in this paper, in addition to the above given review, is two trajectory tracking control laws derived from a nonlinear dual quaternion based model of a fully actuated rigid-body. The control laws are derived from attitude-only quaternion based trajectory tracking laws from literature, highlighting the resemblance between dual quaternion based pose models and quaternion based attitude models. Using a hand-position technique the underactuated trajectory tracking problem is then solved by application of two virtual frames and the extension of the previously derived control laws. The rest of this paper is organised as follows: In Section 2 we present the essential preliminaries with regards to Clifford algebra, quaternions and dual quaternions. In Section 3 the rigid body modelling using the dual quaternion formulation is presented. The trajectory tracking control laws are derived in Section 4 along with proof of stability properties and numerical simulations that demonstrate the theoretical results. Finally, a short conclusion is given in Section 5 and proofs of two Lemmas are presented in Appendix.

2 Preliminaries

2.1 Notation and coordinate reference frames

Throughout this paper scalar values are denoted in normal face, vectors in boldface while matrices are written in capital boldface letters with their dual equivalents being denoted by $\hat{(\cdot)}$. The time derivative is denoted as $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$, and its second derivative as $\ddot{\mathbf{x}} = \frac{d^2\mathbf{x}}{dt}$. The Euclidean norm is denoted by $\|\cdot\|$ while the supremum norm is denoted as $|\cdot|_\infty$. Note that $\mathbf{I}_{n\times n}$ denotes an $n \times n$ identity matrix while

$\mathbf{0}_{n\times m}$ denotes an $n\times m$ matrix of zeros. A function $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if α is strictly increasing, continuous and $\alpha(0)=0$. Moreover, α is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. Vectors are decomposed in different reference frames denoted by superscripts, where $\mathcal{F}^b$ is the body frame, $\mathcal{F}^d$ is the desired frame and $\mathcal{F}^n$ is the North-East-Down (NED) frame which is assumed to be inertial. The rotation matrix from $\mathcal{F}^b$ to $\mathcal{F}^n$ is denoted as $\mathbf{R}_b^n\in SO(3)$, where

$$SO(3):=\{\mathbf{R}\in\mathbb{R}^{3\times 3}:\mathbf{R}^\top\mathbf{R}=\mathbf{I}_{3\times 3},\det(\mathbf{R})=1\} \tag{2.1}$$

is the special orthogonal group of order three. In this work we use unit quaternions to represent rotations, and the equivalent attitude quaternion will be denoted as $\mathbf{q}_{n,b}$. The homogeneous transformation matrix from $\mathcal{F}^b$ to $\mathcal{F}^n$ is denoted as $\mathbf{H}_b^n\in SE(3)$, where

$$SE(3):=\left\{\mathbf{H}\in\mathbb{R}^{4\times 4}:\mathbf{H}=\begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0}_{1\times 3} & 1\end{bmatrix},\mathbf{R}\in SO(3),\mathbf{p}\in\mathbb{R}^3\right\} \tag{2.2}$$

is the group of proper Euclidean motion in three dimensional space. In this work we will use unit dual quaternions to represent transformations, and the equivalent pose dual quaternion is denoted $\hat{\mathbf{q}}_{n,b}$. The angular velocity is denoted $\omega_{b,c}^a$, which is to say the angular velocity of c relative b referenced in a. For any arbitrary vectors $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{R}^3$, we denote the cross-product operator as $\mathbf{S}(\mathbf{v}_1)\mathbf{v}_2=\mathbf{v}_1\times\mathbf{v}_2$.

2.2 Clifford (Geometric) Algebra

As mentioned in the introduction the formal definition of dual quaternions and its algebra can be found in Clifford algebras and a good introduction to the subject can be found in [55]. These are unital and associative algebras over a vector space V with a quadratic form $v^2=Q(v),\ v\in V$. Any Clifford algebra has a set of basis vectors that anti-commute

$$\mathbf{e}_i\mathbf{e}_j+\mathbf{e}_j\mathbf{e}_i=0,\qquad i\neq j \tag{2.3}$$

and square to $1,-1$ or 0. We denote the Clifford algebras by its signature as $Cl_{(p,q,r)}$, where p is the number of generators that square to 1, q the number that square to -1, and r the number that square to 0. The algebra is said to be degenerate if $r\neq 0$. Further, $n=p+q+r$, where n is the dimension of V, the standard basis elements of the algebra are denoted $\mathbf{e}_i$ for $i\in\{0,1\ldots n\}$ where $\mathbf{e}_0=1$ is the scalar. The elements of a Clifford algebra are graded, and the product of basis vectors form different grade elements, sometimes referred to as *monomials*, and in this work we denote them as $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3:=\mathbf{e}_{123}$.
Grade zero elements correspond to the scalar ($\mathbf{e}_0$), grade 1 elements are the basis vectors, grade 2 elements are bi-vectors ($\mathbf{e}_{ij}\ \forall 0<i,j\leq n, i\neq j$) and so on up to grade n elements known as psedoscalars. Together all elements of the algebra make up the canonical basis of the algebra. Given $\mathbf{a}\in Cl_{(p,q,r)}$, its grade k elements we denote $\langle\mathbf{a}\rangle_k$. Any element of the Clifford algebra can be added to any other element which produces a general element called a multivector. The basic operator of Clifford algebra is the linear and invertible geometric product, also called the Clifford product. For any two multivectors $\mathbf{a}$, $\mathbf{b}$, the geometric product is the multiplication of each operand with each others operand [56]. This means that the geometric product results in a linear combination of different basis grade elements, and from this definition one defines a number of other products; the geometric product of a grade k-vector, $\mathbf{a}_k$ with a grade s-vector, $\mathbf{b}_s$, produce [11]

$$\mathbf{a}_k\cdot\mathbf{b}_s:=\langle\mathbf{a}_k\mathbf{b}_s\rangle_0\quad \mathbf{a}_k\lfloor\mathbf{b}_s:=\langle\mathbf{a}_k\mathbf{b}_s\rangle_{k-s}\quad \mathbf{a}_k\rfloor\mathbf{b}_s:=\langle\mathbf{a}_k\mathbf{b}_s\rangle_{s-k}\quad \mathbf{a}_k\wedge\mathbf{b}_s:=\langle\mathbf{a}_k\mathbf{b}_s\rangle_{k+s} \tag{2.4}$$

being the scalar-product, right contraction, left contraction and the associative wedge product, respectively. Note that the right contraction is zero if $k < s$ while the left contraction is zero if $s < k$. In general one defines an *involution* as an operation that maps an operand to itself when applied twice. Three common involutions are the main involution, the reverse and the conjugate. Given a general blade $\mathbf{a}_k$ of grade k these involutions are defined, respectively as

$$\rho(\mathbf{a}_k) = (-1)^k \mathbf{a}_k, \qquad \mathbf{a}_k^\dagger = (-1)^{\frac{k(k-1)}{2}} \mathbf{a}_k, \qquad \mathbf{a}_k^* = (-1)^k \mathbf{a}_k^\dagger. \tag{2.5}$$

Note that the main involution does not affect even grade elements. If in a subalgebra of $Cl_{(p,q,r)}$ all the elements are of even grade we denote it by $Cl^+_{(p,q,r)}$ and it can be shown that any Clifford algebra is isomorphic to the even subalgebra of the algebra with one more generator [28].

2.3 Quaternions

A quaternion can be defined as a hyper-complex number with one real part η and three imaginary parts $\varepsilon = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3]^\top$ [57]. The set of quaternions can then be defined as $\mathbb{H} := \{\mathbf{q} = \eta + \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k : \eta, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}\}$. with the well known quaternion basis elements i, j, k satisfying $i^2 = j^2 = k^2 = ijk = -1$. The connection between quaternions and Clifford algebra can be shown by considering the Clifford algebra $Cl_{(0,2,0)}$. This algebra has canonical basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12},\}$ and a general multivector, $\mathbf{a}$, of this algebra has the form $\mathbf{a} = \eta\mathbf{e}_0 + \varepsilon_1\mathbf{e}_1 + \varepsilon_2\mathbf{e}_2 + \varepsilon_3\mathbf{e}_{12}$. Recalling that $\mathbf{e}_0 = 1$, it may easily be verified that quaternion basis elements i, j, k correspond to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}$, respectively [55] and thus this algebra is isomorphic to the quaternions, i.e. $Cl_{(0,2,0)} \cong \mathbb{H}$. As mentioned in the previous section we also have that $Cl_{(0,2,0)} \cong Cl^+_{(0,3,0)}$ and thus we also have $Cl^+_{(0,3,0)} \cong \mathbb{H}$. It can be seen that quaternions constitute a real vector space which is isomorphic to $\mathbb{R}^4$ through the isomorphism $\theta : \mathbb{H} \to \mathbb{R}^4$ defined as $\theta(\eta + \varepsilon_1 i + \varepsilon_2 j + \varepsilon_3 k) = [\eta\ \varepsilon^\top]^\top$. Taking the geometric product between two quaternions[1], $\mathbf{q} = [\eta_q\ \varepsilon_q^\top]^\top$ and $\mathbf{p} = [\eta_p\ \varepsilon_p^\top]^\top$, one can see by the definitions of the algebra above that

$$\mathbf{p} \otimes \mathbf{q} = \begin{bmatrix} \eta_p\eta_q - \varepsilon_p^\top \varepsilon_q \\ \eta_p\varepsilon_q + \eta_q\varepsilon_p + \mathbf{S}(\varepsilon_p)\varepsilon_q \end{bmatrix}. \tag{2.6}$$

A subset of quaternions is the set with zero imaginary part, denoted as the scalar quaternions $\mathbb{H}_s := \{\mathbf{q} \in \mathbb{H} : \varepsilon = \mathbf{0}\}$, which are useful when representing scalars as quaternions and thus the quaternion norm, i.e. $\|\cdot\|_H : \mathbb{H} \to \mathbb{H}_s$, is defined as $\|\mathbf{q}\|_{\mathbb{H}} = \sqrt{\mathbf{q} \otimes \mathbf{q}^*}$ where the quaternion conjugate can be found by applying (2.5) to a quaternion $\mathbf{q}$ as

$$\mathbf{q}^* = \begin{bmatrix} \eta \\ -\varepsilon \end{bmatrix}. \tag{2.7}$$

A subset of quaternions is the set with zero real part, denoted as pure quaternions[2] $\mathbb{H}_p := \{\mathbf{q} \in \mathbb{H} : \eta = 0\}$. Pure quaternions allow for three dimensional vectors to be represented in the quaternion framework by use of a trivial isomorphism, note also that a nonzero quaternion is pure if and only if it has a negative square, i.e. $\mathbf{q} \otimes \mathbf{q} < 0$. The cross product between two pure quaternions, $\mathbf{q} = [0\ \varepsilon_q^\top]^\top$ and $\mathbf{p} = [0\ \varepsilon_p^\top]^\top$, can be found as

[1] In this work we use the notation $\otimes$ to mean the product between two quaternions, in geometric algebra literature this is usually denoted by juxtaposition.

[2] As is noted in [58] the label *pure* is non discriminating compared to the often used label *vector*. A pure quaternion may be used to represent both vectors (for translation) and bivectors (for rotation).

$$\mathbf{q} \times \mathbf{p} = \frac{1}{2}(\mathbf{q} \otimes \mathbf{p} - \mathbf{p} \otimes \mathbf{q}) = \begin{bmatrix} 0 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{S}(\varepsilon_q) \end{bmatrix} \mathbf{p} := \mathbf{S}_q(\mathbf{q})\mathbf{p}. \tag{2.8}$$

Another subset of quaternions is those that possess the unit norm restriction, i.e. $\|\mathbf{q}\|_{\mathbb{H}} = 1$, which are the unit quaternions that topologically form the $3-sphere$ S^3 in $\mathbb{R}^4$ [59]. The two sets are then defined as

$$\mathbb{H}_u := \{\mathbf{q} \in \mathbb{H} : \|\mathbf{q}\|_{\mathbb{H}} = 1\} \qquad S^3 := \{\mathbf{q} \in \mathbb{R}^4 : \|\mathbf{q}\| = 1\}. \tag{2.9}$$

Under multiplication they form the associative and distributive, but non-abelian Lie Group $Spin(3)$ [55]

$$Spin(3) := \{\mathbf{q} \in \mathbb{H}_u : \mathbf{q} \otimes \mathbf{x} \otimes \mathbf{q}^* \in \mathbb{H}_p \ \forall \mathbf{x} \in \mathbb{H}_p\}. \tag{2.10}$$

This group has its inverse defined by the quaternion conjugate, it is a two-fold covering group of the rotational group defined in (2.1), and thus the map $\phi : Spin(3) \to SO(3)$ is surjective with kernel $\{\pm 1\}$ [58], that is $\phi(\mathbf{q}) = \phi(-\mathbf{q})$. This map can be derived to be

$$\mathbf{R} = \phi(\mathbf{q}) = \mathbf{I}_{3\times 3} + 2\eta \mathbf{S}(\varepsilon) + 2\mathbf{S}^2(\varepsilon). \tag{2.11}$$

Unit quaternions can be used to represent rigid body attitude in three-dimensional space. Attitude kinematics, in the case of one frame rotating relative to another, is defined as

$$\dot{\mathbf{q}}_{n,b} = \frac{1}{2}\mathbf{q}_{n,b} \otimes \omega^b_{n,b}, \tag{2.12}$$

where $\mathbf{q}_{n,b} \in \mathbb{H}$, and $\omega^b_{n,b} \in \mathbb{H}_p$ is the angular velocity from the associated lie algebra $so(3)$. Here one may use the isomorphism from quaternions to real vectors and define an alternative representation as

$$\dot{\mathbf{q}}_{n,b} = \mathbf{T}(\mathbf{q}_{n,b})\omega^b_{n,b}, \qquad \mathbf{T}(\mathbf{q}_{n,b}) = \frac{1}{2}\begin{bmatrix} 0 & -\varepsilon^\top \\ \mathbf{0}_{3\times 1} & \eta \mathbf{I}_{3\times 3} + \mathbf{S}(\varepsilon) \end{bmatrix}, \tag{2.13}$$

where now $\dot{\mathbf{q}}_{n,b} \in \mathbb{R}^4$, $\omega^b_{n,b} \in \mathbb{R}^4$ and $\mathbf{T}(\mathbf{q}_{n,b}) \in \mathbb{R}^{4\times 4}$.

2.4 Dual Quaternions

Proper rigid body motion in 3D Euclidean space consists of rotation and translation, which we describe by dual quaternion algebra. Dual quaternions are a combination of dual numbers and hyper-complex numbers, which can be seen as a quaternion where each element is a dual number or conversely a dual number where each element is a quaternion. For completeness we first define dual numbers as $\mathbb{D} := \{\hat{z} = z_p + \varepsilon z_d : \ z_p, z_d \in \mathbb{R}, \varepsilon \neq 0, \varepsilon^2 = 0\}$ where ε is the dual operator, not to be confused with the quaternion vector element ε. Employing the latter of the above mentioned two ways of constructing a dual quaternion we define the set of dual quaternions as $\mathbb{DH} := \{\hat{\mathbf{q}} = \mathbf{q}_p + \varepsilon \mathbf{q}_d, \ \mathbf{q}_p, \mathbf{q}_d \in \mathbb{H}, \varepsilon \neq 0, \varepsilon^2 = 0\}$ where $\mathbf{q}_p$ is named the primary part and $\mathbf{q}_d$ is dual part of the dual quaternion. To introduce the connection to Clifford algebra consider the degenerate algebra $Cl^+_{(0,3,1)}$ that has the canonical basis $\{\mathbf{e}_0, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{14}, \mathbf{e}_{24}, \mathbf{e}_{34}, \mathbf{e}_{1234}, \}$. A general multivector, $\mathbf{a}$, of this algebra has the form $\mathbf{a} = \eta_p \mathbf{e}_0 + \varepsilon_{p1}\mathbf{e}_{12} + \varepsilon_{p2}\mathbf{e}_{13} + \varepsilon_{p3}\mathbf{e}_{23} + \eta_d \mathbf{e}_{1234} + \varepsilon_{d1}\mathbf{e}_{34} + \varepsilon_{d2}\mathbf{e}_{24} + \varepsilon_{d3}\mathbf{e}_{14}$ and it may be shown how these elements match with the dual quaternion elements, as in Table 1 that can be found in [40]. As with quaternions, dual quaternions can be seen to constitute a real vector space which is isomorphic to $\mathbb{R}^8$ through the isomorphism $\hat{\theta} : \mathbb{DH} \to \mathbb{R}^8$ defined as

$$\hat{\theta}(\mathbf{q}_p + \varepsilon \mathbf{q}_d) = \begin{bmatrix} \mathbf{q}_p \\ \mathbf{q}_d \end{bmatrix}. \tag{2.14}$$

Table 1 Corresponding elements between $Cl^{+}_{(0,3,1)}$ and $\mathbb{DH}$

$Cl^{+}_{(0,3,1)}$	$\mathbb{DH}$	$Cl^{+}_{(0,3,1)}$	$\mathbb{DH}$
$\mathbf{e}_0$	1	$\mathbf{e}_{1234}$	$-\varepsilon$
$\mathbf{e}_{12}$	i	$\mathbf{e}_{34}$	εi
$\mathbf{e}_{13}$	j	$\mathbf{e}_{24}$	$-\varepsilon j$
$\mathbf{e}_{23}$	k	$\mathbf{e}_{14}$	εk

The product of two dual quaternions, $\hat{\mathbf{q}} = \mathbf{q}_p + \varepsilon\mathbf{q}_d$ and $\hat{\mathbf{p}} = \mathbf{p}_p + \varepsilon\mathbf{p}_d$ may be calculated as

$$\hat{\mathbf{q}} \otimes \hat{\mathbf{p}} = \mathbf{q}_p \otimes \mathbf{p}_p + \varepsilon(\mathbf{q}_p \otimes \mathbf{p}_d + \mathbf{q}_d \otimes \mathbf{p}_p). \tag{2.15}$$

Just as the scalar quaternion can represent scalars, so can the scalar dual quaternion represent dual numbers, these are defined as the set $\mathbb{DH}_s := \{\hat{\mathbf{q}} \in \mathbb{DH} : \mathbf{q}_p, \mathbf{q}_d \in \mathbb{H}_s\}$. The dual quaternion norm results in a dual number, i.e. $||\cdot||_{\mathbb{D}} : \mathbb{DH} \to \mathbb{DH}_s$, and is defined as

$$||\hat{\mathbf{q}}||^2_{\mathbb{D}} = \hat{\mathbf{q}} \otimes \hat{\mathbf{q}}^* = \mathbf{q}_p \otimes \mathbf{p}_p^* + \varepsilon(\mathbf{q}_p \otimes \mathbf{p}_d^* + \mathbf{q}_d \otimes \mathbf{p}_p^*), \tag{2.16}$$

where the dual quaternion conjugate can be found, again by applying (2.5) on the dual quaternion multivector, to be

$$\hat{\mathbf{q}}^* = \mathbf{q}_p^* + \varepsilon\mathbf{q}_d^*, \tag{2.17}$$

with the quaternion conjugate given in equation (2.7).

Remark 1. *In order to apply Lyapunov theory we need a norm that maps to $\mathbb{R}$ instead of $\mathbb{D}$, which in practice is achieved by applying the isomporhism in (2.14) and using the Euclidean norm, i.e.* $||\hat{\boldsymbol{q}}|| = \sqrt{\hat{\boldsymbol{q}}^\top \hat{\boldsymbol{q}}},\ \ \hat{\boldsymbol{q}} \in \mathbb{R}^8$.

Just as for the ordinary quaternions we have pure dual quaternions $\mathbb{DH}_p := \{\hat{\mathbf{q}} \in \mathbb{DH} : \mathbf{q}_p, \mathbf{q}_d \in \mathbb{H}_p\}$. Pure dual quaternions allow for six dimensional vectors to be represented is the dual quaternion framework by use of a trivial isomorphism. The cross product between two pure dual quaternions, $\hat{\mathbf{q}} = \mathbf{q}_p + \varepsilon\mathbf{q}_d$ and $\hat{\mathbf{p}} = \mathbf{p}_p + \varepsilon\mathbf{p}_d$, can be calculated as before

$$\hat{\mathbf{q}} \times \hat{\mathbf{p}} = \frac{1}{2}(\hat{\mathbf{q}} \otimes \hat{\mathbf{p}} - \hat{\mathbf{p}} \otimes \hat{\mathbf{q}}) = \begin{bmatrix} \mathbf{S}_q(\mathbf{q}_p) & 0 \\ \mathbf{S}_q(\mathbf{q}_d) & \mathbf{S}_q(\mathbf{q}_p) \end{bmatrix} \hat{\mathbf{p}} = \hat{\mathbf{S}}_q(\hat{\mathbf{q}})\hat{\mathbf{p}}. \tag{2.18}$$

The product between a dual number, $\hat{a} \in \mathbb{D}$, and a pure dual quaternion, $\hat{\mathbf{q}} \in \mathbb{DH}_p$, can be written as

$$\hat{a}\hat{\mathbf{q}} = a_p\mathbf{q}_p + \varepsilon(a_d\mathbf{q}_p + a_p\mathbf{q}_d) = \begin{bmatrix} \mathbf{T}(\mathbf{a}_p) & \mathbf{0}_{4\times4} \\ \mathbf{T}(\mathbf{a}_d) & \mathbf{T}(\mathbf{a}_p) \end{bmatrix} \hat{\mathbf{q}} := \hat{\mathbf{I}}(\hat{a})\hat{\mathbf{q}}, \tag{2.19}$$

where we represent the dual number $\hat{a}$ as a scalar dual quaternion, i.e. $\hat{\mathbf{a}} = \mathbf{a}_p + \varepsilon\mathbf{a}_d \in \mathbb{DH}_s$, and $\mathbf{T}$ is defined in (2.13). The subset of dual quaternions that satisfies the norm constraint $||\hat{\mathbf{q}}||_{\mathbb{D}} = 1 + \varepsilon 0$ is denoted the set of unit dual quaternions; $\mathbb{DH}_u := \{\hat{\mathbf{q}} \in \mathbb{DH} : ||\hat{\mathbf{q}}||_{\mathbb{D}} = 1,\ \mathbf{q}_p \in \mathbb{H}_u, \mathbf{q}_d \in \mathbb{H}\}$; note that from (2.16) the unit norm implies

$$\mathbf{q}_p \otimes \mathbf{q}_p^* = \mathbf{q}_{\mathbf{I}}, \qquad \mathbf{q}_p \otimes \mathbf{q}_d^* + \mathbf{q}_d \otimes \mathbf{q}_p^* = 0, \tag{2.20}$$

where $\mathbf{q}_{\mathbf{I}} = [1\ 0\ 0\ 0]^\top$ is the identity quaternion. In $\mathbb{R}^8$ unit dual quaternions form the group

$$S^3 \ltimes \mathbb{R}^3 := \{\hat{\mathbf{q}} \in \mathbb{R}^8 : \mathbf{q}_p \in S^3, \mathbf{q}_p \otimes \mathbf{q}_d^* + \mathbf{q}_d \otimes \mathbf{q}_p^* = 0\}, \tag{2.21}$$

and it can be shown that under multiplication unit dual quaternions form the group $Spin(3) \ltimes \mathbb{R}^3$

$$Spin(3) \ltimes \mathbb{R}^3 := \{\hat{\mathbf{q}} \in \mathbb{DH}_u : \hat{\mathbf{q}} \otimes \mathbf{x} \otimes \hat{\mathbf{q}}^* \in \mathbb{DH}_p \ \forall \mathbf{x} \in \mathbb{DH}_p\}. \tag{2.22}$$

This group has the inverse defined by the dual quaternion conjugate and is a double cover of the group of proper Euclidean transformations $SE(3)$ defined in (2.2) [28]. As with the map defined between $Spin(3)$ and $SO(3)$ we may define a map $\hat{\phi} : Spin(3) \ltimes \mathbb{R}^3 \to SE(3)$ that is surjective with kernel $\{\pm 1\}$, such that $\hat{\phi}(\hat{\mathbf{q}}) = \hat{\phi}(-\hat{\mathbf{q}})$. In this work we use the map to the adjoint representation of the group, defined in [49] as

$$\hat{\mathbf{H}} = \hat{\phi}(\hat{\mathbf{q}}) := \mathbf{I}_{8\times 8} + 2\hat{\mathbf{I}}(\hat{\eta})\hat{\mathbf{S}}_q(\hat{\varepsilon}) + 2\hat{\mathbf{S}}_q^2(\hat{\varepsilon}) = \begin{bmatrix} 1 & \mathbf{0}_{1\times 3} & 0 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{R} & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} \\ 0 & \mathbf{0}_{1\times 3} & 1 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{S}(\mathbf{p})\mathbf{R} & \mathbf{0}_{3\times 1} & \mathbf{R} \end{bmatrix}, \tag{2.23}$$

where $\hat{\eta} \in \mathbb{D}$ is constructed by the scalar and pseudo-scalar of the dual quaternion while $\hat{\varepsilon} \in \mathbb{DH}_p$ is constructed by the vector parts the dual quaternion. It can be seen that $\hat{\mathbf{H}}$ is the adjoint representation of the group $SE(3)$ represented in $\mathbb{R}^{8\times 8}$.

3 Rigid-Body Modelling

3.1 Kinematics and Dynamics

The pose of the rigid body may be represented by an attitude quaternion and a translation vector using the compact dual quaternion framework as

$$\hat{\mathbf{q}}_{n,b} = \mathbf{q}_{n,b} + \varepsilon\frac{1}{2}\mathbf{p}^n \otimes \mathbf{q}_{n,b} = \mathbf{q}_{n,b} + \varepsilon\frac{1}{2}\mathbf{q}_{n,b} \otimes \mathbf{p}^b, \tag{3.1}$$

where $\mathbf{p}^n, \mathbf{p}^b \in \mathbb{H}_p$ are the rigid body position expressed in $\mathcal{F}^n$ and $\mathcal{F}^b$, respectively. The rigid body kinematics, in terms of one frame transforming relative to another, has a similar relation as that in (2.12),

$$\dot{\hat{\mathbf{q}}}_{n,b} = \frac{1}{2}\hat{\mathbf{q}}_{n,b} \otimes \hat{\omega}^b_{n,b}, \tag{3.2}$$

where $\hat{\mathbf{q}}_{n,b} \in \mathbb{DH}$, and $\hat{\omega}^b_{n,b} \in \mathbb{DH}_p$ is the dual velocity[3] of the body. From screw theory we know that these are elements of the Lie algebra, $se(3)$, to the proper Euclidean group SE(3) [60]. The dual velocity of the body relative the inertial $\mathcal{F}^n$ can then be defined as

$$\hat{\omega}^b_{n,b} = \omega^b_{n,b} + \varepsilon\mathbf{v}^b, \tag{3.3}$$

where $\hat{\omega}^b_{n,b} \in \mathbb{DH}_p$, $\omega^b_{n,b} \in \mathbb{H}_p$ is the angular velocity of the body and $\mathbf{v}^b = \dot{\mathbf{p}}^b + \omega^b_{n,b} \times \mathbf{p}^b$ is the linear velocity. As with the attitude quaternion one may use the isomorphisms to work with elements of $\mathbb{R}^8$ instead of $\mathbb{DH}$, where we define

$$\dot{\hat{\mathbf{q}}}_{n,b} = \hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,b})\hat{\omega}^b_{n,b}, \qquad \hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,b}) = \begin{bmatrix} \mathbf{T}(\mathbf{q}_p) & \mathbf{0}_{4\times 4} \\ \mathbf{T}(\mathbf{q}_d) & \mathbf{T}(\mathbf{q}_p) \end{bmatrix}, \tag{3.4}$$

[3] From screw theory this is known as a twist, or velocity screw, i.e. the angular velocity around an axis and the linear velocity along it.

with $\dot{\hat{\mathbf{q}}}_{n,b} \in \mathbb{R}^8$, $\hat{\omega}^b_{n,b} \in \mathbb{R}^8$, $\hat{\mathbf{T}}(\hat{\mathbf{q}}_{n,b}) \in \mathbb{R}^{8\times 8}$ and $\mathbf{T}(\cdot)$ defined in (2.13).
Further, by expressing the dual momentum, i.e. the co-screw consisting of linear and angular momentum, as an element of $\mathbb{R}^8$ one can relate it to the dual velocity through the anti-diagonal dual inertia matrix as in [50]:

$$\hat{\mathbf{h}}^b = \hat{\mathbf{M}}^b \hat{\omega}^b_{n,b}, \qquad \hat{\mathbf{M}}^b = \begin{bmatrix} 0 & \mathbf{0}_{1\times 3} & 1 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 1} & m\mathbf{I}_3 \\ 1 & \mathbf{0}_{1\times 3} & 0 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{J}^b & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} \end{bmatrix}, \tag{3.5}$$

where $m \in \mathbb{R}$ is the mass and $\mathbf{J}^b \in \mathbb{R}^{3\times 3}$ is the inertia matrix.

Remark 2. *It is not possible to transform the dual inertia matrix using the dual quaternion sandwich product directly. This is due to the fact that inertias are not represented as elements in the algebra. One remedy to this is to double the algebra to $Cl_{(0,6,2)}$ in order to include all possible elements in the algebra as proposed in [60].*

In this work we utilize the transformed dual inertia matrix, thus we include a short technical lemma.

Lemma 3.1. Given an anti-diagonal dual inertia matrix $\hat{\mathbf{M}}^b$ in the reference frame $\mathcal{F}^b$, transforming the matrix into frame $\mathcal{F}^a$ can be done as

$$\hat{\mathbf{M}}^a = \hat{\mathbf{H}}^a_b \hat{\mathbf{M}}^b \hat{\mathbf{H}}^b_a, \tag{3.6}$$

where $\hat{\mathbf{H}}(\cdot)$ is given in (2.23) and $\hat{\mathbf{q}}_{b,a}$ is the dual quaternion representing the pose of frame $\mathcal{F}^a$ relative $\mathcal{F}^b$.

The proof can be found in Appendix A.
The derivative of the transformed dual inertia matrix can be derived using the derivative of $\hat{\mathbf{H}}(\cdot)$, found in [49], to be

$$\dot{\hat{\mathbf{M}}}^a = \hat{\mathbf{M}}^a \hat{\mathbf{S}}_q(\hat{\omega}^a_{b,a}) - \hat{\mathbf{S}}_q(\hat{\omega}^a_{b,a})\hat{\mathbf{M}}^a. \tag{3.7}$$

The rigid body dynamics can now be found by taking the derivative of the dual momentum in (3.5), i.e.

$$\hat{\mathbf{M}}^b \dot{\hat{\omega}}^b_{n,b} = \hat{\mathbf{f}}^b - \hat{\omega}^b_{n,b} \times \hat{\mathbf{M}}^b \hat{\omega}^b_{n,b}, \tag{3.8}$$

where the dual force[4] $\hat{\mathbf{f}}^b \in \mathbb{DH}_p$ is defined in the dual quaternion framework as $\hat{\mathbf{f}}^b = \mathbf{f}^b + \varepsilon\tau^b$ with $\mathbf{f}^b \in \mathbb{H}_p$ representing translation forces in the body frame and $\tau^b \in \mathbb{H}_p$ representing applied moments in the body frame.

3.2 Dynamic model of a quadrotor

Extensive derivation of a quadrotor nonlinear dynamic model can be found in [61, 62, 63]. Roughly speaking, the body frame dual force $\hat{\mathbf{f}}^b$ is for the quadrotor mainly composed of four parts: dual control force $\hat{\mathbf{f}}^b_u = \mathbf{f}^b_u + \varepsilon\tau^b_u$, gravitational force $\hat{\mathbf{f}}^b_g$, dual drag force $\hat{\mathbf{f}}^b_d = \mathbf{f}^b_d + \varepsilon\tau^b_d$, and the gyroscopic torque $\hat{\mathbf{f}}_r = \mathbf{0} + \varepsilon\mathbf{g}^b_r$. Due to blade flapping, a phenomenon thoroughly described in helicopter literature -*cf.* [64, 65], the control, or thrust force, experiences a deflection from the body z-axis resulting in

[4] From screw theory this is known as a wrench, i.e. the combination of force and torque acting on a rigid body

forces in the body horizontal plane when the quadrotor is moving relative to the surrounding air, i.e. $\mathbf{f}_u^b = \mathbf{q}_{b,f} \otimes T\mathbf{e}_3 \otimes \mathbf{q}_{b,f}^b$, where $\mathcal{F}^f$ is a reference frame representing the deflection from the body-frame, $T \in \mathbb{R}$ is the total thrust, and $\mathbf{e}_3 = [0\ 0\ 0\ 1]^\top$. It is common in the literature to model this effect as a drag force [61], i.e. it is assumed that $\mathcal{F}^f = \mathcal{F}^b$, hence the force and torque generated by the propellers in the body frame are calculated as

$$\mathbf{f}_u^b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ C_T(\varpi_1^2 + \varpi_2^2 + \varpi_3^2 + \varpi_4^2) \end{bmatrix}, \qquad \tau_u^b = \begin{bmatrix} 0 \\ C_T l(\varpi_4^2 - \varpi_2^2) \\ C_T l(\varpi_1^2 - \varpi_3^2) \\ C_Q(\varpi_1^2 - \varpi_2^2 + \varpi_3^2 - \varpi_4^2) \end{bmatrix}, \tag{3.9}$$

where $C_T \in \mathbb{R}$ is the thrust coefficient which can be determined through static thrust tests [61], $C_Q \in \mathbb{R}$ is the motor parameter relating the angular velocity of the motor to the rotor torque, $l \in \mathbb{R}$ is the arm length of the quadrotor and $\varpi_i \in \mathbb{R}$, $i = 1,2,3,4$, is the rotational velocity of the i'th rotor. The body frame gravitational force is

$$\hat{\mathbf{f}}_g^b = \hat{\mathbf{M}}^b \hat{\mathbf{q}}_{n,b}^* \otimes \hat{\mathbf{a}}_g^n \otimes \hat{\mathbf{q}}_{n,b}, \tag{3.10}$$

with $\hat{\mathbf{a}}_g^n = [0\ 0\ 0\ 0\ 0\ 0\ 0\ g]^\top$ and g being the gravitational constant. The drag forces associated with quadrotor flight, $\mathbf{f}_d^b$, includes among others the rotor relative momentum drag, rotor blade flapping, induced drag on the rotor for not being able to compensate the thrust imbalance due to blade flapping, and body relative parasitic drag -*cf.* [61]. For this work we include the induced drag, which in body frame can be modelled as

$$\mathbf{f}_d^b = -\mathbf{D}\mathbf{v}^b, \qquad \hat{\mathbf{f}}_d^b = -\hat{\mathbf{D}}\hat{\omega}_{n,b}^b, \qquad \hat{\mathbf{D}} := \begin{bmatrix} \mathbf{0}_{4\times4} & \mathbf{D} \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} \end{bmatrix}, \tag{3.11}$$

where $\mathbf{D} = \mathrm{diag}(0\ \mathrm{d_x}\ \mathrm{d_y}\ 0)$ and $d_x, d_y \in \mathbb{R}$ are the induced drag coefficients. The induced drag on the rotors produce drag torques in body frame due to the displacement of the motors from the center of gravity, however these are considered negligible and hence $\tau_d^b = 0$. Finally, the gyroscopic torque of the rotors on the air frame can be calculated as [66]

$$\mathbf{g}_r^b = -\sum_{i=1}^{4} (-1)^{i+1} \mathbf{S}_q(\omega_{n,b}^b) \mathbf{J}_p \varpi_i, \tag{3.12}$$

where $\mathbf{J}_p \in \mathbb{R}^{4\times4}$ is the moment of inertia of a rotor about its axis and $\varpi_i = [0\ 0\ 0\ \varpi_i]^\top$.
Defining $\hat{\mathbf{f}}_{aux} = \mathbf{f}_g^b + \varepsilon(\mathbf{g}_a^b)$ then allows for the quadrotor nonlinear dynamic model to be stated as

$$\begin{aligned} \dot{\hat{\mathbf{q}}} &= \frac{1}{2}\hat{\mathbf{q}} \otimes \hat{\omega}_{n,b}^b \\ \hat{\mathbf{M}}\dot{\hat{\omega}}_{n,b}^b &= \hat{\mathbf{f}}_u^b + \hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\omega}_{n,b}^b \times \hat{\mathbf{M}}\hat{\omega}_{n,b}^b. \end{aligned} \tag{3.13}$$

4 Controller design

4.1 Problem formulation

The tracking control problem can be stated as; let $\hat{\mathbf{q}}_{n,d} : \mathbb{R}_{\geq 0} \to \mathbb{R}^8$ be a given, two-times continuously differentiable bounded time-varying desired trajectory, i.e.

$$\max\{|\hat{\mathbf{q}}_{\mathrm{n,d}}|_\infty,\ |\hat{\omega}^{\mathrm{d}}_{\mathrm{n,d}}|_\infty,\ |\dot{\hat{\omega}}^{\mathrm{d}}_{\mathrm{n,d}}|_\infty\} \le \beta_{\mathrm{d}}. \tag{4.1}$$

Define the tracking error in dual quaternion coordinates as

$$\hat{\mathbf{q}}_e := \hat{\mathbf{q}}^*_{n,d} \otimes \hat{\mathbf{q}}_{n,b} = \mathbf{q}_e + \varepsilon\frac{1}{2}\mathbf{q}_e \otimes \mathbf{p}^b_e = \mathbf{q}_{e,p} + \varepsilon\mathbf{q}_{e,d} = \begin{bmatrix}\eta_e\\ \varepsilon_e\end{bmatrix} + \varepsilon\begin{bmatrix}\eta_{ed}\\ \varepsilon_{ed}\end{bmatrix}, \tag{4.2}$$

and, due to the double cover $S^3 \ltimes \mathbb{R}^3$ of $SE(3)$, define

$$\hat{\mathbf{q}}_{e\pm} := \begin{bmatrix}(1 \mp \eta_e)\\ \varepsilon_e\end{bmatrix} + \varepsilon\frac{1}{2}\mathbf{q}_e \otimes \mathbf{p}^b_e, \tag{4.3}$$

with error kinematics and dynamics

$$\begin{aligned}\dot{\hat{\mathbf{q}}}_{e\pm} &= \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})\hat{\omega}^b_e\\ \hat{\mathbf{M}}^b\dot{\hat{\omega}}^b_e &= \hat{\mathbf{f}}^b_u + \hat{\mathbf{f}}^b_{aux} - \hat{\mathbf{D}}\hat{\omega}^b_{n,b} - \hat{\mathbf{S}}_q(\hat{\omega}^b_{n,b})\hat{\mathbf{M}}\hat{\omega}^b_{n,b} - \hat{\mathbf{M}}^b\dot{\hat{\omega}}^b_{n,d},\end{aligned} \tag{4.4}$$

where $\hat{\omega}^b_e = \hat{\omega}^b_{n,b} - \hat{\omega}^b_{n,d}$ and

$$\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm}) = \begin{bmatrix}\mathbf{T}_{e\pm}(\mathbf{q}_p) & \mathbf{0}_{4\times4}\\ \mathbf{T}(\mathbf{q}_d) & \mathbf{T}(\mathbf{q}_p)\end{bmatrix}, \qquad \mathbf{T}_{e\pm}(\mathbf{q}_r) = \frac{1}{2}\begin{bmatrix}0 & \pm\varepsilon^\top\\ \mathbf{0}_{3\times1} & \eta\mathbf{I}_{3\times3} + \mathbf{S}(\varepsilon)\end{bmatrix}. \tag{4.5}$$

Then, design a feedback control law, $\hat{\mathbf{f}}^b_u$, that stabilizes the origin for the system (4.4).

Remark 3. *Following [67]*, we define two sets $\hat{\mathbf{q}}_{e+} \in S^3_{e+} \ltimes \mathbb{R}^3 := \{[1-\eta_e,\ \varepsilon_e^\top,\ \mathbf{q}^\top_{e,d}]^\top : \eta_e \ge 0, \hat{\mathbf{q}}_e \in S^3 \ltimes \mathbb{R}^3\}$ and $\hat{\mathbf{q}}_{e-} \in S^3_{e-} \ltimes \mathbb{R}^3 := \{[1+\eta_e,\ \varepsilon_e^\top,\ \mathbf{q}^\top_{e,d}]^\top : \eta_e \le 0, \hat{\mathbf{q}}_e \in S^3 \ltimes \mathbb{R}^3\}$. Thus, $\hat{\mathbf{q}}_{e\pm} \in S^3_{e+} \ltimes \mathbb{R}^3 \cup S^3_{e-} \ltimes \mathbb{R}^3 = S^3_e \ltimes \mathbb{R}^3 := \{[1-|\eta_e|,\ \varepsilon_e^\top,\ \mathbf{q}^\top_{e,d}]^\top : \hat{\mathbf{q}}_e \in S^3 \ltimes \mathbb{R}^3\}$.

Before presenting the main result of this note, we present some technical lemmas that will be useful in the subsequent work

Lemma 4.1. Let $\hat{\mathbf{T}}_e(\cdot)$ be defined in (4.5). Then the following holds

$$4\hat{\mathbf{T}}^\top_e(\hat{\mathbf{q}}_{e\pm})\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm}) = \begin{bmatrix}\mathbf{T}(\mathbf{q}_u) & -\frac{1}{2}\mathbf{S}_q(\mathbf{p}^b_e)\\ \frac{1}{2}\mathbf{S}_q(\mathbf{p}^b_e) & \mathbf{T}(\mathbf{q}_\mathbf{I})\end{bmatrix} := \hat{\mathbf{F}}, \tag{4.6}$$

where $\mathbf{q}_\mathbf{I} \in \mathbb{H}_s$ is the identity quaternion, $\mathbf{p}^b_e \in \mathbb{H}_p$ and $\mathbf{q}_u = (1 + \frac{1}{4}(\mathbf{p}^b_e)^\top\mathbf{p}^b_e)\mathbf{q}_\mathbf{I}$

Proof: We omit the proof for brevity, but the result is readily found by direct computation.

Lemma 4.2. Let $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})$ be defined in (4.5), and let $\mathbf{q}_{e\pm}$ be defined as in (4.3). Then,

$$\begin{aligned}\hat{\mathbf{q}}^\top_{e\pm}\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm}) &= \frac{1}{2}\left[0\ \pm\varepsilon_e^\top\ 0\ \tfrac{1}{2}(\mathbf{p}^b_e)^\top\right] := \frac{1}{2}\tilde{\varepsilon}^\top\\ \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})^\top\hat{\mathbf{q}}_{e\pm} &= \frac{1}{2}\left[0\ \pm\varepsilon_e^\top\ 0\ \tfrac{1}{2}(\mathbf{p}^b_e)^\top\right]^\top := \frac{1}{2}\tilde{\varepsilon}.\end{aligned} \tag{4.7}$$

Proof: The proof can be found in a similar fashion as done for Lemma 3.1 in [54].

Lemma 4.3. Let $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})$ be defined as in (4.5), $\mathbf{q}_{e\pm}$ be defined as in (4.3). Moreover, we define $\Xi = [\mathbf{q}^\top_{e\pm}\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})]^\top$. Then

$$0 \le \|\Xi\|^2 \le \hat{\mathbf{q}}^\top_{e\pm}\hat{\mathbf{q}}_{e\pm}. \tag{4.8}$$

Proof: The proof can be found in [54].

Lemma 4.4. Derivatives of $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})$ satisfies

$$\dot{\hat{\mathbf{T}}}^\top_e(\hat{\mathbf{q}}_{e\pm})\hat{\mathbf{q}}_{e\pm} = \hat{\mathbf{G}}_{e\pm}\hat{\omega}^b_e, \qquad \hat{\mathbf{G}}_{e\pm} = \frac{1}{4}\left(\begin{bmatrix}\pm(\mathbf{T}(\eta_e) + \mathbf{S}_q(\varepsilon_e)) & \mathbf{0}_{4\times4}\\ -\mathbf{S}_q(\mathbf{p}^b_e) & \mathbf{T}(\mathbf{q}_\mathbf{I})\end{bmatrix} - \hat{\mathbf{F}}\right). \tag{4.9}$$

Proof: See Appendix A.

4.2 Control of fully actuated rigid body

One of the main advantages of using the dual quaternion representation of rigid-body kinematic and dynamics is its resemblance to the quaternion attitude kinematics and dynamics, as such one may apply control structures derived for quaternion attitude control to the full 6-DOF control problem. In this section we illustrate this by adapting some of the state feedback quaternion based attitude control laws found in [68] to the 6-DOF trajectory tracking problem presented in Section 4.1 for a fully actuated rigid-body. In this section we assume then that the system is fully actuated, moreover we also assume complete knowledge of system states.

4.2.1 Velocity error sliding surface

Theorem 4.1. Let $\hat{\mathbf{q}}_{eq} \in S_e^3 \ltimes \mathbb{R}^3$ and $\mathrm{sgn}(\eta_{\mathrm{e,p}}(\mathrm{t_0})) = \mathrm{sgn}(\eta_{\mathrm{e,p}}(\mathrm{t}))$ for all $t \geq t_0$[5], let the desired trajectory, $\hat{\mathbf{q}}_{n,d}$, satisfy (4.1). Then the equilibrium points $(\hat{\mathbf{q}}_{e\pm}, \hat{\omega}_e^b) = (\mathbf{0}, \mathbf{0})$ of the system (4.4), in closed-loop with the control law

$$\begin{aligned}
\hat{\mathbf{f}}_u^b &= -\hat{\mathbf{f}}_{aux}^b + \hat{\mathbf{D}}\hat{\omega}_r + \hat{\mathbf{S}}_q(\hat{\omega}_{n,b}^b)\hat{\mathbf{M}}^b\hat{\omega}_{n,b}^b + \hat{\mathbf{M}}^b\dot{\hat{\omega}}_r^b - \hat{\mathbf{K}}^{-1}(k_p\tilde{\varepsilon} + k_d\hat{\mathbf{s}}) \\
\hat{\mathbf{s}} &= \hat{\omega}_{n,b}^b - \hat{\omega}_r^b = \hat{\omega}_e^b + \Gamma\tilde{\varepsilon} \\
\hat{\omega}_r^b &= \hat{\omega}_{n,d}^b - \Gamma\tilde{\varepsilon}
\end{aligned} \tag{4.10}$$

where $\hat{\mathbf{K}}$ is an anti-diagonal identity matrix, $k_p > 0$, $k_d > 0$ and $\Gamma = \Gamma > \mathbf{0}$ are feedback gains, are uniformly asymptotically stable (UAS).

Proof. In the following we only consider, without loss of generality, the positive equilibrium point, i.e. $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e+}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e+})$. The closed-loop kinematics and dynamics, resulting from inserting (4.10) into (4.4), is

$$\begin{aligned}
\dot{\hat{\mathbf{q}}}_{eq} &= \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\hat{\omega}_e^b \\
\hat{\mathbf{M}}^b\dot{\hat{\mathbf{s}}} &= -\hat{\mathbf{D}}\hat{\mathbf{s}} - \hat{\mathbf{K}}^{-1}(k_p\tilde{\varepsilon} + k_d\hat{\mathbf{s}}).
\end{aligned} \tag{4.11}$$

Consider the radially unbounded Lyapunov function candidate

$$V(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) := k_p\hat{\mathbf{q}}_{eq}^\top\hat{\mathbf{q}}_{eq} + \frac{1}{2}(\hat{\mathbf{s}})^\top\hat{\mathbf{K}}\hat{\mathbf{M}}^b\hat{\mathbf{s}}. \tag{4.12}$$

Evaluating the time derivative of V along the closed-loop trajectories of the system generated by (4.11) yields

$$\dot{V} = -k_p\tilde{\varepsilon}^\top\Gamma\tilde{\varepsilon} - k_d\hat{\mathbf{s}}^\top\hat{\mathbf{s}} - \hat{\mathbf{s}}^\top\hat{\mathbf{K}}\hat{\mathbf{D}}\hat{\mathbf{s}}, \tag{4.13}$$

where we have used Lemma 4.2 and the fact that $\hat{\omega}_e^b = \hat{\mathbf{s}} - \Gamma\tilde{\varepsilon}$. Note that $\hat{\mathbf{K}}\hat{\mathbf{D}} = \mathrm{diag}([0\,0\,0\,0\,0\,\mathrm{d_x}\,\mathrm{d_y}\,0])$. We now show that there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(\mathbf{x}) \leq V(\mathbf{x}) \leq \overline{\alpha}(\mathbf{x})$. Defining $\chi = [\hat{\mathbf{q}}_{eq}^\top\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\ (\hat{\omega}_e^c)^\top]^\top$ and utilizing Lemma 4.3, we obtain

$$p_m\|\chi\|^2 \leq V(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) \leq p_M\|\chi\|^2, \tag{4.14}$$

[5] This assumption can be relaxed by employing a hybrid control strategy for the controller, this is however not the main focus of this work.

for some $p_M > p_m > 0$. Thus choosing $\underline{\alpha}(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) = p_m \|\chi\|^2$ and $\overline{\alpha}(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) = p_M \|\chi\|^2$ ensures the existence of such functions. We conclude, by Theorem 4.9 in [69], that the equilibrium point $(\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{T}}_e, \hat{\omega}_e^c) = (\mathbf{0}, \mathbf{0})$ is uniformly asymptotically stable. From the assumption that $\eta_e > 0$ and the unit constraint in (2.21) it can be seen that $\eta_e \to 1$. The proof for the negative equilibrium point, $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e-}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e-})$ is performed in the same way. It follows that the two equilibrium points $\hat{\mathbf{q}}_{eq} \in S_e^3 \ltimes \mathbb{R}^3$ are UAS.

4.2.2 Integrator backstepping

Theorem 4.2. Let $\hat{\mathbf{q}}_{eq} \in S_e^3 \ltimes \mathbb{R}^3$ and $\mathrm{sgn}(\eta_{\mathrm{e,p}}(\mathrm{t}_0)) = \mathrm{sgn}(\eta_{\mathrm{e,p}}(\mathrm{t}))$ for all $t \geq t_0$, and let the desired trajectory, $\hat{\mathbf{q}}_{n,d}$, satisfy (4.1). Then the equilibrium points $(\hat{\mathbf{q}}_{e\pm}, \hat{\omega}_e^b) = (\mathbf{0}, \mathbf{0})$ of the system (4.4), in closed-loop with the control law

$$\begin{aligned} \hat{\mathbf{f}}_u^b &= -\hat{\mathbf{f}}_{aux}^b + \hat{\mathbf{S}}_q(\hat{\omega}_{n,b}^b)\hat{\mathbf{M}}^b\hat{\omega}_{n,b}^b + \hat{\mathbf{D}}(\hat{\omega}_{n,d}^b + \hat{\alpha}) + \hat{\mathbf{M}}^b(\dot{\hat{\omega}}_{n,d}^b - \Gamma(\hat{\mathbf{G}}_\pm + \frac{1}{4}\hat{\mathbf{F}})(\hat{\alpha} + \hat{\mathbf{z}}_2)) \\ &\quad - \hat{\mathbf{K}}^{-1}(k_d\hat{\mathbf{z}}_2 + \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1) \\ \hat{\alpha} &= -\Gamma\hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1 \end{aligned} \tag{4.15}$$

where $\hat{\mathbf{z}}_1 = \hat{\mathbf{q}}_{eq}$ and $\hat{\mathbf{z}}_2 = \hat{\omega}_e^b - \hat{\alpha}$ are auxiliary state variables, $\hat{\mathbf{K}}$ is an anti-diagonal identity matrix, $k_d > 0$ and $\Gamma = \Gamma > \mathbf{0}$ are feedback gains, are uniformly asymptotically stable (UAS).

Proof. In the following we only consider, without loss of generality, the positive equilibrium point, i.e. $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e+}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e+})$. Defining the first backstepping variable as $\hat{\mathbf{z}}_1 := \hat{\mathbf{q}}_{eq}$ we have that the $\hat{\mathbf{z}}_1$ dynamics is $\dot{\hat{\mathbf{z}}}_1 = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\hat{\omega}_e^b$. Introducing the virtual control

$$\hat{\omega}_e^b := \hat{\alpha} + \hat{\mathbf{z}}_2, \tag{4.16}$$

where $\hat{\alpha}$ is a stabilizing function and $\hat{\mathbf{z}}_2$ is the second backstepping variable we define a Lyapunov function candidate as

$$V_1(\hat{\mathbf{z}}_1) := \frac{1}{2}\hat{\mathbf{z}}_1^\top \hat{\mathbf{z}}_1, \tag{4.17}$$

such that after taking the derivative and inserting $\hat{\alpha}$ from (4.15) we obtain

$$\dot{V}_1 = -\hat{\mathbf{z}}_1^\top \hat{\mathbf{T}}_e \Gamma \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1 + \hat{\mathbf{z}}_1^\top \hat{\mathbf{T}}_e \hat{\mathbf{z}}_2, \tag{4.18}$$

which allows one to rewrite the $\hat{\mathbf{z}}_1$ dynamics

$$\dot{\hat{\mathbf{z}}}_1 = -\hat{\mathbf{T}}_e \Gamma \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1 + \hat{\mathbf{T}}_e \hat{\mathbf{z}}_2. \tag{4.19}$$

The second step in the backstepping procedure is then to evaluate the $\hat{\mathbf{z}}_2$ dynamics, which can be found to be

$$\dot{\hat{\mathbf{z}}}_2 = \dot{\hat{\omega}}_e^b - \dot{\hat{\alpha}} = \dot{\hat{\omega}}_{n,b}^b - \dot{\hat{\omega}}_{n,d}^b + \Gamma\dot{\hat{\mathbf{T}}}_e^\top \hat{\mathbf{z}}_1 + \Gamma\hat{\mathbf{T}}_e^\top \dot{\hat{\mathbf{z}}}_1. \tag{4.20}$$

Using Lemma 4.4 and multiplying with the dual inertia matrix we have

$$\begin{aligned} \hat{\mathbf{M}}^b\dot{\hat{\mathbf{z}}}_2 &= \hat{\mathbf{M}}^b\dot{\hat{\omega}}_{n,b}^b - \hat{\mathbf{M}}^b\dot{\hat{\omega}}_{n,d}^b + \hat{\mathbf{M}}^b\Gamma(\hat{\mathbf{G}}_\pm + \frac{1}{4}\hat{\mathbf{F}})(\hat{\alpha} + \hat{\mathbf{z}}_2) \\ &= \hat{\mathbf{f}}_u^b + \hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\mathbf{S}}_q(\hat{\omega}_{n,b}^b)\hat{\mathbf{M}}\hat{\omega}_{n,b}^b - \hat{\mathbf{M}}^b\dot{\hat{\omega}}_{n,d}^b + \hat{\mathbf{M}}^b\Gamma(\hat{\mathbf{G}}_\pm + \frac{1}{4}\hat{\mathbf{F}})(\hat{\alpha} + \hat{\mathbf{z}}_2). \end{aligned} \tag{4.21}$$

Defining a second Lyapunov function candidate as

$$V_2(\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2) := V_1 + \frac{1}{2}\hat{\mathbf{z}}_2^\top \hat{\mathbf{K}}\hat{\mathbf{M}}^b \hat{\mathbf{z}}_2, \tag{4.22}$$

we obtain the derivative as

$$\dot{V}_2 = \dot{V}_1 + \hat{\mathbf{z}}_2^\top \hat{\mathbf{K}}[\hat{\mathbf{f}}_u^b + \hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\mathbf{S}}_q(\hat{\omega}_{n,b}^b)\hat{\mathbf{M}}\hat{\omega}_{n,b}^b - \hat{\mathbf{M}}^b \dot{\hat{\omega}}_{n,d}^b + \hat{\mathbf{M}}^b \Gamma(\hat{\mathbf{G}}_\pm + \frac{1}{4}\hat{\mathbf{F}})(\hat{\alpha} + \hat{\mathbf{z}}_2)]. \tag{4.23}$$

Inserting the control force $\hat{\mathbf{f}}_u^b$ in (4.15) gives

$$\begin{aligned} \dot{V}_2 =& \dot{V}_1 + \hat{\mathbf{z}}_2^\top[-k_d\hat{\mathbf{z}}_2 - \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1] = -\hat{\mathbf{z}}_1^\top \hat{\mathbf{T}}_e \Gamma \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1 + \hat{\mathbf{z}}_1^\top \hat{\mathbf{T}}_e \hat{\mathbf{z}}_2 - k_d \hat{\mathbf{z}}_2^\top \hat{\mathbf{z}}_2 - \hat{\mathbf{z}}_2^\top \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1 \\ =& -\hat{\mathbf{z}}_1^\top \hat{\mathbf{T}}_e \Gamma \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1 - k_d \hat{\mathbf{z}}_2^\top \hat{\mathbf{z}}_2, \end{aligned} \tag{4.24}$$

and the closed-loop kinematics and dynamics

$$\begin{aligned} \dot{\hat{\mathbf{z}}}_1 &= -\hat{\mathbf{T}}_e \Gamma \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1 + \hat{\mathbf{T}}_e \hat{\mathbf{z}}_2 \\ \hat{\mathbf{M}}^b \dot{\hat{\mathbf{z}}}_2 &= -k_d \hat{\mathbf{z}}_2 - \hat{\mathbf{T}}_e^\top \hat{\mathbf{z}}_1. \end{aligned} \tag{4.25}$$

The stability properties of the equilibrium point of the closed-loop system above follows from (4.22) and (4.24), and as with the previous theorem Lemma 4.3 is used to show that there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(\mathbf{x}) \leq V(\mathbf{x}) \leq \overline{\alpha}(\mathbf{x})$, i.e.

$$p_m \|\chi\|^2 \leq V(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) \leq p_M \|\chi\|^2, \tag{4.26}$$

for some $p_M > p_m > 0$ and $\chi = [\hat{\mathbf{z}}_1^\top \hat{\mathbf{T}}_e\ \hat{\mathbf{z}}_2^\top]^\top$. We conclude, by Theorem 4.9 in [69], that the equilibrium point $(\hat{\mathbf{z}}_1^\top \hat{\mathbf{T}}_e, \hat{\mathbf{z}}_2) = (\mathbf{0}, \mathbf{0})$ is uniformly asymptotically stable. By the definition of $\hat{\mathbf{z}}_1 = \hat{\mathbf{q}}_{eq}$ and (4.16), it follows that $(\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{T}}_e, \hat{\omega}_e^c) \to (\mathbf{0}, \mathbf{0})$ asymptotically. Again from the assumption that $\eta_e > 0$ and the unit constraint in (2.21) it can be seen that $\eta_e \to 1$. The proof for the negative equilibrium point, $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e-}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e-})$ is performed in the same way. It follows that the two equilibrium points $\hat{\mathbf{q}}_{eq} \in S_e^3 \ltimes \mathbb{R}^3$ are UAS.

4.3 Underactuated Control

The underactuated control problem is more challenging since the system, in this instance, only have four actuators for six degrees of freedom. Thus, one is only able to track four degrees of freedom and in this work we choose to track three translational degrees and one rotation degree.

Remark 4. *In the problem formulation in subsection (4.1) we do not require the trajectory to satisfy the dynamic model of the system, thus due to the construction of the desired quaternion we in fact impose that the system track six degrees of freedom which strictly speaking makes the problem unsolvable for an underactuated system. Under this formulation, practical stability of the equilibrium points is the best result achievable.*

Remark 5. *In this work we generate $\hat{\boldsymbol{q}}_d$ using (3.1) as a function of a translational trajectory, $\boldsymbol{p}_d^n$, and a desired attitude quaternion generated by a desired body frame yaw angle, ψ_d.*

In light of this, special care has to be taken in order to solve the tracking problem for the underactuated quadrotor platform. In this work we use the hand-position approach [70, 71, 72], by considering a point along the axis of actuation of the underactuated system separated from the origin, its translational motion can be controlled through the rotational actuators. In addition, we introduce a virtual rotational frame that allows the use of dual quaternion coordinates for control, and the result is an augmented system that can be seen as fully actuated in the hand-position point. We now introduce two virtual reference frames $\mathcal{F}^v$ and $\mathcal{F}^c$, both represented by dual quaternions as

$$\hat{\mathbf{q}}_{b,v} = \mathbf{q_I} + \varepsilon\frac{1}{2}\Delta^b \otimes \mathbf{q}_I \qquad \hat{\mathbf{q}}_{v,c} = \mathbf{q}_{v,c} + \varepsilon\frac{1}{2}\mathbf{q}_{v,c} \otimes \mathbf{0}, \tag{4.27}$$

where $\mathbf{0} \in \mathbb{H}_p$ is the zero vector as a pure quaternion and $\Delta^b \in \mathbb{H}_p$ is a constant displacement vector defined as $\Delta^b = [0\ 0\ 0\ \Delta]^\top$. Note that $\hat{\mathbf{q}}_{b,v}$ is a dual quaternion representing a pure translation while $\hat{\mathbf{q}}_{v,c}$ is a dual quaternion representing pure rotation. Using $\mathcal{F}^v$ and $\mathcal{F}^c$ we compose an augmented system, $\hat{\mathbf{q}}_{n,c}$, defined as

$$\hat{\mathbf{q}}_{n,c} = \hat{\mathbf{q}}_{n,b} \otimes \hat{\mathbf{q}}_{b,v} \otimes \hat{\mathbf{q}}_{v,c} = \mathbf{q}_{n,c} + \varepsilon\frac{1}{2}\mathbf{p}_c^n \otimes \mathbf{q}_{n,c}, \tag{4.28}$$

and derive its kinematics

$$\dot{\hat{\mathbf{q}}}_{n,c} = \frac{1}{2}\hat{\mathbf{q}}_{n,c} \otimes \hat{\omega}_{n,c}^c, \tag{4.29}$$

with $\hat{\omega}_{n,c}^c = \hat{\omega}_{n,b}^c + \hat{\omega}_{b,v}^c + \hat{\omega}_{v,c}^c$. Noting that $\hat{\omega}_{b,v}^c = \mathbf{0} + \varepsilon\mathbf{0}$ and taking the derivative of the composed system dual velocity we find

$$\dot{\hat{\omega}}_{n,c}^c = \hat{\mathbf{q}}_{b,c}^* \otimes \dot{\hat{\omega}}_{n,b}^b \otimes \hat{\mathbf{q}}_{b,c} + \hat{\omega}_{n,b}^c \times \hat{\omega}_{b,c}^c + \dot{\hat{\omega}}_{v,c}^c, \tag{4.30}$$

which after inserting the dynamics of (3.13) becomes

$$\dot{\hat{\omega}}_{n,c}^c = \hat{\mathbf{q}}_{b,c}^* \otimes (\hat{\mathbf{M}}^b)^{-1}(\hat{\mathbf{f}}_u^b + \hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\omega}_{n,b}^b \times \hat{\mathbf{M}}^b\hat{\omega}_{n,b}^b) \otimes \hat{\mathbf{q}}_{b,c} + \hat{\omega}_{n,b}^c \times \hat{\omega}_{b,c}^c + \dot{\hat{\omega}}_{v,c}^c. \tag{4.31}$$

Using Lemma 3.1 the transformed dual inertia matrix can now be inserted so that

$$\hat{\mathbf{M}}^c\dot{\hat{\omega}}_{n,c}^c = \hat{\mathbf{H}}_b^c(\hat{\mathbf{q}}_{b,c}^*)(\hat{\mathbf{f}}_u^b + \hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\omega}_{n,b}^b \times \hat{\mathbf{M}}^b\hat{\omega}_{n,b}^b) + \hat{\mathbf{M}}^c(\hat{\omega}_{n,b}^c \times \hat{\omega}_{b,c}^c + \dot{\hat{\omega}}_{v,c}^c), \tag{4.32}$$

where $\hat{\mathbf{M}}^c = \hat{\mathbf{H}}_b^c(\hat{\mathbf{q}}_{b,c}^*)\hat{\mathbf{M}}^b\hat{\mathbf{H}}_c^b(\hat{\mathbf{q}}_{b,c})$. Using the definition of $\hat{\mathbf{H}}(\cdot)$ one can show that

$$\hat{\mathbf{M}}^c = \begin{bmatrix} 0 & \mathbf{0}_{1\times3} & 1 & \mathbf{0}_{1\times3} \\ \mathbf{0}_{3\times1} & m\mathbf{S}(\Delta^c) & \mathbf{0}_{3\times1} & m\mathbf{I}_{3\times3} \\ 1 & \mathbf{0}_{1\times3} & 0 & \mathbf{0}_{1\times3} \\ \mathbf{0}_{3\times1} & \mathbf{J}^c & \mathbf{0}_{3\times1} & -m\mathbf{S}(\Delta^c) \end{bmatrix}, \tag{4.33}$$

where $\mathbf{J}^c = \mathbf{R}_b^c\mathbf{J}^b\mathbf{R}_c^b - m\mathbf{S}^2(\Delta^c)$. This allows a new control force to be defined as

$$\hat{\mathbf{f}}_u^c = \left[\mathbf{R}_b^c(\mathbf{f}_T^b + m\mathbf{S}(\Delta^b)\dot{\omega}_{v,c}^v)\right] + \varepsilon\mathbf{R}_b^c\boldsymbol{\tau}^b, \tag{4.34}$$

where we use the fact that $\dot{\omega}_{v,c}^c = \dot{\omega}_{v,c}^c + \varepsilon\mathbf{0}$. The dynamic equation in (4.32) can now be restated as

$$\hat{\mathbf{M}}^c\dot{\hat{\omega}}_{n,c}^c = \hat{\mathbf{f}}_u^c + \hat{\mathbf{H}}_b^c(\hat{\mathbf{q}}_{b,c}^*)(\hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\omega}_{n,b}^b \times \hat{\mathbf{M}}^b\hat{\omega}_{n,b}^b) + \hat{\mathbf{M}}^c(\hat{\omega}_{n,b}^c \times \hat{\omega}_{b,c}^c) + \hat{\delta}(\hat{\mathbf{f}}_u^c), \tag{4.35}$$

with

$$\hat{\delta}(\hat{\mathbf{f}}_u^c) = \mathbf{0} + \varepsilon\mathbf{J}^c\dot{\omega}_{v,c}^c. \tag{4.36}$$

As can be seen in equation (4.34) the composed augmented system $\hat{\mathbf{q}}_{n,c}$ is fully actuated with regards to the configuration space $SE(3)$, with four real and three virtual actuators. In the following we redefine the control problem for in terms of this augmented system.

4.3.1 Problem definition

Let the pose of the augmented system be composed as in (4.28) and the desired trajectory be defined as before satisfying the boundedness condition (4.1). Further, define the tracking error in dual quaternion coordinates as

$$\hat{\mathbf{q}}_e = \hat{\mathbf{q}}_{n,d}^* \otimes \hat{\mathbf{q}}_{n,c} = \mathbf{q}_e + \varepsilon \frac{1}{2} \mathbf{q}_e \otimes \mathbf{p}_e^b, \tag{4.37}$$

with error kinematics and dynamics

$$\begin{aligned} \dot{\hat{\mathbf{q}}}_{e\pm} &= \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})\hat{\omega}_e^c \\ \hat{\mathbf{M}}^c \dot{\hat{\omega}}_e^c &= \hat{\mathbf{f}}_u^c + \hat{\mathbf{H}}_b^c(\hat{\mathbf{q}}_{b,c}^*)(\hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\omega}_{n,b}^b \times \hat{\mathbf{M}}\hat{\omega}_{n,b}^b) + \hat{\mathbf{M}}^c(\hat{\omega}_{n,b}^c \times \hat{\omega}_{b,c}^c - \dot{\hat{\omega}}_{n,d}^c) + \hat{\delta}(\hat{\mathbf{f}}_u^c), \end{aligned} \tag{4.38}$$

where $\hat{\omega}_e^c = \hat{\omega}_{n,c}^c - \hat{\omega}_{n,d}^c$. Then, design a feedback control law, $\hat{\mathbf{f}}_u^c$, that stabilizes the origin of the system (4.38).

By Proposition 1 in [54] asymptotically stabilizing the origin of the system (4.38) is equivalent to asymptotically stabilizing the ball of radius γ around the origin of system (4.4). The problem at hand can now be solved using the control laws presented in the previous section.

4.3.2 Velocity error sliding surface

Theorem 4.3. Let $\hat{\mathbf{q}}_{eq} \in S_e^3 \ltimes \mathbb{R}^3$ and $\mathrm{sgn}(\eta_{\mathrm{e,p}}(\mathrm{t_0})) = \mathrm{sgn}(\eta_{\mathrm{e,p}}(\mathrm{t}))$ for all $t \geq t_0$, and let the desired trajectory, $\hat{\mathbf{q}}_{n,d}$, satisfy (4.1). Then the equilibrium points $(\hat{\mathbf{q}}_{e\pm}, \hat{\omega}_e^c) = (\mathbf{0}, \mathbf{0})$ of the system (4.38), in closed-loop with the control law

$$\begin{aligned} \hat{\mathbf{f}}_u^c =& -\hat{\mathbf{H}}_b^c(\hat{\mathbf{q}}_{b,c}^*)(\hat{\mathbf{f}}_{aux}^b - \hat{\mathbf{D}}\hat{\omega}_{n,b}^b - \hat{\mathbf{S}}_q(\hat{\omega}_{n,b}^b)\hat{\mathbf{M}}^b\hat{\omega}_{n,b}^b) + \hat{\mathbf{M}}^c(\dot{\hat{\omega}}_r^c - \hat{\mathbf{S}}_q(\hat{\omega}_{n,b}^c)\hat{\omega}_{b,c}^c) \\ & - \hat{\mathbf{K}}^{-1}(k_p\tilde{\varepsilon} + k_d\hat{\mathbf{s}}) - \frac{1}{2}\dot{\hat{\mathbf{M}}}^c\hat{\mathbf{s}} - \hat{\delta}(\hat{\mathbf{f}}_u^c) \\ \hat{\mathbf{s}} =& \hat{\omega}_{n,c}^c - \hat{\omega}_r^c = \hat{\omega}_e^c + \Gamma\tilde{\varepsilon} \\ \hat{\omega}_r^c =& \hat{\omega}_{n,d}^c - \Gamma\tilde{\varepsilon} \end{aligned} \tag{4.39}$$

where $\hat{\mathbf{K}}$ is an anti-diagonal identity matrix, $k_p > 0$, $k_d > 0$ and $\Gamma = \Gamma > \mathbf{0}$ are feedback gains, are uniformly asymptotically stable (UAS).

Proof. Again we only consider, without loss of generality, the positive equilibrium point, i.e. $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e+}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e+})$. The closed-loop kinematics and dynamics, resulting from inserting (4.39) into (4.38), is

$$\begin{aligned} \dot{\hat{\mathbf{q}}}_{eq} &= \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\hat{\omega}_e^c \\ \hat{\mathbf{M}}^b\dot{\hat{\mathbf{s}}} &= -\hat{\mathbf{K}}^{-1}(k_p\tilde{\varepsilon} + k_d\hat{\mathbf{s}}) - \frac{1}{2}\dot{\hat{\mathbf{M}}}^c\hat{\mathbf{s}}. \end{aligned} \tag{4.40}$$

Consider the radially unbounded Lyapunov function candidate

$$V(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) := k_p\hat{\mathbf{q}}_{eq}^\top\hat{\mathbf{q}}_{eq} + \frac{1}{2}(\hat{\mathbf{s}})^\top\hat{\mathbf{K}}\hat{\mathbf{M}}^c\hat{\mathbf{s}}. \tag{4.41}$$

Evaluating the time derivative of V one finds

$$\dot{V} = k_p 2\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{T}}_e \hat{\omega}_e^c + \frac{1}{2}\hat{\mathbf{s}}\hat{\mathbf{K}}\dot{\hat{\mathbf{M}}}^c\hat{\mathbf{s}} + \hat{\mathbf{s}}^\top \hat{\mathbf{K}}\hat{\mathbf{M}}^c\dot{\hat{\mathbf{s}}}, \tag{4.42}$$

and inserting the closed-loop dynamics in (4.40) and using the fact that $\hat{\omega}_e^c = \hat{\mathbf{s}} - \Gamma\tilde{\varepsilon}$ yields

$$\dot{V} = -k_p\tilde{\varepsilon}^\top(\hat{\mathbf{s}} - \Gamma\tilde{\varepsilon}) + \frac{1}{2}\hat{\mathbf{s}}\hat{\mathbf{K}}\dot{\hat{\mathbf{M}}}^c\hat{\mathbf{s}} + \hat{\mathbf{s}}^\top\hat{\mathbf{K}}(-\hat{\mathbf{K}}^{-1}(k_p\tilde{\varepsilon} + k_d\hat{\mathbf{s}}) - \frac{1}{2}\dot{\hat{\mathbf{M}}}^c\hat{\mathbf{s}}) = -k_p\tilde{\varepsilon}^\top\Gamma\tilde{\varepsilon} - k_d\hat{\mathbf{s}}^\top\hat{\mathbf{s}}. \tag{4.43}$$

The proof now follows the same path as in the previous ones, i.e. there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(\mathbf{x}) \leq V(\mathbf{x}) \leq \overline{\alpha}(\mathbf{x})$. Defining $\chi = [\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq})\ (\hat{\omega}_e^c)^\top]^\top$ and utilizing Lemma 4.3, we obtain

$$p_m\|\chi\|^2 \leq V(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) \leq p_M\|\chi\|^2, \tag{4.44}$$

for some $p_M > p_m > 0$. Thus choosing $\underline{\alpha}(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) = p_m\|\chi\|^2$ and $\overline{\alpha}(\hat{\mathbf{q}}_{eq}, \hat{\omega}_e^c) = p_M\|\chi\|^2$ ensures the existence of such functions. We conclude, by Theorem 4.9 in [69], that the equilibrium point $(\hat{\mathbf{q}}_{eq}^\top \hat{\mathbf{T}}_e, \hat{\omega}_e^c) = (\mathbf{0}, \mathbf{0})$ is uniformly asymptotically stable. From the assumption that $\eta_e > 0$ and the unit constraint in (2.21) it can be seen that $\eta_e \to 1$. The proof for the negative equilibrium point, $\hat{\mathbf{q}}_{eq} = \hat{\mathbf{q}}_{e-}$ and $\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{eq}) = \hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e-})$ is performed in the same way. It follows that the two equilibrium points $\hat{\mathbf{q}}_{eq} \in S_e^3 \ltimes \mathbb{R}^3$ are UAS.

4.4 Simulation

We illustrate the performance of the derived state-feedback controllers, both for the fully actuated rigid-body presented in 4.2 and for the quadrotor presented in 4.3. All simulations were performed in MATLAB (R2018a) SIMULINK using a fixed-step Dormand-Prince ODE8 solver with $0.01s$ step-size. Since all simulations are performed in ideal environments without noise they only serve to illustrate the theoretical results. The desired trajectory is a circle with 10 meter radius at an altitude of 10 meters whilst tracking a constant angular velocity around the body z-axis, i.e.

$$\mathbf{p}_d^n(t) = [10sin(0.15t)\ 10cos(0.15t)\ -10], \qquad \omega_{n,d}^d = [0\ 0\ 0.05], \tag{4.45}$$

with initial condition $\mathbf{q}_{n,d} = \mathbf{q}_I$.

Remark 6. *Using (3.1) the desired dual quaternion $\hat{\boldsymbol{q}}_{n,d}$ may easily be constructed. As for the desired dual velocity and acceleration the proper construction in $\mathcal{F}^n$ is made as*

$$\begin{aligned} \hat{\omega}_{n,d}^n &= \omega_{n,d}^n + \varepsilon(\dot{\boldsymbol{p}}_d^n - \boldsymbol{S}_q(\omega_{n,d}^n)\boldsymbol{p}_n^d) \\ \dot{\hat{\omega}}_{n,d}^n &= \dot{\omega}_{n,d}^n + \varepsilon(\ddot{\boldsymbol{p}}_d^n - \boldsymbol{S}_q(\dot{\omega}_{n,d}^n)\boldsymbol{p}_n^d - \boldsymbol{S}_q(\omega_{n,d}^n)\dot{\boldsymbol{p}}_n^d), \end{aligned} \tag{4.46}$$

which can be transformed into $\mathcal{F}^d$ using $\hat{\boldsymbol{q}}_{n,d}$.

The mass and inertia properties of the quadrotor and rigid-body is given as $m = 1.3$kg and $\mathbf{J}^b = \text{diag}\{0.04\ 0.04\ 0.5\}\text{kgm}^2$. The control gains are for all control laws set to $k_p = k_d = 1$ and $\Gamma = \mathbf{I}_{8\times 8}$. The initial condition for the systems is

$$\hat{\mathbf{q}}_{n,b}(t_0) = \mathbf{q}_i + \varepsilon\frac{1}{2}[0\ 5\ 0\ -1]^\top \otimes \mathbf{q}_i \qquad \hat{\omega}_{n,b}^b(t_0) = [0\ 0\ 0\ 0.05\ 0\ 0\ 0\ 0]^\top,$$

with $\mathbf{q}_i = [0.5\ 0.5\ 0.5\ 0.5]^\top$ and for the quadrotor the virtual frames are initialized at

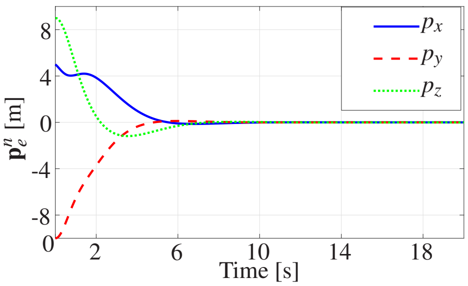

Fig. 1 Position error in $\mathcal{F}^n$ with the controller in (4.10).

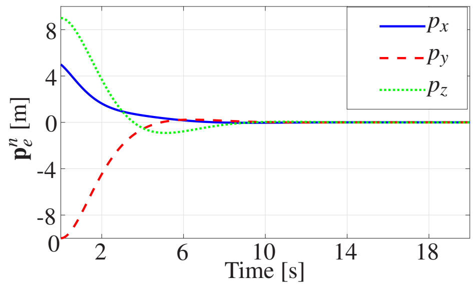

Fig. 2 Position error in $\mathcal{F}^n$ with the controller in (4.15).

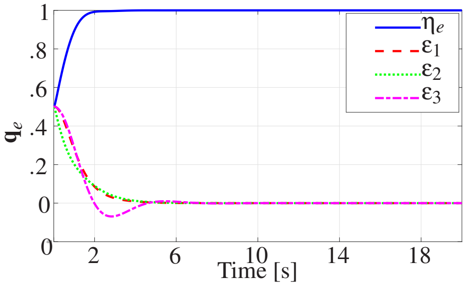

Fig. 3 $\mathbf{q}_e$ with the controller in (4.10).

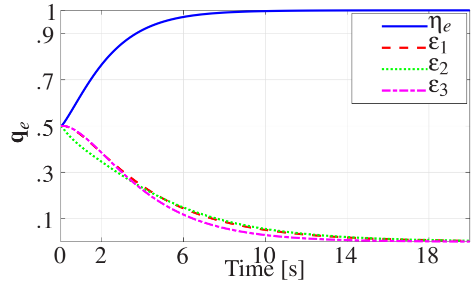

Fig. 4 $\mathbf{q}_e$ with the controller in (4.15).

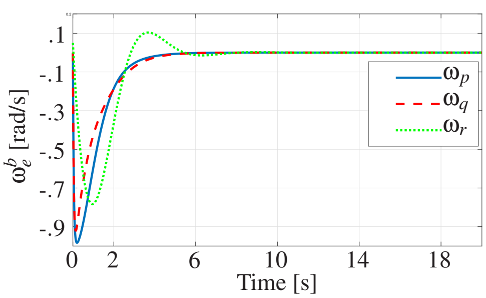

Fig. 5 ω_e^b with the controller in (4.10).

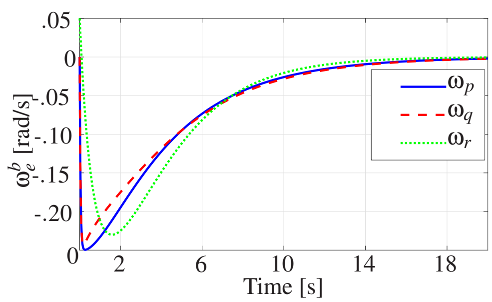

Fig. 6 ω_e^b with the controller in (4.15).

$$\hat{\mathbf{q}}_{b,v}(t_0) = \mathbf{q}_I + \varepsilon\frac{1}{2}\mathbf{q}_I \otimes [0\ 0\ 0\ \Delta]^\top, \qquad \hat{\mathbf{q}}_{v,c}(t_0) = \mathbf{q}_I + \varepsilon\frac{1}{2}\mathbf{q}_I \otimes [0\ 0\ 0\ 0]^\top,$$

with $\Delta = -0.1$. The induced drag coefficients is set to $d_x = d_y = 0.11$ and the gyroscopic torque of the rotors on the airframe is assumed negligible, i.e. $\mathbf{g}_r^b = 0$. First, simulation results are presented for the controller in (4.10) and (4.15) in the case of the fully actuated rigid-body; Figure 1 and 2 present the position error in $\mathcal{F}^n$ for (4.10) and (4.15), respectively. Figure 3 and 4 show how the quaternion error converges to the identity quaternion for both control laws, where it may be seen that the sliding surface controller provides the more aggressive convergence. Figures 5- 8 show the primary and dual part of the dual velocity error $\hat{\omega}_e^b$ for both controllers.

Figures 9-12 present the simulation results for the control law developed for the quadrotor case. Figure 9 show the position error in $\mathcal{F}^n$ for (4.39). Figure 10 show how the quaternion error converges to the identity quaternion while Figures 11 and 12 show the primary and dual part of the dual velocity error $\hat{\omega}_e^c$.

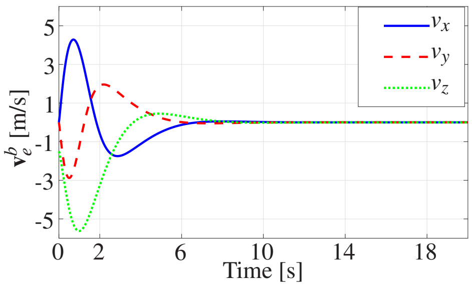

Fig. 7 $\mathbf{v}_e^b$ with the controller in (4.10).

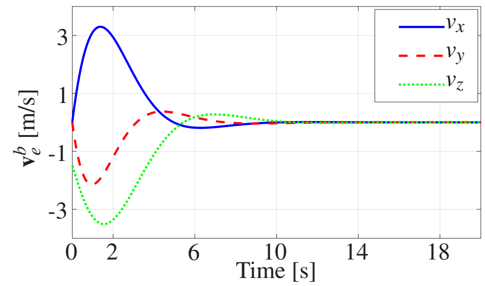

Fig. 8 $\mathbf{v}_e^b$ with the controller in (4.15).

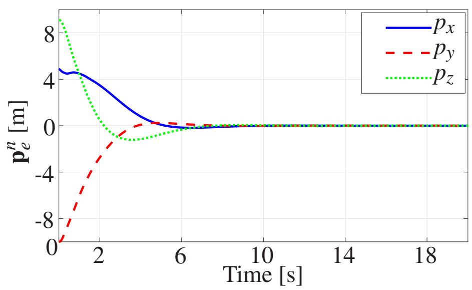

Fig. 9 Position error in $\mathcal{F}^n$ with the controller in (4.39).

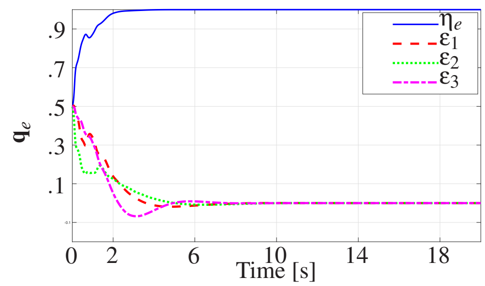

Fig. 10 $\mathbf{q}_e$ with the controller in (4.39).

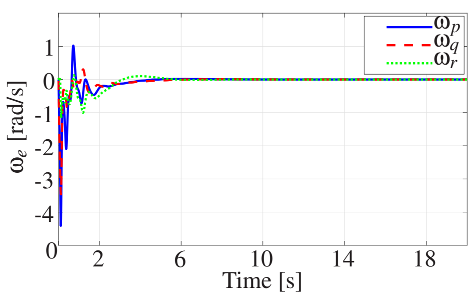

Fig. 11 ω_e^c with the controller in (4.39).

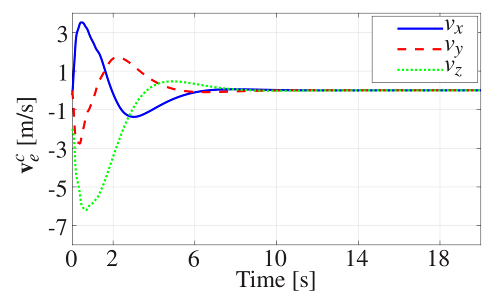

Fig. 12 $\mathbf{v}_e^c$ with the controller in (4.39).

5 Conclusion

In this paper the trajectory tracking problem for both a fully actuated rigid-body and an underactuated quadrotor was solved in the dual quaternion framework. The 6-DOF kinematics and dynamics were derived using dual quaternions and the resemblance to the quaternion formulation of rotational kinematics and dynamics was subsequently used to derive pose control laws for trajectory tracking. A velocity error sliding surface controller and an integrator backstepping controller was derived for the fully actuated rigid-body, and after the introduction of two virtual frames the sliding surface controller was extended to the underactuated case using the hand-position technique. Numerical simulations demonstrate the theoretical results.

Appendix

Proof of lemmas

Proof (Proof of Lemma 3.1). The proof follows from the fact that the kinetic energy must be invariant under transformation, i.e.

$$E_k = \frac{1}{2}(\hat{\omega}^b_{n,b})^\top \hat{\mathbf{K}}\hat{\mathbf{M}}^b \hat{\omega}^b_{n,b} = \frac{1}{2}(\hat{\omega}^a_{n,b})^\top \hat{\mathbf{K}}\hat{\mathbf{M}}^a \hat{\omega}^a_{n,b} \tag{5.1}$$

where $\hat{\mathbf{K}} \in \mathbb{R}^{8\times 8}$ is anti-diagonal identity matrix. Knowing that screws transform under adjoint action, $\hat{\omega}^a_{n,b} = \hat{\mathbf{H}}^a_b \hat{\omega}^b_{n,b}$, we can rewrite the above relation

$$\frac{1}{2}(\hat{\omega}^b_{n,b})^\top \hat{\mathbf{K}}\hat{\mathbf{M}}^b \hat{\omega}^b_{n,b} = \frac{1}{2}(\hat{\omega}^b_{n,b})^\top (\hat{\mathbf{H}}^a_b)^\top \hat{\mathbf{K}}\hat{\mathbf{M}}^a \hat{\mathbf{H}}^a_b \hat{\omega}^b_{n,b} \tag{5.2}$$

which reduces to

$$\begin{aligned} \hat{\mathbf{K}}\hat{\mathbf{M}}^b &= (\hat{\mathbf{H}}^a_b)^\top \hat{\mathbf{K}}\hat{\mathbf{M}}^a \hat{\mathbf{H}}^a_b \\ ((\hat{\mathbf{H}}^a_b)^\top)^{-1} \hat{\mathbf{K}}\hat{\mathbf{M}}^b &= \hat{\mathbf{K}}\hat{\mathbf{M}}^a \hat{\mathbf{H}}^a_b \\ \hat{\mathbf{K}}^{-1}((\hat{\mathbf{H}}^a_b)^\top)^{-1} \hat{\mathbf{K}}\hat{\mathbf{M}}^b (\hat{\mathbf{H}}^a_b)^{-1} &= \hat{\mathbf{M}}^a. \end{aligned} \tag{5.3}$$

By noting that $\hat{\mathbf{K}}^{-1}((\hat{\mathbf{H}}^a_b)^\top)^{-1}\hat{\mathbf{K}} = \hat{\mathbf{H}}^a_b$ and $(\hat{\mathbf{H}}^a_b)^{-1} = \hat{\mathbf{H}}^b_a$ the proof is completed.

Proof (Proof of Lemma 4.4). From Lemma 4.2 we have that

$$\hat{\mathbf{T}}_e^\top(\hat{\mathbf{q}}_{e\pm})\hat{\mathbf{q}}_{e\pm} = \frac{1}{2}\begin{bmatrix} 0 \\ \pm\varepsilon_e \\ 0 \\ \frac{1}{2}\mathbf{p}^b \end{bmatrix} = \frac{1}{2}\tilde{\varepsilon}.$$

Taking the derivative on both sides gives

$$\begin{aligned} \dot{\hat{\mathbf{T}}}_e^\top(\hat{\mathbf{q}}_{e\pm})\hat{\mathbf{q}}_{e\pm} + \hat{\mathbf{T}}_e^\top(\hat{\mathbf{q}}_{e\pm})\dot{\hat{\mathbf{q}}}_{e\pm} &= \frac{1}{2}\dot{\tilde{\varepsilon}} \\ \dot{\hat{\mathbf{T}}}_e^\top(\hat{\mathbf{q}}_{e\pm})\hat{\mathbf{q}}_{e\pm} &= \frac{1}{2}\dot{\tilde{\varepsilon}} - \hat{\mathbf{T}}_e^\top(\hat{\mathbf{q}}_{e\pm})\dot{\hat{\mathbf{q}}}_{e\pm} \end{aligned}$$

and by inserting (4.4) and using the results of Lemma 4.1, we find

$$\dot{\hat{\mathbf{T}}}_e^\top(\hat{\mathbf{q}}_{e\pm})\hat{\mathbf{q}}_{e\pm} = \frac{1}{2}\dot{\tilde{\varepsilon}} - \hat{\mathbf{T}}_e^\top(\hat{\mathbf{q}}_{e\pm})\hat{\mathbf{T}}_e(\hat{\mathbf{q}}_{e\pm})\hat{\omega}^b_e = \frac{1}{2}\dot{\tilde{\varepsilon}} - \frac{1}{4}\hat{\mathbf{F}}\hat{\omega}^b_e.$$

The derivative of $\tilde{\varepsilon}$ can be derived to be

$$\dot{\tilde{\varepsilon}} = \frac{1}{2}\begin{bmatrix} \pm(\mathbf{T}(\eta_e) + \mathbf{S}_q(\varepsilon_e)) & \mathbf{0}_{4\times 4} \\ -\mathbf{S}_q(\mathbf{p}^b_e) & \mathbf{T}(\mathbf{q_I}) \end{bmatrix} \omega^b_e$$

which after insertion concludes the proof.

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