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Some inequalities related to strong convergence of Riesz logarithmic means

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Abstract

In this paper we derive a new strong convergence theorem of Riesz logarithmic means of the one-dimensional Vilenkin–Fourier (Walsh–Fourier) series. The corresponding inequality is pointed out and it is also proved that the inequality is in a sense sharp, at least for the case with Walsh–Fourier series.

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1 Introduction

Concerning definitions used in this introduction we refer to Sect. 2. Weisz [47] proved the boundedness of the maximal operator of Fejér means $\sigma^{\psi,*}$ with respect to bounded Vilenkin systems from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$, for $p > 1/2$. Simon [31] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The corresponding counterexample for $p = 1/2$ is due to Goginava [14]. Moreover, Weisz [50] proved the following result.

Theorem W *The maximal operator of Fejér means $\sigma^{\psi,*}$ is bounded from the Hardy space $H_{1/2}(G_m)$ to the space weak- $L_{1/2}(G_m)$.*

In [35] and [36] it was proved that the maximal operator $\tilde{\sigma}_p^{\psi,}$ defined by*

$$\tilde{\sigma}_p^{\psi,*} := \sup_{n \in \mathbb{N}} \frac{|\sigma_n^{\psi}|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)},$$

where $0 < p \leq 1/2$ and $[1/2 + p]$ denotes the integer part of $1/2 + p$, is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$. Moreover, for any nondecreasing function $\varphi : \mathbb{N}_+ \rightarrow [1, \infty)$ satisfying the condition

$$\overline{\lim}_{n \rightarrow \infty} \frac{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}{\varphi(n)} = +\infty, \tag{1.1}$$

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there exists a martingale $f \in H_p(G_m)$, such that

$$\sup_{n \in \mathbb{N}} \left\| \frac{\sigma_n^\psi f}{\varphi(n)} \right\|_p = \infty.$$

For Walsh–Kaczmarzi system some analogical results were proved in [16] and [37].

Weisz [47] considered the norm convergence of the Fejér means of a Vilenkin–Fourier series and proved the following result.

Theorem W1 (Weisz) *Let $p > 1/2$ and $f \in H_p(G_m)$. Then there exists an absolute constant c_p , depending only on p , such that for all $k = 1, 2, \dots$ and $f \in H_p(G_m)$ the following inequality holds:*

$$\|\sigma_k^\psi f\|_p \leq c_p \|f\|_{H_p(G_m)}.$$

Moreover, in [34] it was proved that the assumption $p > 1/2$ in Theorem W1 is essential. In fact, the following is true.

Theorem T1 *There exists a martingale $f \in H_{1/2}(G_m)$ such that*

$$\sup_{n \in \mathbb{N}} \|\sigma_n^\psi f\|_{1/2} = +\infty.$$

Theorem W1 implies that

$$\frac{1}{n^{2p-1}} \sum_{k=1}^n \frac{\|\sigma_k^\psi f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p(G_m)}^p, \quad 1/2 < p < \infty, n = 1, 2, \dots$$

If Theorem W1 holds for $0 < p \leq 1/2$, then we would have

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^n \frac{\|\sigma_k^\psi f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p(G_m)}^p, \quad 0 < p \leq 1/2, n = 2, 3, \dots \tag{1.2}$$

For the Walsh system in [38] and for the bounded Vilenkin systems in [37] were proved that (1.2) holds, though Theorem T1 is not true for $0 < p < 1/2$.

Some results concerning summability of the Fejér means of a Vilenkin–Fourier series can be found in [10, 12, 16, 25, 28, 30].

The Riesz logarithmic means with respect to the Walsh system was studied by Simon [31], Goginava [15], Gát, Nagy [13] and for Vilenkin systems by Gát [11] and Blahota, Gát [3], Persson, Ragusa, Samko, Wall [26]. Moreover, in [27] it was proved that the maximal operator of the Riesz logarithmic means of a Vilenkin–Fourier series is bounded from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$ when $p > 1/2$ and is not bounded from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$ when $0 < p \leq 1/2$.

In [35] and [36] it was proved that the Riesz logarithmic means has better properties than the Fejér means. In particular, one considered the maximal operator $\tilde{R}_p^{\psi,*}$ of a Riesz logarithmic means $\tilde{R}_p^{\psi,*}$ defined by

$$\tilde{R}_p^{\psi,*} := \sup_{n \in \mathbb{N}} \frac{|R_n^\psi| \log(n+1)}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)},$$

where $0 < p \leq 1/2$ and $[1/2 + p]$ denotes the integer part of $1/2 + p$, which is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.

Moreover, this result is sharp in the following sense: For any nondecreasing function $\varphi : \mathbb{N}_+ \rightarrow [1, \infty)$ satisfying the condition

$$\lim_{n \rightarrow \infty} \frac{(n + 1)^{1/p-2} \log^{2[1/2+p]}(n + 1)}{\varphi(n) \log(n + 1)} = \infty, \tag{1.3}$$

there exists a martingale $f \in H_p(G_m)$, such that

$$\sup_{n \in \mathbb{N}} \left\| \frac{R_n^\psi f}{\varphi(n)} \right\|_p = \infty.$$

The main aim of this paper is to derive a new strong convergence theorem of the Riesz logarithmic means of one-dimensional Vilenkin–Fourier (Walsh–Fourier) series (see Theorem 1). The corresponding inequality is pointed out. The sharpness is proved in Theorem 2, at least for the case with Walsh–Fourier series.

The paper is organized as follows: In Sect. 2 some definitions and notations are presented. The main results are presented and proved in Sect. 3. Section 4 is reserved for some concluding remarks and open problems.

2 Definitions and notations

Let \mathbb{N}_+ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k .

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of the Z_{m_j} .

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is a Haar measure on G_m with $\mu(G_m) = 1$.

If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this paper we discuss only bounded Vilenkin groups.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of G_m , namely

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I}_n := G_m \setminus I_n$.

Let

$$e_n := (0, 0, \dots, x_n = 1, 0, \dots) \in G_m \quad (n \in \mathbb{N}).$$

It is evident that

$$\overline{I}_M = \left(\bigcup_{k=0}^{M-2} \bigcup_{x_k=1}^{m_k-1} \bigcup_{l=k+1}^{M-1} \bigcup_{x_l=1}^{m_l-1} I_{l+1}(x_k e_k + x_l e_l) \right) \cup \left(\bigcup_{k=1}^{M-1} \bigcup_{x_k=1}^{m_k-1} I_M(x_k e_k) \right). \tag{2.1}$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ ($j \in \mathbb{N}$) and only a finite number of the n_j differ from zero. Let $|n| := \max\{j \in \mathbb{N}; n_j \neq 0\}$.

The norm (or quasi-norm when $p < 1$) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p d\mu \right)^{1/p} \quad (0 < p < \infty).$$

The space weak- $L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p(G_m)} := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. Let us define complex valued function $r_k(x) : G_m \rightarrow \mathbb{C}$, the generalized Rademacher functions, as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad (i^2 = -1, x \in G_m, k \in \mathbb{N}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

The Vilenkin systems are orthonormal and complete in $L_2(G_m)$ (for details see e.g. [1]). Specifically, we call this system Walsh–Paley if $m_k = 2$, for all $k \in \mathbb{N}$. In this case we have the dyadic group $G_2 = \prod_{j=0}^{\infty} Z_2$, which is called the Walsh group and the Vilenkin system coincides with the Walsh functions defined by (for details see e.g. [17] and [29])

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}),$$

where $n_k = 0 \vee 1$ and $x_k = 0 \vee 1$.

Now, we introduce analogues of the usual definitions in Fourier analysis.

If $f \in L_1(G_m)$, then we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system ψ (Walsh system w) in the usual manner:

$$\begin{aligned} \widehat{f}^\alpha(k) &:= \int_{G_m} f \overline{\alpha}_k d\mu \quad (\alpha_k = w_k \text{ or } \psi_k) \quad (k \in \mathbb{N}), \\ S_n^\alpha f &:= \sum_{k=0}^{n-1} \widehat{f}^\alpha(k) \alpha_k \quad (\alpha_k = w_k \text{ or } \psi_k) \quad (n \in \mathbb{N}_+, S_0^\alpha f := 0), \\ \sigma_n^\alpha f &:= \frac{1}{n} \sum_{k=0}^{n-1} S_k^\alpha f \quad (\alpha = w \text{ or } \psi) \quad (n \in \mathbb{N}_+), \\ D_n^\alpha &:= \sum_{k=0}^{n-1} \alpha_k \quad (\alpha = w \text{ or } \psi) \quad (n \in \mathbb{N}_+), \\ K_n^\alpha &:= \frac{1}{n} \sum_{k=0}^{n-1} D_k^\alpha \quad (\alpha = w \text{ or } \psi) \quad (n \in \mathbb{N}_+). \end{aligned}$$

It is well known that (see e.g. [1])

$$\sup_{n \in \mathbb{N}} \int_{G_m} |K_n^\alpha| d\mu \leq c < \infty, \quad \text{where } \alpha = w \text{ or } \psi. \tag{2.2}$$

The σ -algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n ($n \in \mathbb{N}$). Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to F_n ($n \in \mathbb{N}$) (for details see e.g. [5, 23, 46]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

In the case $f \in L_1(G_m)$, the maximal functions are also given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p(G_m)} := \|f^*\|_p < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that $S_{M,n}f$ is F_n measurable and the sequence $(S_{M,n}f : n \in \mathbb{N})$ is a martingale. If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Vilenkin–Fourier (Walsh–Fourier) coefficients must be defined in a slightly different manner, namely

$$\widehat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f^{(k)}(x) \overline{\alpha}_i(x) d\mu(x), \quad \text{where } \alpha = w \text{ or } \psi.$$

The Vilenkin–Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M,n}f : n \in \mathbb{N})$ obtained from f .

In the literature, there is the notion of the Riesz logarithmic means of a Fourier series. The n th Riesz logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n^\alpha f := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k^\alpha f}{k}, \quad \text{where } \alpha = w \text{ or } \psi,$$

with

$$l_n := \sum_{k=1}^n \frac{1}{k}.$$

The kernels of Riesz’s logarithmic means are defined by

$$L_n^\alpha := \frac{1}{l_n} \sum_{k=1}^n \frac{D_k^\alpha}{k}, \quad \text{where } (\alpha = w \text{ or } \psi).$$

For the martingale f we consider the following maximal operators:

$$\begin{aligned} \sigma^{\alpha,*} f &:= \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f| \quad (\alpha = w \text{ or } \psi), \\ R^* f &:= \sup_{n \in \mathbb{N}} |R_n^\alpha f| \quad (\alpha = w \text{ or } \psi), \\ \tilde{R}^{\alpha,*} f &:= \sup_{n \in \mathbb{N}} \frac{|R_n^\alpha f|}{\log(n+1)} \quad (\alpha = w \text{ or } \psi), \\ \tilde{R}_p^{\alpha,*} f &:= \sup_{n \in \mathbb{N}} \frac{\log(n+1) |R_n^\alpha f|}{(n+1)^{1/p-2}} \quad (\alpha = w \text{ or } \psi). \end{aligned}$$

A bounded measurable function a is a p -atom, if there exists an interval I , such that

$$\int_I a \, d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

In order to prove our main results we need the following lemma of Weisz (for details see e.g. Weisz [49]).

Proposition 1 *A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p(G_m)$ ($0 < p \leq 1$) if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p -atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers such that for every $n \in \mathbb{N}$*

$$\sum_{k=0}^\infty \mu_k S_{M_n} a_k = f^{(n)} \tag{2.3}$$

and

$$\sum_{k=0}^\infty |\mu_k|^p < \infty.$$

Moreover, $\|f\|_{H_p(G_m)} \sim \inf(\sum_{k=0}^\infty |\mu_k|^p)^{1/p}$, where the infimum is taken over all decompositions of f of the form (2.3).

By using atomic characterization (see Proposition 1) it can be easily proved that the following statement holds (see e.g. Weisz [50]).

Proposition 2 *Suppose that an operator T is sub-linear and for some $0 < p_0 \leq 1$*

$$\int_I |Ta|^{p_0} d\mu \leq c_p < \infty$$

for every p_0 -atom a , where I denotes the support of the atom. If T is bounded from L_{p_1} to L_{p_1} ($1 < p_1 \leq \infty$), then

$$\|Tf\|_{p_0} \leq c_{p_0} \|f\|_{H_{p_0}(G_m)}. \tag{2.4}$$

Let us define classical Hardy spaces (see e.g. [44]). Let $H_p(D)$, $p > 0$ be the one-dimensional complex quasi-Banach space of analytic functions f on the unit disc $D := \{z : |z| < 1\}$ for which

$$\|f\|_{H_p(D)} = \sup_{r < 1} \frac{1}{2\pi} \left(\int_{[-\pi, \pi]} |f(re^{it})|^p dt \right)^{1/p}.$$

Now, we define real Hardy spaces. A real-valued distributions $f(t) \in D'(T)$ belongs to $H_p(T)$ where $T = (-\pi, \pi]$ if and only if there exists a function $F(z) \in H_p(D)$ with the properties $\text{Im}(F(0)) = 0$ and $f(t) = \lim_{r \rightarrow 1} \text{Re } F(re^{it})$ in the sense of distributions. Equipped with quasi-norm $\|f(z)\|_{H_p(T)} = \|F(z)\|_{H_p(D)}$ the class obviously becomes a real quasi-Banach space with quite the same properties as $H_p(D)$. Atomic decomposition of classical Hardy spaces and real Hardy spaces can be found e.g. in Fefferman and Stein [6] (see also Later [19], Torchinsky [44], Wilson [51]).

3 Main results

Our first main result reads as follows.

Theorem 1 *Let $0 < p < 1/2$ and $f \in H_p(G_m)$. Then there exists an absolute constant c_p , depending only on p , such that the inequality*

$$\sum_{n=1}^{\infty} \frac{\log^p n \|R_n^\psi f\|_{H_p(G_m)}^p}{n^{2-2p}} \leq c_p \|f\|_{H_p(G_m)}^p \tag{3.1}$$

holds, where $R_n^\psi f$ denotes the n th Riesz logarithmic mean with respect to the Vilenkin–Fourier series of f .

For the proof of Theorem 1 we will use the following lemmas.

Lemma 1 (see [38]) *Let $x \in I_N(x_k e_k + x_l e_l)$, $1 \leq x_k \leq m_k - 1$, $1 \leq x_l \leq m_l - 1$, $k = 0, \dots, N - 2$, $l = k + 1, \dots, N - 1$. Then*

$$\int_{I_N} |K_n^\psi(x - t)| d\mu(t) \leq \frac{cM_l M_k}{nM_N}, \quad \text{when } n \geq M_N.$$

Let $x \in I_N(x_k e_k)$, $1 \leq x_k \leq m_k - 1$, $k = 0, \dots, N - 1$. Then

$$\int_{I_N} |K_n^\psi(x - t)| d\mu(t) \leq \frac{cM_k}{M_N}, \quad \text{when } n \geq M_N.$$

Lemma 2 (see [39]) Let $x \in I_N(x_k e_k + x_l e_l)$, $1 \leq x_k \leq m_k - 1$, $1 \leq x_l \leq m_l - 1$, $k = 0, \dots, N - 2$, $l = k + 1, \dots, N - 1$. Then

$$\int_{I_N} \sum_{j=M_N+1}^n \frac{|K_j^\psi(x - t)|}{j + 1} d\mu(t) \leq \frac{cM_k M_l}{M_N^2}.$$

Let $x \in I_N(x_k e_k)$, $1 \leq x_k \leq m_k - 1$, $k = 0, \dots, N - 1$. Then

$$\int_{I_N} \sum_{j=M_N+1}^n \frac{|K_j^\psi(x - t)|}{j + 1} d\mu(t) \leq \frac{cM_k}{M_N} J_n.$$

Proof By using an Abel transformation, the kernels of the Riesz logarithmic means can be rewritten as (see also [39])

$$L_n^\psi = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{K_j^\psi}{j + 1} + \frac{K_n^\psi}{l_n}. \tag{3.2}$$

Hence, according to (2.2) we get

$$\sup_{n \in \mathbb{N}} \int_{G_m} |L_n^\alpha| d\mu \leq c < \infty, \quad \text{where } \alpha = w \text{ or } \psi$$

and it follows that R_n^ψ is bounded from L_∞ to L_∞ . By Proposition 2, the proof of Theorem 1 will be complete, if we show that

$$\sum_{n=1}^\infty \frac{\log^p n \int_I |R_n^\psi a|^p d\mu}{n^{2-2p}} \leq c_p < \infty, \quad \text{for } 0 < p < 1/2, \tag{3.3}$$

for every p -atom a , where I denotes the support of the atom.

Let a be an arbitrary p -atom with support I and $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $R_n^\psi a = \sigma_n^\psi(a) = 0$, when $n \leq M_N$. Therefore we suppose that $n > M_N$.

Since $\|a\|_\infty \leq cM_N^2$ if we apply (3.2), then we can conclude that

$$\begin{aligned} & |R_n^\psi a(x)| \\ &= \int_{I_N} |a(t)| |L_n^\psi(x - t)| d\mu(t) \\ &\leq \|a\|_\infty \int_{I_N} |L_n^\psi(x - t)| d\mu(t) \\ &\leq \frac{cM_N^{1/p}}{l_n} \int_{I_N} \sum_{j=M_N+1}^{n-1} \frac{|K_j^\psi(x - t)|}{j + 1} d\mu(t) \\ &\quad + \frac{cM_N^{1/p}}{l_n} \int_{I_N} |K_n^\psi(x - t)| d\mu(t). \end{aligned} \tag{3.4}$$

Let $x \in I_N(x_k e_k + x_l e_l)$, $1 \leq x_k \leq m_k - 1$, $1 \leq x_l \leq m_l - 1$, $k = 0, \dots, N - 2$, $l = k + 1, \dots, N - 1$. From Lemmas 1 and 2 it follows that

$$|R_n^\psi a(x)| \leq \frac{cM_l M_k M_N^{1/p-2}}{\log(n+1)}. \tag{3.5}$$

Let $x \in I_N(x_k e_k)$, $1 \leq x_k \leq m_k - 1$, $k = 0, \dots, N - 1$. Applying Lemmas 1 and 2 we can conclude that

$$|R_n^\psi a(x)| \leq M_N^{1/p-1} M_k. \tag{3.6}$$

By combining (2.1) and (3.4)–(3.6) we obtain

$$\begin{aligned} & \int_{I_N} |R_n^\psi a(x)|^p d\mu(x) \\ &= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j \in \{l+1, \dots, N-1\}}^{m_j-1} \int_{I_N^{k,l}} |R_n^\psi a|^p d\mu + \sum_{k=0}^{N-1} \int_{I_N^{k,N}} |R_n^\psi a|^p d\mu \\ &\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \dots m_{N-1}}{M_N} \frac{(M_l M_k)^p M_N^{1-2p}}{\log^p(n+1)} + \sum_{k=0}^{N-1} \frac{1}{M_N} M_k^p M_N^{1-p} \\ &\leq \frac{cM_N^{1-2p}}{\log^p(n+1)} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^p}{M_l} + \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^p} \\ &\leq \frac{cM_N^{1-2p}}{\log^p(n+1)} + c_p. \end{aligned} \tag{3.7}$$

It is easy to see that

$$\sum_{n=M_N+1}^{\infty} \frac{1}{n^{2-2p}} \leq \frac{c}{M_N^{1-2p}}, \quad \text{for } 0 < p < 1/2. \tag{3.8}$$

By combining (3.7) and (3.8) we get

$$\begin{aligned} & \sum_{n=M_N+1}^{\infty} \frac{\log^p n \int_{I_N} |R_n a|^p d\mu}{n^{2-2p}} \\ &\leq \sum_{n=M_N+1}^{\infty} \left(\frac{c_p M_N^{1-2p}}{n^{2-p}} + \frac{c_p}{n^{2-p}} \right) + c_p \\ &\leq c_p M_N^{1-2p} \sum_{n=M_N+1}^{\infty} \frac{1}{n^{2-2p}} + \sum_{n=M_N+1}^{\infty} \frac{1}{n^{2-p}} + c_p \leq C_p < \infty. \end{aligned}$$

It means that (3.3) holds true and the proof is complete. □

Our next main result shows in particular that the inequality in Theorem 1 is in a special sense sharp at least in the case of Walsh–Fourier series (cf. also Problem 2 in the next section).

Theorem 2 Let $0 < p < 1/2$ and $\Phi : \mathbb{N} \rightarrow [1, \infty)$ be any nondecreasing function, satisfying the condition

$$\lim_{n \rightarrow \infty} \Phi(n) = +\infty. \tag{3.9}$$

Then there exists a martingale $f \in H_p(G_2)$ such that

$$\sum_{n=1}^{\infty} \frac{\log^p n \|R_n^w f\|_p^p \Phi(n)}{n^{2-2p}} = \infty, \tag{3.10}$$

where $R_n^w f$ denotes the n th Riesz logarithmic means with respect to Walsh–Fourier series of f .

Proof It is evident that if we assume that $\Phi(n) \geq cn$, where c is some positive constant then

$$\frac{\log^p n \Phi(n)}{n^{2-2p}} \geq n^{1-2p} \log^p n \rightarrow \infty, \quad \text{as } n \rightarrow \infty,$$

and also (3.10) holds. So, without loss of generality we may assume that there exists an increasing sequence of positive integers $\{\alpha'_k : k \in \mathbb{N}\}$ such that

$$\Phi(\alpha'_k) = o(\alpha'_k), \quad \text{as } k \rightarrow \infty. \tag{3.11}$$

Let $\{\alpha_k : k \in \mathbb{N}\} \subseteq \{\alpha'_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that $\alpha_0 \geq 2$ and

$$\sum_{k=0}^{\infty} \frac{1}{\Phi^{1/2}(2^{2\alpha_k})} < \infty, \tag{3.12}$$

$$\sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta/p}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \leq \frac{2^{2\alpha_{k-1}/p+1}}{\Phi^{1/2p}(2^{2\alpha_{k-1}})}, \tag{3.13}$$

$$\frac{2^{2\alpha_{k-1}/p+1}}{\Phi^{1/2p}(2^{2\alpha_{k-1}})} \leq \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}. \tag{3.14}$$

We note that under condition (3.11) we can conclude that

$$\frac{2^{2\alpha_\eta/p}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \geq \left(\frac{2^{2\alpha_\eta}}{\Phi(2^{2\alpha_\eta})} \right)^{1/2p} \rightarrow \infty, \quad \text{as } \eta \rightarrow \infty$$

and it immediately follows that such an increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$, which satisfies conditions (3.12)–(3.14), can be constructed.

Let

$$f^{(A)}(x) := \sum_{\{k: 2\alpha_k < A\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{1}{\Phi^{1/2p}(2^{2\alpha_k})}$$

and

$$a_k = 2^{2\alpha_k(1/p-1)}(D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}).$$

From (3.12) and Lemma 1 we can conclude that $f = (f^{(n)}, n \in \mathbb{N}) \in H_p(G_2)$.

It is easy to show that

$$\widehat{f}^w(j) = \begin{cases} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})}, & \text{if } j \in \{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1} - 1\}, k \in \mathbb{N}, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1} - 1\}. \end{cases} \tag{3.15}$$

For $n = \sum_{i=1}^s 2^{n_i}$, $n_1 < n_2 < \dots < n_s$ we denote

$$\mathbb{A}_{0,2} := \left\{ n \in \mathbb{N} : n = 2^0 + 2^2 + \sum_{i=3}^{n_s} 2^{n_i} \right\}.$$

Let $2^{2\alpha_k} \leq j \leq 2^{2\alpha_k+1} - 1$ and $j \in \mathbb{A}_{0,2}$. Then

$$R_j^w f = \frac{1}{l_j} \sum_{n=1}^{2^{2\alpha_k-1}} \frac{S_n f}{n} + \frac{1}{l_j} \sum_{n=2^{2\alpha_k}}^j \frac{S_n f}{n} := I + II. \tag{3.16}$$

Let $n < 2^{2\alpha_k}$. Then from (3.13), (3.14) and (3.15) we have

$$\begin{aligned} |S_n^w f(x)| &\leq \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} |\widehat{f}^w(v)| \leq \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \\ &\leq \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta/p}}{\Phi^{1/2p}(2^{2\alpha_\eta})} \leq \frac{2^{2\alpha_{k-1}/p+1}}{\Phi^{1/2p}(2^{2\alpha_{k-1}})} \leq \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}. \end{aligned}$$

Consequently,

$$\begin{aligned} |I| &\leq \frac{1}{l_j} \sum_{n=1}^{2^{2\alpha_k-1}} \frac{|S_n^w f(x)|}{n} \\ &\leq \frac{1}{l_{2^{2\alpha_k}}} \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} \sum_{n=1}^{2^{2\alpha_k-1}} \frac{1}{n} \leq \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}. \end{aligned} \tag{3.17}$$

Let $2^{2\alpha_k} \leq n \leq 2^{2\alpha_k+1} - 1$. Then we have the following:

$$\begin{aligned} S_n^w f &= \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \widehat{f}^w(v) w_v + \sum_{v=2^{2\alpha_k}}^{n-1} \widehat{f}^w(v) w_v \\ &= \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_\eta})} (D_{2^{2\alpha_\eta+1}}^w - D_{2^{2\alpha_\eta}}^w) + \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} (D_n^w - D_{2^{2\alpha_k}}^w). \end{aligned}$$

This gives

$$\begin{aligned}
 II &= \frac{1}{l_j} \sum_{n=2^{2\alpha_k}}^{2^{2\alpha_k+1}} \frac{1}{n} \left(\sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_\eta})} (D_{2^{2\alpha_\eta+1}}^w - D_{2^{2\alpha_\eta}}^w) \right) \\
 &\quad + \frac{1}{l_j} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \sum_{n=2^{2\alpha_k}}^j \frac{(D_n^w - D_{2^{2\alpha_k}}^w)}{n} \\
 &:= II_1 + II_2.
 \end{aligned} \tag{3.18}$$

Let $x \in I_2(e_0 + e_1) \in I_0 \setminus I_1$. We use well-known equalities for Dirichlet kernels (for details see e.g. [17] and [29]): recall that

$$D_{2^n}^w(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{3.19}$$

and

$$D_n^w = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k}^w = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}}^w - D_{2^k}^w), \quad \text{for } n = \sum_{i=0}^{\infty} n_i 2^i, \tag{3.20}$$

so we can conclude that

$$D_n^w(x) = \begin{cases} w_n, & \text{if } n \text{ is odd number,} \\ 0, & \text{if } n \text{ is even number.} \end{cases}$$

Since $\alpha_0 \geq 2, k \in \mathbb{N}$ we obtain $2\alpha_k \geq 4$, for all $k \in \mathbb{N}$ and if we apply (3.19) we get

$$II_1 = 0 \tag{3.21}$$

and

$$II_2 = \frac{1}{l_j} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \sum_{n=2^{2\alpha_k-1}}^{(j-1)/2} \frac{w_{2n+1}}{2n+1} = \frac{1}{l_j} \frac{2^{2\alpha_k(1/p-1)} r_1}{\Phi^{1/2p}(2^{2\alpha_k})} \sum_{n=2^{2\alpha_k-1}}^{(j-1)/2} \frac{w_{2n}}{2n+1}.$$

Let $x \in I_2(e_0 + e_1)$. Then, by the definition of Walsh functions, we get

$$w_{4n+2} = r_1 w_{4n} = -w_{4n}$$

and

$$\begin{aligned}
 |II_2| &= \frac{1}{l_j} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left| \sum_{n=2^{2\alpha_k-1}}^{(j-1)/2} \frac{w_{2n}}{2n+1} \right| \\
 &= \frac{1}{l_j} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left| \frac{w_{j-1}}{j} + \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \left(\frac{w_{4n-4}}{4n-3} + \frac{w_{4n-2}}{4n-1} \right) \right|
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{l_j} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left| \frac{w_{j-1}}{j} + \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \left(\frac{w_{4n-4}}{4n-3} - \frac{w_{4n-2}}{4n-1} \right) \right| \\
 &\geq \frac{c}{\log(2^{2\alpha_k+1})} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\left| \frac{w_{j-1}}{j} \right| - \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} |w_{4n-4}| \left(\frac{1}{4n-3} - \frac{1}{4n-1} \right) \right) \\
 &\geq \frac{1}{4\alpha_k} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\frac{1}{j} - \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \left(\frac{1}{4n-3} - \frac{1}{4n-1} \right) \right). \tag{3.22}
 \end{aligned}$$

By a simple calculation we can conclude that

$$\begin{aligned}
 &\sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \left(\frac{1}{4n-3} - \frac{1}{4n-1} \right) \\
 &= \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \frac{2}{(4n-3)(4n-1)} \\
 &\leq \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \frac{2}{(4n-4)(4n-2)} = \frac{1}{2} \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \frac{1}{(2n-2)(2n-1)} \\
 &\leq \frac{1}{2} \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \frac{1}{(2n-2)(2n-2)} = \frac{1}{8} \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \frac{1}{(n-1)(n-1)} \\
 &\leq \frac{1}{8} \sum_{n=2^{2\alpha_k-2}+1}^{(j-1)/4} \frac{1}{(n-1)(n-2)} = \frac{1}{8} \sum_{l=2^{2\alpha_k-2}+1}^{(j-1)/4} \left(\frac{1}{n-2} - \frac{1}{n-1} \right) \\
 &\leq \frac{1}{8} \left(\frac{1}{2^{2\alpha_k-2}-1} - \frac{4}{j-5} \right) \leq \frac{1}{8} \left(\frac{1}{2^{2\alpha_k-2}-1} - \frac{4}{j} \right).
 \end{aligned}$$

Since $2^{2\alpha_k} \leq j \leq 2^{2\alpha_k+1} - 1$, where $\alpha_k \geq 2$, we obtain

$$\frac{2}{2^{2\alpha_k}-4} \leq \frac{2}{2^4-4} = \frac{1}{6}$$

and

$$\begin{aligned}
 |II_2| &\geq \frac{1}{4\alpha_k} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\frac{1}{j} - \frac{1}{8} \left(\frac{1}{2^{2\alpha_k-2}-1} - \frac{4}{j} \right) \right) \tag{3.23} \\
 &\geq \frac{1}{4\alpha_k} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\frac{3}{2j} - \frac{1}{2^{2\alpha_k+1}-8} \right) \\
 &\geq \frac{1}{4\alpha_k} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\frac{3}{4} \frac{1}{2^{2\alpha_k}} - \frac{1}{2} \frac{1}{2^{2\alpha_k}-4} \right) \\
 &\geq \frac{1}{4\alpha_k} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\frac{1}{4} \frac{1}{2^{2\alpha_k}} + \frac{1}{2} \frac{1}{2^{2\alpha_k}} - \frac{1}{2} \frac{1}{2^{2\alpha_k}-4} \right) \\
 &= \frac{1}{4\alpha_k} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\frac{1}{4} \frac{1}{2^{2\alpha_k}} - \frac{2}{2^{2\alpha_k}(2^{2\alpha_k}-4)} \right) \\
 &\geq \frac{1}{4\alpha_k} \frac{2^{2\alpha_k(1/p-1)}}{\Phi^{1/2p}(2^{2\alpha_k})} \left(\frac{1}{4} \frac{1}{2^{2\alpha_k}} - \frac{1}{6} \frac{1}{2^{2\alpha_k}} \right)
 \end{aligned}$$

$$\geq \frac{1}{48\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} \geq \frac{1}{64\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}.$$

By combining (3.14), (3.16)–(3.23) for $x \in I_2(e_0 + e_1)$ and $0 < p < 1/2$ we find that

$$\begin{aligned} |R_j^w f(x)| &\geq |II_2| - |II_1| - |I| \\ &\geq \frac{1}{64\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} - \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} = \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})}. \end{aligned}$$

Hence,

$$\begin{aligned} &\|R_j^w f\|_{\text{weak-}L_p(G_2)}^p \\ &\geq \frac{1}{128\alpha_k^p} \frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})} \mu \left\{ x \in G_2 : |R_j^w f| \geq \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} \right\}^{1/p} \\ &\geq \frac{1}{128\alpha_k^p} \frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})} \mu \left\{ x \in I_2(e_0 + e_1) : |R_j^w f| \geq \frac{1}{128\alpha_k} \frac{2^{2\alpha_k(1/p-2)}}{\Phi^{1/2p}(2^{2\alpha_k})} \right\} \\ &\geq \frac{1}{128\alpha_k^p} \frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})} (\mu(x \in I_2(e_0 + e_1))) > \frac{1}{516\alpha_k^p} \frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})}. \end{aligned} \tag{3.24}$$

Moreover,

$$\begin{aligned} &\sum_{j=1}^{\infty} \frac{\|R_j^w f\|_{\text{weak-}L_p(G_2)}^p \log^p(j)\Phi(j)}{j^{2-2p}} \\ &\geq \sum_{\{j \in \mathbb{A}_{0,2} : 2^{2\alpha_k} < j \leq 2^{2\alpha_k+1}-1\}} \frac{\|R_j^w f\|_{\text{weak-}L_p}^p \log^p(j)\Phi(j)}{j^{2-2p}} \\ &\geq \frac{c}{\alpha_k^p} \frac{2^{2\alpha_k(1-2p)}}{\Phi^{p/2}(2^{2\alpha_k})} \sum_{\{j \in \mathbb{A}_{0,2} : 2^{2\alpha_k} < j \leq 2^{2\alpha_k+1}-1\}} \frac{\log^p(j)\Phi(j)}{j^{2-2p}} \\ &\geq \frac{c\Phi(2^{2\alpha_k}) \log^p(2^{2\alpha_k})}{\alpha_k^p} \frac{2^{2\alpha_k(1-2p)}}{\Phi^{1/2}(2^{2\alpha_k})} \sum_{\{j \in \mathbb{A}_{0,2} : 2^{2\alpha_k} < j \leq 2^{2\alpha_k+1}-1\}} \frac{1}{j^{2-2p}} \\ &\geq \Phi^{1/2}(2^{2\alpha_k}) \rightarrow \infty, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

The proof is complete. □

4 Final remarks and open problems

In this section we present some final remarks and open problems, which might be interesting for further research. The first problem reads as follows.

Problem 1 For any $f \in H_{1/2}$, is it possible to find strong convergence theorems for Riesz means R_m^w , where $\alpha = w$ or $\alpha = \psi$?

Remark 1 Similar problems for Fejér means with respect to Walsh and Vilenkin systems can be found in [2, 4, 40] (see also [45] and [48]). Our method and estimations of Riesz and Fejér kernels (see Lemmas 1 and 2) do not give an opportunity to prove even similar

strong convergence result as for the case of Fejer means. In particular, for any $f \in H_{1/2}$ is it possible to prove the following inequality:

$$\frac{1}{\log n} \sum_{k=1}^n \frac{\|R_k^\alpha f\|_{1/2}^{1/2}}{k} \leq c \|f\|_{H_{1/2}}^{1/2}, \quad \text{where } \alpha = w \text{ or } \alpha = \psi?$$

It is interesting to generalize Theorem 2 for Vilenkin systems.

Problem 2 For $0 < p < 1/2$ and any nondecreasing function $\Phi : \mathbb{N} \rightarrow [1, \infty)$ satisfying the conditions $\lim_{n \rightarrow \infty} \Phi(n) = +\infty$, is it possible to find a martingale $f \in H_p(G_m)$ such that

$$\sum_{n=1}^{\infty} \frac{\log^p n \|R_n^\psi f\|_p^p \Phi(n)}{n^{2-2p}} = \infty,$$

where $R_n^\psi f$ denotes the n th Riesz logarithmic means with respect to the Vilenkin–Fourier series of f ?

Problem 3 Is it possible to find a martingale $f \in H_{1/2}$, such that

$$\sup_{n \in \mathbb{N}} \|R_n^\alpha f\|_{1/2} = \infty,$$

where $\alpha = w$ or $\alpha = \psi$?

Remark 2 For $0 < p < 1/2$, divergence in the space L_p of Riesz logarithmic means with respect to Walsh and Vilenkin systems of martingale $f \in H_p$ was already proved in [27].

Problem 4 For any $f \in H_p$ ($0 < p \leq 1/2$), is it possible to find necessary and sufficient conditions for the indices k_j for which

$$\|R_{k_j}^\alpha f - f\|_{H_p} \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

where $\alpha = w$ or $\alpha = \psi$?

Remark 3 Similar problem for partial sums and Fejer means with respect to Walsh and Vilenkin systems can be found in Tephnadze [41, 42] and [43].

Problem 5 Is it possible to find necessary and sufficient conditions in terms of the one-dimensional modulus of continuity of martingale $f \in H_p$ ($0 < p \leq 1/2$), for which

$$\|R_j^\alpha f - f\|_{H_p} \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

where $\alpha = w$ or ψ ?

Remark 4 Approximation properties of some summability methods in the classical and real Hardy spaces were considered by Oswald [24], Kryakin and Trebels [18], Storoienko [32, 33] and for martingale Hardy spaces in Fridli, Manchanda and Siddiqi [9] (see also [7, 8]), Nagy [20–22], Tephnadze [41–43].

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