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Differential Invariants of Symplectic and Contact Lie Algebra Actions

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Abstract

In this thesis we consider the equivalence problem for symplectic and conformal symplectic group actions on submanifolds and functions. We solve the equivalence problem for general submanifolds by means of computing differential invariants and describing all the invariants of the associated group action by appealing to the Lie-Tresse theorem.

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Introduction

Consider the problem of determining whether two submanifolds $N_1, N_2 \subseteq M$ are equivalent up to some transformation (change of coordinates). That is, does there exist some diffeomorphism $F : M \to M$ such that $N_1 = F^* N_2$. Locally, this is always the case as manifolds of the same dimension are all locally Euclidean and we consider general diffeomorphisms $F \in \mathrm{Diff}(M)$. The problem at hand in this thesis will be to restrict to a subgroup of $\mathrm{Diff}(M)$ that preserves some additional structure on M and then consider the same problem. In other words, can we transform one submanifold to the other by some restricted change of coordinates? An example of such a question is in Euclidean geometry where one looks to congruence of curves. Curves in Euclidean geometry are distinguished by the square of their curvature. A curve in the plane $y : \mathbb{R} \to \mathbb{R}^2$ has curvature given by

$$\kappa = \frac{y''}{(1 + (y')^2)^{3/2}},$$

for which κ^2 will distinguish plane curves. Meaning if two curves have different square curvature, there exists no Euclidean transformation (isometry) mapping them onto each other. In this thesis we consider the same problem but the transformations will be different and the submanifolds will not only be curves, but higher dimensional submanifolds as well.

Another problem of finding transformations related to submanifolds is the change of coordinates for functions f, or their level curves $\{f = c\}$ (which can be considered submanifolds). Given two functions $f, g : M \to \mathbb{R}$ is there some restricted change of coordinates $F : M \longrightarrow M$, such that $f = F^*g$? This question can be rephrased as a question regarding foliations by hypersurfaces. Is it possible to transform one foliation into the other foliation by a change of coordinates that is subjected to some extra conditions?

The problems introduced are called equivalence problems, which is a standard question in geometry. Can one geometric object be transformed into another by some change of coordinates? We'll try to answer this question in this thesis for symplectic and conformal symplectic transformations. The key to answer the question of equivalence is by invariants, where an invariant is an object which is unaffected by a change of coordinates. Then if two geometric objects have different invariants they can't be equivalent. This will be the approach, namely to compute and understand the invariants of the transformations given.

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Prerequisites

We'll start with a brief introduction to jet spaces, prolongations of vector fields and some notions regarding differential invariants. Most of the theorems regarding differential invariants are also summarized in this chapter. Section 2.1 gives a discussion of jet spaces, which is the setting for most of our work. Section 2.2 introduces prolongation of group actions and vector fields, while section 2.3 and 2.4 introduces differential invariants and the main theorems. In section 2.5 we discuss counting of differential invariants and how to apply our main results.

2.1 Jet Spaces

The discussion of jet spaces is split in two cases. The first being jets of functions and the other being jets of submanifolds. We'll restrict to scalar functions of several variables, which will suffice for our goal. For more details regarding jet spaces, see [Olv95],[Olv00], [KL08], or [KVL86].

2.1.1 Jets of Functions

Let M be an n-dimensional smooth manifold. Denote $C^\infty(M)$ the set of all smooth functions $f : M \to \mathbb{R}$. This forms a commutative unital algebra under addition and multiplication of functions. Consider functions $f \in C^\infty(M)$ for which $f(a) = 0$ for some $a \in M$. These functions forms an ideal, denoted μ_a. The set of functions that vanish at a up to order k also forms an ideal, denoted μ_a^k. Doing this allows the construct

$$J_a^k M = C^\infty(M)/\mu_a^{k+1}, \tag{2.1}$$

which is called jet space of order k at $a \in M$. Elements of $J_a^k M$ are equivalence classes of functions that vanish up to order k at the point a. We say $f \sim g$ if f and g are tangent up to order k. This defines the jet space of functions, denoted $J^k M$, as the following $J^k M = \coprod_{a\in M} J_a^k M$, giving a smooth manifold of dimension $\dim J^k M = n + \binom{n+k}{n}$. For every k and $0 \leq l < k$ there is a natural projection $\pi_{k,l} : J^k M \to J^l M$ defined by $\pi_{k,l}([f]_a^k) = [f]_a^l$. In particular, $J^k M$ is a bundle over $J^l M$ for any $l < k$. We make the convention of writing $M = J^{-1} M$, so that $J^k M$ is also a bundle over M with the projection $\pi([f]_a^k) = a$. This can be summarized as the following tower of jets

$$M = J^{-1}M \xleftarrow{\pi_{-1,0}} J^0 M \xleftarrow{\pi_{0,1}} J^1 M \xleftarrow{\pi_{1,2}} \dots \xleftarrow{\pi_{k-1,k}} J^k M \xleftarrow{\pi_{k,k+1}} \dots,$$

where $\pi_{l,m} \circ \pi_{k,l} = \pi_{k,m}$, for $k > l > m \geq -1$. Let the local coordinates on M be $x^1, \dots, x^n$ and introduce the local coordinates

$$x^i([f]_a^k) = a^i,$$
$$u_\sigma([f]_a^k) = \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a),$$

on $J^k M$ where $\sigma = (i_1, \dots, i_j)$ is a multi-index of length $0 \leq |\sigma| \leq k$. We'll also write u instead of u_0.

2.1.2 Jets of Submanifolds

Let M be a smooth manifold of dimension $n + m$ and consider submanifolds $N \subseteq M$ of dimension n. Two submanifolds $N_1, N_2 \subseteq M$ are considered equivalent if they are tangent up to order k at some point $a \in N_1 \cap N_2$. This defines an equivalence relation on the set of all submanifolds. We'll denote this equivalence class as $[N]_a^k$ for submanifolds that are tangent up to order k at the point $a \in M$ and refer to the class $[N]_a^k$ as the k-jet of N at the point a. Denote $J_a^k(M, n)$ as the set of all k-jets at the point a of dimension n. Doing this we can define $J^k(M, n) = \coprod_{a\in M} J_a^k(M, n)$ as the space of all k-jets of submanifolds. This comes equipped with the structure of a smooth manifold of dimension $\dim J^k(M, n) = n + m\binom{n+k}{k}$. The jet space $J^k(M, n)$ carries a natural projection map $\pi_{k,l}$ defined by $\pi_{k,l}([N]_a^k) = [N]_a^l$ for $k > l \geq 0$, making $\pi_{k,l}$ smooth bundles. As before, this gives a tower structure of bundles

$$J^0(M, n) \xleftarrow{\pi_{0,1}} J^1(M, n) \xleftarrow{\pi_{1,2}} \dots \xleftarrow{\pi_{k-1,k}} J^k(M, n) \xleftarrow{\pi_{k,k+1}} \dots$$

similar to the situation for jets of functions. Given $N \subseteq M$ there is a natural embedding $j_k : N \hookrightarrow J^k(M,n)$ defined by $j_k(N) = [N]^k_a$ for $a \in N$. In many cases we discuss J^kM and $J^k(M,n)$ at the same time, when this occurs we'll use the notation J^k instead when there is no difference between the two cases. The tower structure associated with the jet bundles allows the construction of the inverse limit of the jet bundles, $J^\infty = \varprojlim J^k$, which is needed later.

2.2 Prolongation

In this section we'll discuss prolongation, mainly of vector fields, but also prolongation of group actions. Prolongations allow us to extend group actions and vector fields on M to J^k.

2.2.1 Prolongation of Group Actions

Recall that a Lie group action is a Lie group homomorphism $\Phi : G \longrightarrow \mathrm{Diff}(M)$, where M is a smooth manifold and $\mathrm{Diff}(M)$ denotes the set of diffeomorphisms of M, which forms a group under composition. Given any $g \in G$, we then view g as a diffeomorphism of M. This action is written as $g \cdot p$ for $g \in G$ and $p \in M$. Any $g \in G$ is called a point transformation since it takes points $p \in M$ and sends them to $g \cdot p \in M$. Given $g \in G$, then Φ_g is taken as a diffeomorphism of M, then the idea of prolongation is to construct a diffeomorphism $\Phi^{(k)}_g : J^k \longrightarrow J^k$ called the k-th order prolongation. The prolongation of Φ_g in local coordinates can be expressed as

$$\Phi^{(k)}_g([N]^k_p) = [\Phi_g(N)]^k_{\Phi_g(p)}, \tag{2.2}$$

where N is a submanifold $N \subseteq M$ while for functions $f : M \longrightarrow \mathbb{R}$ the action is by the pullback $g \cdot f = g^* f$.

The bundle J^k comes equipped with an additional structure called the Cartan distribution. The map $j^k : N \longrightarrow J^k(M,n)$ takes submanifolds of M into $J^k(M,n)$, in particular consider the tangent space of this submanifold at the point $a_k \in J^k(M,n)$, define

$$L(a_{k+1}) = T_{a_k} j^k(N) \subseteq T_{a_k} J^k(M,n). \tag{2.3}$$

This does not depend on the choice of N but it is dependent on $a_{k+1} = [N]^{k+1}_a$ since we need information about higher jets to describe the tangent space. Define the Cartan distribution as

$$C_k(a_k) = \mathrm{span}\{L(a_{k+1}) \mid a_{k+1} \in \pi^{-1}_{k+1,k}(a_k)\} \subseteq TJ^k,$$

that is, all tangent spaces to prolonged functions/submanifolds. The Cartan distribution can be described as

$$C_k = \langle \mathcal{D}^{(k)}_{x^i}, \partial_{u_\sigma} \mid |\sigma| = k \rangle, \tag{2.4}$$

with $\mathcal{D}^{(k)}_{x^i} = \partial_{x^i} + \sum_{|\tau|<k} u^j_{\tau+1_i} \partial_{u^j_\tau}$. A local diffeomorphism of J^k that preserves the Cartan distribution is called a **Lie transformation** and in fact the prolongations $\Phi^{(k)}_g$ are Lie transformations. If we consider jet spaces of single variable functions, in particular J^1M, then J^1M is an odd-dimensional manifold and the Cartan distribution reduces the following contact structure

$$C_1 = \langle \mathcal{D}^{(1)}_{x^i} = \partial_{x^i} + u\partial_{u_i}, \partial_u \rangle. \tag{2.5}$$

A **contact structure** on an n-dimensional manifold is a codimension 1 distribution $\Pi \subseteq TM$ that is completely non-integrable. In particular J^1M is a contact manifold. In many cases it is desirable to work with the annihilator of C_1 instead. The annihilator is

$$\operatorname{Ann} C_1 = \langle \omega = du - u_i dx^i \rangle, \tag{2.6}$$

where we call ω a contact form. A diffeomorphism $\phi : J^1M \to J^1M$ that preserves the contact structure, that is, $\phi^*\omega = \lambda\omega$, is called a **contact transformation**. The next theorem gives a description of transformations that preserve structures in jet spaces.

Theorem 2.2.1. *(Lie-Bäcklund)*. Suppose M is an n-dimensional manifold. Let J^k denote the k-th jet bundle of M. Then any Lie transformation of J^k is the prolongation of

(a) $m \geq 2$; local point transformation, $\Phi_g : M \to M$,

(b) $m = 1$; local contact transformation $\phi : J^1 \to J^1$.

where m indicates the number of dependent variables.

The Lie-Bäcklund theorem then classifies all diffeomorphisms for jets of functions due to the restriction of only considering one dependent variable.

2.2.2 Prolongation of Vector Fields

Let M be a smooth manifold of dimension n and $X \in \mathcal{D}(J^0)$ be a vector field, given as

$$X = a^i \partial_{x^i} + b_j \partial_{u_j},$$

in local coordinates, with x^i independent and u_j dependent. We'd like to "lift" the vector field X to a vector field on J^k, denoted as $X^{(k)}$. This is called the k-th prolongation of X. The bundle J^k comes naturally equipped with the Cartan distribution, so the prolongation has to preserve the Cartan distribution, that is

$$\mathcal{L}_{X^{(k)}} C_k \equiv 0 \mod C_k. \tag{2.7}$$

Due to the Lie-Bäcklund theorem every vector field that preserves the Cartan distribution is the prolongation of either a point transformation on M or a contact transformation on J^1. Due to this we have two cases to consider when dealing with the prolongation of vector fields. The prolongation of vector fields can be computed explicitly by the requirement of preserving the Cartan distribution. In local coordinates:

$$X^{(k)} = a^i \mathcal{D}^{(k+1)}_{x^i} + \sum_{|\sigma| \leq k} \mathcal{D}_\sigma(\varphi^j) \partial_{u^j_\sigma} \tag{2.8}$$

where $\varphi = (\varphi^1, \ldots, \varphi^m)$ and $\varphi^j = b^j - a^i u^j_i$. The function φ is the generating function for the prolongation for $m \geq 2$. The case of contact transformations takes the form

$$X^{(k)}_f = -\partial_{u_i}(f) \mathcal{D}^{(k+1)}_{x^i} + \sum_{|\sigma| \leq k} \mathcal{D}_\sigma(f) \partial_{u_\sigma} \tag{2.9}$$

for $f = -a^i u_i$ when $m = 1$.

2.3 Differential Invariants

In this section we'll state some definitions, techniques for computation and theorems related to our main topic of differential invariants. Differential invariants were introduced by Sophus Lie in [Lie80] to study local transformation groups and they provide useful information regarding the equivalence problem.

2.3.1 Definitions

Definition 2.3.1. Let G be a Lie group acting on a smooth manifold M. Denote the action as $g \cdot p$ for $g \in G$ and $p \in M$. A function $I : M \to \mathbb{R}$ is called an **invariant** if

$$I(g \cdot p) = I(p) \tag{2.10}$$

for all $g \in G$. If I is an invariant of the k-th prolonged group action we call I a **differential invariant of order** k.

If I and J are differential invariants of the same order, then so are $I + J$ and IJ. In particular, the set of all differential invariants forms a commutative algebra over $\mathbb{R}$. Denote the algebra of k-th order differential invariants by $\mathcal{A}_k$. A differential invariant of order k is also a differential invariant of order $k+1$, so $\mathcal{A}_k \subseteq \mathcal{A}_{k+1}$. We can then consider the set of all differential invariants of arbitrary order, defined and denoted as $\mathcal{A} = \varinjlim \mathcal{A}_k \subseteq C^\infty(J^\infty)$. This gives a filtration of $\mathcal{A}$ as

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_k \subseteq \mathcal{A}_{k+1} \subseteq \ldots$$

which we'd like to describe. In classical invariant theory the invariant functions separate the orbits of the group action under some regularity assumptions, see [Olv99]. This fact will be the main tool to solve the equivalence problem, so understanding differential invariants is essential. Our goal in this thesis is then to find necessary and sufficient conditions to completely describe the orbits of the group action and find all necessary invariants. In classical invariant theory for reductive group actions there exists a finite generating set for the algebra of invariants, the semisimple case is due to Hilbert, (see [Hil93]), and the general reductive case is due to Mumford, (see [MFK94]). If $I_1, I_2, \ldots, I_p$ are differential invariants then for any smooth function H depending on p variables we have that $H(I_1, I_2, \ldots, I_p)$ is also a differential invariant whenever the composition is defined. Because of this some invariants are not necessary. In particular, we would like to have an independent set of generators. For independent invariants we use the following definition.

Definition 2.3.2. The k-th order differential invariants $I_1, I_2, \ldots, I_p$ are called **independent invariants** if they are independent as functions on J^k in the usual sense.

In our situation $\mathcal{A}$ is not finitely generated in the usual sense since by prolongation new independent invariants are obtained (more on this later). Because of this there is an infinite amount of independent differential invariants on J^∞. The algebra $\mathcal{A}$ is however, finitely generated in the sense of the Lie-Tresse theorem (discussed below). The problem at hand is how to find the differential invariants and to determine necessary and sufficient conditions for these to generate the whole algebra of differential invariants.

2.3.2 Finding Differential Invariants

The homomorphism $\Phi : G \hookrightarrow \mathrm{Diff}(M)$ allows us to consider $g \in G$ as a diffeomorphism of M. The condition for being an invariant can then be rephrased as the pullback $g^* I = I$. For G, a connected Lie group, we can use the exponential map to write $g(t) = \exp(Xt)$ for some $X \in \mathrm{Lie}(G) = \mathfrak{g}$. This corresponds to a one-parameter subgroup associated to G, which is the flow associated to the vector field X. Rewrite $g^*I = I$ and taking a derivative gives

$$\frac{d}{dt}\bigg|_{t=0} \exp(Xt)^* I = \mathcal{L}_X I = 0, \quad \text{for all } X \in \mathfrak{g}. \tag{2.11}$$

This gives an equation to actually solve for invariants, given some Lie algebra $\mathfrak{g}$. Choose a basis for the Lie algebra $\mathfrak{g}$, then solve Eq. (2.11) on the chosen basis. This is sufficient due to the linearity of the Lie derivative. Vector field prolongation is a Lie algebra homomorphism (see [Olv00] Thm 2.39. p. 115) so this gives a criterion to find differential invariants of order k. We have

$$\mathcal{L}_{X^{(k)}} I = 0, \quad \text{for all } X \in \mathfrak{g}, \tag{2.12}$$

where $X^{(k)}$ denotes the k-th order prolongation of X. Using Eq. (2.12) we can compute differential invariants by solving a linear PDE system on the unknown function I depending on the variables on J^k. This is our first approach to finding differential invariants, but as we shall see in the next chapter it is not sufficient for computing them in numerous cases due to computation time and complexity of the equations needed to be solved.

Let $\mathcal{A} \subseteq C^\infty(J^\infty)$ denote the algebra of differential invariants and $\nabla : \mathcal{A} \to \mathcal{A}$ be a derivation of $\mathcal{A}$. Derivations of $\mathcal{A}$ forms a module over $\mathcal{A}$ and a Lie subalgebra of $\mathcal{D}(J^\infty)$ under the usual Lie bracket. Denote this $\mathcal{A}$-module by $\mathcal{M} \subseteq \mathcal{D}(J^\infty)$.

Definition 2.3.3. A derivation $\nabla \in \mathcal{M}$ is called an **invariant derivation** if it is G-invariant. That is, for all $g \in G$ we have $g_*^{(k+1)}\nabla = \nabla g_*^{(k)}$ for all k starting from the order of the coefficients of ∇.

As above, for G a connected Lie group, we can use the infinitesimal approach by writing $g = d/dt \exp(tX)|_{t=0}$. This yields

$$\mathcal{L}_{X^{(k)}} \nabla = [X^{(k)}, \nabla] = 0, \quad \text{for all } X \in \mathfrak{g}. \tag{2.13}$$

Linearity allows us to check (2.13) by choosing a basis for the Lie algebra $\mathfrak{g}$ and letting the commutator act on the local coordinate functions. Invariant derivations play an important role in the statement of the Lie-Tresse theorem, acting as part of the generating set for $\mathcal{A}$.

Lemma 2.3.4. If I is a differential invariant of order k and ∇ is an invariant derivation, then $\nabla(I)$ is a differential invariant of order $k+1$.

Proof. Let ∇ be an invariant derivation, and consider the following commutative diagram:

$$\begin{tikzcd} C^\infty(J^\infty) \arrow[r, "\mathcal{L}_{X^{(\infty)}}"] \arrow[d, "\nabla"] & C^\infty(J^\infty) \arrow[d, "\nabla"] \\ C^\infty(J^\infty) \arrow[r, "\mathcal{L}_{X^{(\infty)}}"] & C^\infty(J^\infty) \end{tikzcd}$$

An invariant derivation acting on a function I, increases the jet order of the function, while the Lie derivative preserves the jet order. The restriction of the diagram to some k gives the commutative diagram:

$$\begin{tikzcd} C^\infty(J^k) \arrow[r, "\mathcal{L}_{X^{(k)}}"] \arrow[d, "\nabla"] & C^\infty(J^k) \arrow[d, "\nabla"] \\ C^\infty(J^{k+1}) \arrow[r, "\mathcal{L}_{X^{(k+1)}}"] & C^\infty(J^{k+1}). \end{tikzcd}$$

A function I, is a differential invariant, if $\mathcal{L}_{X^{(k)}} I = 0$. Take I to be a differential invariant of order k, then by commutativity of the diagram

$$\nabla(\mathcal{L}_{X^{(k)}} I) = \mathcal{L}_{X^{(k+1)}}(\nabla(I)).$$

Because $\mathcal{L}_{X^{(k)}} I = 0$, it follows by commutativity that $\mathcal{L}_{X^{(k+1)}}(\nabla(I)) = 0$. ■

According to Lemma 2.3.4 it is possible to obtain new invariants from any given set of invariants by applying invariant derivations. Any given set of invariants can produce new differential invariants of any order by repeated application of invariant derivations. Therefore, the algebra of differential invariants can't be finitely generated in the usual sense, unless all the new differential invariants obtained can be expressed by the previous invariants. This is not the case since the new invariants are of higher order. Another way to construct differential invariants by using invariant derivations is provided by the next proposition.

Proposition 2.3.5. If ∇ is an invariant derivation and I is a differential invariant, both of order k. Then $I\nabla$ is also an invariant derivation, in particular if $\nabla_1, \ldots, \nabla_l$ are linearly independent invariant derivations, then $\nabla = I^i \nabla_i$ is also an invariant derivation if and only if I^i are all differential invariants.

In particular, we can compute all Lie brackets of our invariant derivations and write the results in terms of our original invariant derivations. The coefficients forming the linear combinations are then differential invariants by Proposition 2.3.5.

2.4 The Lie-Tresse Theorem

In this section we'll state the Lie-Tresse theorem, but for the purposes of the computations and hypothesis of the theorem it is necessary to restrict the attention to a class of simpler Lie groups and actions. We'll assume that the Lie group is algebraic, that is, it is given by algebraic equations as a subset $G \subseteq \mathrm{GL}(n)$, for some $n \in \mathbb{N}$. We'll also make some restrictions of the action of G on M. Let $G^{(k)}$ denote the prolongation of the group G to J^k and denote $G_a^{(k)} = \{\varphi \in G^{(k)} \mid \varphi(a) = a\}$ the stabilizer of $a \in M$. This forms a subgroup of $G^{(k)}$ which also acts on J^k. Now we can describe algebraic actions of G on M.

Definition 2.4.1. The action of G on M is called an **algebraic action** if the stabilizer $G_a^{(k)}$ is an algebraic group that acts algebraically on J_a^k for any $a \in M$.

Remark. Here $G_a^{(k)}$ acting algebraically on J_a^k means that the action is described by algebraic equations, meaning either polynomial or rational equations.

The prolongation of an algebraic action is again algebraic (see [KL16]). In particular we only need to check that $G_a^{(1)}$ acts algebraically on J_a^1. The algebra of differential invariants $\mathcal{A}$ is not finitely generated in the usual sense, but in the sense of the Lie-Tresse theorem which we are now ready to state.

Theorem 2.4.2. (*Kruglikov-Lychagin*). Let G be an algebraic Lie group acting on a smooth manifold M. Denote $\mathcal{A}$ as the algebra of differential invariants. If the action of G on M is both algebraic and transitive, there exists a finite number of differential invariants $I_1, I_2, \ldots, I_p$ and a finite number of invariant derivations $\nabla_1, \nabla_2, \ldots, \nabla_q$ such that, any $I \in \mathcal{A}$ is a polynomial of the form $\nabla_J I_i$, with $1 \leq i \leq p$, $J = (j_1, \ldots, j_r)$, and all the coefficients are rational functions of I_i. Further, on generic points the differential invariants separate the regular orbits.

This is the main tool to be used in understanding the algebra of differential invariants, which in turn sheds light on the equivalence problem. The idea of a finiteness theorem was introduced by Lie in [Lie80], then Tresse demonstrated and argued for it in [Tre93] and some partial proofs for a micro-local version are done in [Ovs82],[Kum74] and [KL06]. For more details and a complete proof of the global version can be found in [KL16]. The theorem doesn't state how many of each are sufficient. This is the topic for the next section.

2.5 Generators and Differential Syzygies

According to the Lie-Tresse theorem, there exists a finite number of differential invariants and invariant derivations to generate the whole algebra of differential invariants. The goal of this section is to answer the following two questions in the case of Lie group actions:

(1) How many differential invariants are necessary?

(2) How many invariant derivations do we need?

2.5.1 Counting Invariants and Invariant Derivations

Recall that we're considering a Lie group action $G \hookrightarrow \mathrm{Diff}(M)$ so any $g \in G$ gives a diffeomorphism of M. On the other hand we'd like to consider the infinitesimal approach, that given a Lie algebra $\mathfrak{g}$, we can construct a Lie algebra action. A **Lie algebra action** is a Lie algebra homomorphism $\mathfrak{g} \hookrightarrow \mathcal{D}(M)$. In particular, given any $X \in \mathfrak{g}$ as abstract data for some Lie algebra, we can construct a corresponding vector field on M by:

$$\hat{X} = \frac{d}{dt}\bigg|_{t=0} (\exp(tX) \cdot p), \quad \text{for } p \in M, \tag{2.14}$$

called the **infinitesimal generator** of the group action. Here $\hat{X}$ is a vector field on M. We'll typically call $X \in \mathfrak{g}$ a vector field on M using this identification and we'll do so from now on.

Given a Lie algebra action $\phi : \mathfrak{g} \hookrightarrow \mathcal{D}(M)$ on the manifold M, which takes $X \in \mathfrak{g}$ to $\phi(X) \in \mathcal{D}(M)$. As with group actions, we'll abuse the notation and just write X and treat it as a vector field on M. This action induces a local Lie group action by the exponential map. Then understanding the orbits, $\mathcal{O}_p = \{g \cdot p \mid g \in G\}$, of the action is closely related to the number of invariants for the action. Under a Lie group action the orbits are submanifolds of M. If the corresponding group acting is connected, so are the orbits. This will always be the case for us since our actions are defined by the Lie algebra, which gets mapped to an open neighborhood around unity in the corresponding Lie group. Differential invariants are constant on the orbits of the action. In our case the action will be given in terms of the Lie algebra, which determines a distribution on M of rank $\dim \mathfrak{g}$. Due to this, Frobenius' theorem sheds light on the structure of the orbits. Recall that a distribution Π is called **integrable** if there exists some submanifold $N \subseteq M$ such that $\Pi = TN$. A distribution is called involutive if $X, Y \in \Pi$ implies $[X, Y] \in \Pi$.

Theorem 2.5.1. (*Frobenius*). Let M be a manifold of dimension n and $\Pi \subseteq TM$ be a distribution of rank $r < n$ everywhere. If Π is involutive, then there exists integrable submanifolds corresponding to Π, moreover there exists local coordinates such that $\Pi = \langle \partial_{x^1}, \ldots, \partial_{x^r} \rangle$ for which $x^{r+1} = c_{r+1}, \ldots, x^n = c_n$ are integral submanifolds of Π.

Due to the Frobenius' Theorem we can deduce some restrictions on the number of independent differential invariants. The orbits of the group action are integral submanifolds for the Lie algebra $\mathfrak{g}$ viewed as a distribution on J^k. In fact, take a basis for $\mathfrak{g}$ at $p \in M$, say $\langle X_1|_p, X_2|_p, \ldots, X_r|_p \rangle$. Then these vector fields span the tangent space of the orbit at p in M. This also applies to the prolongation of $\mathfrak{g}$. Due to this the dimension of the orbits in J^k are equal to the dimension of $\mathfrak{g}^{(k)}|_p$.[1] The action of the associated group is transitive if $\mathfrak{g}_p = TM|_p$ for every $p \in M$. This gives the dimension of the orbits of the action on J^k. To compute it, it is sufficient to compute the rank of $\mathfrak{g}^{(k)}$ in J^k. Let s_k denote the maximal generic orbit dimension of the action of $\mathfrak{g}^{(k)}$ on J^k. As noted above, this is $s_k = \dim \mathfrak{g}^{(k)}|_{p_k} \subseteq TJ^k|_{p_k}$ for $p_k \in J^k$. We'll only consider the discussion micro-locally, that is in some neighborhood $p_k \in U \subseteq J^k$ to avoid singularities in the orbits. Then s_k can be computed as the rank of the Jacobian matrix of the vector fields $X_1^{(k)}, \ldots, X_r^{(k)}$, having the coefficients as entries. In accordance with the Lie-Tresse theorem a transitive action is needed on the base space M. This simple check of the rank for the vector fields takes care of this requirement. The next theorem is a consequence of the Frobenius' theorem.

Theorem 2.5.2. Let G be a Lie group acting freely on the n-dimensional manifold M with s-dimensional orbits. Then at any point $p \in M$ there exists local independent invariants $I_1, \ldots, I_{n-s}$ defined in a neighborhood of p.

Proof. The proof uses the Frobenius' Theorem and can be found in [Olv00], see theorem 2.17. p. 86. ■

In particular, for a free action the orbits have the same dimension, so we can determine the number of independent invariants in an open neighborhood $U \subseteq J^k$, that is micro-locally. Let i_k be the number of invariants of order k, then by Theorem 2.5.2, we have

$$i_k = \dim J^k - s_k \tag{2.15}$$

number of differential invariants of order k. This count includes all invariants of any order less than or equal to k (recall that any invariant of order $k-1$ is also an invariant of order k). Let j_k denote the number of independent

1. This is true if the action is free.

differential invariants of order k that are not of order less than k. This is simply $j_k = i_k - i_{k-1}$, or alternatively

$$j_k = \dim J^k - s_k - (\dim J^{k-1} - s_{k-1}). \tag{2.16}$$

If r is the dimension of $\mathfrak{g}$ and at some step l in the prolongation the orbit dimension is $s_l = r$, then for all $k > l$ it must be the case that $s_k = r$. The reason being that the rank of $\mathfrak{g}$ as a distribution can't increase beyond the dimension of $\mathfrak{g}$ as a Lie algebra. We call r the **stable orbit dimension** and l the **order of stabilization**. This gives a very simple way of counting the number of independent differential invariants.

Proposition 2.5.3. Let l be the order of stabilization for the action. Then for all $k > l$ there exists $j_k = \dim J^k - \dim J^{k-1}$ independent differential invariants of order k.

A more detailed exposition on the dimension count can be found in [Olv95].

Remark. Although this is done micro-locally it holds in fact globally, whenever the hypothesis of Theorem 2.4.2 holds, see [KL16].

This answers the first question of this section. The second question is answered in the next theorem.

Theorem 2.5.4. Let G be a Lie group acting on a manifold M under the assumptions of Theorem 2.4.2. Then there exists a finite number $\nabla_1, \nabla_2, \ldots, \nabla_n$ of invariant derivations, where $n = \dim M$.

Proof. See [KL16] Theorem 21. p. 1391. ■

Remark. The discussion in [KL16] focuses on differential equations embedded as submanifolds in J^k. The results holds for our cases when the differential equation is taken to be empty.

In our setup the groups acting are finite-dimensional Lie groups, so it is always the case that the algebra is infinite.

2.5.2 Differential Syzygies

To close off the chapter we give a brief discussion of differential syzygies. If $\mathcal{A}$ is the algebra of differential invariants, then by Theorem 2.4.2 it is finitely generated by $I_1, \ldots, I_s, \nabla_1, \ldots, \nabla_n$, for some $s, n \geq 1$. A **differential syzygy** is a relation among these generators. That is, an expression of the form $F(\nabla_{J_1}(I_{i_1}), \ldots, \nabla_{J_n}(I_{i_n})) = 0$, where F is a function taking a finite amount of arguments and $J_1, \ldots, J_n$ are multi-indices. Write $\mathcal{A} = \langle I_1, \ldots, I_s, \nabla_1, \ldots, \nabla_n \mid F(\nabla_{J_1}(I_{i_1}), \ldots, \nabla_{J_n}(I_{i_n}))\rangle$ to express the generators for the algebra and the differential syzygies. If no differential syzygies exist the algebra is said to be freely generated by $I_1, \ldots, I_s, \nabla_1, \ldots, \nabla_n$, written as $\mathcal{A} = \langle I_1, \ldots, I_s, \nabla_1, \ldots, \nabla_n\rangle$. Considering the algebra of differential invariants as being finitely generated by I_i and ∇_j, then the set of differential syzygies forms a module over this space, a so called D-module. This module of differential syzygies is also finitely generated. For more on differential syzygies, a proof of finiteness and D-modules see [KL16].

3

Computation of Differential Invariants

In this chapter the theory introduced in Chapter 2 is put to use by doing some actual computations. The approach is to try and use Eq. (2.12) to find invariants and Eq. (2.13) to determine invariant derivations and then appeal to the Lie-Tresse theorem and the dimension count to describe the whole algebra of differential invariants. However, as we'll see, this is problematic in some cases. All computations were done in Maple 2018.

Convention: All differential invariants are denoted by I with a subscript. The subscript consists of a number and a letter. The number reflects the order of the invariant, while the letter is arbitrary and only there to distinguish invariants of the same order. If no letter is given there is only 1 invariant on the corresponding jet space.

The dimension formulas for the jet spaces are stated here again for easy reference. Have

$$\dim J^k M = n + \binom{n+k}{k} \tag{3.1}$$

for jets of functions with $\dim M = n$. While for jets of submanifolds of dimension n the formula is

$$\dim J^k(M,n) = n + m\binom{n+k}{k} \tag{3.2}$$

for $\dim M = n + m$.

3.1 Review of Symplectic and Contact Geometry

To start off, we review some notions from symplectic and contact geometry. The discussion is brief, but more details can be found in [KLR07].

3.1.1 Symplectic Geometry

Recall that a **symplectic manifold** is a smooth even-dimensional manifold, equipped with a nondegenerate closed 2-form ω. Let $M = \mathbb{R}^{2n}$, viewed as a symplectic manifold with local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$. Take $\omega = \sum_i dx^i \wedge dy^i$ as the standard symplectic structure. Locally every symplectic manifold takes this form. Consider transformations $\varphi \in \mathrm{Diff}(M)$, such that $\varphi^*\omega = \omega$. That is, symmetries of the symplectic form (also called **symplectomorphisms**). This forms an infinite-dimensional group of symmetries with the group operation being composition of maps. Instead of working on the group level we can pass to the infinitesimal setting to obtain linear equations through the Lie derivative

$$\mathcal{L}_X\omega = \frac{d}{dt}\bigg|_{t=0} \varphi_t^*\omega = 0, \tag{3.3}$$

where φ_t is the associated flow of X. Therefore, we may look to vector fields $X \in \mathcal{D}(M)$, such that $\mathcal{L}_X\omega = 0$. This forms an infinite-dimensional Lie algebra of vector fields, called Hamiltonian vector fields. Denote it by $\mathfrak{h}$. For our purposes we'd like to consider finite-dimensional subalgebras of this Lie algebra. The infinite-dimensional Lie algebra corresponding to the Hamiltonian vectors fields are generated by smooth functions $f \in C^\infty(M)$. In fact we have a Lie algebra isomorphism, $\mathfrak{h} \simeq C^\infty(M)$, given in local coordinates as

$$f \mapsto X_f = -\partial_{y^i}(f)\partial_{x^i} + \partial_{x^i}(f)\partial_{y^i}, \tag{3.4}$$

for the case $M = \mathbb{R}^{2n}$. The induced Lie bracket on the algebra of smooth functions is called the Poisson bracket, and is typically denoted as $\{f, g\}$. It is defined by $\{f, g\} = h$, where the functions correspond to the Lie bracket for $[X_f, X_g] = X_h$, where X_f, X_g, X_h are all Hamiltonian vector fields. In local coordinates

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i} \right). \tag{3.5}$$

To obtain a finite-dimensional subalgebra, consider $\mathcal{P}^{(n)}$, that is the space of homogeneous forms of degree n. Take $f \in \mathcal{P}^{(2)}$ to be a quadratic form on M. Differentiating a quadratic form gives a linear function and multiplying two linear functions gives a quadratic form, thus $\{f,g\}$ is another quadratic form. In particular, the space of quadratic forms is closed under the Poisson bracket. This gives a finite-dimensional Poisson subalgebra $\mathcal{P}^{(2)} \subseteq C^\infty(M)$. By Eq. (3.4) we get an isomorphic Lie algebra of Hamiltonian vector fields generated by quadratic forms, which we'll denote as $\mathfrak{g}$. Then we've obtained a finite-dimensional subalgebra of $\mathfrak{h}$ with dimension $\dim \mathfrak{g} = n(2n+1)$, which is the same dimension as the space of quadratic forms on $M = \mathbb{R}^{2n}$. The Lie algebra $\mathfrak{g}$ consists of linear Hamiltonian vector fields, so it is the standard representation of the Lie algebra $\mathfrak{sp}(2n;\mathbb{R})$.

3.1.2 Contact Geometry

Let M be a smooth odd-dimensional manifold, then M is a **contact manifold** if M is equipped with a **contact structure**. A contact structure is a codimension 1 maximally non-integrable distribution $\Pi \subseteq TM$. An equivalent formulation is that M is equipped with a 1-form α, for which $\Pi = \ker \alpha$ is a codimension 1 distribution such that $d\alpha|_\Pi$ is nondegenerate. Let $M = \mathbb{R}^{2n+1}$ with local coordinates $x^1, \ldots, x^n, u, p_1, \ldots, p_n$, then $\alpha = du - p_i dx^i$ is a contact structure on M. Locally, all contact manifolds take this form. Now, consider symmetries of α, that is, transformations $\varphi \in \mathrm{Diff}(M)$, such that $\varphi^*\alpha = \lambda\alpha$, for some $\lambda \in C^\infty(M)$. [1] This gives an infinite-dimensional group as in the symplectic case. Converting the problem to the infinitesimal version yields

$$\mathcal{L}_X \alpha = \frac{d}{dt}\Big|_{t=0} \varphi_t^* \alpha \equiv 0, \quad \mod \langle \alpha \rangle. \tag{3.6}$$

where φ_t is the flow associated to X. This forms a Lie algebra of infinitesimal symmetries generated by a function $f \in C^\infty(M)$, and can be written in local coordinates as

$$X_f = -\partial_{p_i}(f)\mathcal{D}_{x^i} + f\partial_u + \mathcal{D}_{x^i}(f)\partial_{p_i}, \tag{3.7}$$

where $\mathcal{D}_{x^i} = \partial_{x^i} + \sum_{j,\sigma} u^j_{\sigma+1_i} \partial_{u^j_\sigma}$. The vector field X_f is called a **contact vector field**. The Lie algebra of infinitesimal symmetries is then isomorphic as a Lie algebra with $C^\infty(M)$, with the induced Lie bracket coming from $[X_f, X_g] = X_h$. The bracket, given as $[f,g] = h$ is called the **Lagrange bracket**. In local coordinates it takes the form

$$[f,g] = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i} \right) + \sum_{i=1}^n p_i \left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial p_i} \right) + \left(f\frac{\partial g}{\partial u} - g\frac{\partial f}{\partial u} \right).$$

As in the symplectic case we want to consider finite-dimensional subalgebras. Consider the subspace $\mathcal{P}^{(2)} \subseteq C^\infty(M)$ as before, however this can't be closed

1. This is an annihilator to a distribution, so we have to preserve α up to scale.

under the Lagrange bracket. For example, take $f = x^2$ and $g = u^2$, then $[f, g] = 2x^2u - 2xu^2$, which is of degree 3. To keep the degree fixed, we can avoid this issue by taking u to have weight 2. That is, take f to be a quadratic form in the variables x^i, p_i and of first order in u, but of weight 2. This gives a subalgebra that is closed under the Lagrange bracket, so the map $f \mapsto X_f$ gives an isomorphism of Lie algebras. The Lie algebra of vector fields $\mathfrak{g}$ is a Lie algebra of dimension $\dim \mathfrak{g} = \binom{2n+2-1}{2} + 1 = n(2n+1) + 1$.

3.2 Symplectic Computations in 2 Dimensions

Now we're in a position to start doing some actual computations, and we start with the symplectic case. Let $M = \mathbb{R}^2$, taken as a symplectic manifold with $\omega = dx \wedge dy$ as the symplectic structure. Take (x^2, xy, y^2) as a basis for the space of quadratic forms. The corresponding Hamiltonian vector fields are then:

$$\begin{aligned} X_1 &= 2x\partial_y, \\ X_2 &= -x\partial_x + y\partial_y, \\ X_3 &= -2y\partial_x, \end{aligned}$$

generated by x^2, xy and y^2 respectively. Computing the nonzero Lie brackets yields the following structure relations

$$\begin{aligned} [X_1, X_2] &= 2X_1, \\ [X_1, X_3] &= 4X_2, \\ [X_2, X_3] &= 2X_3. \end{aligned}$$

Let $\mathfrak{g} = \langle X_1, X_2, X_3 \rangle$ denote this Lie algebra, then the Levi Decomposition in this case shows that the radical part is 0, therefore $\mathfrak{g}$ is semisimple and by construction the Lie algebra is $\mathfrak{sp}(2; \mathbb{R})$ which is isomorphic to $\mathfrak{sl}(2; \mathbb{R})$.

3.2.1 Differential Invariants, Part 1.1: Curves

Let the Lie algebra $\mathfrak{g}$ be defined as above. Induce an action on $M = \mathbb{R}^2$, this action induces an action on the curves in M. The induced action on the curves in M induce an action on the corresponding jet spaces associated to the curves. Let x, y be local coordinates on $J^0(M, 1)$, where x is taken to be independent and y dependent, meaning the curves will be represented as $y = y(x)$. To describe the algebra of differential invariants under the action of $\mathfrak{g} = \langle X_1, X_2, X_3 \rangle$ on $J^0(M, 1)$, we'll apply the Lie-Tresse theorem, but firstly we have to check if the algebraic criteria of the problem holds. Recall that we require the group and all its prolongations to act algebraically on M and the action on the base M has to be transitive. Prolongation of an algebraic action is algebraic, so the only check needed is G itself and the first prolongation acting on $J^1(M,1)$. The Lie group associated to the Lie algebra $\mathfrak{sl}(2;\mathbb{R})$ is $G = \mathrm{SL}(2;\mathbb{R}) \subseteq \mathrm{GL}(2;\mathbb{R})$.[2] This is an algebraic group, being described by linear equations and the condition that $ad - bc = 1$ all of which are algebraic equations. Take $g \in G$, then g corresponds to matrix with entries a, b, c, d such that $ad - bc = 1$. The action $\Phi : G \times M \longrightarrow M$ can then be supplied with some $g \in G$ to obtain a diffeomorphism of M. That is $\Phi(g,p) = \Phi_g(p) \in \mathrm{Diff}(M)$. The action on $J^0(M,1)$ can then be written explicitly as

$$\Phi_g(x,y) = (ax+by, cx+dy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

The first prolongation is then

$$\Phi_g^{(1)}(x,y,y_1) = \left(ax+by, cx+dy, \frac{dy_1+c}{by_1+a} \right).$$

To check that the stabilizer of $\Phi_g^{(1)}$ at a generic point p acts algebraically on $J^1(M,1)$, take a generic point $0 \neq p \in J^0(M,1)$. The group acts transitively on $J^0(M,1) \setminus \{0\}$, so the point p can be taken to be $p = (1,0)$ as a representative for a generic orbit. Thus, we can compute the stabilizer of $p = (1,0)$ to check the criterion for the action being algebraic. The criteria for being the stabilizer of $p = (1,0)$ becomes $\Phi_g(1,0) = (a,c)$, which gives $a = 1$, $c = 0$ and $d = 1$ by the condition that $ad - bc = 1$. The stabilizer of p, denoted $\Phi_{g;p}$ is then of the form

$$\Phi_{g;p} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where b is arbitrary. Then the prolongation of $\Phi_{g;p}$ takes the form

$$\Phi_{g;p}^{(1)} = \left(x+by, y, \frac{y_1}{by_1+1} \right).$$

The action of the stabilizer on $J^1(M,1)$ consists of rational equations, so the action is algebraic and the Lie-Tresse theorem applies if we restrict to $J^0(M,1) \setminus 0$.

The Lie algebra $\mathfrak{g}$ acts almost transitively on $J^0(M,1)$ (the action is transitive on $J^0(M,1) \setminus 0$), so there are no differential invariants of order 0 on generic orbits. Prolonging the action to $J^1(M,1)$ the Lie algebra $\mathfrak{g}^{(1)}$ has rank 3 on

2. Exponentiating the Lie algebra gives a neighborhood around unity in the corresponding group. By doing finite products between elements obtained from the Lie algebra we can generate all of $\mathrm{SL}(2;\mathbb{R})$.

generic points, which is equal to the dimension of $J^1(M, 1)$, meaning there are no differential invariants of order 1. By Proposition 2.5.3 we have reached the stable orbit dimension, which is 3, so for $k \geq 2$ the orbits have dimension $s_k = 3$. The order of stabilization is then 1, so by Theorem 2.5.2 the number of independent differential invariants of order k is $j_k = \dim J^k - \dim J^{k-1} = 1$ for $k \geq 2$.

Jet Level	j_k
0	0
1	0
$k \geq 2$	1

The dimension of $J^2(M, 1)$ is 4, so there should be one invariant. To compute it, prolong the vector fields in accordance with Eq. (2.9) and solve Eq. (2.12) with $I = I(x, y, y_1, y_{1,1})$,

$$\mathcal{L}_{X_i^{(2)}} I(x, y, y_1, y_{1,1}) = 0.$$

The solution is then the first invariant

$$I_2 = \frac{y_{1,1}}{(xy_1 - y)^3}. \tag{3.8}$$

The dimension count then guarantees this is the only invariant needed. To understand the whole algebra of differential invariants, which shall be denoted by $\mathcal{A}$ the Lie-Tresse theorem is applied. According to the Lie-Tresse theorem there should exist invariant derivations and differential invariants which together suffice to generate the whole algebra of differential invariants. In accordance with the dimension count there should be only 1 functionally independent invariant of order 2 which is already computed. By prolongation we'll obtain one new invariant for every $k > 2$. Computing all of them is of course ineffective and will not give a complete description of the all invariants since there are infinitely many of them. To avoid this the attempt is to find invariant derivations instead. By Theorem 2.5.4 only one invariant derivation should suffice since the algebra of invariants is infinite due to the group acting being finite-dimensional. To find an invariant derivation solve Eq. (2.13) with $\nabla = Q(x, y, y_1)\mathcal{D}_x$. This gives

$$Q(x, y, y_1) = \frac{C}{xy_1 - y}, \tag{3.9}$$

where C is some nonzero constant. Taking ∇ as simple as possible by letting $C = 1$. The resulting invariant derivation is then

$$\nabla = \frac{1}{xy_1 - y}\mathcal{D}_x. \tag{3.10}$$

By Lemma 2.3.4 it is possible to obtain differential invariants of higher order by applying invariant derivations. Thus, $I_3 = \nabla(I_2)$, is a differential invariant of order 3. By the dimension count there should be one functionally independent differential invariant of order 3, which is the one obtained. In general there is one functionally independent differential invariant of order k, and it is computed as $I_k = \nabla^{k-2}(I_2)$. In accordance with the Lie-Tresse theorem, the algebra of differential invariants is generated by I_2 and ∇ with no differential syzygies between the generators. Thus, the algebra $\mathcal{A} = \langle I_2, \nabla \rangle$ is free on I_2 and ∇. To summarize:

Differential Invariants	Invariant Derivations
$I_2 = \frac{y_{1,1}}{(xy_1-y)^3}$	$\nabla = \frac{1}{xy_1-y}\mathcal{D}_x$

Later when discussing curves in dimension 4 we develop another, more geometric way to obtaining everything needed to understand the algebra of curves in $\mathbb{R}^2$. See section 3.4.1 for the geometric method.

3.2.2 Differential Invariants, Part 1.2: Functions

Take $M = \mathbb{R}^2$ with local coordinates x, y as a symplectic manifold with the symplectic form $\omega = dx \wedge dy$. Let J^0M be the jet space of $\mathbb{R}$-valued functions on M with local coordinates x, y, u. Take $\mathfrak{g} = \langle X_1, X_2, X_3 \rangle$ as defined above, and induce an action on $M = \mathbb{R}^2$ which is prolonged trivially to J^0M. In this case x and y are independent and u is considered dependent. To understand the algebra of differential invariants we want to apply the Lie-Tresse theorem, but to do this we need to verify the algebraic conditions. Firstly, the group is the same as in the last computation, so the group is algebraic. The action in this case is the same, but the prolongation is different. Given $g \in G$ the action $\Phi : G \times M \longrightarrow M$ is

$$\Phi_g(x, y) = (ax + by, cx + dy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for $ad - bc = 1$. On J^0M the action is prolonged trivially, so that

$$\Phi_g^{(0)}(x, y, u) = (ax + by, cx + dy, u).$$

As above the stabilizer of a generic point, say $p = (1, 0)$ is

$$\Phi_{g;p} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

The prolongation of the stabilizer is then

$$\Phi^{(1)}_{g;p} = (x + by, y, u, u_1, -bu_1 + u_2)\,.$$

This is an algebraic action, being defined by polynomials, so the hypothesis for the Lie-Tresse theorem holds and we can proceed as before.

Before computing the differential invariants we investigate the number of differential invariants for each k. On $M \setminus \{0\}$ the action is transitive so there are no invariants. On J^0M we have $\dim J^0M = 3$, but the rank of $g^{(0)}$ is 2, so there is one invariant on J^0M, namely $I_0 = u$ by construction. In accordance with our notation, we have $i_0 = 1$ and $j_0 = 1$. On J^1M the orbit dimension is 3, so there are $j_1 = i_1 - i_0$ independent first order differential invariants. In this case $i_1 = \dim J^1M - s_1 = 2$. In particular, $j_1 = i_1 - i_0 = 1$, so there is one differential invariant of order 1. At this stage the stable orbit dimension has been reached, so for $k \geq 2$, there is $j_k = \dim J^k - \dim J^{k-1} = k + 1$ independent invariants of order k.

Jet Level	j_k
0	1
1	1
$k \geq 2$	$k+1$

To compute differential invariants of order k the approach is to solve Eq. (2.12)

$$\mathcal{L}_{X_i^{(k)}} I = 0, \quad \text{for } i = 1, 2, 3,$$

where $I \in C^\infty(J^kM)$. Doing this for $k = 0, 1, 2$ gives the following differential invariants

$$\begin{aligned}
I_0 &= u,\\
I_1 &= xu_1 + yu_2,\\
I_{2a} &= x^2u_{1,1} + 2xyu_{1,2} + y^2u_{2,2},\\
I_{2b} &= xu_2u_{1,1} - yu_1u_{2,2} + (yu_2 - xu_1)u_{1,2},\\
I_{2c} &= u_1^2u_{2,2} - 2u_1u_2u_{1,2} + u_2^2u_{1,1}.
\end{aligned}$$

To determine the whole algebra of differential invariants it is necessary to determine some invariant derivations. Due to Theorem 2.5.4, there shall be 2 of them. Take an invariant derivation as $\nabla = \sum_{i=1}^n Q^i \mathcal{D}_{x^i}$ where Q^i are functions on 1-jets. Solve

$$[X^{(k)}, \nabla] = 0,$$

on the local coordinates. In this case $n = 2$ so a general invariant derivation takes the form $\nabla = Q(x, y, u, u_1, u_2)\mathcal{D}_x + R(x, y, u, u_1, u_2)\mathcal{D}_y$. Plugging this into Eq. (2.13) and solving for Q and R gives

$$\begin{aligned} Q(x, y, u, u_1, u_2) &= -F_1(u, xu_1 + yu_2)u_2 + F_2(u, xu_1 + yu_2)x, \\ R(x, y, u, u_1, u_2) &= F_1(u, xu_1 + yu_2)u_1 + F_2(u, xu_1 + yu_2)y. \end{aligned}$$

Choose F_1 and F_2 as simple as possible, say either 0 or 1. This gives the two necessary invariant derivations

$$\begin{aligned} \nabla_1 &= x\mathcal{D}_x + y\mathcal{D}_y, \\ \nabla_2 &= -u_2\mathcal{D}_x + u_1\mathcal{D}_y. \end{aligned}$$

Let $\mathcal{A}$ denote the algebra of differential invariants. Then using the differential invariants and invariant derivations we can construct a generating set for $\mathcal{A}$. The invariant I_0 can be used to get higher order invariants. Applying ∇_1, ∇_2 to I_0 would yield two new first order invariants, but there should be only one independent invariant, therefore there must be some differential syzygy. The differential syzygy is easy in this case, since $\nabla_2(I_0) = 0$, while $I_1 = \nabla_1(I_0)$. Then everything is obtained on J^1M. On J^2M the invariants I_{2a}, I_{2b} and I_{2c} are all independent which is verified by the rank of the following matrix:

$$\begin{pmatrix} \partial_{u_{1,1}} I_{2a} & \partial_{u_{1,2}} I_{2a} & \partial_{u_{2,2}} I_{2a} \\ \partial_{u_{1,1}} I_{2b} & \partial_{u_{1,2}} I_{2b} & \partial_{u_{2,2}} I_{2b} \\ \partial_{u_{1,1}} I_{2c} & \partial_{u_{1,2}} I_{2c} & \partial_{u_{2,2}} I_{2c} \end{pmatrix} = \begin{pmatrix} x^2 & 2xy & y^2 \\ xu_2 & -xu_1 + yu_2 & -yu_1 \\ u_2^2 & -2u_1u_2 & u_1^2 \end{pmatrix},$$

which is 3 on generic points, so these are functionally independent. However, the invariants I_{2a} and I_{2b} can be expressed using I_0 and ∇_1 and ∇_2 as

$$\begin{aligned} I_{2a} &= \nabla_1^2(I_0) - \nabla_1(I_0), \\ I_{2b} &= -\nabla_2(\nabla_1)(I_0). \end{aligned}$$

In particular, I_{2a} and I_{2b} are not needed as generators for the algebra of differential invariants. On the level of 2-jets we only get two new differential invariant by applying ∇_1 and ∇_2, hence I_{2c} is necessary since 3 is needed. Applying derivations again yields six invariants on 3-jets, but only 4 are necessary and independent by the dimension count. Therefore, there shall be some differential syzygies, and in fact, there are two of them. The first differential syzygy comes from the commutator, $[\nabla_1, \nabla_2]$. This is due to Proposition 2.3.5 which also gives a way to obtain it. The commutator yields another invariant derivation, which should be expressible through the ones we already have, that is, it should exists coefficients c_1, c_2 such that

$$[\nabla_1, \nabla_2] = c_1\nabla_1 + c_2\nabla_2.$$

Also by Proposition 2.3.5, the coefficients are invariant. Apply this to the coordinates x, y gives a linear system which can be solved by standard methods.

$$\begin{aligned} [\nabla_1, \nabla_2](x) &= c_1\nabla_1(x) + c_2\nabla_2(x), \\ [\nabla_1, \nabla_2](y) &= c_1\nabla_1(y) + c_2\nabla_2(y). \end{aligned}$$

The coefficients c_1 and c_2, being invariants themselves can be expressed by the invariants we have computed. To find the relations between these coefficients and the invariants we have is discussed below. This gives the first differential syzygy

$$I_1[\nabla_1, \nabla_2] = I_{2b}\nabla_1 + (I_{2a} - I_1)\nabla_2.$$

This differential syzygy gives that $\nabla_2(I_{2a})$ and $\nabla_1(I_{2b})$ are functionally dependent since $\nabla_2(I_{2a})$ and $\nabla_1(I_{2b})$ come from I_0 by applying ∇_1 and ∇_2, that is they are of the form $\nabla_2(\nabla_1(I_0))$ and $\nabla_1(\nabla_2(I_0))$. The second differential syzygy is harder to obtain and comes from a relation between the remaining 5 third order invariants.

Any function on J^k correspond to a differential operator (possibly nonlinear) and the behavior can be understood by looking at the top terms in derivatives. The **symbol** operator accomplishes this. The map $\sigma_k : C^\infty(J^kM) \longrightarrow S^kTM$ is called the symbol, defined by $C^\infty(J^kM) \ni F \mapsto d_{a_k}F|_{F(a_k)}$, where $F(a_k) = \ker(d\pi_{k,k-1} : T_{a_k}J^kM \longrightarrow T_{a_{k-1}}J^{k-1}M) \simeq S^kT^*M$. The result after applying the symbol to a function on J^kM is symmetric polynomial in the basis ∂_x, ∂_y (in this case since $M = \mathbb{R}^2$ with coordinates x, y). As an example, take the invariant I_{2a}, then

$$\sigma_2(I_{2a}) = \sigma_2(x^2u_{1,1} + 2xyu_{1,2} + y^2u_{2,2}) = x^2\partial_x^2 + 2xy\partial_x\partial_y + y^2\partial_y^2.$$

In this case the highest order terms are linear, so the coefficients of each term can be computed by differentiation with respect to the jet variables $u_{1,1}, u_{1,2}, u_{2,2}$. If the invariant I_{2a} included lower order terms, say we compute $\sigma_2(I_{2a} + I_1)$, then the answer is the same since lower order terms are automatically 0.

Write the invariants of the form[3]

$$\alpha_1\nabla_1(I_{2a}) + \alpha_2\nabla_2(I_{2a}) + \alpha_3\nabla_2(I_{2b}) + \alpha_4\nabla_1(I_{2c}) + \alpha_5\nabla_2(I_{2c}),$$

for some arbitrary coefficients α_i. Then compute the symbol for this combination and set it to 0. Setting the symbol to 0 implies that the highest order (in this case 3) terms vanish, so that the remainder in the original expression is of order 2. The symbol is computed by differentiation with respect to the jet coordinates if the expressions are linear in the highest order. If not, they must be linearized first. Computing this gives the coefficients for the symbol which is a cubic form in the basis ∂_x, ∂_y. To make this cubic form 0 we require all the coefficients to vanish. Setting this system of equations to 0 allows us to solve for all α_i's.

$$\sigma_3(\alpha_1\nabla_1(I_{2a}) + \alpha_2\nabla_2(I_{2a}) + \alpha_3\nabla_2(I_{2b})) + \alpha_4\nabla_1(I_{2a}) + \alpha_5\nabla_2(I_{2c})) = 0.$$

3. $\nabla_1(I_{2b})$ is thrown away due to the syzygy relating it to $\nabla_2(I_{2a})$

In this case $\alpha_1 = \alpha_2 = \alpha_5 = 0$ and $\alpha_3 = \alpha_4$. Take $\alpha_3 = 1$ and we obtain a relation

$$\nabla_2(I_{2b}) + \nabla_1(I_{2a}) = \mu(x, y, u, u_1, u_2, u_{1,1}, u_{1,2}, u_{2,2}).$$

All 3-jets have been eliminated and what remains is nonlinear in the 2-jets, so the same method will not work if applied again. To remedy this, take the invariants on 2-jets, that is I_{2a}, I_{2b}, I_{2c} and solve them for $u_{1,1}, u_{1,2}, u_{2,2}$ and substitute into μ. Now the 2-jets have been eliminated from the problem. Do the same for I_1 and u_1, u_2 and so on. This gives the last syzygy after substituting back I_{2a}, I_{2b}, I_{2c}. The differential syzygy turns out to be:

$$(\nabla_2(I_{2b}) + \nabla_1(I_{2a}))I_1 - (3I_{2a} - I_1)I_{2c} - 3I_{2b}^2 = 0.$$

Define the following

$$\begin{aligned}
\mathcal{R}_1 &= \nabla_2(I_0),\\
\mathcal{R}_2 &= I_1[\nabla_1, \nabla_2] - I_{2b}\nabla_1 - (I_{2a} - I_1)\nabla_2,\\
\mathcal{R}_3 &= (\nabla_2(I_{2b}) + \nabla_1(I_{2a}))I_1 - (3I_{2a} - I_1)I_{2c} - 3I_{2b}^3.
\end{aligned}$$

where I_1, I_{2a}, I_{2b} are defined by the relations to I_0 and ∇_1, ∇_2 as above. Then, the algebra of differential invariants is generated as $\mathcal{A} = \langle I_0, I_{2c}, \nabla_1, \nabla_2 \,|\, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \rangle$. In summary:

Differential Invariants	Derivations
$I_0 = u$ $I_1 = xu_1 + yu_2$ $I_{2a} = x^2u_{1,1} + 2xyu_{1,2} + y^2u_{2,2}$ $I_{2b} = (yu_2 - xu_1)u_{1,2}$ $\quad +xu_2u_{1,1} - yu_1u_{2,2}$ $I_{2c} = u_1^2u_{2,2} - 2u_1u_2u_{1,2} + u_2^2u_{1,1}$	$\nabla_1 = x\mathcal{D}_x + y\mathcal{D}_y$ $\nabla_2 = -u_2\mathcal{D}_x + u_1\mathcal{D}_y$

3.3 Contact Computations in 3 Dimensions

Taking $M = \mathbb{R}^3$, then the space of quadratic forms has a basis (x^2, p^2, u, xp), with u having weight 2. This gives the following vector fields:

$$\begin{aligned} X_1 &= x^2\partial_u + 2x\partial_p, \\ X_2 &= -2p\partial_x - p^2\partial_u, \\ X_3 &= 2u\partial_u + 2p\partial_p, \\ X_4 &= -x\partial_x + p\partial_p, \end{aligned}$$

corresponding to $x^2, p^2, 2u, xp$, respectively. [4] This results in the following nonzero commutator relations:

$$\begin{aligned} [X_1, X_2] &= 4X_4, \\ [X_1, X_3] &= 2X_1, \\ [X_1, X_4] &= 2X_1, \\ [X_2, X_3] &= -2X_2, \\ [X_2, X_4] &= -2X_2. \end{aligned}$$

Let $\mathfrak{g} = \langle X_1, X_2, X_3, X_4\rangle$ denote this Lie algebra. The Levi decomposition of $\mathfrak{g}$ gives that $\langle X_3 - X_4\rangle$ is the radical and $\langle X_1, X_2, X_4\rangle$ is semisimple. The semisimple part is $\mathfrak{sl}(2;\mathbb{R})$ by checking the signature of the Killing form. The whole Lie algebra corresponds to $\mathfrak{g} \simeq \mathfrak{sl}(2;\mathbb{R}) \oplus \mathbb{R} \simeq \mathfrak{gl}(2;\mathbb{R})$. In particular, $X_3 - X_4$ corresponds to the center of $\mathfrak{gl}(2;\mathbb{R})$ and $\langle X_1, X_2, X_4\rangle$ is the usual $\mathfrak{sl}(2;\mathbb{R})$.

The corresponding group G is a subgroup of $\mathrm{GL}(2;\mathbb{R})$, and in fact a subgroup of the connected component of $\mathrm{GL}(2;\mathbb{R})$ that contains the identity. Doing multiplication between the elements in the neighborhood of the identity generates the whole connected component of $\mathrm{GL}(2;\mathbb{R})$ having positive determinant. Denote this subgroup of $\mathrm{GL}(2;\mathbb{R})$ as $\mathrm{GL}_+(2;\mathbb{R})$. The action of the group $G = \mathrm{GL}_+(2;\mathbb{R})$ is not the standard representation of G as the action is on $\mathbb{R}^3$. However, the action on x and p is the standard representation of $\mathrm{GL}(2;\mathbb{R})$. To determine the action $\Phi : G \times M \longrightarrow M$ on the u coordinate we require that for any $g \in G$ we have $\Phi_g^*\alpha = \lambda\alpha$, where $\alpha = du - pdx$ is the standard contact structure on $\mathbb{R}^3$. Write the action as $\Phi_g(x, u, p) = (ax + bp, f(x, u, p), cx + dp)$ where $g \in \mathrm{GL}(2;\mathbb{R})$ is a matrix having entries a, b, c, d for which $ad - bc \neq 0$. Then solve $\Phi_g^*\alpha = \lambda\alpha$. From this we obtain the PDE system

$$\begin{aligned} f_x(x, u, p) &= -\lambda p + acx + adp, \\ f_u(x, u, p) &= \lambda, \\ f_p(x, u, p) &= bcx + bdp. \end{aligned}$$

4. $2u$ is just for convenience.

Eliminate λ and solve the PDE system. This determines the action of $G = \mathrm{GL}_+(2;\mathbb{R})$ on u, so the complete action on $\mathbb{R}^3$ is

$$\Phi(x,u,p) = (ax+bp, (ad-bc)\left(u + \frac{xp}{2}\right) + \frac{(ax+bp)(cx+dp)}{2}, cx+dp). \quad (3.11)$$

for $ad-bc \neq 0$. The action obtained is algebraic, given by polynomial equations hence by taking the Zariski closure the equations extends to the whole group $G = \mathrm{GL}(2;\mathbb{R})$.

3.3.1 Differential Invariants, Part 2.1: Curves

Take $M = \mathbb{R}^3$ and let $\mathfrak{g} = \langle X_1, X_2, X_3, X_4 \rangle$ be the Lie algebra as described above with coordinates $x \mapsto t, u \mapsto x$ and $p \mapsto y$. Induce an action of $\mathfrak{g}$ on M and in turn on $J^0(M,1) = \mathbb{R}^3$, where $J^0(M,1)$ has local coordinates t,x,y, with t being the independent variable and x,y considered dependent. The setup is then a Lie group acting on $\mathbb{R}^3$, which induces an action on the curves in $\mathbb{R}^3$ which gives a corresponding action on the jets of these curves, that is $J^k(M,1)$. As before, to understand the whole algebra of differential invariants it is necessary to find a complete generating set consisting of differential invariants and invariant derivations, which for a transitive algebraic action will then suffice to generate everything by the Lie-Tresse theorem. The group acting is $\mathrm{GL}(2;\mathbb{R})$ which is defined through polynomial equations, so it is algebraic, thus the only verification needed is the algebraic nature of the stabilizer on $J^1(M,1)$. The action $\Phi : G \times M \longrightarrow M$ is defined in Eq. (3.11), so in the new coordinates, it reads

$$\Phi_g(t,x,y) = \left(at+by, (ad-bc)\left(x - \frac{yt}{2}\right) + \frac{(at+by)(ct+dy)}{2}, ct+dy\right),$$

provided $ad-bc \neq 0$, where g corresponds to a 2×2 invertible matrix having a,b,c,d as entries. To compute the stabilizer, choose a generic point. In this case the point $p = (1,1,0)$ is a generic point. Computing the stabilizer of p gives

$$\Phi_{g;p} = \left(t+by, \left(x - \frac{yt}{2}\right) + \frac{(t+by)y}{2}, y\right)$$

where b is arbitrary. The prolongation of $\Phi_{g;p}$ is then

$$\Phi^{(1)}_{g;p} = \left(t+by, \left(x - \frac{yt}{2}\right) + \frac{(t+by)y}{2}, y, \frac{byy_1 + x_1}{by_1+1}, \frac{y_1}{by_1+1}\right),$$

for b arbitrary. The action consists of rational functions on the coordinates, so the action is algebraic, allowing the use of the Lie-Tresse theorem.

The action of $\mathfrak{g}$ on $J^0(M,1) = \mathbb{R}^3$ is transitive on generic points (the action has open dense orbits) since the rank of $\mathfrak{g}$ on $J^0(M,1)$ is 3. Prolonging the action to $J^1(M,1)$ gives that $\dim J^1(M,1) = 5$, while the rank of $\mathfrak{g}^{(1)}$ is 4, meaning there is one invariant of order 1, so $i_1 = j_1 = 1$. To compute it, solve Eq. (2.12) with $I = I(t,x,y,x_1,y_1)$. This gives

$$I_1 = \frac{y_1 t - y}{y - x_1}.$$

Due to Proposition 2.5.3 the prolongation has reached the stable orbit dimension, so for $k \geq 2$, the number of invariants become $j_k = \dim J^k(M,1) - \dim J^{k-1}(M,1) = 2$.

Jet Level	j_k
0	0
1	1
k	2

Solving Eq. (2.12) on second jets gives two differential invariants

$$\begin{aligned} I_{2a} &= \frac{y_{1,1}\,(ty-2x)^2}{(y-x_1)^3}, \\ I_{2b} &= \frac{\left(\left(ty_{1,1} - x_{1,1} + y_1\right)y - t\left(y_{1,1}x_1 - y_1 x_{1,1} + y_1^2\right)\right)(ty-2x)}{(y-x_1)^2\,(y_1 t - y)}, \end{aligned}$$

which is all that is needed. Using Eq. (2.13) gives the necessary invariant derivation

$$\nabla = \frac{ty-2x}{y-x_1}\mathcal{D}_t.$$

In this case only one invariant derivation is necessary by Theorem 2.5.4. Applying ∇ to I_{2a} and I_{2b} gives two new invariants on $J^3(M,1)$ which is exactly the number needed. This generates everything, so the algebra of differential invariants is freely generated as $\mathcal{A} = \langle I_1, I_{2a}, \nabla \rangle$. In this case I_{2b} is expressible by I_1 and ∇ hence it is not needed as a generator. In summary:

Differential Invariants	Invariant Derivations
$I_1 = \frac{y_1 t - y}{y - x_1}$ $I_{2a} = \frac{y_{1,1}(ty-2x)^2}{(y-x_1)^3}$ $I_{2b} = \frac{\left(\left(ty_{1,1}-x_{1,1}+y_1\right)y - t\left(y_{1,1}x_1 - y_1x_{1,1}+y_1^2\right)\right)(ty-2x)}{(y-x_1)^2(y_1t-y)}$	$\nabla = \frac{ty-2x}{y-x_1}\mathcal{D}_t$

3.3.2 Differential Invariants, Part 2.2: Surfaces

Let $M = \mathbb{R}^3$ and induce an action of the Lie algebra $\mathfrak{g}$. In this case, we consider jets of submanifolds of dimension 2, that is, surfaces in $\mathbb{R}^3$. The action of $\mathfrak{g}$

on M induces an action on the surfaces and therefore also their jets. Let the local coordinates on $J^0(M,1)$ be t, s, x, where x is considered the dependent variable.

To check that the action is algebraic, take a generic point, say $p = (1,1,0)$. Then compute the stabilizer. This was done in the previous computation. The prolongation of the stabilizer in this case is different, it is

$$\Phi^{(1)}_{g;p} = \left(t + bx, \left(s - \frac{xt}{2}\right) + \frac{(t+bx)x}{2}, x, \frac{x_1}{bxx_2 + bx_1 + 1}, \frac{x_2}{bxx_2 + bx_1 + 1}\right),$$

where b is arbitrary. This defines an algebraic action on $J^1(M,2)$ since all entries consists of rational functions. Having verified this we can move on to the computations as the Lie-Tresse theorem is guaranteed to hold on generic points.

Letting $\mathfrak{g}$ act on $J^0(M,2)$ gives no invariants, since the rank of $\mathfrak{g}$ is 3 at generic points. On $J^1(M,2)$ the rank of $\mathfrak{g}^{(1)}$ is 4, while $\dim J^1(M,1) = 5$, so there is one invariant. Solving Eq. (2.12) with $I = I(t,s,x,x_1,x_2)$ gives

$$I_1 = \frac{(-xx_2 - x_1)t + x}{x_2(-tx + 2s)}$$

so $i_1 = j_1 = 1$. The prolongation of the algebra has reached its stable orbit dimension of 4, and in accordance with Proposition 2.5.3, it follows that $j_k = \dim J^k - \dim J^{k-1} = k+1$, for $k \geq 2$. Then on second jets the expected number of new independent invariants of order 2 is $j_2 = 3$. In general we have:

Jet Level	j_k
0	0
1	1
$k \geq 2$	$k+1$

Again, solving Eq. (2.12) on the level of second jets gives

$$I_{2a} = \frac{-x_2^3 st - t^2 x_2^2 x_{1,1} + 2tx_{1,2}\left(x_1 t - x\right)x_2 - x_{2,2}\left(x_1 t - x\right)^2}{\left(2s - tx\right)x_2^3},$$
$$I_{2b} = \frac{-x_2^3 xt - tx_1 x_2^2 - t\left(xx_{1,2} + x_{1,1}\right)x_2 + \left(xx_{2,2} + x_{1,2}\right)\left(x_1 t - x\right)}{x_2^2\left(txx_2 + x_1 t - x\right)},$$
$$I_{2c} = \frac{\left(x^2 x_{2,2} + \left(x_2^2 + 2x_{1,2}\right)x + x_2 x_1 + x_{1,1}\right)\left(2s - tx\right)}{x_2\left(txx_2 + x_1 t - x\right)^2}.$$

The Jacobian of these has rank 3, hence all of them are independent. However, some of them can be express using invariant derivations and differential invariants of lower order. The invariants I_{2a} and I_{2b} are not needed as generators since these are obtainable through I_1 and ∇_1, ∇_2. The invariant derivations are computed to be

$$\nabla_1 = \frac{1}{x_2}\mathcal{D}_t + \frac{x}{x_2}\mathcal{D}_s,$$
$$\nabla_2 = t\mathcal{D}_t + \frac{(x - tx_1)}{x_2}\mathcal{D}_s.$$

On 2-jets we have 3 independent second order invariants and by applying invariant derivations gives a total of 6 invariants of order 3. The dimension count gives that there are only 4 independent third order differential invariants, so there should be some differential syzygies. In fact, there are two of them. The first one comes from the commutator of ∇_1 and ∇_2 and the second is a relation among the third order invariants. To compute them the same method as before is used, which is discussed in detail in section 3.2.2. Skipping the details the differential syzygies are computed to be

$$I_1[\nabla_1, \nabla_2] - (I_1 - I_{2a})\,\nabla_1 + I_1(I_{2b} + 1)\nabla_2 = 0,$$
$$\nabla_1(I_{2a}) + I_1\nabla_2(I_{2b}) + (I_1 + I_{2b} + 1)I_{2a} + (2I_{2b} + 1)I_1 = 0.$$

To shorten things, denote $\mathcal{R}_1 = I_1[\nabla_1, \nabla_2] - (I_1 - I_{2a})\,\nabla_1 + I_1(I_{2b} + 1)\nabla_2$ and $\mathcal{R}_2 = \nabla_1(I_{2a}) + I_1\nabla_2(I_{2b}) + (I_1 + I_{2b} + 1)I_{2a} + (2I_{2b} + 1)I_1$. Then the algebra of differential invariants is generated as $\mathcal{A} = \langle I_1, I_{2c}, \nabla_1, \nabla_2 \mid \mathcal{R}_1, \mathcal{R}_2\rangle$.

Differential Invariants	Invariants Derivations
$I_1 = \frac{(-xx_2 - x_1)t + x}{x_2(-tx+2s)}$ $I_{2a} = \frac{-x_2^3 st - t^2 x_2^2 x_{1,1} + 2t x_{1,2}(x_1 t - x)x_2 - x_{2,2}(x_1 t - x)^2}{(2s - tx)x_2^3}$ $I_{2b} = \frac{-x_2^3 xt - tx_1 x_2^2 - t\big(x x_{1,2} + x_{1,1}\big)x_2 + \big(x x_{2,2} + x_{1,2}\big)(x_1 t - x)}{x_2^2(txx_2 + x_1 t - x)}$ $I_{2c} = \frac{\big(x^2 x_{2,2} + \big(x_2^2 + 2x_{1,2}\big)x + x_2 x_1 + x_{1,1}\big)(2s - tx)}{x_2(txx_2 + x_1 t - x)^2}$	$\nabla_1 = \frac{1}{x_2}\mathcal{D}_t + \frac{x}{x_2}\mathcal{D}_s$ $\nabla_2 = t\mathcal{D}_t + \frac{(x - tx_1)}{x_2}\mathcal{D}_s$

3.3.3 Differential Invariants, Part 2.3: Functions

Let $M = \mathbb{R}^3$ with local coordinates x, u, p and take the contact vector fields generated by $x^2, p^2, 2u, xp$ as above. Prolong the vector fields trivially to $J^0M = \mathbb{R}^3(x, u, p) \times \mathbb{R}(y)$, leaving the fiber coordinate y fixed. Firstly, to check that the action is algebraic is the same procedure as the previous computations, so the details are omitted. The action on J^0M is

$$\Phi_g(x, u, p, y) = \left(ax + by, (ad - bc)\big(u + \frac{xp}{2}\big) + \frac{(ax + bp)(cx + dp)}{2}, cx + dp, y\right).$$

for some $g \in G$. Clearly, $I_0 = y$ is an invariant of order 0. Prolonging the vector fields to J^1M yields that $\mathfrak{g}^{(1)}$ has rank 4, so that $s_1 = 4$. The dimension of J^1M is $\dim J^1M = 7$, so $i_1 = 3$. This gives that $j_1 = i_1 - i_0 = 2$. In this case $s_1 = 4$, which is the dimension of $\mathfrak{g}$, so the action has reached the stable orbit dimension. By Proposition 2.5.3 the number of independent differential invariants of order k is $j_k = \dim J^k - \dim J^{k-1} = \binom{n+k-1}{k}$ for $k \geq 2$. Computing this gives $j_k = \frac{1}{2}(k+2)(k+1)$.

Jet Level	j_k
0	1
1	2
$k \geq 2$	$\frac{1}{2}(k+2)(k+1)$

On the level of 1-jets the PDE system is able to be solved by Eq. (2.12) using Maple, which gives

$$\begin{aligned} I_{1a} &= (px - 2u)y_2, \\ I_{1b} &= (xy_2 + y_3)p + xy_1. \end{aligned}$$

It is also possible to obtain all invariant derivations on the level of 1-jets. These are computed to be

$$\begin{aligned} \nabla_1 &= x\mathcal{D}_x + 2u\mathcal{D}_u + p\mathcal{D}_p \\ \nabla_2 &= (px - 2u)\mathcal{D}_u \\ \nabla_3 &= (xy_2 + y_3)(px - 2u)\mathcal{D}_x - xy_1(px - 2u)\mathcal{D}_u - y_1(px - 2u)\mathcal{D}_p \end{aligned}$$

By applying ∇_1, ∇_2 and ∇_3 to I_0 we obtain 3 differential invariants of order 1, but only 2 of them are independent, meaning there must be some differential syzygy. In this case it is easy to find, due to $\nabla_3(I_0) = 0$. Set $\mathcal{R}_1 = \nabla_3(I_0)$. This concludes to story of J^1M. Moving on to J^2M there are $j_2 = 6$ independent second order differential invariants by the dimension count. However, finding these is the issue to due computation time in solving the corresponding PDE system. Therefore, a new approach is needed which we now discuss.

The Method of Moving Frames

The approach to finding the invariants needed is called **the method of moving frames**. A more detailed description of the method can be found in [Olv99]. Let G be an r-dimensional Lie group acting freely and regularly on the n-dimensional manifold M. Let the action $\Phi : G \times M \longrightarrow M$ be described by $\Phi_g(a) = \Phi(g, a) = (\Phi_1(g, a), \Phi_2(g, a), \ldots, \Phi_n(g, a))$, for $a \in M$. Under these hypothesis the method of moving frames can be applied and begins by solving the **normalization equations**:

$$\Phi_1(g, a) = c_1, \ \Phi_2(g, a) = c_2, \ \ldots, \ \Phi_r(g, a) = c_r.$$

If a group element $g \in G$ is written in terms of local coordinates $g = (g_1, \ldots, g_r)$ on G, then the normalization equations can be solved for these local coordinates in terms of the coordinates on M. Having solved for the parameters $g_1, g_2, \ldots, g_r$, the next step is to substitute them into a group element. This gives a map $\psi : M \to G$, called a **moving frame**. Having eliminated the parameters of the Lie group, the first r coordinates of M was used, then the remaining $(n-r)$ coordinates can be used to construct a complete set of invariants for the group action. That is, the set

$$\Phi_{r+1}(\psi(a), a), \ldots, \Phi_n(\psi(a), a)$$

forms a complete set of independent invariants for the group action.

To apply the method we first prolong the action to J^1M. Here the action is free and regular on generic orbits, so the method can be applied. Let $\Phi_g^{(1)}$ be the first prolongation of the action defined above. The normalization equations can be taken to be [5]

$$X = (\Phi_g^{(1)})^*(x),\ U = (\Phi_g^{(1)})^*(u),\ P = (\Phi_g^{(1)})^*(p),\ Y_1 = (\Phi_g^{(1)})^*(y_1).$$

This makes it possible to solve for a, b, c, d, for a generic choice of $X, U, P, Y1$, which gives a moving frame $\psi : J^1M \to G$. To obtain the necessary second order invariants, define $T(a) = \Phi_g^{(2)}(\psi(a), a)$. Then the set

$$\{T^*(y_{1,1}), T^*(y_{1,2}), T^*(y_{1,3}), T^*(y_{2,2}), T^*(y_{2,3}), T^*(y_{3,3})\},$$

forms an independent set of second order invariants. The normalization parameters $X, U, P, Y1$ has to be chosen in such a way that the normalization equations has a solution. Here the coefficients are arbitrary, but can be treated as constants and the coefficients of these will be invariants. What remains is how to choose the invariants when the coefficients are arbitrary and this can be done by elimination. The first step is to choose an invariant, but we already have a total of 6 second order invariants which are obtained by applying invariant derivations to I_{1a} and I_{1b}. This set has rank 5, so there is one missing. To find it we go through the invariants obtained by the method of moving frames and check the independence, if we find one that is independent, we are done. [6] The invariants found are summarized in the table below. On 2-jets there are a total of 6 independent differential invariants and by applying invariant derivations we get a total of 18 third order differential invariants, but by our dimension count, there are only 10 independent third order differential invariants, so

5. We can't take the equation $Y = (\Phi^{(1)})(y)$, since y is an invariant, meaning the equation is independent of the parameters of the group.
6. Alternatively, one can pick a generic point and solve the normalization equations with $X, U, P, Y1$ chosen to be this point. The invariants become longer if this is done, unless one finds a magic point that simplifies everything.

there must be some differential syzygies. There is always a differential syzygy between the invariant derivations due to Proposition 2.3.5 by computing the commutators between all invariant derivations. In this case $[\nabla_1, \nabla_2] = 0$, which gives the first differential syzygy. The other differential syzygies can be found by following the method discussed previously for functions on $\mathbb{R}^2$, see section 3.2.2. All the differential syzygies coming from the commutators are

$$\begin{aligned}
\mathcal{R}_2 &= [\nabla_1, \nabla_2] \\
\mathcal{R}_3 &= (I_{1a} + I_{1b})[\nabla_1, \nabla_3] + I_{2c}(\nabla_1 + \nabla_2) - (I_{2a} + I_{2b})\nabla_3 \\
\mathcal{R}_4 &= (I_{1a} + I_{1b})[\nabla_2, \nabla_3] - (I_{1b}(I_{1a} + I_{1b}) - I_{2e})\nabla_1 + (I_{1a}(I_{1a} + I_{1b}) + I_{2e})\nabla_2 \\
&\quad - (I_{2b} + I_{2d} - 2(I_{1a} + I_{1b}))\nabla_3
\end{aligned}$$

The remaining differential syzygies is found by the symbol method discussed earlier and reducing the lower order remainder by eliminating jet variables in favor for the invariants.

$$\begin{aligned}
\mathcal{R}_5 &= (I_{1a} + I_{1b})(\nabla_3(I_{2b}) - \nabla_1(I_{2e})) - (I_{2c} - I_{2e})I_{2b} + I_{2a}I_{2e} - I_{2c}I_{2d} \\
\mathcal{R}_6 &= (I_{1a} + I_{1b})(\nabla_3(I_{2c}) - \nabla_1(I_{2f})) - 3I_{2c}^2 - (I_{1a}^2 + I_{1a}I_{1b} + 3I_{2e})I_{2c} + 3I_{2f}(I_{2a} + I_{2b}) \\
\mathcal{R}_7 &= (I_{1a} + I_{1b})(-\nabla_3(I_{2e}) + \nabla_2(I_{2f})) - I_{1b}^4 - 4I_{1a}I_{1b}^3 - (5I_{1a}^2 + 2I_{2c})I_{1b}^2 \\
&\quad - (2I_{1a}^3 + (2I_{2c} - 3I_{2e})I_{1a} + 4I_{2f})I_{1b} + 3I_{2e}I_{1a}^2 + 4I_{2f}I_{1a} + 3I_{2e}^2 \\
&\quad + 3I_{2c}I_{2e} - 3I_{2f}(I_{2b} + I_{2d})
\end{aligned}$$

The differential syzygies are these expressions set to 0. Then the algebra of differential invariants is generated as
$\mathcal{A} = \langle I_0, I_{2f}, \nabla_1, \nabla_2, \nabla_3 \mid \mathcal{R}_i = 0, i = 1 \ldots 7 \rangle$.

Differential Invariants
$I_0 = y$
$I_{1a} = y_3 p + 2u y_2 + y_1 x$
$I_{1b} = y_2 \beta$
$I_{2a} = p^2 y_{3,3} + (4y_{2,3}u + 2x y_{1,3} + y_3)p + 4u^2 y_{2,2}$ $+(4x y_{1,2} + 4y_2)u + x(x y_{1,1} + y_1)$
$I_{2b} = (px - 2u)(y_{2,3}p + 2y_{2,2}u + x y_{1,2} + 2y_2)$
$I_{2c} = -(px - 2u)((y_{1,2}y_1 - y_{1,1}y_2)x^2 + ((y_{2,3}p + 2y_{2,2}u + y_2 + y_{1,3})y_1$ $-p y_2 y_{1,3} - 2u y_2 y_{1,2} - y_3 y_{1,1})x + (p y_{3,3} + 2y_{2,3}u)y_1$ $-y_3(p y_{1,3} + 2y_{1,2}u))$
$I_{2d} = (px - 2u)(-2y_2 + (px - 2u)y_{2,2})$
$I_{2e} = -(px - 2u)(p(y_1 y_{2,2} - y_2 y_{1,2})x^2 + ((y_{2,3}y_1 - y_2^2 - y_3 y_{1,2})p$ $+(-2y_{2,2}u - y_2)y_1 + 2u y_2 y_{1,2})x - p y_2 y_3 - 2u(y_{2,3}y_1 - y_3 y_{1,2}))$
$I_{2f} = p^2 x^4 y_1^2 y_{2,2} - 2p^2 x^4 y_1 y_2 y_{1,2} + p^2 x^4 y_2^2 y_{1,1} + 2p^2 x^3 y_1^2 y_{2,3}$ $-p^2 x^3 y_1 y_2^2 - 2p^2 x^3 y_1 y_2 y_{1,3} - 2p^2 x^3 y_1 y_3 y_{1,2} + 2p^2 x^3 y_2 y_3 y_{1,1}$

$$
\begin{array}{l}
-4pux^3y_1^2y_{2,2} + 8pux^3y_1y_2y_{1,2} - 4pux^3y_2^2y_{1,1} + p^2x^2y_1^2y_{3,3} \\
-p^2x^2y_1y_2y_3 - 2p^2x^2y_1y_3y_{1,3} + p^2x^2y_3^2y_{1,1} - 8pux^2y_1^2y_{2,3} \\
+4pux^2y_1y_2^2 + 8pux^2y_1y_2y_{1,3} + 8pux^2y_1y_3y_{1,2} - 8pux^2y_2y_3y_{1,1} \\
+4u^2x^2y_1^2y_{2,2} - 8u^2x^2y_1y_2y_{1,2} + 4u^2x^2y_2^2y_{1,1} - 4puxy_1^2y_{3,3} \\
+4puxy_1y_2y_3 + 8puxy_1y_3y_{1,3} - 4puxy_3^2y_{1,1} + 8u^2xy_1^2y_{2,3} \\
-4u^2xy_1y_2^2 - 8u^2xy_1y_2y_{1,3} - 8u^2xy_1y_3y_{1,2} + 8u^2xy_2y_3y_{1,1} \\
+4u^2y_1^2y_{3,3} - 4u^2y_1y_2y_3 - 8u^2y_1y_3y_{1,3} + 4u^2y_3^2y_{1,1}
\end{array}
$$

where $\beta = px - 2u$.

Invariant Derivations
$\nabla_1 = x\mathcal{D}_x + 2u\mathcal{D}_u + p\mathcal{D}_p,$ $\nabla_2 = \beta\mathcal{D}_u,$ $\nabla_3 = (xy_2 + y_3)\beta\mathcal{D}_x - xy_1\beta\mathcal{D}_u - y_1\beta\mathcal{D}_p.$

3.4 Symplectic Computations in 4 Dimensions

In 4-dimensions we take $M = \mathbb{R}^4$ with local coordinates x^1, x^2, y^1, y^2 and the symplectic form as

$$\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2.$$

Our vector fields are symmetries of ω, which are computed in terms of Eq. (3.4) for some function $f \in C^\infty(M)$. To get a finite-dimensional Lie algebra to work with, take f to be quadratic in the base coordinates. This gives the following Hamiltonian vector fields

$$\begin{aligned}
X_1 &= 2x^1\partial_{y^1} & X_6 &= -x^2\partial_{x^1} + y^1\partial_{y^2} \\
X_2 &= -x^1\partial_{x^1} + y^1\partial_{y^1} & X_7 &= -y^2\partial_{x^1} - y^1\partial_{x^2} \\
X_3 &= x^2\partial_{y^1} + x^1\partial_{y^2} & X_8 &= 2x^2\partial_{y^2} \\
X_4 &= -x^1\partial_{x^2} + y^2\partial_{y^1} & X_9 &= -x^2\partial_{x^2} + y^2\partial_{y^2} \\
X_5 &= -2y^1\partial_{x^1} & X_{10} &= -2y^2\partial_{x^2}.
\end{aligned}$$

Denote this Lie algebra by $\mathfrak{g} = \langle X_1, \ldots, X_{10}\rangle$. The Lie algebra is 10-dimensional, semisimple and non-compact, and it corresponds to the Lie algebra $\mathfrak{g} \simeq \mathfrak{sp}(4;\mathbb{R})$. The Lie group corresponding to this Lie algebra is $\mathrm{Sp}(4;\mathbb{R})$ obtained by finite products of elements in a neighborhood of unity. The group can be expressed as a 4×4 matrix of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

for which X shall satisfy $X^T \Omega X = \Omega$, where Ω is the standard matrix representation for the symplectic form in the coordinates x^1, x^2, y^1, y^2. In these coordinates the matrix takes the form

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I denotes the 2×2 identity matrix. This gives a restriction on A, B, C, D and they have to satisfy

$$\begin{aligned}
-C^T A + A^T C &= 0, \\
-C^T B + A^T D &= I, \\
-D^T A + B^T C &= -I \\
-D^T B + B^T D &= 0.
\end{aligned}$$

This gives a parametrization of $\mathrm{Sp}(4;\mathbb{R})$ in terms of 10 parameters. The group is algebraic being described by polynomial equations and the action of $\mathbf{Sp}(4;\mathbb{R})$ on $M = \mathbb{R}^4$ is the standard representation. To check that the action is algebraic, pick a generic point, say $p = (1, 0, 0, 0)$ and compute the stabilizer of this point by the procedure as introduced before. The method is the same so the details are omitted in this cases.

3.4.1 Differential Invariants, Part 3.1: Curves

Take the Lie algebra $\mathfrak{g}$ as defined above. Let $\mathfrak{g}$ act on $M = \mathbb{R}^4$, having local coordinates t, x, y, z. The action on M induces an action on submanifolds of dimension 1, that is unparameterized curves in M. This induces an action on all the corresponding jet spaces $J^k(M, 1)$. Take the coordinates t, x, y, z on $J^0(M, 1)$ and consider t as independent and x, y, z as dependent. In this setup, computing the necessary invariants by solving Eq. (2.12) is problematic due to the complexity of the associated PDE system, due to Maple being unable to find the necessary invariants to completely describe the whole algebra of differential invariants. The method of moving frames is another issue, due to the group having 10 parameters and the normalization equations being polynomial equations. The approach to circumvent all this is by finding the invariants and derivations by a more geometric approach.

Firstly, let's count the number of invariants. The algebra has dimension $\dim \mathfrak{g} = 10$, while $\dim J^0(M, 1) = 4$, so no invariants here as the action is transitive on $M \setminus \{0\}$. Likewise, $\dim J^1(M, 1) = 7$ and the rank of $\mathfrak{g}^{(1)}$ is 7, hence no invariants. The second jet space $J^2(M, 1)$ has dimension 10, but the algebra has rank 9, so there is one invariant. In our notation, set $i_2 = j_2 = 1$. On $J^3(M, 1)$ the dimension is 13, so there are 3 differential invariants, so $i_3 = 3$, but only 2 of them are of order 3, as $j_3 = i_3 - i_2 = 2$. For $J^3(M, 1)$, the algebra has rank of 10, hence the stable orbit dimension has been reached, so for $k \geq 4$, there are $j_k = 3$ new independent differential invariants. The conclusion is that $j_4 = 3$. To generate everything a complete independent set of invariants is needed on the level of 4-jets. By applying invariant derivations one obtains 3 new invariants which gives everything on the next level of jets. Then the Lie-Tresse theorem will guarantee that this generates everything. In summary:

Jet Level	j_k
0	0
1	0
2	1
3	2
$k \geq 4$	3

Now to discuss the approach. The setup is $G = \mathrm{Sp}(4; \mathbb{R})$ acting on $M = \mathbb{R}^4$, which induces an action on unparametrized curves in M, which induces an action on k-jets of curves. On $J^0(M, 1)$ the group preserves the symplectic form $\omega = dt \wedge dy + dx \wedge dz$, but the group is also linear, so the vector space structure of $J^0(M, 1)$ is preserved as well. In particular, the origin is preserved, so we can form a vector from the origin to any point $p = (t, x, y, z) \in J^0(M, 1)$. Denote the corresponding vector by $v_0 = (t, x, y, z)$.
Next, on the level of 1-jets $J^1(M, 1)$, which is the space of 1-jets of unparametrized

curves. Take some unparameterized curve and pick a parameterization $c = c(t) = (x(t), y(t), z(t))$, then the tangent vector at any point of this curve can be computed as $w_1 = \mathcal{D}_t^{(1)} = \partial_t + x_1\partial_x + y_1\partial_y + z_1\partial_z$, but this is not a canonical choice of a tangent vector due to it coming from a specific choice of a parameterization. Any tangent vector has the form $v_1 = \beta w_1$, for some constant β. The question is how to determine the tangent vector canonically, i.e. to make v_1 invariant. The fact that ω is invariant gives a way to fix β. The condition $\omega(v_0, v_1) = 1$ allows to solve for β by normalization, giving $\beta = 1/(ty_1+xz_1-x_1z-y)$. Then v_1 is obtained as $v_1 = \beta(\partial_t+x_1\partial_x+y_1\partial_y+z_1\partial_z)$. This yields the first and only needed invariant derivation by to Theorem 2.5.4, so

$$\nabla = \frac{1}{(ty_1 + xz_1 - x_1z - y)}\mathcal{D}_t.$$

The approach going forward is in some sense the same. There is some freedom associated to a parameterization of a given curve. Therefore, fixing this freedom in a canonical way using the symplectic form gives vectors that are invariant. The vectors that are left untouched in the process can then be evaluated using the symplectic form to obtain differential invariants.

Using the parameterization $c = c(t)$ from above, the first normalization is associated to the change of parameterization and the effect it has on the tangent vector. If $c = c(\tau)$ is another parameterization, then the relation between the tangent vectors are

$$\frac{dc}{dt} = \frac{d\tau}{dt}\frac{dc}{d\tau},$$

by the chain rule. This can be written as $w_1 = k_1v_1$, for $d\tau/dt = k_1$. The vector w_1 is not canonical, but rather convenient. The vector w_1 is associated with the specific choice of parameterization we made at the start. The parameterization $c = c(t)$ is not canonical, but simple, and easy to compute with. Having the simple parameterization allows to compute all derivatives of $c(t)$, which we'll need. The chain rule then relates it to the other parameterization $c(\tau)$. Doing this will give some parameters, which can be fixed in a natural way by the symplectic form. This is the process of normalization, which will suffice to get everything. The relation $w_1 = k_1v_1$ is the same as above, only $\beta = 1/k_1$, so the solution to finding v_1 is already done.

Look to $J^2(M, 1)$, then the change of parameterization on 2-jets becomes

$$\frac{d^2c}{dt^2} = \frac{d^2c}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 + \frac{dc}{d\tau}\frac{d^2\tau}{dt^2}.$$

Denote $v_2 = d^2c/d\tau^2$, $w_2 = d^2c/dt^2$ and $d^2\tau/dt^2 = k_2$. The equation becomes

$$w_2 = v_2k_1^2 + v_1k_2.$$

Using the parameterization for $c(t)$ it is a simple computation to get w_2, namely $w_2 = (0, x_{1,1}, y_{1,1}, z_{1,1})$. It is then possible to solve for v_2 as

$$v_2 = \frac{w_2 - v_1 k_2}{k_1^2}.$$

Then k_2 can be fixed by $\omega(v_0, v_2) = 0$. [7] This uniquely determines v_2, which can now be used to find the first differential invariant. In fact, $I_2 = \omega(v_1, v_2)$ is a differential invariant of order 2. In coordinates

$$I_2 = \omega(v_1, v_2) = \frac{x_1 z_{1,1} - z_1 x_{1,1} + y_{1,1}}{(ty_1 + xz_1 - zx_1 - y)^3}. \tag{3.12}$$

This gives the only second order invariant needed by our counting. There are 2 independent third order invariants by our dimension count. The first can be obtained by $\nabla(I_2)$, but to find the second one the normalization method must be applied. To find the missing invariants, apply the normalization approach on 3-jets. The change of parameterization is

$$\frac{d^3c}{dt^3} = \frac{d^3c}{d\tau^3}\left(\frac{d\tau}{dt}\right)^3 + 3\frac{d^2c}{d\tau^2}\frac{d\tau}{dt}\frac{d^2\tau}{dt^2} + \frac{dc}{d\tau}\frac{d^3\tau}{dt^3}.$$

Again, rewrite it in more a simpler form as

$$w_3 = v_3 k_1^3 + 3k_1 k_2 v_2 + k_3 v_1.$$

The unknown here is k_3, since the vector w_3 is easily computed to be $w_3 = (0, x_{1,1,1}, y_{1,1,1}, z_{1,1,1})$. Then k_3 can be determined by normalization. That is, k_3 can be fixed by $\omega(v_0, v_3) = 0$, where

$$v_3 = \frac{w_3 - 3k_1 k_2 v_2 - k_3 v_1}{k_1^3}.$$

This uniquely determines v_3, which allows the computation of two differential invariants. These are

$$\begin{aligned} I_{3a} &= \omega(v_1, v_3), \\ I_{3b} &= \omega(v_2, v_3). \end{aligned}$$

The actual invariants can be found in the Appendix. This gives the missing third order differential invariants. To get everything however, it requires another invariant on the level of 4-jets. The method is the same, the chain rule gives

$$\frac{d^4c}{dt^4} = \frac{d^4c}{d\tau^4}\left(\frac{d\tau}{dt}\right)^4 + 6\frac{d^3c}{d\tau^3}\left(\frac{d\tau}{dt}\right)^2\frac{d^2\tau}{dt^2} + \frac{d^2c}{d\tau^2}\left(4\frac{d\tau}{dt}\frac{d^3\tau}{dt^3} + 3\left(\frac{d^2\tau}{dt^2}\right)^2\right) + \frac{dc}{d\tau}\frac{d^4\tau}{dt^4},$$

7. The equation is affine in v_2 so it can be normalized to give 0. Previously it was a scaling so 0 couldn't be used.

which can be written as

$$w_4 = v_4k_1^4 + 6v_3k_1^2k_2 + v_2\left(4k_1k_3 + 3k_2^2\right) + v_1k_4.$$

Find k_4 by $\omega(v_0, v_4) = 0$. This uniquely determines v_4, so the invariants of order 4 are then found by

$$\begin{aligned} I_{4a} &= \omega(v_1, v_4), \\ I_{4b} &= \omega(v_2, v_4), \\ I_{4c} &= \omega(v_3, v_4). \end{aligned}$$

These are independent, but I_{4a} and I_{4b} can be expressed by the invariants of order 3 and the invariant derivation. The invariants I_{3a} and I_{3b} are independent, but I_{3a} can be expressed through $\nabla(I_2)$, so it is not needed. This gives the necessary invariants to generate the whole algebra of differential invariants. The algebra is freely generated as $\mathcal{A} = \langle I_2, I_{3b}, I_{4c}\nabla\rangle$.

Remark. The method described here can be applied to the case of curves in 2-dimensions. This gives a geometric description of the invariants. The results are summarized in the Appendix.

3.4.2 Differential Invariants, Part 3.2: Surfaces

Take $M = \mathbb{R}^4$ and look to submanifolds of dimension 2, so $J^0(M,2) = \mathbb{R}^4(t,s,x,y)$ where t,s are considered independent and x,y taken as dependent. Let the Lie algebra $\mathfrak{g} = \mathfrak{sp}(4;\mathbb{R})$ be as above and induce an action on M. As for curves the cases of surfaces also suffers from the complexity issue, in that Maple is unable to obtain all necessary invariants. The approach to finding differential invariants is therefore done geometrically.

Letting $\mathfrak{g}$ act on $J^0(M,2)$ yields no invariants as the algebra has rank 4 on generic points, which is the dimension of $J^0(M,2)$, so in accordance with Theorem 2.5.2, there are no invariants and the action is transitive on $J^0(M,2)\setminus\{0\}$. Prolonging to $J^1(M,2)$ the dimension is $\dim J^1(M,2) = 8$ and the rank of the algebra is 8, so no invariants at this stage. On $J^2(M,2)$ the dimension is 14, while the algebra has rank 10. This is the stable orbit dimension, so the order of stabilization is 2. Hence, for $k \geq 3$, there must be $j_k = \dim J^k(M,2) - \dim J^{k-1}(M,2) = 2k+2$ independent differential invariants of order k. On the level of 2-jets there are 4 independent differential invariants and by Theorem 2.5.4 there should be 2 invariant derivations. Applying invariant derivations to the differential invariants gives a total of 8 possible invariants. According to our counting there should be $j_3 = 8$ invariants, so we expect this to generate everything.

Jet Level	j_k
0	0
1	0
2	4
$k \geq 3$	$2k+2$

Having done the counting we can proceed with the geometric approach. The Lie group acting on M coming from the Lie algebra is the linear group $G = \mathrm{Sp}(4;\mathbb{R})$. Let it act on $J^0(M,2)$, which is the space of unparameterized surfaces in M. This space can be identified with M itself and carries the symplectic form $\omega = dt \wedge dx + ds \wedge dy$, which is invariant under the group. Consider a point $p \in J^0(M,2)$ as $p = (t,s,x,y)$. The origin is fixed by the group so we can construct a vector $v_0 = (t,s,x,y)$, as before. Take p to be a point on some surface Σ, described by $\Sigma = \{f = 0, g = 0\}$ with $f = x - \xi(t,s)$ and $g = y - \eta(t,s)$. Then the tangent space to Σ at p is spanned by the following vector fields

$$T_p\Sigma = \langle \mathcal{D}_x^{(1)}, \mathcal{D}_y^{(1)} \rangle = \langle \partial_t + x_1\partial_x + y_1\partial_y, \partial_s + x_2\partial_x + y_2\partial_y \rangle.$$

Equivalently, $T_p\Sigma = \mathrm{Ann}(d_pf, d_pg)$, where $d_pf = dx - \xi_1 dt - \xi_2 ds = dx - x_1 dt - x_2 ds$ and similarly for $d_pg = dy - \eta_1 dt - \eta_2 ds = dy - y_1 dt - y_2 ds$. The restriction of ω to $T_p\Sigma$ has rank 2 on generic 1-jets, so $T_p\Sigma$ is a symplectic subspace of dimension 2. Then the orthogonal complement, denoted and defined as $T_p\Sigma^{\perp\omega} = \{w \in T_pM \mid \omega(v,w) = 0, \text{ for } v \in T_p\Sigma\}$, is also a symplectic vector space with the restriction of ω as a symplectic form. Therefore, on generic 1-jets there is a canonical splitting as $T_pM = T_p\Sigma \oplus T_p\Sigma^{\perp\omega}$.[8] Using these two planes we can decompose v_0 as $v_0 = v_0^{\parallel} + v_0^{\perp}$, where $v_0^{\parallel} \in T_p\Sigma$ and $v_0^{\perp} \in T_p\Sigma^{\perp\omega}$.[9] There exists natural projections π_1, π_2 for which $\pi_1 : T_pM \longrightarrow T_p\Sigma$ and $\pi_2 : T_pM \longrightarrow T_p\Sigma^{\perp\omega}$, such that $v_0^{\parallel} = \pi_1(v_0)$ and $v_0^{\perp} = \pi_2(v_0)$. At this step, there is $T_p\Sigma$ with a vector $v_0^{\parallel}$ and a symplectic form $\omega|_{T_p\Sigma}$, this is everything coming from the 1-jets and our setup.

Moving on to 2-jets there is more structure on the tangent space. Take the defining functions f and g and change them by defining $F = \alpha f + \beta g$ and $G = \gamma f + \delta g$, where $\alpha, \beta, \gamma, \delta$ are arbitrary functions that satisfy $\alpha\delta - \beta\gamma \neq 0$ at p. Have $F(p) = G(p) = 0$ since $f(p) = g(p) = 0$, for $p \in \Sigma$. The surface Σ can then be described equivalently as $\Sigma = \{F = 0, G = 0\}$. The tangent space can be described as the annihilator of the differentials to the defining functions. The change of defining functions results in the following change in

8. Note that the vectors are not orthogonal in the usual sense. There is no metric here to define a notion of angle or even length.
9. The notation is borrowed from Riemannian geometry by analogy, even though the meaning here is not the same.

the differentials

$$d_pF = \alpha(p)d_pf + \beta(p)d_pg,$$
$$d_pG = \gamma(p)d_pf + \delta(p)d_pg.$$

Restricting to $T_p\Sigma$ gives $d_pF = 0$ and $d_pG = 0$. Therefore, the tangent space is still the same. Apply the symmetric differential to dF and dG and restrict to $T_p\Sigma$. This gives

$$d_p^2F = \alpha(p)d_p^2f + \beta(p)d_p^2g,$$
$$d_p^2G = \gamma(p)d_p^2f + \delta(p)d_p^2g.$$

This gives a 2-dimensional space of quadratic forms associated to the tangent space $T_p\Sigma$. The quadratic forms d_p^2f and d_p^2g form a basis for the space of quadratic forms on $T_p\Sigma$, but this is not a canonical basis. The goal going forward is to find a canonical basis $Q_1, Q_2 \in \langle d_p^2f|_{T_p\Sigma}, d_p^2g|_{T_p\Sigma}\rangle$. Firstly, the quadratic form Q_1 is not well defined, being dependent on two arbitrary functions. To fix one of the parameters set $Q_1(v_0^{\|}, v_0^{\|}) = 0$. This condition ensures that Q_1 has a Lorentzian signature or is degenerate, and in the generic case it is non-degenerate. The vector $v_0^{\|}$ is a null-like vector in the sense of Lorentzian geometry. Now, Q_1 is well-defined up to scale. A Lorentzian metric on the plane has two independent null-like vectors and this second null-like vector will be the second invariant derivation. Take a vector $w^{\|} \in T_p\Sigma$. At this stage this vector is undefined, but the condition $\omega(v_0^{\|}, w^{\|}) = 1$ assures that the vector is linearly independent from $v_0^{\|}$ and defined up to the change $w^{\|} \mapsto w^{\|} + kv_0^{\|}$ for some constant k. The vector can be used to determine Q_1 uniquely. The first requirement is that $w^{\|}$ should be a null-like vector with respect to Q_1. That is, $Q_1(w^{\|}, w^{\|}) = 0$. This makes it possible to determine $w^{\|}$ uniquely, due to $Q_1(w^{\|}, w^{\|}) = 0$ implying that any scaling of Q_1 is also 0 on $w^{\|}$. The vector $w^{\|}$ is not a scalar multiple of $v_0^{\|}$ so the last condition $Q_1(v_0^{\|}, w^{\|}) = 1$ makes it possible to determine Q_1 uniquely.

The quadratic form Q_1, corresponds to a 1-form, $\sigma_1 \in \mathrm{Ann}(T_p\Sigma)$. The 1-form σ_1 is not uniquely defined, having two parameters, but the condition $Q_1 = d_p\sigma_1$ uniquely determines σ_1, since knowing the coefficients of either σ_1 or Q_1 determines the other due to the coefficients being the same. Finally, our first differential invariant is then found to be

$$I_{2a} = \sigma_1(v_0^{\perp}).$$

The 1-form σ_1 satisfies $\sigma_1(v^{\|}) = 0$ and $\sigma_1(w^{\|}) = 0$, since $\sigma_1 \in \mathrm{Ann}(T_p\Sigma)$. Pick another 1-form σ_2 in the same space. Then σ_2 depends on two arbitrary functions. To determine it uniquely we follow the method above, but in reverse, in that we find σ_2 first. If we can determine the coefficient functions for σ_2, then

$Q_2 = d_p\sigma_2$ is determined uniquely, having the same coefficients as σ_2. Take a vector $w^\perp \in T_p\Sigma^{\perp\omega}$, which can be chosen to be independent from $v_0^\perp$ by the condition that $\omega(v_0^\perp, w^\perp) = 1$. This determines $w^\perp$ up to the transformation $w^\perp \mapsto w^\perp + lv_0^\perp$, for some constant l. The condition $\sigma_1(w^\perp) = 0$ will then fix $w^\perp$ uniquely.

To determine σ_2 uniquely can be done by the conditions that $\sigma_2(v_0^\perp) = 0$ and $\sigma_2(w^\perp) = 1$. This gives another unique 1-form which is independent from σ_1. This allows the computation of the corresponding quadratic form $Q_2 = d_p\sigma_2$. Finally, the remaining differential invariants are

$$\begin{aligned} I_{2b} &= Q_2(v_0^\parallel, v_0^\parallel), \\ I_{2c} &= Q_2(v_0^\parallel, w^\parallel), \\ I_{2d} &= Q_2(w^\parallel, w^\parallel). \end{aligned}$$

The vectors $v_0^\parallel$ and $w^\parallel$ are tangent vectors to Σ so they correspond to the invariant derivations ∇_1, ∇_2. A summary of the invariants can be found in the Appendix. The algebra is generated by these invariants and the two invariant derivations ∇_1, ∇_2 with some differential syzygies, which are omitted due to complexity in the computation and the overall length of the expressions.

3.4.3 Differential Invariants, Part 3.3: Hypersurfaces

Let $M = \mathbb{R}^4$ and consider submanifolds of dimension 3, that is hypersurfaces. Take $J^0(M, 1) = \mathbb{R}^4(x, y, z, u)$ where x, y, z are considered independent and u dependent. Induce a Lie algebra action of $\mathfrak{g} = \mathfrak{sp}(4; \mathbb{R})$ on M, this in turn induces an action of hypersurfaces in M, and therefore also all the associated jet spaces, $J^k(M, 3)$. As is the cases above the computation time is a problem for finding everything that is needed. Maple is able to compute some of the invariants and derivations, but not all of them, thus a more geometric approach is needed. Before going through the method we investigate the number of invariants needed.

There are no invariants on $J^0(M, 3)$ since the algebra has rank 4, which is the dimension of $J^0(M, 3)$. On $J^1(M, 3)$ the dimension is 7, but no invariants since the rank of the Lie algebra is 7, so the action is transitive on generic points. The orbit of stabilization is reached on $J^2(M, 3)$. The rank is 10 and $\dim J^2(M, 3) = 13$, hence we expect $j_2 = 3$ independent differential invariants. For $k > 2$, the number of independent differential invariants is $j_k = \frac{1}{2}(k^2 + 3k + 2)$. In particular, $j_3 = 10$. The number of invariant derivations in this setup is 3, so this generates a total number of 9 invariants by applying invariant derivations and by Proposition 2.3.5, we get that $[\nabla_i, \nabla_j] = I_{ij}^k \nabla_k$ yields a maximum number of

9 differential invariants of order 3. The total is then 18, which should suffice to obtain 10 which are independent to generate everything.

Jet Level	j_k
0	0
1	0
2	3
$k \geq 3$	$\frac{1}{2}(k^2 + 3k + 2)$

The group corresponding to the Lie algebra is $G = \mathrm{Sp}(4; \mathbb{R})$, so it preserves the symplectic form on M given in the local coordinates as $\omega = dx \wedge dz + dy \wedge du$. The Lie group is a linear group so the vector space structure of $\mathbb{R}^4$ is also preserved. Pick a point $p \in J^0(M, 3)$ as $p = (x, y, z, u)$, then we can form a vector from the origin to this point, which we'll denote by $v_0 = (x, y, z, u)$. Consider a parameterization of a hypersurface $\Sigma = \{u = u(x, y, z)\}$, given by some function $u(x, y, z)$. The tangent space is spanned by the vectors

$$T_p\Sigma = \langle \partial_x + u_1\partial_u, \partial_y + u_2\partial_u, \partial_z + u_3\partial_u \rangle = \langle \mathcal{D}_x^{(1)}, \mathcal{D}_y^{(1)}, \mathcal{D}_z^{(1)} \rangle.$$

Now, consider the orthogonal complement to $T_p\Sigma$ with respect to ω as defined in the previous computation. The basis for the tangent space of Σ is not canonical, but the span, which is the tangent space itself is something geometric and independent of coordinates. Using the basis chosen we can compute $T_p\Sigma^{\perp\omega}$. Let $w = a\partial_x + b\partial_y + c\partial_z + d\partial_u$ be some vector in $J^0(M, 3)$, then $w \in T_p\Sigma^{\perp\omega}$ if $\omega(w, \mathcal{D}_x^{(1)}) = \omega(w, \mathcal{D}_y^{(1)}) = \omega(w, \mathcal{D}_z^{(1)}) = 0$. Doing this defines w up to scale. Write w in coordinates and denote it as w_1. Then

$$w_1 = -u_3\partial_x + \partial_y + u_1\partial_z + u_2\partial_u,$$

so $T_p\Sigma^{\perp\omega} = \langle w_1 \rangle$. The vector w_1 is only determined up to scale, so to fix the scale, we use ω to normalize it. Define $v_1 = k_1 w_1$, and normalize by $\omega(v_0, v_1) = 1$. Doing this gives $k_1 = 1/(xu_1 + yu_2 + zu_3 - u)$, so the canonical vector v_1 becomes

$$v_1 = \frac{1}{xu_1 + yu_2 + zu_3 - u}(-u_3\partial_x + \partial_y + u_1\partial_z + u_2\partial_u). \tag{3.13}$$

This vector field is tangent to the hypersurface so it is horizontal, thus it can be rewritten in terms of the total derivative, this yields the first invariant derivation and the formula is:

$$\nabla_1 = \frac{-u_3\mathcal{D}_x + \mathcal{D}_y + u_1\mathcal{D}_z}{xu_1 + yu_2 + zu_3 - u}. \tag{3.14}$$

Recall that the hypersurface is defined by $u = u(x, y, z)$, so introduce $q = -u + u(x, y, z)$, then $\Sigma = \{q = 0\}$. The tangent space can be described by the kernel of the following 1-form $dq = -du + u_1dx + u_2dy + u_3dz$, which is the 1-form corresponding to the tangent vector fields to the surface. In other words, $T_p\Sigma = \ker dq$. What happens when we change the defining function q for Σ? The hypersurface Σ is defined by $\Sigma = \{q = 0\}$, so what happens if we introduce a nonzero factor f on Σ such that $f|_\Sigma \neq 0$? That is, $\Sigma = \{fq = 0\}$. Due to f being nonzero on Σ this defines the same hypersurface since the zero set is the same. For convenience set $q' = fq$. What happens to the tangent space $T_p\Sigma$ when we introduce a factor of f? The tangent space can be described by $T_p\Sigma = \ker dq$. For q' we have $dq' = dfq + fdq$, then if $w' \in T_p\Sigma$, then $dq'(w') = q(p)df(w') + f(p)dq(w') = 0$, since $q(p) = 0$ on Σ and dq on $T_p\Sigma$ is 0. Therefore, the tangent space of $\{q = 0\}$ and $\{q' = 0\}$ are the defined by the same equation $\{dq = 0\} = \{dq' = 0\}$.

Recall the second symmetric differential introduced during the discussion of surfaces in 4-dimensions. The defining function for the hypersurface has a corresponding symmetric differential. The relation between q and q' is computed as

$$d^2q' = d(d(fq)) = d(qdf + fdq) = qd^2f + 2dfdq + fd^2q.$$

Restricting this quadratic form to Σ it simplifies to

$$d_p^2q' = 2d_pfd_pq + f(p)d_p^2q,$$

since q is 0 on Σ. Restricting to the tangent space of Σ, where the quadratic form is defined, gives

$$d^2q'\big|_{T_p\Sigma} = f(p)d^2q\big|_{T_p\Sigma}.$$

In particular, the quadratic form is defined on $T_p\Sigma$ up to scale, call this factor k_2. If we recall the first differential, the same scaling factor popped up, since $d_pq' = f(p)d_pq$, so $d_pq' = k_2d_pq$. This can be used to determine k_2. Evaluating d_pq on $T_p\Sigma$ gives 0 by the definition of d_pq. Thus, pick a natural vector not in the tangent space. There are two candidates for such a vector, these being v_0 and v_1. The orthogonal complement of the tangent space is in fact a subspace of the tangent space, so $d_pq(v_1) = 0$. The vector v_0 is not in the tangent space, so it can be used to solve for k_2. Normalizing $d_pq'(v_0) = 1$, gives $k_2 = 1/dq(v_0)$ for generic 1-jets. Therefore, $d^2q' = 1/dq(v_0)d^2q$. Then $q = -u + u(x, y, z)$ can be used to actually compute the quadratic form. Doing this yields

$$Q = d^2q' = \frac{u_{1,1}dx^2 + 2u_{1,2}dxdy + 2u_{1,3}dxdz + u_{2,2}dy^2 + 2u_{2,3}dydz + u_{3,3}dz^2}{xu_1 + yu_2 + zu_3 - u}.$$

The first invariant is then computed by

$$I_{2a} = Q(v_1, v_1) = \frac{u_1^2u_{3,3} - 2u_1u_3u_{1,3} + u_3^2u_{1,1} + 2u_1u_{2,3} - 2u_3u_{1,2} + u_{2,2}}{(xu_1 + yu_2 + zu_3 - u)^3}.$$

On the tangent space $T_p\Sigma$ there is the invariant vector v_1 (the first invariant derivation), the symmetric 2-form Q, the 2-form $\omega|_{T_p\Sigma}$ which has rank 2, the 1-forms $\alpha = \omega(v_0, \cdot)$ and $\beta = Q(v_1, \cdot)$. The 1-forms α and β turn out to be independent. All of this gives a canonical splitting of the tangent space as $T_p\Sigma = \langle v_1 \rangle \oplus \Pi$, where $\Pi = \ker(\alpha)$. Clearly $v_1 \notin \ker(\alpha)$, since $\omega(v_0, v_1) = 1$ by the normalization. Hence every vector in $T_p\Sigma$ can be expressed through v_1 and Π. The dimension of the tangent space is 3, or more generally odd-dimensional, so $\omega|_{T_p\Sigma}$ is degenerate. Using the splitting we can compute $\omega|_{T_p\Sigma}$ in this basis, which yields that the kernel of ω is in fact $\langle v_1 \rangle$. In particular, the restriction $\omega|_\Pi$ is nondegenerate. Using all this information we're able to construct two more invariants and two more invariant derivations.

Take $w_2 \in \Pi \cap \ker(\beta)$. The vector w_2 is only defined up to scale. Take $v_2 = k_3 w_2$, then normalize to determine k_3 by $Q(v_2, v_2) = 1$. Then $k_3^2 = 1/Q(w_2, w_2)$. The second invariant derivation is then $v_2 = w_2/\sqrt{Q(w_2, w_2)}$. [10] Take $v_3 \in \Pi$ such that $\langle v_3 \rangle = \langle v_2 \rangle^{\perp Q}$, this guarantees that v_2 and v_3 are independent. The next step is to normalize, which is done by $\omega(v_2, v_3) = 1$, when setting $v_3 = k_4 w_3$. [11] Then we get two differential invariants by $I_{2b} = Q(v_1, v_3)$ and $I_{2c} = Q(v_3, v_3)$. A calculation of the rank of the corresponding Jacobi matrix shows that they are independent. The algebra of differential invariants is generated as $\mathcal{A} = \langle I_{2a}, I_{2b}, I_{2c}, \nabla_1, \nabla_2, \nabla_3 \mid \mathcal{R} \rangle$, here $\mathcal{R}$ represents the differential syzygies as there shall be some of them. The vectors v_1, v_2 and v_3 are all tangent vectors, hence theycan be written in terms of total derivatives.

Differential Invariants	Invariant Derivations
$I_{2a} = Q(v_1, v_1)$	$\nabla_1 = v_1$
$I_{2b} = Q(v_1, v_3)$	$\nabla_2 = v_2$
$I_{2c} = Q(v_3, v_3)$	$\nabla_3 = v_3$

The actual formulas are found in the Appendix.

3.4.4 Differential Invariants, Part 3.4: Functions

Let $M = \mathbb{R}^4$ be a symplectic manifold with local coordinates x^1, x^2, y^1, y^2 and symplectic form $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. Induce an action of $\mathfrak{g}$ on M and consider jets of functions $f : M \longrightarrow \mathbb{R}$, and prolong it trivially to J^0M with u as the fiber coordinate. Once again, the approach here is geometric due to difficulties for Maple in completely solving Eq. (2.12).

10. Technically the vector is only defined up to sign due to the appearance of the square root, hence all that follows from it is also only defined up to sign. For the differential invariants however, we can square them to get rid of this. To remove this ambiguity in the vector itself one can multiply by a suitable power of some invariant to remove it.
11. One could alternatively normalize by $Q(v_1, v_3) = 1$, then the invariant is $\omega(v_2, v_3)$.

The action of $\mathfrak{g}$ is transitive on $M \setminus \{0\}$, so no invariants here. The rank of the algebra on J^0M is 4, but $\dim J^0M = 5$, so there is one invariant. The invariant is trivially $I_0 = u$ by construction, nevertheless $i_0 = j_0 = 1$. Prolonging to J^1M the algebra has rank 8 and the dimension is $\dim J^1M = 9$, so $j_1 = 1$. Continue to J^2M, the dimension becomes $\dim J^2M = 19$, and the algebra has rank 10. Therefore, there are 9 invariants, but only 7 second order differential invariants, so $j_2 = 7$. The stable orbit dimension has been reached, so for $k \geq 3$, there are $j_k = \dim J^kM - \dim J^{k-1}M = \frac{1}{6}(k^3 + 6k^2 + 11k + 6)$ independent differential invariants or order k for $k \geq 3$. To summmarize:

Jet Level	j_k
0	1
1	1
2	7
$k \geq 3$	$\frac{1}{6}(k^3 + 6k^2 + 11k + 6)$

To start of the geometric approach consider the base manifold M and look for invariants. As before the symplectic form $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$ is an invariant 2-form and the first invariant derivation is also here, namely $\nabla_1 = x^1\mathcal{D}_{x^1} + x^2\mathcal{D}_{x^2} + y^1\mathcal{D}_{y^1} + y^2\mathcal{D}_{y^2}$, being the centralizer of the group action. This is everything on M, while on J^0M the function u is an invariant of order 0 by construction. On J^1M the invariant $I_1 = \nabla_1(I_0)$ shows up, which is computed to be

$$I_1 = \nabla_1(I_0) = x^1u_1 + x^2u_2 + y^1u_3 + y^2u_4.$$

The function u is invariant, so du is an invariant 1-form. The second invariant derivation is found as $\nabla_2 = \omega^{-1}du$. On J^2M there is $Q = d^2u$, which is an invariant quadratic form. Define $A = \omega^{-1}Q$, giving an invariant endomorphism of $\pi_2^*(TM)$ (the pullback bundle of TM by $\pi_2 : J^2M \to M$). The remaining invariant derivations are then $\nabla_3 = A\nabla_2$ and $\nabla_4 = A^2\nabla_2$. This is all the invariant derivations needed by Theorem 2.5.4. At this stage we have the following invariant objects on the level of 2-jets associated to J^2M, these being $Q = d^2u$ and $A = \omega^{-1}Q$. There are also the invariant derivations $\nabla_1, \nabla_2, \nabla_3$ and ∇_4. Doing invariant operations with these objects we can produce the necessary 7 independent invariants on J^2M. All second order differential invariants can then be constructed by

$$\begin{aligned} I_{2a} &= \nabla_1(I_1), & I_{2e} &= \operatorname{tr}(A^2),\\ I_{2b} &= \nabla_2(I_1), & I_{2f} &= \det A,\\ I_{2c} &= \nabla_3(I_1), & I_{2g} &= \omega(\nabla_2, \nabla_3),\\ I_{2d} &= \nabla_4(I_1), \end{aligned}$$

with $I_1 = \nabla_1(I_0)$ as above. All of these invariants are independent and forms a complete set of second order differential invariants. The actual formulas for the invariants can be found in the Appendix. There are 7 second order differential invariants and by applying invariant derivations we obtain 28 third order differential invariants, which should be sufficient to obtain 20 independent third order differential invariants, and in fact, it is. The action is algebraic and transitive on $M \setminus \{0\}$, so by the Lie-Tresse theorem the algebra is generated as $\mathcal{A} = \langle I_0, I_{2e}, I_{2f}, I_{2g}, \nabla_1, \nabla_2, \nabla_3, \nabla_4 | \nabla_2(I_0) = \nabla_4(I_0) = 0, \mathcal{R}\rangle$. As before, the $\mathcal{R}$ represents the missing differential syzygies, which are omitted.

Remark. The method used here can also be used to find all invariants and derivations in the case of jets of functions from section 3.2.2. This gives a geometric description which is summarized in the Appendix.

3.5 The Solution to the Equivalence Problem

The computations done allows a solution to the equivalence problem for submanifolds and foliations in $\mathbb{R}^d$ for $d = 2, 3, 4$ under symplectic and conformal symplectic actions. Recall that two submanifolds $N_1, N_2 \subseteq M$ (or foliations) are said to be equivalent under a Lie group action if $N_1 = \Phi_g^* N_2$, for all $g \in G$ given some action $\Phi : G \times M \longrightarrow M$. Let $\mathcal{A}$ be the algebra of differential invariants associated to G acting on M and thereby on the submanifolds of M and their jets. Let the algebra be generated as $\mathcal{A} = \langle I_1, \ldots, I_k, \nabla_1, \ldots, \nabla_n\rangle$. Then we define the signature of a submanifold $N \subseteq M$ as a map (or rather the image of this map) $\Psi : N \longrightarrow \mathbb{R}^K$, where $K = k + nk$ (or possibly a smaller number of scalar invariant generators for $\mathcal{A}$). Defined by $\Psi(a) = (I_1(a), \ldots, I_p(a), \nabla_i I_j(a))$ for $a \in N$, where the image is $S_N = \{(I_1(a), \ldots, I_p(a), \nabla_i I_j(a)) \mid a \in N, i = 1, \ldots, n, j = 1, \ldots, k\} \subseteq \mathbb{R}^K$. Then two generic submanifolds N_1, N_2 are equivalent under the action of G if and only if $S_{N_1} = S_{N_2}$. This solves the equivalence problem for all the cases we've considered.

As an example, take two curves $\alpha_1, \alpha_2 : \mathbb{R} \longrightarrow \mathbb{R}^2$ and consider the equivalence problem under a symplectic group action (which is the group $\mathrm{SL}(2; \mathbb{R})$ in this case). That is, does there exist some change of coordinates F (preserving $\omega = dx \wedge dy$), such that $\alpha_1 = F^* \alpha_2$? The question can be solved by checking the signature of the two curves. Recall that the algebra of differential invariants in this case is generated as $\mathcal{A} = \langle I_2, \nabla\rangle$, thus the signatures are $S_{\alpha_i} = \{I_2(a), \nabla I_2(a) \mid a \in \alpha_i\}$ for $i = 1, 2$. If the signatures are identically equal as a subset of $\mathbb{R}^2$, then the answer to the equivalence problem is yes, such F exists.

In this thesis we considered symplectic manifolds of dimension 2 and 4 and the geometric approach can be generalized (with some more work) to higher dimensions. The case of curves in $\mathbb{R}^{2d}$ is easy to generalize as the approach is exactly the same. For hypersurfaces or rather codimension 1 submanifolds the approach is more or less the same and can by solved in the same manner, although the computations will get harder but in theory is do able. The case of functions for symplectic manifolds can be treated effectively in all higher dimensions by a method analogous to the method used above. However, submanifolds of arbitrary dimensions in general symplectic and contact manifolds do not have a uniform classification.

A

List of Invariants

A.1 Differential Invariants in 2-dimensions

A.1.1 Jets of Submanifolds: Curves

Symplectic manifold $M = \mathbb{R}^2(x, y)$ with the symplectic form $\omega = dx \wedge dy$.
Independent coordinate: x.
Dependent coordinate: y.
Algebra is freely generated as $\mathcal{A} = \langle I_2, \nabla \rangle$.

Differential invariant
$I_2 = \frac{y_{1,1}}{(xy_1 - y)^3}$

where $I_{2a} = \omega(v_1, v_2)$, with v_1, v_2 being constructed by the method analogously as for curves in 4-dimensions.

Invariant Derivation
$\nabla = \frac{1}{xy_1 - y} \mathcal{D}_x$

A.1.2 Jets of Functions

Symplectic manifold $M = \mathbb{R}^2(x, y)$ with the symplectic form $\omega = dx \wedge dy$.
Independent coordinates: x, y.
Dependent coordinate: u.

Algebra is generated as $\mathcal{A} = \langle I_0, I_{2c}, \nabla_1, \nabla_2 \mid \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \rangle$.
Where the differential syzygies are

$$\mathcal{R}_1 = \nabla_2(I_0),$$
$$\mathcal{R}_2 = I_1[\nabla_1, \nabla_2] - I_{2b}\nabla_1 - (I_{2a} - I_1)\nabla_2,$$
$$\mathcal{R}_3 = (\nabla_2(I_{2b}) + \nabla_1(I_{2a}))I_1 - (3I_{2a} - I_1)I_{2c} - 3I_{2b}^3.$$

Differential Invariants
$I_0 = u$
$I_1 = xu_1 + yu_2$
$I_{2a} = x^2u_{1,1} + 2xyu_{1,2} + y^2u_{2,2}$
$I_{2b} = (yu_2 - xu_1)u_{1,2}$ $+xu_2u_{1,1} - yu_1u_{2,2}$
$I_{2c} = u_1^2u_{2,2} - 2u_1u_2u_{1,2} + u_2^2u_{1,1}$

Alternatively, the geometric approach yields

Differential Invariants
$I_0 = u$
$I_1 = xu_1 + yu_2$
$I_{2a} = u_{1,1}x^2 + (2yu_{1,2} + u_1)x + y(yu_{2,2} + u_2)$
$I_{2b} = (u_1u_{1,2} - u_2u_{1,1})x + y(u_1u_{2,2} - u_2u_{1,2})$
$I_{2c} = u_{1,1}u_{2,2} - u_{1,2}^2$

The geometric description is $I_{2a} = \nabla_1(I_1), I_{2b} = \nabla_2(I_1)$ and $I_{2c} = \det A$, with $A = \omega^{-1}Q$ being defined in the same way as the 4-dimensional case.

The invariant derivations are

Invariant derivations
$\nabla_1 = x\mathcal{D}_x + y\mathcal{D}_y$
$\nabla_2 = -u_2\mathcal{D}_x + u_1\mathcal{D}_y$

A.2 Invariants in 3-dimensions

A.2.1 Jet of Submanifolds: Curves

Contact manifold $M = \mathbb{R}^3(t, x, y)$ with the contact form $\alpha = dx - ydt$.
Independent coordinate: t.
Dependent coordinates: x, y.
Algebra is freely generated as $\mathcal{A} = \langle I_1, I_{2a}, \nabla \rangle$.

Differential Invariants	Invariant Derivations
$I_1 = \frac{y_1 t - y}{y - x_1}$ $I_{2a} = \frac{y_{1,1}(ty-2x)^2}{(y-x_1)^3}$ $I_{2b} = \frac{\left(\left(ty_{1,1}-x_{1,1}+y_1\right)y - t\left(y_{1,1}x_1 - y_1 x_{1,1} + y_1^2\right)\right)(ty-2x)}{(y-x_1)^2(y_1 t - y)}$	$\nabla = \frac{ty-2x}{y-x_1}\mathcal{D}_t$

A.2.2 Jet of Submanifolds: Surfaces

Contact manifold $M = \mathbb{R}^3(t, s, x)$ with the contact form $\alpha = ds - xdt$.
Independent coordinates: t, s.
Dependent coordinate: x.
Algebra is generated as $\mathcal{A} = \langle I_1, I_{2c}, \nabla_1, \nabla_2 \mid \mathcal{R}_1, \mathcal{R}_2 \rangle$.
Where the differential syzygies are

$$\mathcal{R}_1 = I_1[\nabla_1, \nabla_2] - (I_1 - I_{2a})\nabla_1 + I_1(I_{2b} + 1)\nabla_2$$
$$\mathcal{R}_2 = \nabla_1(I_{2a}) + I_1\nabla_2(I_{2b}) + (I_1 + I_{2b} + 1)I_{2a} + (2I_{2b} + 1)I_1$$

Differential Invariants	Invariants Derivations
$I_1 = \frac{(-xx_2 - x_1)t + x}{x_2(-tx+2s)}$ $I_{2a} = \frac{-x_2^3 st - t^2 x_2^2 x_{1,1} + 2tx_{1,2}(x_1 t - x)x_2 - x_{2,2}(x_1 t - x)^2}{(2s-tx)x_2^3}$ $I_{2b} = \frac{-x_2^3 xt - tx_1 x_2^2 - t\left(xx_{1,2} + x_{1,1}\right)x_2 + \left(xx_{2,2} + x_{1,2}\right)(x_1 t - x)}{x_2^2(txx_2 + x_1 t - x)}$ $I_{2c} = \frac{\left(x^2 x_{2,2} + \left(x_2^2 + 2x_{1,2}\right)x + x_2 x_1 + x_{1,1}\right)(2s-tx)}{x_2(txx_2 + x_1 t - x)^2}$	$\nabla_1 = \frac{1}{x_2}\mathcal{D}_t + \frac{x}{x_2}\mathcal{D}_s$ $\nabla_2 = t\mathcal{D}_t + \frac{(x - tx_1)}{x_2}\mathcal{D}_s$

A.2.3 Jets of Functions

Contact manifold $M = \mathbb{R}^3(x, u, p)$ with the contact form $\alpha = du - pdx$.
Independent coordinates: x, u, p.
Dependent coordinate: y.

Algebra is generated as $\mathcal{A} = \langle I_0, I_{2f}, \nabla_1, \nabla_2, \nabla_3 \mid \mathcal{R}_i, i = 1..7\rangle$.

Differential Invariants
$I_0 = y$
$I_{1a} = y_3 p + 2uy_2 + y_1 x$ $I_{1b} = y_2\beta$
$I_{2a} = p^2 y_{3,3} + (4y_{2,3}u + 2xy_{1,3} + y_3)p + 4u^2 y_{2,2}$ $+(4xy_{1,2} + 4y_2)u + x(xy_{1,1} + y_1)$ $I_{2b} = \beta(y_{2,3}p + 2y_{2,2}u + xy_{1,2} + 2y_2)$ $I_{2c} = -\beta((y_{1,2}y_1 - y_{1,1}y_2)x^2 + ((y_{2,3}p + 2y_{2,2}u + y_2 + y_{1,3})y_1$ $-py_2y_{1,3} - 2uy_2y_{1,2} - y_3y_{1,1})x + (py_{3,3} + 2y_{2,3}u)y_1$ $-y_3(py_{1,3} + 2y_{1,2}u))$ $I_{2d} = \beta(-2y_2 + \beta y_{2,2})$ $I_{2e} = -\beta(p(y_1y_{2,2} - y_2y_{1,2})x^2 + ((y_{2,3}y_1 - y_2^2 - y_3y_{1,2})p$ $+(-2y_{2,2}u - y_2)y_1 + 2uy_2y_{1,2})x - py_2y_3 - 2u(y_{2,3}y_1 - y_3y_{1,2}))$ $I_{2f} = p^2x^4y_1^2y_{2,2} - 2p^2x^4y_1y_2y_{1,2} + p^2x^4y_2^2y_{1,1} + 2p^2x^3y_1^2y_{2,3}$ $-p^2x^3y_1y_2^2 - 2p^2x^3y_1y_2y_{1,3} - 2p^2x^3y_1y_3y_{1,2} + 2p^2x^3y_2y_3y_{1,1}$ $-4pux^3y_1^2y_{2,2} + 8pux^3y_1y_2y_{1,2} - 4pux^3y_2^2y_{1,1} + p^2x^2y_1^2y_{3,3}$ $-p^2x^2y_1y_2y_3 - 2p^2x^2y_1y_3y_{1,3} + p^2x^2y_3^2y_{1,1} - 8pux^2y_1^2y_{2,3}$ $+4pux^2y_1y_2^2 + 8pux^2y_1y_2y_{1,3} + 8pux^2y_1y_3y_{1,2} - 8pux^2y_2y_3y_{1,1}$ $+4u^2x^2y_1^2y_{2,2} - 8u^2x^2y_1y_2y_{1,2} + 4u^2x^2y_2^2y_{1,1} - 4puxy_1^2y_{3,3}$ $+4puxy_1y_2y_3 + 8puxy_1y_3y_{1,3} - 4puxy_3^2y_{1,1} + 8u^2xy_1^2y_{2,3}$ $-4u^2xy_1y_2^2 - 8u^2xy_1y_2y_{1,3} - 8u^2xy_1y_3y_{1,2} + 8u^2xy_2y_3y_{1,1}$ $+4u^2y_1^2y_{3,3} - 4u^2y_1y_2y_3 - 8u^2y_1y_3y_{1,3} + 4u^2y_3^2y_{1,1}$

where $\beta = px - 2u$.

Invariant Derivations
$\nabla_1 = x\mathcal{D}_x + 2u\mathcal{D}_u + p\mathcal{D}_p,$ $\nabla_2 = \beta\mathcal{D}_u,$ $\nabla_3 = (xy_2 + y_3)\beta\mathcal{D}_x - xy_1\beta\mathcal{D}_u - y_1\beta\mathcal{D}_p.$

Differential syzygies:

$$
\begin{aligned}
\mathcal{R}_1 &= \nabla_3(I_0)\\
\mathcal{R}_2 &= [\nabla_1, \nabla_2]\\
\mathcal{R}_3 &= (I_{1a} + I_{1b})[\nabla_1, \nabla_3] + I_{2c}(\nabla_1 + \nabla_2) - (I_{2a} + I_{2b})\nabla_3\\
\mathcal{R}_4 &= (I_{1a} + I_{1b})[\nabla_2, \nabla_3] - (I_{1b}(I_{1a} + I_{1b}) - I_{2e})\nabla_1 + (I_{1a}(I_{1a} + I_{1b}) + I_{2e})\nabla_2\\
&\quad - (I_{2b} + I_{2d} - 2(I_{1a} + I_{1b}))\nabla_3\\
\mathcal{R}_5 &= (I_{1a} + I_{1b})(\nabla_3(I_{2b}) - \nabla_1(I_{2e})) - (I_{2c} - I_{2e})I_{2b} + I_{2a}I_{2e} - I_{2c}I_{2d}\\
\mathcal{R}_6 &= (I_{1a} + I_{1b})(\nabla_3(I_{2c}) - \nabla_1(I_{2f})) - 3I_{2c}^2 - (I_{1a}^2 + I_{1a}I_{1b} + 3I_{2e})I_{2c} + 3I_{2f}(I_{2a} + I_{2b})\\
\mathcal{R}_7 &= (I_{1a} + I_{1b})(-\nabla_3(I_{2e}) + \nabla_2(I_{2f})) - I_{1b}^4 - 4I_{1a}I_{1b}^3 - (5I_{1a}^2 + 2I_{2c})I_{1b}^2\\
&\quad - (2I_{1a}^3 + (2I_{2c} - 3I_{2e})I_{1a} + 4I_{2f})I_{1b} + 3I_{2e}I_{1a}^2 + 4I_{2f}I_{1a} + 3I_{2e}^2\\
&\quad + 3I_{2c}I_{2e} - 3I_{2f}(I_{2b} + I_{2d})
\end{aligned}
$$

A.3 Invariants in 4-dimensions

A.3.1 Jet of Submanifolds: Curves

Symplectic manifold $M = \mathbb{R}^4(t, x, y, z)$ with the symplectic form $\omega = dt \wedge dy + dx \wedge dz$.
Independent coordinate: t.
Dependent coordinates: x, y, z.
Algebra is generated freely as $\mathcal{A} = \langle I_2, I_{3b}, I_{4c}, \nabla \rangle$.

The jet notation in the table is changed to make it shorter. The subscript indicates on which level of jets the function is defined, in other words how many derivatives are taken.

Differential Invariants
$I_2 = \gamma^3(x_1z_2 - z_1x_2 + y_2)$
$\begin{aligned} I_{3b} = -\gamma^6(&tx_1y_2z_3 - tx_1y_3z_2 - tx_2y_1z_3 + tx_2y_3z_1 + tx_3y_1z_2 - tx_3y_2z_1 \\ &-xy_2z_3 + xy_3z_2 + x_2yz_3 - x_2y_3z - x_3yz_2 + x_3y_2z) \end{aligned}$
$\begin{aligned} I_{4c} = -\gamma^{10}(&t^3x_1y_1^2y_3z_4 - t^3x_1y_1^2y_4z_3 - 3t^3x_1y_1y_2^2z_4 + 4t^3x_1y_1y_2y_3z_3 \\ &+3t^3x_1y_1y_2y_4z_2 - 4t^3x_1y_1y_3^2z_2 + 3t^3x_1y_2^3z_3 - 3t^3x_1y_2^2y_3z_2 + 3t^3x_2y_1^2y_2z_4 \\ &-4t^3x_2y_1^2y_3z_3 - 3t^3x_2y_1y_2^2z_3 - 3t^3x_2y_1y_2y_4z_1 + 4t^3x_2y_1y_3^2z_1 \\ &+3t^3x_2y_2^2y_3z_1 - t^3x_3y_1^3z_4 + 4t^3x_3y_1^2y_3z_2 + t^3x_3y_1^2y_4z_1 + 3t^3x_3y_1y_2^2z_2 \\ &-4t^3x_3y_1y_2y_3z_1 - 3t^3x_3y_2^3z_1 + t^3x_4y_1^3z_3 - 3t^3x_4y_1^2y_2z_2 - t^3x_4y_1^2y_3z_1 \\ &+3t^3x_4y_1y_2^2z_1 - 3t^2xx_1y_1y_2z_2z_4 + 4t^2xx_1y_1y_2z_3^2 + 2t^2xx_1y_1y_3z_1z_4 \\ &-4t^2xx_1y_1y_3z_2z_3 - 2t^2xx_1y_1y_4z_1z_3 + 3t^2xx_1y_1y_4z_2^2 - 3t^2xx_1y_2^2z_1z_4 \\ &+6t^2xx_1y_2^2z_2z_3 + 4t^2xx_1y_2y_3z_1z_3 - 6t^2xx_1y_2y_3z_2^2 + 3t^2xx_1y_2y_4z_1z_2 \\ &-4t^2xx_1y_3^2z_1z_2 + 3t^2xx_2y_1^2z_2z_4 - 4t^2xx_2y_1^2z_3^2 + 3t^2xx_2y_1y_2z_1z_4 \\ &-6t^2xx_2y_1y_2z_2z_3 - 3t^2xx_2y_1y_4z_1z_2 + 6t^2xx_2y_2y_3z_1z_2 - 3t^2xx_2y_2y_4z_1^2 \\ &+4t^2xx_2y_3^2z_1^2 - 2t^2xx_3y_1^2z_1z_4 + 4t^2xx_3y_1^2z_2z_3 - 4t^2xx_3y_1y_2z_1z_3 \\ &+6t^2xx_3y_1y_2z_2^2 + 4t^2xx_3y_1y_3z_1z_2 + 2t^2xx_3y_1y_4z_1^2 - 6t^2xx_3y_2^2z_1z_2 \\ &-4t^2xx_3y_2y_3z_1^2 + 2t^2xx_4y_1^2z_1z_3 - 3t^2xx_4y_1^2z_2^2 - 2t^2xx_4y_1y_3z_1^2 \\ &+3t^2xx_4y_2^2z_1^2 - 2t^2x_1^2y_1y_3zz_4 + 2t^2x_1^2y_1y_4zz_3 + 3t^2x_1^2y_2^2zz_4 \\ &-4t^2x_1^2y_2y_3zz_3 - 3t^2x_1^2y_2y_4zz_2 + 4t^2x_1^2y_3^2zz_2 + 4t^2x_1x_2y_1y_3zz_3 \\ &-3t^2x_1x_2y_1y_4zz_2 - 6t^2x_1x_2y_2^2zz_3 + 6t^2x_1x_2y_2y_3zz_2 + 3t^2x_1x_2y_2y_4zz_1 \\ &-4t^2x_1x_2y_3^2zz_1 + 2t^2x_1x_3y_1^2zz_4 - 4t^2x_1x_3y_1y_2zz_3 - 2t^2x_1x_3y_1y_4zz_1 \\ &+4t^2x_1x_3y_2y_3zz_1 - 2t^2x_1x_4y_1^2zz_3 + 3t^2x_1x_4y_1y_2zz_2 + 2t^2x_1x_4y_1y_3zz_1 \\ &-3t^2x_1x_4y_2^2zz_1 - 3t^2x_2^2y_1^2zz_4 + 6t^2x_2^2y_1y_2zz_3 + 3t^2x_2^2y_1y_4zz_1 \\ &-6t^2x_2^2y_2y_3zz_1 + 4t^2x_2x_3y_1^2zz_3 - 6t^2x_2x_3y_1y_2zz_2 - 4t^2x_2x_3y_1y_3zz_1 \\ &+6t^2x_2x_3y_2^2zz_1 + 3t^2x_2x_4y_1^2zz_2 - 3t^2x_2x_4y_1y_2zz_1 - 4t^2x_3^2y_1^2zz_2 \\ &+4t^2x_3^2y_1y_2zz_1 - 3tx^2x_1y_2z_1z_2z_4 + 4tx^2x_1y_2z_1z_3^2 + 3tx^2x_1y_2z_2^2z_3 \\ &+tx^2x_1y_3z_1^2z_4 - 4tx^2x_1y_3z_1z_2z_3 - 3tx^2x_1y_3z_2^3 - tx^2x_1y_4z_1^2z_3 \\ &+3tx^2x_1y_4z_1z_2^2 + 3tx^2x_2y_1z_1z_2z_4 - 4tx^2x_2y_1z_1z_3^2 - 3tx^2x_2y_1z_2^2z_3 \end{aligned}$

$$
\begin{aligned}
&+4tx^2x_2y_3z_1^2z_3+3tx^2x_2y_3z_1z_2^2-3tx^2x_2y_4z_1^2z_2-tx^2x_3y_1z_1^2z_4\\
&+4tx^2x_3y_1z_1z_2z_3+3tx^2x_3y_1z_2^3-4tx^2x_3y_2z_1^2z_3-3tx^2x_3y_2z_1z_2^2+tx^2x_3y_4z_1^3\\
&+tx^2x_4y_1z_1^2z_3-3tx^2x_4y_1z_1z_2^2+3tx^2x_4y_2z_1^2z_2-tx^2x_4y_3z_1^3+3txx_1^2y_2zz_2z_4\\
&-4txx_1^2y_2zz_3^2-2txx_1^2y_3zz_1z_4+4txx_1^2y_3zz_2z_3+2txx_1^2y_4zz_1z_3-3txx_1^2y_4zz_2^2\\
&-3txx_1x_2y_1zz_2z_4+4txx_1x_2y_1zz_3^2+3txx_1x_2y_2zz_1z_4-6txx_1x_2y_2zz_2z_3\\
&-4txx_1x_2y_3zz_1z_3+6txx_1x_2y_3zz_2^2+2txx_1x_3y_1zz_1z_4-4txx_1x_3y_1zz_2z_3\\
&+4txx_1x_3y_3zz_1z_2-2txx_1x_3y_4zz_1^2-2txx_1x_4y_1zz_1z_3+3txx_1x_4y_1zz_2^2\\
&-3txx_1x_4y_2zz_1z_2+2txx_1x_4y_3zz_1^2-3txx_2^2y_1zz_1z_4+6txx_2^2y_1zz_2z_3\\
&-6txx_2^2y_3zz_1z_2+3txx_2^2y_4zz_1^2+4txx_2x_3y_1zz_1z_3-6txx_2x_3y_1zz_2^2\\
&+6txx_2x_3y_2zz_1z_2-4txx_2x_3y_3zz_1^2+3txx_2x_4y_1zz_1z_2-3txx_2x_4y_2zz_1^2\\
&-4txx_3^2y_1zz_1z_2+4txx_3^2y_2zz_1^2+tx_1^3y_3z^2z_4-tx_1^3y_4z^2z_3-3tx_1^2x_2y_2z^2z_4\\
&+3tx_1^2x_2y_4z^2z_2-tx_1^2x_3y_1z^2z_4+4tx_1^2x_3y_2z^2z_3-4tx_1^2x_3y_3z^2z_2+tx_1^2x_3y_4z^2z_1\\
&+tx_1^2x_4y_1z^2z_3-tx_1^2x_4y_3z^2z_1+3tx_1x_2^2y_1z^2z_4+3tx_1x_2^2y_2z^2z_3-3tx_1x_2^2y_3z^2z_2\\
&-3tx_1x_2^2y_4z^2z_1-4tx_1x_2x_3y_1z^2z_3+4tx_1x_2x_3y_3z^2z_1-3tx_1x_2x_4y_1z^2z_2\\
&+3tx_1x_2x_4y_2z^2z_1+4tx_1x_3^2y_1z^2z_2-4tx_1x_3^2y_2z^2z_1-3tx_2^3y_1z^2z_3+3tx_2^3y_3z^2z_1\\
&+3tx_2^2x_3y_1z^2z_2-3tx_2^2x_3y_2z^2z_1-t^2xy_1^2y_3z_4+t^2xy_1^2y_4z_3+3t^2xy_1y_2^2z_4\\
&-4t^2xy_1y_2y_3z_3-3t^2xy_1y_2y_4z_2+4t^2xy_1y_3^2z_2-3t^2xy_2^3z_3+3t^2xy_2^2y_3z_2\\
&-2t^2x_1yy_1y_3z_4+2t^2x_1yy_1y_4z_3+3t^2x_1yy_2^2z_4-4t^2x_1yy_2y_3z_3-3t^2x_1yy_2y_4z_2\\
&+4t^2x_1yy_3^2z_2-6t^2x_2yy_1y_2z_4+8t^2x_2yy_1y_3z_3+3t^2x_2yy_2^2z_3+3t^2x_2yy_2y_4z_1\\
&-4t^2x_2yy_3^2z_1+3t^2x_2y_1y_2y_4z-4t^2x_2y_1y_3^2z-3t^2x_2y_2^2y_3z+3t^2x_3yy_1^2z_4\\
&-8t^2x_3yy_1y_3z_2-2t^2x_3yy_1y_4z_1-3t^2x_3yy_2^2z_2+4t^2x_3yy_2y_3z_1\\
&-t^2x_3y_1^2y_4z+4t^2x_3y_1y_2y_3z+3t^2x_3y_2^3z-3t^2x_4yy_1^2z_3+6t^2x_4yy_1y_2z_2\\
&+2t^2x_4yy_1y_3z_1-3t^2x_4yy_2^2z_1+t^2x_4y_1^2y_3z-3t^2x_4y_1y_2^2z+3tx^2y_1y_2z_2z_4\\
&-4tx^2y_1y_2z_3^2-2tx^2y_1y_3z_1z_4+4tx^2y_1y_3z_2z_3+2tx^2y_1y_4z_1z_3-3tx^2y_1y_4z_2^2\\
&+3tx^2y_2^2z_1z_4-6tx^2y_2^2z_2z_3-4tx^2y_2y_3z_1z_3+6tx^2y_2y_3z_2^2-3tx^2y_2y_4z_1z_2\\
&+4tx^2y_3^2z_1z_2+3txx_1yy_2z_2z_4-4txx_1yy_2z_3^2-2txx_1yy_3z_1z_4+4txx_1yy_3z_2z_3\\
&+2txx_1yy_4z_1z_3-3txx_1yy_4z_2^2+2txx_1y_1y_3zz_4-2txx_1y_1y_4zz_3-3txx_1y_2^2zz_4\\
&+4txx_1y_2y_3zz_3+3txx_1y_2y_4zz_2-4txx_1y_3^2zz_2-6txx_2yy_1z_2z_4+8txx_2yy_1z_3^2\\
&-3txx_2yy_2z_1z_4+6txx_2yy_2z_2z_3+3txx_2yy_4z_1z_2-3txx_2y_1y_2zz_4-4txx_2y_1y_3zz_3\\
&+6txx_2y_1y_4zz_2+6txx_2y_2^2zz_3-12txx_2y_2y_3zz_2+3txx_2y_2y_4zz_1-4txx_2y_3^2zz_1\\
&+4txx_3yy_1z_1z_4-8txx_3yy_1z_2z_3+4txx_3yy_2z_1z_3-6txx_3yy_2z_2^2-4txx_3yy_3z_1z_2\\
&-2txx_3yy_4z_1^2+8txx_3y_1y_2zz_3-4txx_3y_1y_3zz_2-2txx_3y_1y_4zz_1+6txx_3y_2^2zz_2\\
&+4txx_3y_2y_3zz_1-4txx_4yy_1z_1z_3+6txx_4yy_1z_2^2+2txx_4yy_3z_1^2-3txx_4y_1y_2zz_2\\
&+2txx_4y_1y_3zz_1-3txx_4y_2^2zz_1+2tx_1^2yy_3zz_4-2tx_1^2yy_4zz_3-4tx_1x_2yy_3zz_3\\
&+3tx_1x_2yy_4zz_2-3tx_1x_2y_2y_4z^2+4tx_1x_2y_3^2z^2-4tx_1x_3yy_1zz_4+4tx_1x_3yy_2zz_3\\
&+2tx_1x_3yy_4zz_1+2tx_1x_3y_1y_4z^2-4tx_1x_3y_2y_3z^2+4tx_1x_4yy_1zz_3-3tx_1x_4yy_2zz_2\\
&-2tx_1x_4yy_3zz_1-2tx_1x_4y_1y_3z^2+3tx_1x_4y_2^2z^2+6tx_2^2yy_1zz_4-6tx_2^2yy_2zz_3\\
&-3tx_2^2yy_4zz_1-3tx_2^2y_1y_4z^2+6tx_2^2y_2y_3z^2-8tx_2x_3yy_1zz_3+6tx_2x_3yy_2zz_2\\
&+4tx_2x_3yy_3zz_1+4tx_2x_3y_1y_3z^2-6tx_2x_3y_2^2z^2-6tx_2x_4yy_1zz_2+3tx_2x_4yy_2zz_1\\
&+3tx_2x_4y_1y_2z^2+8tx_3^2yy_1zz_2-4tx_3^2yy_2zz_1-4tx_3^2y_1y_2z^2+3x^3y_2z_1z_2z_4\\
&-4x^3y_2z_1z_3^2-3x^3y_2z_2^2z_3-x^3y_3z_1^2z_4+4x^3y_3z_1z_2z_3+3x^3y_3z_2^3+x^3y_4z_1^2z_3\\
&-3x^3y_4z_1z_2^2-3x^2x_1y_2zz_2z_4+4x^2x_1y_2zz_3^2+2x^2x_1y_3zz_1z_4-4x^2x_1y_3zz_2z_3\\
&-2x^2x_1y_4zz_1z_3+3x^2x_1y_4zz_2^2-3x^2x_2yz_1z_2z_4+4x^2x_2yz_1z_3^2+3x^2x_2yz_2^2z_3
\end{aligned}
$$

$$
\begin{aligned}
&-3x^2x_2y_2zz_1z_4 + 6x^2x_2y_2zz_2z_3 - 4x^2x_2y_3zz_1z_3 - 9x^2x_2y_3zz_2^2 + 6x^2x_2y_4zz_1z_2 \\
&+x^2x_3yz_1^2z_4 - 4x^2x_3yz_1z_2z_3 - 3x^2x_3yz_2^3 + 8x^2x_3y_2zz_1z_3 + 3x^2x_3y_2zz_2^2 \\
&-4x^2x_3y_3zz_1z_2 - x^2x_3y_4zz_1^2 - x^2x_4yz_1^2z_3 + 3x^2x_4yz_1z_2^2 - 3x^2x_4y_2zz_1z_2 \\
&+x^2x_4y_3zz_1^2 - xx_1^2y_3z^2z_4 + xx_1^2y_4z^2z_3 + 3xx_1x_2yzz_2z_4 - 4xx_1x_2yzz_3^2 \\
&+3xx_1x_2y_2z^2z_4 + 4xx_1x_2y_3z^2z_3 - 6xx_1x_2y_4z^2z_2 - 2xx_1x_3yzz_1z_4 \\
&+4xx_1x_3yzz_2z_3 - 8xx_1x_3y_2z^2z_3 + 4xx_1x_3y_3z^2z_2 + 2xx_1x_3y_4z^2z_1 \\
&+2xx_1x_4yzz_1z_3 - 3xx_1x_4yzz_2^2 + 3xx_1x_4y_2z^2z_2 - 2xx_1x_4y_3z^2z_1 \\
&+3xx_2^2yzz_1z_4 - 6xx_2^2yzz_2z_3 - 3xx_2^2y_2z^2z_3 + 9xx_2^2y_3z^2z_2 - 3xx_2^2y_4z^2z_1 \\
&-4xx_2x_3yzz_1z_3 + 6xx_2x_3yzz_2^2 - 6xx_2x_3y_2z^2z_2 + 4xx_2x_3y_3z^2z_1 \\
&-3xx_2x_4yzz_1z_2 + 3xx_2x_4y_2z^2z_1 + 4xx_3^2yzz_1z_2 - 4xx_3^2y_2z^2z_1 + x_1^2x_3yz^2z_4 \\
&-x_1^2x_3y_4z^3 - x_1^2x_4yz^2z_3 + x_1^2x_4y_3z^3 - 3x_1x_2^2yz^2z_4 + 3x_1x_2^2y_4z^3 \\
&+4x_1x_2x_3yz^2z_3 - 4x_1x_2x_3y_3z^3 + 3x_1x_2x_4yz^2z_2 - 3x_1x_2x_4y_2z^3 - 4x_1x_3^2yz^2z_2 \\
&+4x_1x_3^2y_2z^3 + 3x_2^3yz^2z_3 - 3x_2^3y_3z^3 - 3x_2^2x_3yz^2z_2 + 3x_2^2x_3y_2z^3 \\
&+2txyy_1y_3z_4 - 2txyy_1y_4z_3 - 3txyy_2^2z_4 + 4txyy_2y_3z_3 + 3txyy_2y_4z_2 \\
&-4txyy_3^2z_2 + tx_1y^2y_3z_4 - tx_1y^2y_4z_3 + 3tx_2y^2y_2z_4 - 4tx_2y^2y_3z_3 \\
&-3tx_2yy_2y_4z + 4tx_2yy_3^2z - 3tx_3y^2y_1z_4 + 4tx_3y^2y_3z_2 + tx_3y^2y_4z_1z \\
&+2tx_3yy_1y_4z - 4tx_3yy_2y_3 + 3tx_4y^2y_1z_3 - 3tx_4y^2y_2z_2 - tx_4y^2y_3z_1 \\
&-2tx_4yy_1y_3z + 3tx_4yy_2^2z - 3x^2yy_2z_2z_4 + 4x^2yy_2z_3^2 + 2x^2yy_3z_1z_4 \\
&-4x^2yy_3z_2z_3 - 2x^2yy_4z_1z_3 + 3x^2yy_4z_2^2 - 2xx_1yy_3zz_4 + 2xx_1yy_4zz_3 \\
&+3xx_2y^2z_2z_4 - 4xx_2y^2z_3^2 + 3xx_2yy_2zz_4 + 4xx_2yy_3zz_3 - 6xx_2yy_4zz_2 \\
&-2xx_3y^2z_1z_4 + 4xx_3y^2z_2z_3 - 8xx_3yy_2zz_3 + 4xx_3yy_3zz_2 + 2xx_3yy_4zz_1 \\
&+2xx_4y^2z_1z_3 - 3xx_4y^2z_2^2 + 3xx_4yy_2zz_2 - 2xx_4yy_3zz_1 + 2x_1x_3y^2zz_4 \\
&-2x_1x_3yy_4z^2 - 2x_1x_4y^2zz_3 + 2x_1x_4yy_3z^2 - 3x_2^2y^2zz_4 + 3x_2^2yy_4z^2 \\
&+4x_2x_3y^2zz_3 - 4x_2x_3yy_3z^2 + 3x_2x_4y^2zz_2 - 3x_2x_4yy_2z^2 - 4x_3^2y^2zz_2 \\
&+4x_3^2yy_2z^2 - xy^2y_3z_4 + xy^2y_4z_3 + x_3y^3z_4 - x_3y^2y_4z - x_4y^3z_3 + x_4y^2y_3z)
\end{aligned}
$$

with the invariant derivation

Invariant Derivation
$\nabla = \gamma \mathcal{D}_t$

where $\gamma = 1/(ty_1 + xz_1 - x_1z - y)$.

A.3.2 Jet of Submanifolds: Surfaces

Symplectic manifold $M = \mathbb{R}^4(t, s, x, y)$ with the symplectic form $\omega = dt \wedge dx + ds \wedge dy$.
Independent coordinates: t, s.
Dependent coordinates: x, y.
Algebra is generated as $\mathcal{A} = \langle I_{2a}, I_{2b}, I_{2c}, I_{2d}, \nabla_1, \nabla_2, \nabla_3 \mid \mathcal{R} \rangle$.
Again, $\mathcal{R}$ represents the unknown differential syzygies.

Differential Invariants
$I_{2a} = (-s^3x_2^2y_1^2y_{2,2} + 2s^3x_2^2y_1y_2y_{1,2} - s^3x_2^2y_2^2y_{1,1} + s^3x_2x_{1,1}y_2^3$ $-2s^3x_2x_{1,2}y_1y_2^2 + s^3x_2x_{2,2}y_1^2y_2 + s^3x_2y_1^3y_{2,2} - 2s^3x_2y_1^2y_2y_{1,2}$ $+s^3x_2y_1y_2^2y_{1,1} - s^3x_{1,1}y_1y_2^3 + 2s^3x_{1,2}y_1^2y_2^2 - s^3x_{2,2}y_1^3y_2$ $-2s^2tx_1x_2^2y_1y_{2,2} + 2s^2tx_1x_2^2y_2y_{1,2} - 2s^2tx_1x_2x_{1,2}y_2^2 + 2s^2tx_1x_2x_{2,2}y_1y_2$ $+s^2tx_1x_2y_1^2y_{2,2} - s^2tx_1x_2y_2^2y_{1,1} + 2s^2tx_1x_{1,2}y_1y_2^2 - 2s^2tx_1x_{2,2}y_1^2y_2$ $+s^2tx_1y_1^3y_{2,2} - 2s^2tx_1y_1^2y_2y_{1,2} + s^2tx_1y_1y_2^2y_{1,1} + 2s^2tx_2^3y_1y_{1,2}$ $-2s^2tx_2^3y_2y_{1,1} + 2s^2tx_2^2x_{1,1}y_2^2 - 2s^2tx_2^2x_{1,2}y_1y_2 - 2s^2tx_2^2y_1^2y_{1,2}$ $+2s^2tx_2^2y_1y_2y_{1,1} - s^2tx_2x_{1,1}y_1y_2^2 + s^2tx_2x_{2,2}y_1^3 - s^2tx_{1,1}y_1^2y_2^2$ $+2s^2tx_{1,2}y_1^3y_2 - s^2tx_{2,2}y_1^4 - st^2x_1^2x_2^2y_{2,2} + st^2x_1^2x_2x_{2,2}y_2$ $-st^2x_1^2x_2y_1y_{2,2} + 2st^2x_1^2x_2y_2y_{1,2} - st^2x_1^2x_{2,2}y_1y_2 + 2st^2x_1^2y_1^2y_{2,2}$ $-2st^2x_1^2y_1y_2y_{1,2} + 2st^2x_1x_2^3y_{1,2} - 2st^2x_1x_2^2x_{1,2}y_2 - 2st^2x_1x_2^2y_2y_{1,1}$ $+2st^2x_1x_2x_{2,2}y_1^2 - 2st^2x_1x_2y_1^2y_{1,2} + 2st^2x_1x_2y_1y_2y_{1,1} + 2st^2x_1x_{1,2}y_1^2y_2$ $-2st^2x_1x_{2,2}y_1^3 - st^2x_2^4y_{1,1} + st^2x_2^3x_{1,1}y_2 + st^2x_2^3y_1y_{1,1} + st^2x_2^2x_{1,1}y_1y_2$ $-2st^2x_2^2x_{1,2}y_1^2 - 2st^2x_2x_{1,1}y_1^2y_2 + 2st^2x_2x_{1,2}y_1^3 - t^3x_1^3x_2y_{2,2}$ $+t^3x_1^3y_1y_{2,2} + 2t^3x_1^2x_2^2y_{1,2} + t^3x_1^2x_2x_{2,2}y_1 - 2t^3x_1^2x_2y_1y_{1,2}$ $-t^3x_1^2x_{2,2}y_1^2 - t^3x_1x_2^3y_{1,1} - 2t^3x_1x_2^2x_{1,2}y_1 + t^3x_1x_2^2y_1y_{1,1}$ $+2t^3x_1x_2x_{1,2}y_1^2 + t^3x_2^3x_{1,1}y_1 - t^3x_2^2x_{1,1}y_1^2 + 2s^2xx_2^2y_1y_{2,2}$ $-2s^2xx_2^2y_2y_{1,2} + 2s^2xx_2x_{1,2}y_2^2 - 2s^2xx_2x_{2,2}y_1y_2 - s^2xx_2y_1^2y_{2,2}$ $+s^2xx_2y_2^2y_{1,1} - 2s^2xx_{1,2}y_1y_2^2 + 2s^2xx_{2,2}y_1^2y_2 - s^2xy_1^3y_{2,2}$ $+2s^2xy_1^2y_2y_{1,2} - s^2xy_1y_2^2y_{1,1} - 2s^2x_2^2yy_1y_{1,2} + 2s^2x_2^2yy_2y_{1,1}$ $-3s^2x_2x_{1,1}yy_2^2 + 4s^2x_2x_{1,2}yy_1y_2 - s^2x_2x_{2,2}yy_1^2 + 2s^2x_2yy_1^2y_{1,2}$ $-2s^2x_2yy_1y_2y_{1,1} + 3s^2x_{1,1}yy_1y_2^2 - 4s^2x_{1,2}yy_1^2y_2 + s^2x_{2,2}yy_1^3$ $+2stxx_1x_2^2y_{2,2} - 2stxx_1x_2x_{2,2}y_2 + 2stxx_1x_2y_1y_{2,2} - 4stxx_1x_2y_2y_{1,2}$ $+2stxx_1x_{2,2}y_1y_2 - 4stxx_1y_1^2y_{2,2} + 4stxx_1y_1y_2y_{1,2} - 2stxx_2^3y_{1,2}$ $+2stxx_2^2x_{1,2}y_2 + 2stxx_2^2y_2y_{1,1} - 2stxx_2x_{2,2}y_1^2 + 2stxx_2y_1^2y_{1,2}$ $-2stxx_2y_1y_2y_{1,1} - 2stxx_{1,2}y_1^2y_2 + 2stxx_{2,2}y_1^3 - 2stx_1x_2^2yy_{1,2}$ $+4stx_1x_2x_{1,2}yy_2 - 2stx_1x_2x_{2,2}yy_1 + 2stx_1x_2yy_2y_{1,1} - 4stx_1x_{1,2}yy_1y_2$ $+2stx_1x_{2,2}yy_1^2 + 2stx_1yy_1^2y_{1,2} - 2stx_1yy_1y_2y_{1,1} + 2stx_2^3yy_{1,1}$ $-4stx_2^2x_{1,1}yy_2 + 2stx_2^2x_{1,2}yy_1 - 2stx_2^2yy_1y_{1,1} + 2stx_2x_{1,1}yy_1y_2$ $+2stx_{1,1}yy_1^2y_2 - 2stx_{1,2}yy_1^3 + 3t^2xx_1^2x_2y_{2,2} - 3t^2xx_1^2y_1y_{2,2}$ $-4t^2xx_1x_2^2y_{1,2} - 2t^2xx_1x_2x_{2,2}y_1 + 4t^2xx_1x_2y_1y_{1,2} + 2t^2xx_1x_{2,2}y_1^2$ $+t^2xx_2^3y_{1,1} + 2t^2xx_2^2x_{1,2}y_1 - t^2xx_2^2y_1y_{1,1} - 2t^2xx_2x_{1,2}y_1^2$ $-t^2x_1^2x_2x_{2,2}y - 2t^2x_1^2x_2yy_{1,2} + t^2x_1^2x_{2,2}yy_1 + 2t^2x_1^2yy_1y_{1,2}$ $+2t^2x_1x_2^2x_{1,2}y + 2t^2x_1x_2^2yy_{1,1} - 2t^2x_1x_2yy_1y_{1,1} - 2t^2x_1x_{1,2}yy_1^2$ $-t^2x_2^3x_{1,1}y - t^2x_2^2x_{1,1}yy_1 + 2t^2x_2x_{1,1}yy_1^2 - sx^2x_2^2y_{2,2}$ $+sx^2x_2x_{2,2}y_2 - sx^2x_2y_1y_{2,2} + 2sx^2x_2y_2y_{1,2} - sx^2x_{2,2}y_1y_2 + 2sx^2y_1^2y_{2,2}$ $-2sx^2y_1y_2y_{1,2} + 2sxx_2^2yy_{1,2} - 4sxx_2x_{1,2}yy_2 + 2sxx_2x_{2,2}yy_1$ $-2sxx_2yy_2y_{1,1} + 4sxx_{1,2}yy_1y_2 - 2sxx_{2,2}yy_1^2 - 2sxyy_1^2y_{1,2}$ $+2sxyy_1y_2y_{1,1} - sx_2^2y^2y_{1,1} + 3sx_2x_{1,1}y^2y_2 - 2sx_2x_{1,2}y^2y_1$ $+sx_2y^2y_1y_{1,1} - 3sx_{1,1}y^2y_1y_2 + 2sx_{1,2}y^2y_1^2 - 3tx^2x_1x_2y_{2,2}$ $+3tx^2x_1y_1y_{2,2} + 2tx^2x_2^2y_{1,2} + tx^2x_2x_{2,2}y_1 - 2tx^2x_2y_1y_{1,2}$

$$-tx^2x_{2,2}y_1^2 + 2txx_1x_2x_{2,2}y + 4txx_1x_2yy_{1,2} - 2txx_1x_{2,2}yy_1$$
$$-4txx_1yy_1y_{1,2} - 2txx_2^2x_{1,2}y - 2txx_2^2yy_{1,1} + 2txx_2yy_1y_{1,1}$$
$$+2txx_{1,2}yy_1^2 - 2tx_1x_2x_{1,2}y^2 - tx_1x_2y^2y_{1,1} + 2tx_1x_{1,2}y^2y_1$$
$$+tx_1y^2y_1y_{1,1} + 2tx_2^2x_{1,1}y^2 - tx_2x_{1,1}y^2y_1 - tx_{1,1}y^2y_1^2$$
$$+x^3x_2y_{2,2} - x^3y_1y_{2,2} - x^2x_2x_{2,2}y - 2x^2x_2yy_{1,2} + x^2x_{2,2}yy_1$$
$$+2x^2yy_1y_{1,2} + 2xx_2x_{1,2}y^2 + xx_2y^2y_{1,1} - 2xx_{1,2}y^2y_1 - xy^2y_1y_{1,1}$$
$$-x_2x_{1,1}y^3 + x_{1,1}y^3y_1)/(s^2x_{1,1}y_1y_2y_{2,2} - s^2x_{1,1}y_2^2y_{1,2}$$
$$-s^2x_{1,2}y_1^2y_{2,2} + s^2x_{1,2}y_2^2y_{1,1} + s^2x_{2,2}y_1^2y_{1,2} - s^2x_{2,2}y_1y_2y_{1,1}$$
$$+stx_1x_{1,1}y_2y_{2,2} - 2stx_1x_{1,2}y_1y_{2,2} + 2stx_1x_{2,2}y_1y_{1,2} - stx_1x_{2,2}y_2y_{1,1}$$
$$+stx_2x_{1,1}y_1y_{2,2} - 2stx_2x_{1,1}y_2y_{1,2} + 2stx_2x_{1,2}y_2y_{1,1} - stx_2x_{2,2}y_1y_{1,1}$$
$$-t^2x_1^2x_{1,2}y_{2,2} + t^2x_1^2x_{2,2}y_{1,2} + t^2x_1x_2x_{1,1}y_{2,2} - t^2x_1x_2x_{2,2}y_{1,1}$$
$$-t^2x_2^2x_{1,1}y_{1,2} + t^2x_2^2x_{1,2}y_{1,1} - sxx_{1,1}y_2y_{2,2} + 2sxx_{1,2}y_1y_{2,2} - 2sxx_{2,2}y_1y_{1,2}$$
$$+sxx_{2,2}y_2y_{1,1} - sx_{1,1}yy_1y_{2,2} + 2sx_{1,1}yy_2y_{1,2} - 2sx_{1,2}yy_2y_{1,1} + sx_{2,2}yy_1y_{1,1}$$
$$+2txx_1x_{1,2}y_{2,2} - 2txx_1x_{2,2}y_{1,2} - txx_2x_{1,1}y_{2,2} + txx_2x_{2,2}y_{1,1} - tx_1x_{1,1}yy_{2,2}$$
$$+tx_1x_{2,2}yy_{1,1} + 2tx_2x_{1,1}yy_{1,2} - 2tx_2x_{1,2}yy_{1,1} - x^2x_{1,2}y_{2,2} + x^2x_{2,2}y_{1,2}$$
$$+xx_{1,1}yy_{2,2} - xx_{2,2}yy_{1,1} - x_{1,1}y^2y_{1,2} + x_{1,2}y^2y_{1,1})$$

$$I_{2b} = \Big(((y_1^2y_{2,2} - 2y_1y_2y_{1,2} + y_2^2y_{1,1})x_2 + 2x_{1,2}y_1y_2^2 - x_{2,2}y_1^2y_2$$
$$-y_2^3x_{1,1})s^3 + (((-2y_1y_{1,2} + 2y_2y_{1,1})x_2^2 + ((2y_{2,2}x_1 + 2x_{1,2}y_2)y_1$$
$$-2y_{1,2}x_1y_2 - 2y_2^2x_{1,1})x_2 - x_{2,2}y_1^3 + (y_{2,2}x_1 + 2x_{1,2}y_2)y_1^2$$
$$-2y_2((x_{2,2} + y_{1,2})x_1 + 1/2y_2x_{1,1})y_1 + 2x_1y_2^2(x_{1,2} + 1/2y_{1,1}))t$$
$$+((-2y_{2,2}x + 2y_{1,2}y)y_1 - 2y_2(-y_{1,2}x + yy_{1,1}))x_2 + (-y_{2,2}x + x_{2,2}y)y_1^2$$
$$-4y_2((-1/2x_{2,2} - 1/2y_{1,2})x + x_{1,2}y)y_1 - 2y_2^2((x_{1,2} + 1/2y_{1,1})x$$
$$-3/2yx_{1,1}))s^2 + ((y_{1,1}x_2^3 + (-2x_1y_{1,2} - y_2x_{1,1})x_2^2 + (2x_{1,2}y_1^2$$
$$+(-2x_1y_{1,2} - 2y_2x_{1,1})y_1 + 2(1/2y_{2,2}x_1 + y_2(x_{1,2} + y_{1,1}))x_1)x_2$$
$$+2x_1(-x_{2,2}y_1^2 + (y_{2,2}x_1 + x_{1,2}y_2)y_1 - 1/2x_1y_2(x_{2,2} + 2y_{1,2})))t^2$$
$$+((y_{1,2}x - 2yy_{1,1})x_2^2 + ((2y_{1,2}x - 2x_{1,2}y)y_1 + (-2y_{2,2}x + 2y_{1,2}y)x_1$$
$$-2y_2((x_{1,2} + y_{1,1})x - 2yx_{1,1}))x_2 + (2xx_{2,2} - 2x_{1,2}y)y_1^2$$
$$+((-4y_{2,2}x + 2y(x_{2,2} + y_{1,2}))x_1 - 2y_2(x_{1,2}x - yx_{1,1}))y_1$$
$$-4x_1y_2((-1/2x_{2,2} - y_{1,2})x + y(x_{1,2} + 1/2y_{1,1})))t + (y_{2,2}x^2 - 2xyy_{1,2}$$
$$+y^2y_{1,1})x_2 + (2y_{2,2}x^2 - 2y(x_{2,2} + y_{1,2})x + 2x_{1,2}y^2)y_1$$
$$+4((-1/4x_{2,2} - 1/2y_{1,2})x^2 + y(x_{1,2} + 1/2y_{1,1})x - 3/4y^2x_{1,1})y_2)s$$
$$+((x_1y_{1,1} - x_{1,1}y_1)x_2^2 + (-2x_1^2y_{1,2} + 2x_1x_{1,2}y_1)x_2 - x_{2,2}x_1^2y_1$$
$$+y_{2,2}x_1^3)t^3 + ((-y_{1,1}x + yx_{1,1})x_2^2 + ((-2x_{1,2}x + 2yx_{1,1})y_1 - 2x_1(-2y_{1,2}x$$
$$+y(x_{1,2} + y_{1,1})))x_2 - 2x_1((-xx_{2,2} + x_{1,2}y)y_1 - 1/2x_1(-3y_{2,2}x$$
$$+y(x_{2,2} + 2y_{1,2}))))t^2 + ((-2y_{1,2}x^2 + 2y(x_{1,2} + y_{1,1})x - 2y^2x_{1,1})x_2$$
$$+(-x^2x_{2,2} + 2xx_{1,2}y - y^2x_{1,1})y_1 + 2x_1(3/2y_{2,2}x^2 - y(x_{2,2} + 2y_{1,2})x$$
$$+y^2(x_{1,2} + 1/2y_{1,1})))t - x^3y_{2,2} + y(x_{2,2} + 2y_{1,2})x^2 - 2y^2(x_{1,2}$$
$$+1/2y_{1,1})x + x_{1,1}y^3\Big)/\Big((x_2 - y_1)^3\Big)$$

$$I_{2c} = \Big(- x_{2,2}^2y_{1,1}y_1^2y_2 + (-y_{1,1}^2x_2y_2^2 + (y_2^3x_{1,1} + x_2y_1^2y_{2,2} + 2y_1y_2(x_2y_{1,2}$$
$$+x_{1,2}y_2))y_{1,1} + (y_1^2y_2y_{2,2} - 4y_1y_2^2y_{1,2})x_{1,1}$$
$$+2y_{1,2}y_1^2(-x_2y_{1,2} + x_{1,2}y_2))x_{2,2} + (x_2x_{1,1}y_2^2y_{2,2} - 4x_2x_{1,2}y_1y_2y_{2,2}$$
$$+2x_2x_{1,2}y_2^2y_{1,2} - 2x_{1,2}^2y_2^3)y_{1,1} - y_{2,2}y_2^3x_{1,1}^2 + (-y_{2,2}^2x_2y_1^2$$

$$
\begin{aligned}
&+2y_1y_2(x_2y_{1,2}+x_{1,2}y_2)y_{2,2}+2x_{1,2}y_{1,2}y_2^3-2y_{1,2}^2x_2y_2^2)x_{1,1}\\
&-2x_{1,2}y_{2,2}y_1^2(-x_2y_{1,2}+x_{1,2}y_2))s^3+(((-2x_1y_1y_2-y_1^3)y_{1,1}x_{2,2}^2\\
&+((-x_1y_2^2-2x_2^2y_2)y_{1,1}^2+(2y_2^2(x_2+1/2y_1)x_{1,1}+2y_1x_1(x_2\\
&+1/2y_1)y_{2,2}+2x_2^2y_1y_{1,2}+2y_2(x_1y_{1,2}+x_{1,2}y_1)x_2+2y_2(x_{1,2}x_1y_2\\
&+y_{1,2}x_1y_1+x_{1,2}y_1^2))y_{1,1}+((2x_1y_1y_2+y_1^3)y_{2,2}-4y_{1,2}y_2(x_1y_2\\
&+x_2y_1+y_1^2))x_{1,1}+4(x_{1,2}x_1y_2+1/2x_{1,2}y_1^2-y_{1,2}x_1x_2\\
&-1/2y_{1,2}x_1y_1)y_1y_{1,2})x_{2,2}+(y_{2,2}y_2(x_1y_2+2x_2^2)x_{1,1}-4x_{1,2}((x_1x_2y_2+x_1y_1y_2\\
&+x_2^2y_1)y_{2,2}+(-y_{1,2}x_2^2+x_{1,2}x_2y_2+1/2y_2(-x_1y_{1,2}+x_{1,2}y_1))y_2))y_{1,1}\\
&-2y_{2,2}y_2^2(x_2+1/2y_1)x_{1,1}^2+(-2y_1x_1(x_2+1/2y_1)y_{2,2}^2+(2x_2^2y_1y_{1,2}\\
&+2y_2(x_1y_{1,2}+x_{1,2}y_1)x_2+2y_2(x_{1,2}x_1y_2+y_{1,2}x_1y_1+x_{1,2}y_1^2))y_{2,2}\\
&+4(-y_{1,2}x_2^2+x_{1,2}x_2y_2+1/2y_2(-x_1y_{1,2}+x_{1,2}y_1))y_2y_{1,2})x_{1,1}-4(x_{1,2}x_1y_2\\
&+1/2x_{1,2}y_1^2-y_{1,2}x_1x_2-1/2y_{1,2}x_1y_1)y_1y_{2,2}x_{1,2})t+2y_1y_{1,1}(xy_2\\
&+1/2yy_1)x_{2,2}^2+((xy_2^2+2x_2yy_2)y_{1,1}^2+(-3y_2^2yx_{1,1}-2y_1x(x_2+1/2y_1)y_{2,2}\\
&-2y_{1,2}(xy_2+yy_1)x_2-2((y_{1,2}x+2x_{1,2}y)y_1+x_{1,2}xy_2)y_2)y_{1,1}\\
&+((-2xy_1y_2-yy_1^2)y_{2,2}+4y_{1,2}y_2(xy_2+2yy_1))x_{1,1}\\
&-4y_1y_{1,2}(-y_{1,2}xx_2+(1/2x_{1,2}y-1/2y_{1,2}x)y_1+x_{1,2}xy_2))x_{2,2}\\
&+(-y_{2,2}y_2(xy_2+2yx_2)x_{1,1}+6x_{1,2}(((2/3xy_2+2/3yy_1)x_2\\
&+2/3xy_1y_2)y_{2,2}+(-2/3y_{1,2}yx_2+y_2(x_{1,2}y-1/3y_{1,2}x))y_2))y_{1,1}\\
&+3y_{2,2}yy_2^2x_{1,1}^2+(2y_1x(x_2+1/2y_1)y_{2,2}^2+(-2y_{1,2}(xy_2+yy_1)x_2\\
&-2((y_{1,2}x+2x_{1,2}y)y_1+x_{1,2}xy_2)y_2)y_{2,2}-6x_{1,2}y_{1,2}yy_2^2\\
&+2y_{1,2}^2xy_2^2+4y_{1,2}^2yx_2y_2)x_{1,1}+4y_1y_{2,2}x_{1,2}(-y_{1,2}xx_2\\
&+(1/2x_{1,2}y-1/2y_{1,2}x)y_1+x_{1,2}xy_2))s^2\\
&+((-y_{1,1}x_1(x_1y_2+2y_1^2)x_{2,2}^2+((-2x_1x_2y_2-x_2^3)y_{1,1}^2\\
&+(x_2y_2(x_2+2y_1)x_{1,1}+x_1^2(x_2+2y_1)y_{2,2}+2x_1x_2^2y_{1,2}\\
&+(2x_{1,2}x_1y_2+2y_{1,2}x_1y_1+2x_{1,2}y_1^2)x_2+2x_1y_2(x_1y_{1,2}\\
&+x_{1,2}y_1))y_{1,1}+((x_1^2y_2+2x_1y_1^2)y_{2,2}-4((x_1y_2+y_1^2)x_2\\
&+x_1y_1y_2)y_{1,2})x_{1,1}+2y_{1,2}x_1(-y_{1,2}x_1x_2+x_{1,2}x_1y_2-2y_{1,2}x_1y_1\\
&+2x_{1,2}y_1^2))x_{2,2}+((2x_1x_2y_2+x_2^3)y_{2,2}x_{1,1}-2x_{1,2}(2x_1(x_1y_2+x_2^2\\
&+x_2y_1)y_{2,2}+(-y_{1,2}x_2^2+x_{1,2}x_2y_2+2y_2(-x_1y_{1,2}\\
&+x_{1,2}y_1))x_2))y_{1,1}-y_{2,2}x_2y_2(x_2+2y_1)x_{1,1}^2+(-x_1^2(x_2+2y_1)y_{2,2}^2\\
&+(2x_1x_2^2y_{1,2}+(2x_{1,2}x_1y_2+2y_{1,2}x_1y_1+2x_{1,2}y_1^2)x_2\\
&+2x_1y_2(x_1y_{1,2}+x_{1,2}y_1))y_{2,2}+2(-y_{1,2}x_2^2+x_{1,2}x_2y_2\\
&+2y_2(-x_1y_{1,2}+x_{1,2}y_1))x_2y_{1,2})x_{1,1}-2x_{1,2}y_{2,2}x_1(-y_{1,2}x_1x_2\\
&+x_{1,2}x_1y_2-2y_{1,2}x_1y_1+2x_{1,2}y_1^2))t^2+(2y_{1,1}(xx_1y_2\\
&+xy_1^2+yx_1y_1)x_{2,2}^2+((2xx_2y_2+2yx_1y_2+2x_2^2y)y_{1,1}^2\\
&+(-4y_2(x_2+1/2y_1)yx_{1,1}-2xx_1(x_2+2y_1)y_{2,2}\\
&-2xx_2^2y_{1,2}+((-2y_{1,2}x-2x_{1,2}y)y_1-2x_{1,2}xy_2-2y_{1,2}yx_1)x_2\\
&-2x_{1,2}yy_1^2+(-2x_{1,2}xy_2-2y_{1,2}yx_1)y_1-4x_1y_2(y_{1,2}x+x_{1,2}y))y_{1,1}\\
&+((-2xx_1y_2-2xy_1^2-2yx_1y_1)y_{2,2}+4((xy_2+yy_1)x_2+xy_1y_2\\
&+2yx_1y_2+yy_1^2)y_{1,2})x_{1,1}-4(-xx_1x_2y_{1,2}+x_{1,2}xy_1^2\\
&+x_1(-2y_{1,2}x+x_{1,2}y)y_1+x_{1,2}xx_1y_2)y_{1,2})x_{2,2}\\
&+(-2y_{2,2}(xx_2y_2+yx_1y_2+x_2^2y)x_{1,1}+8((1/2xx_2^2\\
&+(1/2xy_1+1/2yx_1)x_2+x_1(xy_2+1/2yy_1))y_{2,2}-1/2y_{1,2}yx_2^2
\end{aligned}
$$

$$+y_2(x_{1,2}y - 1/2y_{1,2}x)x_2 + 1/2yy_2(-x_1y_{1,2} + x_{1,2}y_1))x_{1,2})y_{1,1}$$
$$+4y_{2,2}y_2(x_2 + 1/2y_1)yx_{1,1}^2 + (2xx_1(x_2 + 2y_1)y_{2,2}^2 + (-2xx_2^2y_{1,2}$$
$$+((-2y_{1,2}x - 2x_{1,2}y)y_1 - 2x_{1,2}xy_2 - 2y_{1,2}yx_1)x_2 - 2x_{1,2}yy_1^2$$
$$+(-2x_{1,2}xy_2 - 2y_{1,2}yx_1)y_1 - 4x_1y_2(y_{1,2}x + x_{1,2}y))y_{2,2} - 8(-1/2y_{1,2}yx_2^2$$
$$+y_2(x_{1,2}y - 1/2y_{1,2}x)x_2 + 1/2yy_2(-x_1y_{1,2} + x_{1,2}y_1))y_{1,2})x_{1,1}$$
$$+4y_{2,2}x_{1,2}(-xx_1x_2y_{1,2} + x_{1,2}xy_1^2 + x_1(-2y_{1,2}x + x_{1,2}y)y_1$$
$$+x_{1,2}xx_1y_2))t - y_{1,1}x(xy_2 + 2yy_1)x_{2,2}^2 + ((-2xyy_2 - x_2y^2)y_{1,1}^2$$
$$+(3y_2x_{1,1}y^2 + x^2(x_2 + 2y_1)y_{2,2} + 2xx_2yy_{1,2} + (2xyy_{1,2} + 2x_{1,2}y^2)y_1$$
$$+4y_2xx_{1,2}y + 2y_2x^2y_{1,2})y_{1,1} + ((x^2y_2 + 2xyy_1)y_{2,2} - 8y_{1,2}y(xy_2$$
$$+1/2yy_1))x_{1,1} + 2xy_{1,2}(-y_{1,2}xx_2 + (-2y_{1,2}x + 2x_{1,2}y)y_1 + x_{1,2}xy_2))x_{2,2}$$
$$+(2(xy_2 + 1/2yx_2)y_{2,2}yx_{1,1} - 6x_{1,2}(2/3x(xy_2 + yx_2 + yy_1)y_{2,2}$$
$$+(-1/3y_{1,2}yx_2 + y_2(x_{1,2}y - 2/3y_{1,2}x))y))y_{1,1} - 3y_{2,2}y^2y_2x_{1,1}^2$$
$$+(-x^2(x_2 + 2y_1)y_{2,2}^2 + (2xx_2yy_{1,2} + (2xyy_{1,2} + 2x_{1,2}y^2)y_1$$
$$+4y_2xx_{1,2}y + 2y_2x^2y_{1,2})y_{2,2} + 6x_{1,2}y_{1,2}y^2y_2$$
$$-4y_{1,2}^2xyy_2 - 2y_{1,2}^2y^2x_2)x_{1,1} - 2y_{2,2}xx_{1,2}(-y_{1,2}xx_2$$
$$+(-2y_{1,2}x + 2x_{1,2}y)y_1 + x_{1,2}xy_2))s + (-x_{2,2}^2y_{1,1}x_1^2y_1$$
$$+(-y_{1,1}^2x_1x_2^2 + (x_2^2x_{1,1}y_1 + y_{2,2}x_1^3 + 2x_1x_2(x_1y_{1,2} + x_{1,2}y_1))y_{1,1}$$
$$+(x_1^2y_1y_{2,2} - 4x_1x_2y_1y_{1,2})x_{1,1} + 2x_{1,2}y_{1,2}x_1^2y_1$$
$$-2y_{1,2}^2x_1^3)x_{2,2} + (y_{2,2}x_1x_2^2x_{1,1} - 2(2y_{2,2}x_1^2$$
$$+x_2(-x_1y_{1,2} + x_{1,2}y_1))x_{1,2}x_2)y_{1,1} - y_{2,2}x_2^2y_1x_{1,1}^2$$
$$+(-y_{2,2}^2x_1^3 + 2x_1x_2(x_1y_{1,2} + x_{1,2}y_1)y_{2,2}$$
$$+2y_{1,2}x_2^2(-x_1y_{1,2} + x_{1,2}y_1))x_{1,1} - 2x_{1,2}y_{2,2}x_1^2(-x_1y_{1,2}$$
$$+x_{1,2}y_1))t^3 + (2x_1y_{1,1}(xy_1 + 1/2yx_1)x_{2,2}^2 + ((xx_2^2 + 2x_1x_2y)y_{1,1}^2$$
$$+(-yx_2(x_2 + 2y_1)x_{1,1} - 3xx_1^2y_{2,2} + (-2x_{1,2}xy_1 - 2x_1(2y_{1,2}x + x_{1,2}y))x_2$$
$$-2yx_1(x_1y_{1,2} + x_{1,2}y_1))y_{1,1} + ((-2xx_1y_1 - x_1^2y)y_{2,2} + 4((xy_1 + yx_1)x_2$$
$$+yx_1y_1)y_{1,2})x_{1,1} - 4x_1y_{1,2}(x_{1,2}xy_1 + 1/2x_1(-3y_{1,2}x + x_{1,2}y)))x_{2,2}$$
$$+(-y_{2,2}x_2(x_2x + 2yx_1)x_{1,1} + 2((4xx_1x_2 + 2x_1^2y)y_{2,2} + ((-y_{1,2}x + x_{1,2}y)x_2$$
$$+2y(-x_1y_{1,2} + x_{1,2}y_1))x_2)x_{1,2})y_{1,1} + y_{2,2}yx_2(x_2 + 2y_1)x_{1,1}^2$$
$$+(3y_{2,2}^2xx_1^2 + ((-2x_{1,2}xy_1 - 2x_1(2y_{1,2}x + x_{1,2}y))x_2 - 2yx_1(x_1y_{1,2}$$
$$+x_{1,2}y_1))y_{2,2} - 2((-y_{1,2}x + x_{1,2}y)x_2 + 2y(-x_1y_{1,2} + x_{1,2}y_1))x_2y_{1,2})x_{1,1}$$
$$+4y_{2,2}x_1x_{1,2}(x_{1,2}xy_1 + 1/2x_1(-3y_{1,2}x + x_{1,2}y)))t^2 + (-y_{1,1}x(xy_1 + 2yx_1)x_{2,2}^2$$
$$+((-2xx_2y - x_1y^2)y_{1,1}^2 + (2(x_2 + 1/2y_1)y^2x_{1,1} + 3x_1x^2y_{2,2} + (2x^2y_{1,2}$$
$$+2xx_{1,2}y)x_2 + 2(x_{1,2}xy_1 + x_1(2y_{1,2}x + x_{1,2}y))y)y_{1,1} + ((x^2y_1 + 2xx_1y)y_{2,2}$$
$$-4y_{1,2}y(x_2x + xy_1 + yx_1))x_{1,1} + 2y_{1,2}x(-3xx_1y_{1,2} + x_{1,2}xy_1 + 2x_1x_{1,2}y))x_{2,2}$$
$$+(2y_{2,2}(x_2x + 1/2yx_1)yx_{1,1} - 4x_{1,2}((x^2x_2 + 2xx_1y)y_{2,2} + ((-y_{1,2}x + x_{1,2}y)x_2$$
$$+1/2y(-x_1y_{1,2} + x_{1,2}y_1))y))y_{1,1} - 2y_{2,2}(x_2 + 1/2y_1)y^2x_{1,1}^2 + (-3y_{2,2}^2x^2x_1$$
$$+((2x^2y_{1,2} + 2xx_{1,2}y)x_2 + 2(x_{1,2}xy_1 + x_1(2y_{1,2}x + x_{1,2}y))y)y_{2,2}$$
$$+4((-y_{1,2}x + x_{1,2}y)x_2 + 1/2y(-x_1y_{1,2} + x_{1,2}y_1))y_{1,2}y)x_{1,1}$$
$$-2x_{1,2}y_{2,2}x(-3xx_1y_{1,2} + x_{1,2}xy_1 + 2x_1x_{1,2}y))t + x^2x_{2,2}^2yy_{1,1} + (y_{1,1}^2xy^2$$
$$+(-x^3y_{2,2} - 2x^2yy_{1,2} - 2xx_{1,2}y^2 - x_{1,1}y^3)y_{1,1}$$
$$+(-x^2yy_{2,2} + 4xy^2y_{1,2})x_{1,1} - 2x_{1,2}y_{1,2}x^2y + 2y_{1,2}^2x^3)x_{2,2}$$
$$+(4x^2x_{1,2}yy_{2,2} - xx_{1,1}y^2y_{2,2} - 2xx_{1,2}y^2y_{1,2} + 2x_{1,2}^2y^3)y_{1,1}$$

$$
\begin{aligned}
&+x_{1,1}^2 y^3 y_{2,2} + (y_{2,2}^2 x^3 + (-2x^2 y y_{1,2} - 2x x_{1,2} y^2) y_{2,2} \\
&-2x_{1,2} y_{1,2} y^3 + 2y_{1,2}^2 x y^2) x_{1,1} + 2x_{1,2} y_{2,2} x^2 (-y_{1,2} x + x_{1,2} y)) / \big(2(((y_1 y_2 y_{2,2} \\
&-y_2^2 y_{1,2}) x_{1,1} + (y_1^2 y_{1,2} - y_1 y_2 y_{1,1}) x_{2,2} + x_{1,2}(-y_1^2 y_{2,2} \\
&+y_2^2 y_{1,1})) s^2 + ((((x_1 y_2 + x_2 y_1) y_{2,2} - 2y_{1,2} x_2 y_2) x_{1,1} \\
&+((-x_1 y_2 - x_2 y_1) y_{1,1} + 2y_{1,2} x_1 y_1) x_{2,2} + 2x_{1,2}(-x_1 y_1 y_{2,2} \\
&+x_2 y_2 y_{1,1})) t + ((-x y_2 - y y_1) y_{2,2} + 2y_{1,2} y y_2) x_{1,1} \\
&+((x y_2 + y y_1) y_{1,1} - 2y_{1,2} x y_1) x_{2,2} - 2x_{1,2}(-x y_1 y_{2,2} + y y_2 y_{1,1})) s \\
&+((x_1 x_2 y_{2,2} - x_2^2 y_{1,2}) x_{1,1} + (x_1^2 y_{1,2} - x_1 x_2 y_{1,1}) x_{2,2} \\
&+x_{1,2}(-x_1^2 y_{2,2} + x_2^2 y_{1,1})) t^2 + (((-x x_2 - x_1 y) y_{2,2} \\
&+2y_{1,2} y x_2) x_{1,1} + ((x x_2 + x_1 y) y_{1,1} - 2y_{1,2} x x_1) x_{2,2} \\
&-2x_{1,2}(-x x_1 y_{2,2} + x_2 y y_{1,1})) t + (x y y_{2,2} - y^2 y_{1,2}) x_{1,1} \\
&+(x^2 y_{1,2} - x y y_{1,1}) x_{2,2} + x_{1,2}(-x^2 y_{2,2} + y^2 y_{1,1}))(x_2 - y_1)^2\big)
\end{aligned}
$$

$$
\begin{aligned}
I_{2d} = 2\Big(&((1/2 y_{1,1} y_2^2 + 1/2 y_{2,2} y_1^2 - y_{1,2} y_1 y_2) x_2 \\
&+y_2(-1/2 y_2^2 x_{1,1} + x_{1,2} y_1 y_2 - 1/2 x_{2,2} y_1^2)) s^3 + (((-y_1 y_{1,2} + y_2 y_{1,1}) x_2^2 \\
&+((y_{2,2} x_1 + x_{1,2} y_2) y_1 - y_{1,2} x_1 y_2 - y_2^2 x_{1,1}) x_2 - 1/2 x_{2,2} y_1^3 \\
&+(x_{1,2} y_2 + 1/2 y_{2,2} x_1) y_1^2 - y_2((x_{2,2} + y_{1,2}) x_1 + 1/2 y_2 x_{1,1}) y_1 \\
&+x_1 y_2^2 (x_{1,2} + 1/2 y_{1,1})) t + ((-y_{2,2} x + y_{1,2} y) y_1 - y_2(-x y_{1,2} + y y_{1,1})) x_2 \\
&+(-1/2 y_{2,2} x + 1/2 x_{2,2} y) y_1^2 - 2y_2((-1/2 x_{2,2} - 1/2 y_{1,2}) x + x_{1,2} y) y_1 \\
&-((x_{1,2} + 1/2 y_{1,1}) x - 3/2 y x_{1,1}) y_2^2) s^2 + ((1/2 y_{1,1} x_2^3 + (-y_{1,2} x_1 \\
&-1/2 y_2 x_{1,1}) x_2^2 + (x_{1,2} y_1^2 + (-y_{1,2} x_1 - y_2 x_{1,1}) y_1 + x_1(1/2 y_{2,2} x_1 \\
&+y_2(x_{1,2} + y_{1,1}))) x_2 + (-x_{2,2} y_1^2 + (y_{2,2} x_1 + x_{1,2} y_2) y_1 - 1/2 x_1 y_2 (x_{2,2} \\
&+2y_{1,2})) x_1) t^2 + ((x y_{1,2} - y y_{1,1}) x_2^2 + ((x y_{1,2} - x_{1,2} y) y_1 + (-y_{2,2} x + y_{1,2} y) x_1 \\
&-y_2((x_{1,2} + y_{1,1}) x - 2y x_{1,1})) x_2 + (x x_{2,2} - x_{1,2} y) y_1^2 + \\
&((-2y_{2,2} x + y(x_{2,2} + y_{1,2})) x_1 - y_2(x x_{1,2} - y x_{1,1})) y_1 \\
&-2y_2 x_1((-1/2 x_{2,2} - y_{1,2}) x + y(x_{1,2} + 1/2 y_{1,1}))) t \\
&+(-x y y_{1,2} + 1/2 y^2 y_{1,1} + 1/2 x^2 y_{2,2}) x_2 + (x^2 y_{2,2} - y(x_{2,2} + y_{1,2}) x \\
&+x_{1,2} y^2) y_1 + 2y_2((-1/4 x_{2,2} - 1/2 y_{1,2}) x^2 + y(x_{1,2} + 1/2 y_{1,1}) x \\
&-3/4 x_{1,1} y^2)) s + ((1/2 y_{1,1} x_1 - 1/2 y_1 x_{1,1}) x_2^2 + x_1(-y_{1,2} x_1 + x_{1,2} y_1) x_2 \\
&-1/2 x_1^2 (-y_{2,2} x_1 + x_{2,2} y_1)) t^3 + ((-1/2 y_{1,1} x + 1/2 y x_{1,1}) x_2^2 \\
&+((-x x_{1,2} + y x_{1,1}) y_1 - (-2x y_{1,2} + y(x_{1,2} + y_{1,1})) x_1) x_2 \\
&-x_1((-x x_{2,2} + x_{1,2} y) y_1 - 1/2(-3y_{2,2} x + y(x_{2,2} + 2y_{1,2})) x_1)) t^2 \\
&+((-y_{1,2} x^2 + y(x_{1,2} + y_{1,1}) x - x_{1,1} y^2) x_2 + (-1/2 x^2 x_{2,2} \\
&-1/2 x_{1,1} y^2 + x x_{1,2} y) y_1 + (3/2 x^2 y_{2,2} - y(x_{2,2} + 2y_{1,2}) x + y^2 (x_{1,2} \\
&+1/2 y_{1,1})) x_1) t - 1/2 y_{2,2} x^3 + 1/2 y(x_{2,2} + 2y_{1,2}) x^2 \\
&-y^2 (x_{1,2} + 1/2 y_{1,1}) x + 1/2 y^3 x_{1,1}\Big) (x_{1,2}^2 y_{1,1} y_{2,2} - y_{1,2}(y_{2,2} x_{1,1} \\
&+x_{2,2} y_{1,1}) x_{1,2} + x_{2,2} y_{1,2}^2 x_{1,1} + 1/4(-y_{2,2} x_{1,1} + x_{2,2} y_{1,1})^2) \\
&/\Big((x_2 - y_1)\Big(((y_1 y_2 y_{2,2} - y_2^2 y_{1,2}) x_{1,1} + (y_1^2 y_{1,2} - y_1 y_2 y_{1,1}) x_{2,2} \\
&+x_{1,2}(-y_{2,2} y_1^2 + y_{1,1} y_2^2)) s^2 + ((((x_1 y_2 + x_2 y_1) y_{2,2} - 2y_{1,2} x_2 y_2) x_{1,1} \\
&+((-x_1 y_2 - x_2 y_1) y_{1,1} + 2y_{1,2} x_1 y_1) x_{2,2} + 2x_{1,2}(-x_1 y_1 y_{2,2} + x_2 y_2 y_{1,1})) t \\
&+((-x y_2 - y y_1) y_{2,2} + 2y_{1,2} y y_2) x_{1,1} + ((x y_2 + y y_1) y_{1,1} - 2y_{1,2} x y_1) x_{2,2} \\
&-2x_{1,2}(-x y_1 y_{2,2} + y y_2 y_{1,1})) s + ((x_1 x_2 y_{2,2} - x_2^2 y_{1,2})
\end{aligned}
$$

$x_{1,1} + (x_1^2 y_{1,2} - x_1 x_2 y_{1,1}) x_{2,2} + (-x_1^2 y_{2,2} + x_2^2 y_{1,1}) x_{1,2}) t^2$
$+(((-x x_2 - x_1 y) y_{2,2} + 2 y_{1,2} y x_2) x_{1,1} + ((x x_2 + x_1 y) y_{1,1} - 2 y_{1,2} x x_1) x_{2,2}$
$-2 x_{1,2} (-x x_1 y_{2,2} + x_2 y y_{1,1})) t + (x y y_{2,2} - y^2 y_{1,2}) x_{1,1} + (y_{1,2} x^2$
$-x y y_{1,1}) x_{2,2} + x_{1,2}(-x^2 y_{2,2} + y^2 y_{1,1})\Big)^2\Big)$

A.3.3 Jet of Submanifolds: Hypersurfaces

Symplectic manifold $M = \mathbb{R}^4(x, y, z, u)$ with the symplectic form $\omega = dx \wedge dz + dy \wedge du$.
Independent coordinates: x, y, z.
Dependent coordinate: u.
Algebra is generated as $\mathcal{A} = \langle I_{2a}, I_{2b}, I_{2c}, \nabla_1, \nabla_2, \nabla_3 \mid \mathcal{R} \rangle$.
The $\mathcal{R}$ represents the unknown differential syzygies.

Differential Invariants
$I_{2a} = \gamma^3 (u_1^2 u_{3,3} - 2 u_1 u_3 u_{1,3} + u3^2 u_{1,1} + 2 u_1 u_{2,3} - 2 u_3 u_{1,2} + u_{2,2})$
$I_{2b} = \gamma^7 (y^2 u_1^4 u_{2,2} u_{3,3}^2 - y^2 u_1^4 u_{2,3}^2 u_{3,3} - 2 y^2 u_1^3 u_2 u_{1,2} u_{3,3}^2 + 2 y^2 u_1^3 u_2 u_{1,3} u_{2,3} u_{3,3}$ $+2 y^2 u_1^3 u_3 u_{1,2} u_{2,3} u_{3,3} - 4 y^2 u_1^3 u_3 u_{1,3} u_{2,2} u_{3,3} + 2 y^2 u_1^3 u_3 u_{1,3} u_{2,3}^2$ $+y^2 u_1^2 u_2^2 u_{1,1} u_{3,3}^2 - y^2 u_1^2 u_2^2 u_{1,3}^2 u_{3,3} - 2 y^2 u_1^2 u_2 u_3 u_{1,1} u_{2,3} u_{3,3}$ $+6 y^2 u_1^2 u_2 u_3 u_{1,2} u_{1,3} u_{3,3} - 4 y^2 u_1^2 u_2 u_3 u_{1,3}^2 u_{2,3} + 2 y^2 u_1^2 u_3^2 u_{1,1} u_{2,2} u_{3,3}$ $-y^2 u_1^2 u_3^2 u_{1,1} u_{2,3}^2 - y^2 u_1^2 u_3^2 u_{1,2}^2 u_{3,3} - 4 y^2 u_1^2 u_3^2 u_{1,2} u_{1,3} u_{2,3} + 4 y^2 u_1^2 u_3^2 u_{1,3}^2 u_{2,2}$ $-2 y^2 u_1 u_2^2 u_3 u_{1,1} u_{1,3} u_{3,3} + 2 y^2 u_1 u_2^2 u_3 u_{1,3}^3 - 2 y^2 u_1 u_2 u_3^2 u_{1,1} u_{1,2} u_{3,3}$ $+6 y^2 u_1 u_2 u_3^2 u_{1,1} u_{1,3} u_{2,3} - 4 y^2 u_1 u_2 u_3^2 u_{1,2} u_{1,3}^2 + 2 y^2 u_1 u_3^3 u_{1,1} u_{1,2} u_{2,3}$ $-4 y^2 u_1 u_3^3 u_{1,1} u_{1,3} u_{2,2} + 2 y^2 u_1 u_3^3 u_{1,2}^2 u_{1,3} + y^2 u_2^2 u_3^2 u_{1,1}^2 u_{3,3} - y^2 u_2^2 u_3^2 u_{1,1} u_{1,3}^2$ $-2 y^2 u_2 u_3^3 u_{1,1}^2 u_{2,3} + 2 y^2 u_2 u_3^3 u_{1,1} u_{1,2} u_{1,3} + y^2 u_3^4 u_{1,1}^2 u_{2,2} - y^2 u_3^4 u_{1,1} u_{1,2}^2$ $+2 x y u_1^3 u_{1,2} u_{2,3} u_{3,3} - 2 x y u_1^3 u_{1,3} u_{2,2} u_{3,3} - 2 x y u_1^2 u_2 u_{1,1} u_{2,3} u_{3,3}$ $+2 x y u_1^2 u_2 u_{1,2} u_{1,3} u_{3,3} + 2 x y u_1^2 u_3 u_{1,1} u_{2,2} u_{3,3} - 2 x y u_1^2 u_3 u_{1,2}^2 u_{3,3}$ $-4 x y u_1^2 u_3 u_{1,2} u_{1,3} u_{2,3} + 4 x y u_1^2 u_3 u_{1,3}^2 u_{2,2} + 4 x y u_1 u_2 u_3 u_{1,1} u_{1,3} u_{2,3}$ $-4 x y u_1 u_2 u_3 u_{1,2} u_{1,3}^2 + 2 x y u_1 u_3^2 u_{1,1} u_{1,2} u_{2,3} - 6 x y u_1 u_3^2 u_{1,1} u_{1,3} u_{2,2}$ $+4 x y u_1 u_3^2 u_{1,2}^2 u_{1,3} - 2 x y u_2 u_3^2 u_{1,1}^2 u_{2,3} + 2 x y u_2 u_3^2 u_{1,1} u_{1,2} u_{1,3}$ $+2 x y u_3^3 u_{1,1}^2 u_{2,2} - 2 x y u_3^3 u_{1,1} u_{1,2}^2 + 2 y^2 u_1^3 u_{2,2} u_{2,3} u_{3,3} - 2 y^2 u_1^3 u_{2,3}^3$ $-4 y^2 u_1^2 u_2 u_{1,2} u_{2,3} u_{3,3} + 4 y^2 u_1^2 u_2 u_{1,3} u_{2,3}^2 - 2 y^2 u_1^2 u_3 u_{1,2} u_{2,2} u_{3,3}$ $+6 y^2 u_1^2 u_3 u_{1,2} u_{2,3}^2 - 4 y^2 u_1^2 u_3 u_{1,3} u_{2,2} u_{2,3} + 2 y^2 u_1 u_2^2 u_{1,1} u_{2,3} u_{3,3}$ $-2 y^2 u_1 u_2^2 u_{1,3}^2 u_{2,3} - 4 y^2 u_1 u_2 u_3 u_{1,1} u_{2,3}^2 + 4 y^2 u_1 u_2 u_3 u_{1,2}^2 u_{3,3}$ $+2 y^2 u_1 u_3^2 u_{1,1} u_{2,2} u_{2,3} - 6 y^2 u_1 u_3^2 u_{1,2}^2 u_{2,3} + 4 y^2 u_1 u_3^2 u_{1,2} u_{1,3} u_{2,2}$ $-2 y^2 u_2^2 u_3 u_{1,1} u_{1,2} u_{3,3} + 2 y^2 u_2^2 u_3 u_{1,2} u_{1,3}^2 + 4 y^2 u_2 u_3^2 u_{1,1} u_{1,2} u_{2,3}$ $-4 y^2 u_2 u_3^2 u_{1,2}^2 u_{1,3} - 2 y^2 u_3^3 u_{1,1} u_{1,2} u_{2,2} + 2 y^2 u_3^3 u_{1,2}^3 - 2 y z u_1^3 u_{2,2} u_{3,3}^2$ $+2 y z u_1^3 u_{2,3}^2 u_{3,3} + 2 y z u_1^2 u_2 u_{1,2} u_{3,3}^2 - 2 y z u_1^2 u_2 u_{1,3} u_{2,3} u_{3,3}$

$$
\begin{aligned}
&-2yzu_1^2u_3u_{1,2}u_{2,3}u_{3,3} + 6yzu_1^2u_3u_{1,3}u_{2,2}u_{3,3} - 4yzu_1^2u_3u_{1,3}u_{2,3}^2\\
&-4yzu_1u_2u_3u_{1,2}u_{1,3}u_{3,3} + 4yzu_1u_2u_3u_{1,3}^2u_{2,3} - 2yzu_1u_3^2u_{1,1}u_{2,2}u_{3,3}\\
&+2yzu_1u_3^2u_{1,1}u_{2,3}^2 + 4yzu_1u_3^2u_{1,2}u_{1,3}u_{2,3} - 4yzu_1u_3^2u_{1,3}^2u_{2,2}\\
&+2yzu_2u_3^2u_{1,1}u_{1,2}u_{3,3} - 2yzu_2u_3^2u_{1,1}u_{1,3}u_{2,3} - 2yzu_3^3u_{1,1}u_{1,2}u_{2,3}\\
&+2yzu_3^3u_{1,1}u_{1,3}u_{2,2} + 2yuu_1^3u_{1,2}u_{3,3}^2 - 2yuu_1^3u_{1,3}u_{2,3}u_{3,3}\\
&-2yuu_1^2u_2u_{1,1}u_{3,3}^2 + 2yuu_1^2u_2u_{1,3}^2u_{3,3} + 2yuu_1^2u_3u_{1,1}u_{2,3}u_{3,3}\\
&-6yuu_1^2u_3u_{1,2}u_{1,3}u_{3,3} + 4yuu_1^2u_3u_{1,3}^2u_{2,3} + 4yuu_1u_2u_3u_{1,1}u_{1,3}u_{3,3}\\
&-4yuu_1u_2u_3u_{1,3}^3 + 2yuu_1u_3^2u_{1,1}u_{1,2}u_{3,3} - 6yuu_1u_3^2u_{1,1}u_{1,3}u_{2,3}\\
&+4yuu_1u_3^2u_{1,2}u_{1,3}^2 - 2yuu_2u_3^2u_{1,1}^2u_{3,3} + 2yuu_2u_3^2u_{1,1}u_{1,3}^2\\
&+2yuu_3^3u_{1,1}^2u_{2,3} - 2yuu_3^3u_{1,1}u_{1,2}u_{1,3} + x^2u_1^2u_{1,1}u_{2,3}^2\\
&-2x^2u_1^2u_{1,2}u_{1,3}u_{2,3} + x^2u_1^2u_{1,3}^2u_{2,2} - 2x^2u_1u_3u_{1,1}u_{1,3}u_{2,2}\\
&+2x^2u_1u_3u_{1,2}^2u_{1,3} + x^2u_3^2u_{1,1}^2u_{2,2} - x^2u_3^2u_{1,1}u_{1,2}^2 + 4xyu_1^2u_{1,2}u_{2,3}^2\\
&-4xyu_1^2u_{1,3}u_{2,2}u_{2,3} - 2xyu_1u_2u_{1,1}u_{2,2}u_{3,3} - 2xyu_1u_2u_{1,1}u_{2,3}^2\\
&+2xyu_1u_2u_{1,2}^2u_{3,3} + 2xyu_1u_2u_{1,3}^2u_{2,2} + 4xyu_1u_3u_{1,1}u_{2,2}u_{2,3}\\
&-8xyu_1u_3u_{1,2}^2u_{2,3} + 4xyu_1u_3u_{1,2}u_{1,3}u_{2,2} + 4xyu_2u_3u_{1,1}u_{1,2}u_{2,3}\\
&-4xyu_2u_3u_{1,2}^2u_{1,3} - 4xyu_3^2u_{1,1}u_{1,2}u_{2,2} + 4xyu_3^2u_{1,2}^3\\
&-2xzu_1^2u_{1,2}u_{2,3}u_{3,3} + 2xzu_1^2u_{1,3}u_{2,2}u_{3,3} - 2xzu_1u_3u_{1,1}u_{2,2}u_{3,3}\\
&+2xzu_1u_3u_{1,1}u_{2,3}^2 + 2xzu_1u_3u_{1,2}^2u_{3,3} - 2xzu_1u_3u_{1,3}^2u_{2,2}\\
&-2xzu_3^2u_{1,1}u_{1,2}u_{2,3} + 2xzu_3^2u_{1,1}u_{1,3}u_{2,2} + 2xuu_1^2u_{1,1}u_{2,3}u_{3,3}\\
&-2xuu_1^2u_{1,2}u_{1,3}u_{3,3} - 4xuu_1u_3u_{1,1}u_{1,3}u_{2,3} + 4xuu_1u_3u_{1,2}u_{1,3}^2\\
&+2xuu_3^2u_{1,1}^2u_{2,3} - 2xuu_3^2u_{1,1}u_{1,2}u_{1,3} + y^2u_1^2u_{2,2}^2u_{3,3}\\
&-y^2u_1^2u_{2,2}u_{2,3}^2 - 2y^2u_1u_2u_{1,2}u_{2,2}u_{3,3} + 2y^2u_1u_2u_{1,3}u_{2,2}u_{2,3}\\
&+2y^2u_1u_3u_{1,2}u_{2,2}u_{2,3} - 2y^2u_1u_3u_{1,3}u_{2,2}^2 + y^2u_2^2u_{1,1}u_{2,3}^2\\
&+y^2u_2^2u_{1,2}^2u_{3,3} - 2y^2u_2^2u_{1,2}u_{1,3}u_{2,3} - 2y^2u_2u_3u_{1,1}u_{2,2}u_{2,3}\\
&+2y^2u_2u_3u_{1,2}u_{1,3}u_{2,2} + y^2u_3^2u_{1,1}u_{2,2}^2 - y^2u_3^2u_{1,2}^2u_{2,2}\\
&-4yzu_1^2u_{2,2}u_{2,3}u_{3,3} + 4yzu_1^2u_{2,3}^3 + 4yzu_1u_2u_{1,2}u_{2,3}u_{3,3}\\
&-4yzu_1u_2u_{1,3}u_{2,3}^2 + 4yzu_1u_3u_{1,2}u_{2,2}u_{3,3} - 8yzu_1u_3u_{1,2}u_{2,3}^2\\
&+4yzu_1u_3u_{1,3}u_{2,2}u_{2,3} - 2yzu_2u_3u_{1,1}u_{2,2}u_{3,3} + 2yzu_2u_3u_{1,1}u_{2,3}^2\\
&-2yzu_2u_3u_{1,2}^2u_{3,3} + 2yzu_2u_3u_{1,3}^2u_{2,2} + 4yzu_3^2u_{1,2}^2u_{2,3}\\
&-4yzu_3^2u_{1,2}u_{1,3}u_{2,2} + 4yuu_1^2u_{1,2}u_{2,3}u_{3,3} - 4yuu_1^2u_{1,3}u_{2,3}^2\\
&-4yuu_1u_2u_{1,1}u_{2,3}u_{3,3} + 4yuu_1u_2u_{1,3}^2u_{2,3} + 4yuu_1u_3u_{1,1}u_{2,3}^2\\
&-4yuu_1u_3u_{1,2}^2u_{3,3} + 4yuu_2u_3u_{1,1}u_{1,2}u_{3,3} - 4yuu_2u_3u_{1,2}u_{1,3}^2\\
&-4yuu_3^2u_{1,1}u_{1,2}u_{2,3} + 4yuu_3^2u_{1,2}^2u_{1,3} + z^2u_1^2u_{2,2}u_{3,3}^2\\
&-z^2u_1^2u_{2,3}^2u_{3,3} - 2z^2u_1u_3u_{1,3}u_{2,2}u_{3,3} + 2z^2u_1u_3u_{1,3}u_{2,3}^2\\
&+z^2u_3^2u_{1,2}^2u_{3,3} - 2z^2u_3^2u_{1,2}u_{1,3}u_{2,3} + z^2u_3^2u_{1,3}^2u_{2,2}\\
&-2zuu_1^2u_{1,2}u_{3,3}^2 + 2zuu_1^2u_{1,3}u_{2,3}u_{3,3} + 4zuu_1u_3u_{1,2}u_{1,3}u_{3,3}\\
&-4zuu_1u_3u_{1,3}^2u_{2,3} - 2zuu_3^2u_{1,1}u_{1,2}u_{3,3} + 2zuu_3^2u_{1,1}u_{1,3}u_{2,3}\\
&+u^2u_1^2u_{1,1}u_{3,3}^2 - u^2u_1^2u_{1,3}^2u_{3,3} - 2u^2u_1u_3u_{1,1}u_{1,3}u_{3,3}\\
&+2u^2u_1u_3u_{1,3}^3 + u^2u_3^2u_{1,1}^2u_{3,3} - u^2u_3^2u_{1,1}u_{1,3}^2\\
&+2x^2u_1u_{1,1}u_{2,2}u_{2,3} - 2x^2u_1u_{1,2}^2u_{2,3} - 2x^2u_3u_{1,1}u_{1,2}u_{2,2}\\
&+2x^2u_3u_{1,2}^3 + 2xyu_1u_{1,2}u_{2,2}u_{2,3} - 2xyu_1u_{1,3}u_{2,2}^2
\end{aligned}
$$

$$
\begin{aligned}
& -2xyu_2u_{1,1}u_{2,2}u_{2,3} + 2xyu_2u_{1,2}u_{1,3}u_{2,2} + 2xyu_3u_{1,1}u_{2,2}^2 \\
& -2xyu_3u_{1,2}^2u_{2,2} - 4xzu_1u_{1,2}u_{2,3}^2 + 4xzu_1u_{1,3}u_{2,2}u_{2,3} \\
& +4xzu_3u_{1,2}^2u_{2,3} - 4xzu_3u_{1,2}u_{1,3}u_{2,2} + 2xuu_1u_{1,1}u_{2,2}u_{3,3} \\
& +2xuu_1u_{1,1}u_{2,3}^2 - 2xuu_1u_{1,2}^2u_{3,3} - 2xuu_1u_{1,3}^2u_{2,2} \\
& -4xuu_3u_{1,1}u_{1,2}u_{2,3} + 4xuu_3u_{1,2}^2u_{1,3} - 2yzu_1u_{2,2}^2u_{3,3} \\
& +2yzu_1u_{2,2}u_{2,3}^2 + 2yzu_2u_{1,2}u_{2,2}u_{3,3} - 2yzu_2u_{1,3}u_{2,2}u_{2,3} \\
& -2yzu_3u_{1,2}u_{2,2}u_{2,3} + 2yzu_3u_{1,3}u_{2,2}^2 + 2yuu_1u_{1,2}u_{2,2}u_{3,3} \\
& -2yuu_1u_{1,3}u_{2,2}u_{2,3} - 2yuu_2u_{1,1}u_{2,3}^2 - 2yuu_2u_{1,2}^2u_{3,3} \\
& +4yuu_2u_{1,2}u_{1,3}u_{2,3} + 2yuu_3u_{1,1}u_{2,2}u_{2,3} - 2yuu_3u_{1,2}u_{1,3}u_{2,2} \\
& +2z^2u_1u_{2,2}u_{2,3}u_{3,3} - 2z^2u_1u_{2,3}^3 - 2z^2u_3u_{1,2}u_{2,2}u_{3,3} \\
& +2z^2u_3u_{1,2}u_{2,3}^2 - 4zuu_1u_{1,2}u_{2,3}u_{3,3} + 4zuu_1u_{1,3}u_{2,3}^2 \\
& +2zuu_3u_{1,1}u_{2,2}u_{3,3} - 2zuu_3u_{1,1}u_{2,3}^2 + 2zuu_3u_{1,2}^2u_{3,3} \\
& -2zuu_3u_{1,3}^2u_{2,2} + 2u^2u_1u_{1,1}u_{2,3}u_{3,3} - 2u^2u_1u_{1,3}^2u_{2,3} \\
& -2u^2u_3u_{1,1}u_{1,2}u_{3,3} + 2u^2u_3u_{1,2}u_{1,3}^2 + x^2u_{1,1}u_{2,2}^2 \\
& -x^2u_{1,2}^2u_{2,2} - 2xzu_{1,2}u_{2,2}u_{2,3} + 2xzu_{1,3}u_{2,2}^2 + 2xuu_{1,1}u_{2,2}u_{2,3} \\
& -2xuu_{1,2}u_{1,3}u_{2,2} + z^2u_{2,2}^2u_{3,3} - z^2u_{2,2}u_{2,3}^2 \\
& -2zuu_{1,2}u_{2,2}u_{3,3} + 2zuu_{1,3}u_{2,2}u_{2,3} + u^2u_{1,1}u_{2,3}^2 \\
& +u^2u_{1,2}^2u_{3,3} - 2u^2u_{1,2}u_{1,3}u_{2,3} \\
I_{2c} &= \gamma^4(y^2u_1^2u_{2,2}u_{3,3} - y^2u_1^2u_{2,3}^2 - 2y^2u_1u_2u_{1,2}u_{3,3} + 2y^2u_1u_2u_{1,3}u_{2,3} \\
& +2y^2u_1u_3u_{1,2}u_{2,3} - 2y^2u_1u_3u_{1,3}u_{2,2} + y^2u_2^2u_{1,1}u_{3,3} \\
& -y^2u_2^2u_{1,3}^2 - 2y^2u_2u_3u_{1,1}u_{2,3} + 2y^2u_2u_3u_{1,2}u_{1,3} + y^2u_3^2u_{1,1}u_{2,2} \\
& -y^2u_3^2u_{1,2}^2 + 2xyu_1u_{1,2}u_{2,3} - 2xyu_1u_{1,3}u_{2,2} - 2xyu_2u_{1,1}u_{2,3} \\
& +2xyu_2u_{1,2}u_{1,3} + 2xyu_3u_{1,1}u_{2,2} - 2xyu_3u_{1,2}^2 - 2yzu_1u_{2,2}u_{3,3} \\
& +2yzu_1u_{2,3}^2 + 2yzu_2u_{1,2}u_{3,3} - 2yzu_2u_{1,3}u_{2,3} - 2yzu_3u_{1,2}u_{2,3} \\
& +2yzu_3u_{1,3}u_{2,2} + 2yuu_1u_{1,2}u_{3,3} - 2yuu_1u_{1,3}u_{2,3} - 2yuu_2u_{1,1}u_{3,3} \\
& +2yuu_2u_{1,3}^2 + 2yuu_3u_{1,1}u_{2,3} - 2yuu_3u_{1,2}u_{1,3} + x^2u_{1,1}u_{2,2} - u_{1,2}^2x^2 \\
& -2xzu_{1,2}u_{2,3} + 2xzu_{1,3}u_{2,2} + 2xuu_{1,1}u_{2,3} - 2u_{1,3}xuu_{1,2} + z^2u_{2,2}u_{3,3} \\
& -z^2u_{2,3}^2 - 2zuu_{1,2}u_{3,3} + 2zuu_{1,3}u_{2,3} + u^2u_{1,1}u_{3,3} - u_{1,3}^2u^2)
\end{aligned}
$$

with $\gamma = 1/(xu_1 + yu_2 + zu_3 - u)^3$.

A.3.4 Jet of Functions

Symplectic manifold $M = \mathbb{R}^4(x^1, x^2, y^1, y^2)$ with the symplectic form $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$.
Independent coordinates: x^1, x^2, y^1, y^2.
Dependent coordinate: u.
Algebra is generated as $\mathcal{A} = \langle I_0, I_{2e}, I_{2f}, I_{2g}, \nabla_1, \nabla_2, \nabla_3, \nabla_4 \mid \nabla_2(I_0) = \nabla_4(I_0) =$

$0, \mathcal{R}\rangle$.

The formulas are summarized below, with the notation $x^1 = x1,\ x^2 = x2,\ y^1 = y1,\ y^2 = y2$.

Differential Invariants
$I_0 = u$
$I_1 = x1u_1 + x2u_2 + y1u_3 + y2u_4$
$\begin{aligned} I_{2a} = {} & x1^2u_{1,1} + (2x2u_{1,2} + 2y1u_{1,3} + 2y2u_{1,4} + u_1)x1 + x2^2u_{2,2} \\ & +y1^2u_{3,3} + \left(2y2u_{3,4} + u_3\right) y1 + y2 \left(y2u_{4,4} + u_4\right) \\ & +(2y1u_{2,3} + 2y2u_{2,4} + u_2)x2 \end{aligned}$
$\begin{aligned} I_{2b} = {} & (x1u_{1,3} + x2u_{2,3} + y1u_{3,3} + y2u_{3,4})u_1 \\ & +(x1u_{1,4} + x2u_{2,4} + y1u_{3,4} + y2u_{4,4})u_2 \\ & -(x1u_{1,1} + x2u_{1,2} + y1u_{1,3} + y2u_{1,4})u_3 \\ & -(x1u_{1,2} + x2u_{2,2} + y1u_{2,3} + y2u_{2,4})u_4 \end{aligned}$
$\begin{aligned} I_{2c} = {} & u_1^2u_{3,3} + (2u_{3,4}u_2 - 2u_{1,3}u_3 - 2u_{2,3}u_4 + (x1u_{3,4} + x2u_{3,3})u_{1,2} \\ & +(y2u_{3,3} - x1u_{2,3})u_{1,4} + ((-u_{1,3} - u_{2,4})x2 - y2u_{4,4})u_{2,3} \\ & +(x2u_{2,2} - y2(u_{1,3} - u_{2,4}))u_{3,4} - x1(-u_{1,1}u_{3,3} + u_{1,3}^2))u_1 + u_2^2u_{4,4} \\ & +(-2u_{1,4}u_3 - 2u_{2,4}u_4 + (x1u_{4,4} + x2u_{3,4})u_{1,2} \\ & +(-x2u_{2,3} + (-u_{1,3} - u_{2,4})x1 - y1u_{3,3})u_{1,4} \\ & +u_{4,4}y1u_{2,3} + (x1u_{1,1} + y1(u_{1,3} - u_{2,4}))u_{3,4} + x2(u_{2,2}u_{4,4} - u_{2,4}^2))u_2 \\ & +u_3^2u_{1,1} + (2u_{1,2}u_4 + (y1u_{3,4} + (-u_{1,3} + u_{2,4})x2 + y2u_{4,4})u_{1,2} + y2u_{1,1}u_{3,4} \\ & +(-y1u_{2,3} - x2u_{2,2} - y2(u_{1,3} + u_{2,4}))u_{1,4} + x2u_{1,1}u_{2,3} - y1(u_{1,3}^2 - u_{1,1}u_{3,3}))u_3 \\ & +(u_{2,2}u_4 + (y2u_{3,4} + (u_{1,3} - u_{2,4})x1 + y1u_{3,3})u_{1,2} + (x1\ u_{2,2} - y2u_{2,3})u_{1,4} \\ & + \left(-x1u_{1,1} - y1(u_{1,3} + u_{2,4})\right) u_{2,3} + u_{2,2}u_{3,4}y1 + y2(u_{2,2}u_{4,4} - u_{2,4}^2))u_4 \end{aligned}$
$\begin{aligned} I_{2d} = {} & x1u_1u_{1,1}u_{1,3}u_{3,3} + 2x1u_1u_{1,2}u_{1,3}u_{3,4} + x1u_1u_{1,2}u_{2,3}u_{4,4} - x1u_1u_{1,2}u_{2,4}u_{3,4} \\ & -x1u_1u_{1,3}^3 - 2x1u_1u_{1,3}u_{1,4}u_{2,3} + x1u_1u_{1,4}u_{2,2}u_{3,4} - x1u_1u_{1,4}u_{2,3}u_{2,4} \\ & +x1u_2u_{1,1}u_{1,4}u_{3,3} - x1u_2u_{1,1}u_{2,3}u_{4,4} + x1u_2u_{1,1}u_{2,4}u_{3,4} + x1u_2u_{1,2}u_{1,3}u_{4,4} \\ & +x1u_2u_{1,2}u_{1,4}u_{3,4} - x1u_2u_{1,3}^2u_{1,4} - x1u_2u_{1,3}u_{1,4}u_{2,4} - x1u_2u_{1,4}^2u_{2,3} \\ & +x1u_2u_{1,4}u_{2,2}u_{4,4} - x1u_2u_{1,4}u_{2,4}^2 - x1u_3u_{1,1}^2u_{3,3} - 2x1u_3u_{1,1}u_{1,2}u_{3,4} \\ & +x1u_3u_{1,1}u_{1,3}^2 + 2x1u_3u_{1,1}u_{1,4}u_{2,3} - x1u_3u_{1,2}^2u_{4,4} + 2x1u_3u_{1,2}u_{1,4}u_{2,4} \\ & -x1u_3u_{1,4}^2u_{2,2} - x1u_4u_{1,1}u_{1,2}u_{3,3} - x1u_4u_{1,1}u_{2,2}u_{3,4} + x1u_4u_{1,1}u_{2,3}u_{2,4} \\ & -x1u_4u_{1,2}^2u_{3,4} + x1u_4u_{1,2}u_{1,3}^2 - x1u_4u_{1,2}u_{1,3}u_{2,4} + x1u_4u_{1,2}u_{1,4}u_{2,3} \\ & -x1u_4u_{1,2}u_{2,2}u_{4,4} + x1u_4u_{1,2}u_{2,4}^2 + x1u_4u_{1,3}u_{1,4}u_{2,2} + x2u_1u_{1,1}u_{2,3}u_{3,3} \\ & +x2u_1u_{1,2}u_{2,3}u_{3,4} + x2u_1u_{1,2}u_{2,4}u_{3,3} - x2u_1u_{1,3}^2u_{2,3} + x2u_1u_{1,3}u_{2,2}u_{3,4} \\ & -x2u_1u_{1,3}u_{2,3}u_{2,4} - x2u_1u_{1,4}u_{2,2}u_{3,3} - x2u_1u_{1,4}u_{2,3}^2 + x2u_1u_{2,2}u_{2,3}u_{4,4} \\ & -x2u_1u_{2,3}u_{2,4}^2 + x2u_2u_{1,1}u_{2,3}u_{3,4} - x2u_2u_{1,2}u_{1,3}u_{3,4} + x2u_2u_{1,2}u_{1,4}u_{3,3} \\ & +2x2u_2u_{1,2}u_{2,4}u_{3,4} - x2u_2u_{1,3}u_{1,4}u_{2,3} - 2x2u_2u_{1,4}u_{2,3}u_{2,4} + x2u_2u_{2,2}u_{2,4}u_{4,4} \\ & -x2u_2u_{2,4}^3 - x2u_3u_{1,1}u_{1,2}u_{3,3} - x2u_3u_{1,1}u_{2,2}u_{3,4} + x2u_3u_{1,1}u_{2,3}u_{2,4} \\ & -x2u_3u_{1,2}^2u_{3,4} + x2u_3u_{1,2}u_{1,3}^2 - x2u_3u_{1,2}u_{1,3}u_{2,4} + x2u_3u_{1,2}u_{1,4}u_{2,3} \\ & -x2u_3u_{1,2}u_{2,2}u_{4,4} + x2u_3u_{1,2}u_{2,4}^2 + x2u_3u_{1,3}u_{1,4}u_{2,2} - x2u_4u_{1,1}u_{2,3}^2 \end{aligned}$

$$
\begin{array}{l}
-x2u_4u_{1,2}{}^2u_{3,3} + 2\,x2\,u_4u_{1,2}u_{1,3}u_{2,3} - 2\,x2\,u_4u_{1,2}u_{2,2}u_{3,4} + 2\,x2\,u_4u_{1,4}u_{2,2}u_{2,3} \\
-x2\,u_4u_{2,2}{}^2u_{4,4} + x2\,u_4u_{2,2}u_{2,4}{}^2 + y1\,u_1u_{1,1}u_{3,3}{}^2 + 2\,y1\,u_1u_{1,2}u_{3,3}u_{3,4} \\
-y1\,u_1u_{1,3}{}^2u_{3,3} - 2\,y1\,u_1u_{1,4}u_{2,3}u_{3,3} + y1\,u_1u_{2,2}u_{3,4}{}^2 + y1\,u_1u_{2,3}{}^2u_{4,4} \\
-2\,y1\,u_1u_{2,3}u_{2,4}u_{3,4} + y1u_2u_{1,1}u_{3,3}u_{3,4} + y1\,u_2u_{1,2}u_{3,3}u_{4,4} + y1\,u_2u_{1,2}u_{3,4}{}^2 \\
-y1\,u_2u_{1,3}{}^2u_{3,4} - y1\,u_2u_{1,3}u_{2,3}u_{4,4} + y1\,u_2u_{1,3}u_{2,4}u_{3,4} - y1\,u_2u_{1,4}u_{2,3}u_{3,4} \\
-y1\,u_2u_{1,4}u_{2,4}u_{3,3} + y1\,u_2u_{2,2}u_{3,4}u_{4,4} - y1\,u_2u_{2,4}{}^2u_{3,4} - y1\,u_3u_{1,1}u_{1,3}u_{3,3} \\
-2\,y1\,u_3u_{1,2}u_{1,3}u_{3,4} - y1\,u_3u_{1,2}u_{2,3}u_{4,4} + y1\,u_3u_{1,2}u_{2,4}u_{3,4} + y1\,u_3u_{1,3}{}^3 \\
+2\,y1\,u_3u_{1,3}u_{1,4}u_{2,3} - y1\,u_3u_{1,4}u_{2,2}u_{3,4} + y1\,u_3u_{1,4}u_{2,3}u_{2,4} - y1\,u_4u_{1,1}u_{2,3}u_{3,3} \\
-y1\,u_4u_{1,2}u_{2,3}u_{3,4} - y1\,u_4u_{1,2}u_{2,4}u_{3,3} + y1\,u_4u_{1,3}{}^2u_{2,3} - y1\,u_4u_{1,3}u_{2,2}u_{3,4} \\
+y1\,u_4u_{1,3}u_{2,3}u_{2,4} + y1\,u_4u_{1,4}u_{2,2}u_{3,3} + y1\,u_4u_{1,4}u_{2,3}{}^2 - y1\,u_4u_{2,2}u_{2,3}u_{4,4} \\
+y1\,u_4u_{2,3}u_{2,4}{}^2 + y2\,u_1u_{1,1}u_{3,3}u_{3,4} + y2\,u_1u_{1,2}u_{3,3}u_{4,4} + y2\,u_1u_{1,2}u_{3,4}{}^2 \\
-y2\,u_1u_{1,3}{}^2u_{3,4} - y2\,u_1u_{1,3}u_{2,3}u_{4,4} + y2\,u_1u_{1,3}u_{2,4}u_{3,4} - y2\,u_1u_{1,4}u_{2,3}u_{3,4} \\
-y2\,u_1u_{1,4}u_{2,4}u_{3,3} + y2\,u_1u_{2,2}u_{3,4}u_{4,4} - y2\,u_1u_{2,4}{}^2u_{3,4} + y2\,u_2u_{1,1}u_{3,4}{}^2 \\
+2\,y2\,u_2u_{1,2}u_{3,4}u_{4,4} - 2\,y2\,u_2u_{1,3}u_{1,4}u_{3,4} + y2\,u_2u_{1,4}{}^2u_{3,3} - 2\,y2\,u_2u_{1,4}u_{2,3}u_{4,4} \\
+y2\,u_2u_{2,2}u_{4,4}{}^2 - y2\,u_2u_{2,4}{}^2u_{4,4} - y2\,u_3u_{1,1}u_{1,4}u_{3,3} + y2\,u_3u_{1,1}u_{2,3}u_{4,4} \\
-y2\,u_3u_{1,1}u_{2,4}u_{3,4} - y2\,u_3u_{1,2}u_{1,3}u_{4,4} - y2\,u_3u_{1,2}u_{1,4}u_{3,4} + y2\,u_3u_{1,3}{}^2u_{1,4} \\
+y2\,u_3u_{1,3}u_{1,4}u_{2,4} + y2\,u_3u_{1,4}{}^2u_{2,3} - y2\,u_3u_{1,4}u_{2,2}u_{4,4} + y2\,u_3u_{1,4}u_{2,4}{}^2 \\
-y2\,u_4u_{1,1}u_{2,3}u_{3,4} + y2\,u_4u_{1,2}u_{1,3}u_{3,4} - y2\,u_4u_{1,2}u_{1,4}u_{3,3} - 2\,y2\,u_4u_{1,2}u_{2,4}u_{3,4} \\
+y2\,u_4u_{1,3}u_{1,4}u_{2,3} + 2\,y2\,u_4u_{1,4}u_{2,3}u_{2,4} - y2\,u_4u_{2,2}u_{2,4}u_{4,4} + y2\,u_4u_{2,4}{}^3
\end{array}
$$

$$
\begin{aligned}
I_{2e} &= -2u_{1,1}u_{3,3} - 4u_{1,2}u_{3,4} + 2u_{1,3}^2 + 4u_{1,4}u_{2,3} - 2u_{2,2}u_{4,4} + 2u_{2,4}^2 \\
I_{2f} &= u_{1,1}u_{2,2}u_{3,3}u_{4,4} - u_{1,1}u_{2,2}u_{3,4}^2 - u_{1,1}u_{2,3}^2u_{4,4} + 2u_{1,1}u_{2,3}u_{2,4}u_{3,4} \\
&\quad -u_{1,1}u_{2,4}^2u_{3,3} - u_{1,2}^2u_{3,3}u_{4,4} + u_{1,2}^2u_{3,4}^2 + 2u_{1,2}u_{1,3}u_{2,3}u_{4,4} - 2u_{1,2}u_{1,3}u_{2,4}u_{3,4} \\
&\quad -2u_{1,2}u_{1,4}u_{2,3}u_{3,4} + 2u_{1,2}u_{1,4}u_{2,4}u_{3,3} - u_{1,3}^2u_{2,2}u_{4,4} + u_{1,3}^2u_{2,4}^2 \\
&\quad +2u_{1,3}u_{1,4}u_{2,2}u_{3,4} - 2u_{1,3}u_{1,4}u_{2,3}u_{2,4} - u_{1,4}^2u_{2,2}u_{3,3} + u_{1,4}^2u_{2,3}^2 \\
I_{2g} &= u_1^2u_{3,3} + 2u_1u_2u_{3,4} - 2u_1u_3u_{1,3} - 2u_1u_4u_{2,3} + u_2^2u_{4,4} \\
&\quad -2u_2u_3u_{1,4} - 2u_2u_4u_{2,4} + u_3^2u_{1,1} + 2u_3u_4u_{1,2} + u_4^2u_{2,2}
\end{aligned}
$$

with the invariant derivations

Invariant Derivations
$\nabla_1 = x1\mathcal{D}_{x1} + x2\mathcal{D}_{x2} + y1\mathcal{D}_{y1} + y2\mathcal{D}_{y2}$
$\nabla_2 = -u_3\mathcal{D}_{x1} - u_4\mathcal{D}_{x2} + u_1\mathcal{D}_{y1} + u_2\mathcal{D}_{y2}$
$\nabla_3 = (-u_{3,3}u_1 - u_{3,4}u_2 + u_{1,3}u_3 + u_{2,3}u_4)\mathcal{D}_{x1}$ $+(-u_{3,4}u_1 - u_{4,4}u_2 + u_{1,4}u_3 + u_4u_{2,4})\mathcal{D}_{x2}$ $+(u_{1,3}u_1 + u_{1,4}u_2 - u_{1,1}u_3 - u_{1,2}u_4)\mathcal{D}_{y1}$ $+(u_{2,3}u_1 + u_{2,4}u_2 - u_{1,2}u_3 - u_{2,2}u_4)\mathcal{D}_{y2}$
$\nabla_4 = \big(\big(u_{4,4}u_2 - u_{1,4}u_3 - u_4(u_{2,4} + u_{1,3})\big)\,u_{2,3}$ $+\big((-u_{2,4} + u_{1,3})u_2 + u_{1,2}u_3 + u_{2,2}u_4\big)\,u_{3,4}$ $-u_2u_{1,4}u_{3,3} + (u_{1,1}u_{3,3} - u_{1,3}^2)u_3 + u_4u_{1,2}u_{3,3})\mathcal{D}_{x1}$ $+((u_{3,3}u_1 + (-u_{2,4} - u_{1,3})u_3 - u_{2,3}u_4)u_{1,4}$ $+((u_{2,4} - u_{1,3})u_1 + u_{1,1}u_3 + u_{1,2}u_4)u_{3,4}$

$$
\begin{aligned}
&-u_1u_{2,3}u_{4,4} + u_3u_{1,2}u_{4,4} - u_4(-u_{2,2}u_{4,4} + u_{2,4}^2))\mathcal{D}_{x2}\\
&+((-u_{3,4}u_1 - u_{4,4}u_2 + u_4(u_{2,4} - u_{1,3}))u_{1,2}\\
&+(u_{2,3}u_1 + (u_{2,4} + u_{1,3})u_2 - u_{2,2}u_4)u_{1,4}\\
&+(-u_{1,1}u_{3,3} + u_{1,3}^2)u_1 - u_{1,1}(u_{3,4}u_2 - u_{2,3}u_4))\mathcal{D}_{y1}\\
&+((-u_{3,3}u_1 - u_{3,4}u_2 - u_3(u_{2,4} - u_{1,3}))u_{1,2})\\
&+((u_{2,4} + u_{1,3})u_1 + u_{1,4}u_2 - u_{1,1}u_3)u_{2,3}\\
&-u_1u_{2,2}u_{3,4} + (-u_{2,2}u_{4,4} + u_{2,4}^2)u_2 + u_3u_{1,4}u_{2,2})\mathcal{D}_{y2}
\end{aligned}
$$

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