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RESEARCH PAPER

ASYMPTOTICS OF FUNDAMENTAL SOLUTIONS FOR TIME FRACTIONAL EQUATIONS WITH CONVOLUTION KERNELS

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Abstract

The paper deals with the large time asymptotic of the fundamental solution for a time fractional evolution equation with a convolution type operator. In this equation we use a Caputo time derivative of order $\alpha \in (0, 1)$, and assume that the convolution kernel of the spatial operator is symmetric, integrable and shows a super-exponential decay at infinity. Under these assumptions we describe the point-wise asymptotic behavior of the fundamental solution in all space-time regions.

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1. **Introduction and main results**

A random time change in Markov processes is motivated by several reasons. First of all, such change will destroy (in general) the Markov property of the process. The latter is important in the study of biological models where the Markov dynamics is a quite rough approximation to realistic behaviour. Actually, it is one of possible realizations of a general concept of biological times specific for such models.

In many areas of theoretical and experimental physics we meet a notion of sub-diffusion behavior in stochastic dynamics. In particular, that is true for dynamics in some composite or fractal media. The random time

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techniques give a possibility to realize such sub-diffusion asymptotic in concrete model situations.

And finally, the random time change in Markov processes is an interesting and reach source of problems inside of stochastic analysis.

The general framework for a random time change can be described briefly as the following scheme. Let $\{X_t, t \geq 0; P_x, x \in E\}$ be a strong Markov process in a phase space E. Denote T_t its transition semigroup (in a proper Banach space) and L the generator of this semigroup. Let $S_t, t \geq 0$, be a subordinator (i.e., a non-decreasing real-valued Lévy process) with $S_0 = 0$ and the Laplace exponent Φ:

$$\mathbf{E}e^{-\lambda S_t} = e^{-t\Phi(\lambda)}, \;\; t, \, \lambda > 0.$$

We assume that S_t is independent of X_t.

Denote by E_t, $t > 0$, the inverse subordinator and introduce the time changed process $Y_t = X_{E_t}$. We are interested in the time evolution

$$v(x,t) = \mathbf{E}^x[f(Y_t)]$$

for a given initial function f. Note that taking informally $f = \delta$ we arrive at the fundamental solution of the related evolution problem. It is well known, see e.g. [13], [3], that $v(t,x)$ is the unique strong solution to the following Cauchy problem

$$\mathbb{D}_t^{(k)} v(x,t) = Lv(x,t) \;\; v(x,0) = f(x).$$

Here we use a generalized fractional derivative

$$\mathbb{D}_t^{(k)}\phi(t) = \frac{d}{dt}\int_0^t k(t-s)(\phi(s) - \phi(0))ds$$

with a kernel k uniquely defined by Φ.

Let $u(x,t)$ be the solution to a similar Cauchy problem but with the ordinary time derivative. In stochastic terminology, it is the solution to the forward Kolmogorov equation corresponding to the process X_t. Under quite general assumptions there is a nice and essentially obvious relation between these evolutions:

$$v(x,t) = \int_0^\infty u(x,s)G_t(s)\, ds,$$

where $G_t(s)$ is the density of E_t. Of course, we may have similar relations for fundamental solutions to considered equations, for the backward Kolmogorov equations or time evolutions of other related quantities. This technical relation between the random time change and evolution equations with fractal derivatives is an important technical background in the study of resulting processes.

Having in mind the analysis of the influence of the random time change on the asymptotic properties of $v(x,t)$, we may hope that the latter formula gives all necessary technical equipments. Unfortunately, the situation is essentially more complicated. The point is about the density $G_t(s)$, in general, our knowledge for a generic subordinator is very poor. There are two particular cases in which the asymptotic analysis was already realized. First of all, it is the situation of so-called stable subordinators. Starting with pioneering works by Meerschaert and his collaborators, this case was studied in details [1], [9].

Another case is related to a scaling property assumed for Φ, see [4]. It is, nevertheless, difficult to give an interpretation of this scaling assumption in terms of the subordinator.

The problem of asymptotic behaviour of a solution to a fractional evolution equation includes two essentially different aspects. On the one hand, we should choose certain class of random times. Another point is a particular type of Markov processes we start with. In this paper we restrict ourself to the situation of inverse stable subordinators as random times. Initial Markov processes that we consider are pure jump homogeneous Markov processes also known as compound Poisson processes or random walks in $\mathbb{R}^d$ with continuous time. More precisely, we will be concerned with the time asymptotic of corresponding fundamental solutions or, that is the same, related heat kernels.

Our goal is to describe the large time behavior of the time fractional nonlocal heat kernel $w_\alpha(x,t)$, $0<\alpha<1$, that is a solution of the following fractional time parabolic problem:

$$\begin{cases} \partial_t^\alpha w_\alpha = a * w_\alpha - w_\alpha \\ w_\alpha|_{t=0} = \delta_0 \end{cases}, \tag{1.1}$$

where ∂_t^α is the Caputo derivative in time of order $\alpha \in (0,1)$) (see e.g. the books [7, 12] and $a(x)$ is a convolution kernel. We assume that $a(x) \geq 0$; $a(x) = a(-x)$; $a(x) \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, and

$$\int_{\mathbb{R}^d} a(x)dx = 1.$$

We assume additionally that the convolution kernel $a(x)$ satisfies for some $p>1$ the following condition

$$0 \leq a(x) \leq C_1 e^{-b|x|^p}. \tag{1.2}$$

Denote by $u(x,t)$ the fundamental solution of a nonlocal heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = a * u - u \\ u|_{t=0} = \delta_0 \end{cases}. \tag{1.3}$$

Then

$$u(x,t) \;=\; e^{-t}\delta_0(x) \;+\; q(x,t) \tag{1.4}$$

with

$$q(x,t) \;=\; \sum_{k=1}^{\infty} \frac{t^k\, e^{-t}}{k!}\; a^{*k}(x). \tag{1.5}$$

The function $q(x,t)$ is the regular part of the nonlocal heat kernel $u(x,t)$.

The solution $w_\alpha(x,t)$ of (1.1) can be expressed in terms of the heat kernel $u(x,t)$ and the density of the inverse α-stable subordinator. Namely, $w_\alpha(x,t)$ admits the following representation, see e.g. [3], [4],

$$w_\alpha(x,t) = \int_0^\infty u(x,r) d_r \mathbb{P}(S_r \ge t) = \int\limits_0^\infty u(x,r) G_t^\alpha(r) dr, \tag{1.6}$$

where $S = \{S_r,\ r \ge 0\}$ is the α-stable subordinator with the Laplace transform $\mathbb{E} e^{-\lambda S_r} = e^{-r\lambda^\alpha}$, and

$$G_t^\alpha(r) = d_r \Pr\{V_t^{(\alpha)} \le r\} \tag{1.7}$$

is the density of the inverse α-stable subordinator $V_t^{(\alpha)}$. Using the representation for the Laplace transform of $G_t^\alpha(r)$ (see e.g. [14]):

$$\mathcal{L}(G_t^\alpha(r)) = E_\alpha(-\lambda t^\alpha), \quad \text{with} \quad E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha\, j + 1)},$$

being the 1-parameter Mittag-Leffler function (see e.g. in books [7, 12]), and by the properties of the Laplace transform we get for every $k = 0, 1, 2, \ldots$

$$\int\limits_0^\infty G_t^\alpha(r)\, r^k\, e^{-r} dr \;=\; (-1)^k \frac{\partial^k}{\partial \lambda^k} E_\alpha(-\lambda t^\alpha)|_{\lambda=1} \;=\; t^{\alpha k}\, E_\alpha^{(k)}(-t^\alpha). \tag{1.8}$$

By relations (1.4)-(1.6) we have

$$w_\alpha(x,t) = \delta_0(x) \int\limits_0^\infty G_t^\alpha(r)\, e^{-r} dr + \sum_{k=1}^{\infty} \frac{a^{*k}(x)}{k!} \int\limits_0^\infty G_t^\alpha(r)\, r^k\, e^{-r} dr. \tag{1.9}$$

Consequently representations (1.8) and (1.9) imply the following formula for $w_\alpha(x,t)$:

$$w_\alpha(x,t) \;=\; E_\alpha(-t^\alpha)\delta_0(x) \;+\; p_\alpha(x,t), \tag{1.10}$$

where the function $p_\alpha(x,t)$ defined by

$$p_\alpha(x,t) \;=\; \sum_{k=1}^{\infty} \frac{a^{*k}(x)}{k!}\; t^{\alpha k}\, E_\alpha^{(k)}(-t^\alpha) \tag{1.11}$$

is the regular part of $w_\alpha(x,t)$. Let us notice, that in the case $\alpha = 1$ with $E_1(z) = e^z$, we obtain solution (1.4), i.e. $w_1(x,t) = u(x,t)$, and $p_1(x,t) = q(x,t)$.

Unfortunately, the elegant formula (1.10) could not help much with describing point-wise asymptotics for $p_\alpha(x,t)$, and we choose in this paper an other way of studying the asymptotic behavior of $p_\alpha(x,t)$ which is based on the detailed asymptotic analysis of the function $q(x,t)$ that was done in our previous paper [6].

Denote by $g_{\alpha,r}(s),\ s\ge 0$, the density of the α-stable subordinator S_r. The process S_r has the following self-similarity property:

the distribution of S_r is the same as the distribution of $r^{1/\alpha}S_1$.

Consequently,

$$g_{\alpha,r}(s) = r^{-1/\alpha} g_\alpha(sr^{-1/\alpha}),\ s\ge 0, \tag{1.12}$$

where $g_\alpha(s)=g_{\alpha,1}(s)$ is the density of the α-stable law with Laplace transform

$$\int_0^\infty e^{-\lambda s} g_\alpha(s)ds = e^{-\lambda^\alpha}.$$

In addition, the density $g_\alpha(s),\ s\ge 0$ has the following asymptotics, see e.g. [11], [14]:

$$\begin{aligned} g_\alpha(s) &\sim K_\alpha \Big(\frac{\alpha}{s}\Big)^{\frac{2-\alpha}{2(1-\alpha)}} \exp\Big\{-|1-\alpha|\Big(\frac{s}{\alpha}\Big)^{\frac{\alpha}{\alpha-1}}\Big\}, \quad \text{as } s\to 0+;\\ g_\alpha(s) &\sim \frac{\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1}, \quad \text{as } s\to+\infty, \end{aligned} \tag{1.13}$$

with $K_\alpha = \big(2\pi\alpha(1-\alpha)\big)^{-\frac12}$. Then the density $G_t^\alpha(r)$ of the inverse α-stable subordinator $V_t^{(\alpha)}$ that was introduced in (1.7) has the form

$$G_t^\alpha(r) \ = \ \frac{1}{\alpha}\, t\, r^{-1-\frac{1}{\alpha}}\, g_\alpha(tr^{-\frac{1}{\alpha}}), \tag{1.14}$$

see e.g. [9], [11]. The relation (1.9) implies that the regular part $p=p_\alpha$ of the fundamental solution w_α of the time fractional equation can be written as

$$p(x,t) \ = \ \int_0^\infty q(x,r) d_r\mathbb{P}(S_r\ge t) \ = \ \int_0^\infty q(x,r) G_t^\alpha(r)dr. \tag{1.15}$$

In what follows for the sake of brevity we use the notation $p(\cdot)$ instead of $p_\alpha(\cdot)$. Using (1.14) and the change variables $z=tr^{-1/\alpha}$ one can rearrange equality (1.15) as

$$p(x,t) \ = \ \int_0^\infty g_\alpha(z) q\big(x,\frac{t^\alpha}{z^\alpha}\big)dz. \tag{1.16}$$

Make in the integral on the right-hand side the change variables

$$s = z^{-\alpha} \tag{1.17}$$

and denote

$$\hat g_\alpha(s) = g_\alpha(z)|_{z=s^{-1/\alpha}}, \quad W_\alpha(s) = \frac{1}{\alpha} s^{-\frac{1}{\alpha}-1}\hat g_\alpha(s). \tag{1.18}$$

Here, $W_\alpha(s)$ is the (one-parameter) Wright function

$$W_\alpha(s) = \sum_{j=0}^{\infty} \frac{s^j}{j!\,\Gamma(\alpha j+1)}.$$

The detailed properties of the Wright function can be found say in: [5], in the books [7, 12], and for its relations with the (multi-index) Mittag-Leffler functions, and with the generalized Wright ${}_p\Psi_q$-function via the Laplace transform, see for example in [8, (10),(11)].

By (1.18) representation (1.16) takes the form:

$$p(x,t) \ = \ \int_0^\infty \frac{1}{\alpha} s^{-\frac{1}{\alpha}-1} \hat{g}_\alpha(s) q(x, t^\alpha s) ds \ = \ \int_0^\infty W_\alpha(s) q(x, t^\alpha s) ds. \quad (1.19)$$

Notice that in the new variable s defined in (1.17) even for small s such that $s \gg t^{-\alpha}$ the behaviour of the function $q(x, t^\alpha s)$ is governed by the large time asymptotics of the function $q(x,\tau)$.

Moreover, the asymptotic formulae in (1.13) imply the following asymptotics for the function $W_\alpha(s)$:

$$\begin{aligned} W_\alpha(s) \ &\sim \ c_1(\alpha)\, s^{\frac{1}{2(1-\alpha)}-1} \exp\{-c_2(\alpha) s^{\frac{1}{1-\alpha}}\}, \quad \text{as } \ s \to \infty; \\ W_\alpha(s) \ &\to \ \tfrac{\alpha}{\Gamma(1-\alpha)} \tfrac{1}{\alpha}, \quad \text{as } \ s \to 0+, \end{aligned} \qquad (1.20)$$

with $c_2(\alpha) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$. It readily follows from (1.20) that the function $W_\alpha(s)$ has a finite positive limit as $s \to 0+$, and $\int_0^\infty W_\alpha(s) ds = 1$ since W_α is a probability density.

Representation (1.19) and the asymptotic formulae in (1.20) allow one to study the large time behaviour of $p(x,t)$. It turns out that the asymptotics of $p(t,x)$ depends crucially on the ratio between $|x|$ and t. We consider separately the following regions:

- $|x|$ is bounded
- (Subnormal deviations) $1 \ll |x| \ll t^{\frac{\alpha}{2}}$, or equivalently, there exists an increasing function $r(t)$, $r(0)=0$, $\lim\limits_{t\to\infty} r(t) = +\infty$ such that $r(t) \le |x| \le (r(t)+1)^{-1} t^{\alpha/2}$ for all sufficiently large t.
- (Normal deviations) $x = vt^{\alpha/2}(1+o(1))$, where v is an arbitrary vector in $\mathbb{R}^d \setminus \{0\}$.
- (Moderate deviations) $x = vt^\beta(1+o(1))$ with $\frac{\alpha}{2} < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$.
- (Large deviations) $x = vt(1+o(1))$ with $v \in \mathbb{R}^d \setminus \{0\}$.
- (Extra large deviations) $|x| \gg t$, i.e. $\lim\limits_{t\to\infty} \frac{|x(t)|}{t} = \infty$.

Remark 1.1. Notice that for any positive function $r(t)$ such that $r(t) \to \infty$ and $r(t)t^{-\alpha/2} \to 0$, as $t \to \infty$, the set $\{(x,t) : r(t) < |x| < (1+r(t))^{-1}t^{\frac{\alpha}{2}}$ belongs to the region of subnormal deviations $\{(x,t) \in \mathbb{R}^d \times (0,+\infty) : 1 \ll |x| \ll t^{\frac{\alpha}{2}}\}$.

Denote

$$\Psi(v,s) = \frac{1}{|\det \sigma|^{1/2}(2\pi s)^{d/2}} \exp\Big(- \frac{(\sigma^{-1}v,v)}{s}\Big), \tag{1.21}$$

where

$$\sigma^{ij} = \int\limits_{\mathbb{R}^d} z^i z^j a(z)\, dz.$$

Theorem 1.1. *For the function $p(x,t)$ the following asymptotic relations hold as $t \to \infty$:*

1) If $|x|$ is bounded, then

$$\begin{aligned} &\mathsf{c}_- t^{-\frac{\alpha}{2}} \le p(x,t) \le \mathsf{c}_+ t^{-\frac{\alpha}{2}} && \text{if } d=1,\\ &\mathsf{c}_- t^{-\alpha}\log t \le p(x,t) \le \mathsf{c}_+ t^{-\alpha}\log t && \text{if } d=2,\\ &\mathsf{c}_- t^{-\alpha} \le p(x,t) \le \mathsf{c}_+ t^{-\alpha} && \text{if } d\ge 3. \end{aligned} \tag{1.22}$$

2) If $1 \ll |x| \ll t^{\frac{\alpha}{2}}$, then

$$\begin{aligned} &\mathsf{c}_- t^{-\frac{\alpha}{2}} \le p(x,t) \le \mathsf{c}_+ t^{-\frac{\alpha}{2}} && \text{if } d=1,\\ &\mathsf{c}_- t^{-\alpha}\log\Big(\tfrac{t^\alpha}{|x|^2}\Big) \le p(x,t) \le \mathsf{c}_+ t^{-\alpha}\log\Big(\tfrac{t^\alpha}{|x|^2}\Big) && \text{if } d=2,\\ &\mathsf{c}_- t^{-\alpha}|x|^{2-d} \le p(x,t) \le \mathsf{c}_+ t^{-\alpha}|x|^{2-d} && \text{if } d\ge 3. \end{aligned} \tag{1.23}$$

3) If $x = vt^{\alpha/2}(1+o(1))$ with $v \in \mathbb{R}^d \setminus \{0\}$, then

$$p(t^{\alpha/2}v,t) = t^{-\frac{d\alpha}{2}} \int\limits_0^\infty W_\alpha(s)\Psi(v,s)\, ds\, \big(1+o(1)\big). \tag{1.24}$$

4) If $x = vt^{\beta}(1+o(1))$ with $\frac{\alpha}{2} < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$, then

$$p(x,t) = \exp\Big\{ - K_v\, t^{\frac{2\beta-\alpha}{2-\alpha}}(1+o(1))\Big\} \tag{1.25}$$

with the constant

$$K_v = (2-\alpha)\alpha^{\frac{\alpha}{2-\alpha}}\ \Big(\frac{1}{2}(\sigma^{-1}v,v)\Big)^{\frac{1}{2-\alpha}}.$$

5) If $x = vt(1+o(1))$ with $v \in \mathbb{R}^d \setminus \{0\}$, then

$$p(x,t) = \exp\Big\{ - \mathtt{F}(v)t(1+o(1))\Big\}, \tag{1.26}$$

the function $\mathtt{F}$ will be introduced in (5.5).

6) If $|x| \gg t$, then

$$p(x,t) \le \exp\Big\{ -\mathsf{c}_+ |x| \left(\log\left|\tfrac{x}{t}\right|\right)^{\frac{p-1}{p}} \Big\}. \tag{1.27}$$

REMARK 1.2. Observe that the region of large deviations $\{(x,t) : |x| \sim t\}$ for the time fractional heat kernel studied in this work is the same as that for the heat kernel of equation (1.3).

Notice also that in the region of extra large deviations $|x| \gg t$ the asymptotic upper bound (1.27) is similar to that obtained in [6] for $q(x,t)$.

2. Subnormal deviation region

In this section we deal with the region $\{(x,t) : |x| \ll t^{\frac{\alpha}{2}}\}$. We consider separately the cases of bounded $|x|$ and growing $|x|$.

2.1. The case of bounded $|x|$.

In this case

$$\begin{aligned} q(x,t^\alpha s) &\le C_1 \min\big\{t^\alpha s;\ (t^\alpha s)^{-\frac{d}{2}}\big\}, \\ q(x,t^\alpha s) &\ge C_2 \min\big\{t^\alpha s;\ (t^\alpha s)^{-\frac{d}{2}}\big\} \end{aligned} \tag{2.1}$$

with some constants $C_1, C_2 > 0$. Indeed, the estimate by $t^\alpha s$ holds for small value of $\tau = t^\alpha s$, while the estimate $(t^\alpha s)^{-\frac{d}{2}}$ holds for large $\tau = t^\alpha s$.

Using representation (1.19) we get

$$p(x,t) = \int_0^\infty W_\alpha(s) q(x,t^\alpha s)ds \le \tilde{C}_1 \int_0^{t^{-\alpha}} t^\alpha s ds + C_1 \int_{t^{-\alpha}}^\infty W_\alpha(s)(t^\alpha s)^{-\frac{d}{2}} ds. \tag{2.2}$$

The analogous estimate from below holds with an other constant, as follows from (2.1). Let us estimate the second integral in (2.2):

$$t^{-\frac{\alpha d}{2}} \int_{t^{-\alpha}}^\infty W_\alpha(s) s^{-\frac{d}{2}} ds. \tag{2.3}$$

Using the properties of the function $W_\alpha(s)$ we get for all $d \ne 2$

$$\int_{t^{-\alpha}}^1 W_\alpha(s) s^{-\frac{d}{2}} ds + \int_1^\infty W_\alpha(s) s^{-\frac{d}{2}} ds = c_3 t^{-\alpha + \frac{d\alpha}{2}} + c_4, \tag{2.4}$$

and for $d = 2$:

$$\int_{t^{-\alpha}}^1 W_\alpha(s) s^{-\frac{d}{2}} ds + \int_1^\infty W_\alpha(s) s^{-\frac{d}{2}} ds = c_5 \alpha \ln t + c_6. \tag{2.5}$$

Here c_j are constants. Combining (2.2) - (2.5) we obtain the asymptotics (1.22) for $p(x,t)$.

2.2. **The case $1 \ll |x| \ll t^{\frac{\alpha}{2}}$.**

Here we study the asymptotic behaviour of $p(x,t)$ in the region $\{(x,t) \in \mathbb{R}^d \times (0,+\infty) \,:\, 1 \ll |x| \ll t^{\frac{\alpha}{2}}\}$ as $t \to \infty$.

THEOREM 2.1. *Let $r = r(t)$ be an increasing function such that $r(0) = 0$ and $\lim\limits_{t\to\infty} r(t) = +\infty$. Then for all sufficiently large t and for all $x \in \mathbb{R}^d$ such that $r(t) \le |x| \le (r(t)+1)^{-1} t^{\frac{\alpha}{2}}$, we have*

$$\begin{aligned} &\mathsf{c}_- t^{-\frac{\alpha}{2}} \le p(x,t) \le \mathsf{c}_+ t^{-\frac{\alpha}{2}}, && \text{if } d = 1,\\ &\mathsf{c}_- t^{-\alpha} \log\Big(\tfrac{t^\alpha}{|x|^2}\Big) \le p(x,t) \le \mathsf{c}_+ t^{-\alpha} \log\Big(\tfrac{t^\alpha}{|x|^2}\Big), && \text{if } d = 2,\\ &\mathsf{c}_- t^{-\alpha}|x|^{2-d} \le p(x,t) \le \mathsf{c}_+ t^{-\alpha}|x|^{2-d}, && \text{if } d \ge 3. \end{aligned} \tag{2.6}$$

P r o o f. Our arguments rely on the following statement.

PROPOSITION 2.1. *There exist positive constants $c_j > 0$, $j = 1, 2, 3, 4$, such that for all sufficiently large $s > 0$ and $x \in \{x \in \mathbb{R}^d \,:\, |x| \le s\}$ we have*

$$c_1 s^{-\frac{d}{2}} \exp\Big(-c_2 \frac{|x|^2}{s}\Big) \le q(x,s) \le c_3 s^{-\frac{d}{2}} \exp\Big(-c_4 \frac{|x|^2}{s}\Big). \tag{2.7}$$

The proof of this proposition is postponed till Appendix.

Let us consider the case $d \ge 3$. We turn now to the upper bound in (2.6) and consider separately the intervals $J_1 = (0, |x|t^{-\alpha})$, $J_2 = (|x|t^{-\alpha}, |x|^{\frac{3}{2}}t^{-\alpha})$ and $J_3 = (|x|^{\frac{3}{2}}t^{-\alpha}, +\infty)$.

By the same arguments as in the proof of Theorem 3.2 in [6, Section 3.4] one can derive that

$$q(x, st^\alpha) \le \exp(-c|x|) \qquad \text{for all } \ s \in J_1$$

with some $c > 0$. This implies the inequality

$$\int_{J_1} W_\alpha(s) q(x, st^\alpha) ds \le t^{-\alpha}|x| \exp(-c|x|) \le ct^{-\alpha}|x|^{2-d}. \tag{2.8}$$

According to Proposition 2.1, for all $s \in J_2$

$$q(x, st^\alpha) \le c_3 (st^\alpha)^{-\frac{d}{2}} \exp\Big(-c_4 \frac{|x|^2}{st^\alpha}\Big) \le \exp\Big(-c_4 |x|^{\frac{1}{2}}\Big).$$

Therefore,

$$\int_{J_2} W_\alpha(s) q(x, st^\alpha) ds \le t^{-\alpha}|x|^{\frac{3}{2}} \exp(-c_4|x|^{\frac{1}{2}}) \le ct^{-\alpha}|x|^{2-d}. \tag{2.9}$$

Using one more time Proposition 2.1, we obtain

$$\int\limits_{J_3} W_\alpha(s)q(x,st^\alpha)ds \le \int\limits_{|x|^{\frac{3}{2}}t^{-\alpha}}^{\infty} c_3(st^\alpha)^{-\frac{d}{2}}\exp\big(-c_4\frac{|x|^2}{st^\alpha}\big)ds$$

$$= t^{-\alpha}|x|^{2-d}\int\limits_{|x|^{-\frac{1}{2}}}^{\infty} c_3 s^{-\frac{d}{2}}\exp\big(-\frac{c_4}{s}\big)ds \le t^{-\alpha}|x|^{2-d}\int\limits_0^\infty c_3 s^{-\frac{d}{2}}\exp\big(-\frac{c_4}{s}\big)ds.$$

Combining the latter estimate with (2.8) and (2.9) yields the desired upper bound in (2.6).

In order to obtain the lower bound in (2.6) we estimate from below the contribution of the interval $s\in(t^{-\alpha}|x|^2, 2t^{-\alpha}|x|^2)$ as follows

$$\int\limits_{|x|^2t^{-\alpha}}^{2|x|^2t^{-\alpha}} W_\alpha(s)q(x,st^\alpha)ds \ge c_5\int\limits_{|x|^2t^{-\alpha}}^{2|x|^2t^{-\alpha}} (st^\alpha)^{-\frac{d}{2}}\exp\big(-c_2\frac{|x|^2}{st^\alpha}\big)ds$$

$$= c_5 t^{-\alpha}|x|^{2-d}\int\limits_1^2 s^{-\frac{d}{2}}\exp\big(-\frac{c_2}{s}\big)ds.$$

This implies the required lower bound.

The cases $d=1$ and $d=2$ can be considered in a similar way. □

3. **Normal deviations region**

In this section we assume that $x = vt^{\alpha/2}$.

THEOREM 3.1. *Under our standing assumptions on* $a(\cdot)$ *for any* $v\in\mathbb{R}^d\setminus\{0\}$ *we have*

$$p(t^{\alpha/2}v,t) = t^{-\frac{d\alpha}{2}}\int\limits_0^\infty W_\alpha(s)\Psi(v,s)\,ds\,\big(1+o(1)\big), \tag{3.1}$$

where $o(1)$ *tends to zero as* $t\to\infty$.

P r o o f. In representation (1.19) it is convenient to divide the integration interval into three parts, $J_1 = (0,\frac14 t^{-\alpha/2})$, $J_2 = (\frac14 t^{-\alpha/2},\delta)$ and $J_3 = (\delta,+\infty)$, where δ is a sufficiently small number that will be chosen later.

Step 1. We first estimate the contribution of J_3. According to [2, Theorem 19.1] we have

$$\lim_{n\to\infty}\max_{v\in\mathbb{R}^d}\big|n^{d/2}a^{*n}\big(\sqrt{n}v\big)-\Psi(v,1)\big| = 0, \tag{3.2}$$

where the function Ψ was defined in (1.21). This implies in the standard way that for any $\delta > 0$

$$\lim_{t\to\infty} \sup_{s\geq\delta,\, v\in\mathbb{R}^d} \left|t^{\frac{d\alpha}{2}} q\big(t^{\frac{\alpha}{2}}v, st^{\alpha}\big) - \Psi(v,s)\right| = 0. \tag{3.3}$$

See the proof of relation (3.3) in Appendix. By the Lebesgue theorem

$$t^{\frac{d\alpha}{2}} \int_{\delta}^{\infty} W_\alpha(s) q\big(t^{\frac{\alpha}{2}}v, st^{\alpha}\big)\, ds \longrightarrow \int_{\delta}^{\infty} W_\alpha(s)\Psi(v,s)\, ds \tag{3.4}$$

for each $v \in \mathbb{R}^d$, as $t \to \infty$. Consequently,

$$\int_{\delta}^{\infty} W_\alpha(s) q\big(t^{\frac{\alpha}{2}}v, st^{\alpha}\big)\, ds = t^{-\frac{d\alpha}{2}} \int_{\delta}^{\infty} W_\alpha(s)\Psi(v,s)\, ds \,\big(1 + o(1)\big), \tag{3.5}$$

where $o(1)$ tends to zero as $t \to \infty$. Observe also that

$$\int_{0}^{\delta} W_\alpha(s)\Psi(v,s)\, ds \ \to 0, \quad \text{as } \delta \to 0. \tag{3.6}$$

Step 2. Next we are going to show that the contribution of the interval J_2 is getting negligible as $\delta \to 0$. To this end we prove that

$$q(t^{\alpha/2}v, st^{\alpha}) \leq C(v) t^{-\frac{\alpha d}{2}} \qquad \text{for all } \ s \in J_2 \tag{3.7}$$

with some constant $C(v)$ that might depend on v. The proof relies on the representation formula for $q(x,t)$ in (1.5). In order to extract the terms that provide the main contribution to the sum in the representation of $q(t^{\alpha/2}v, st^{\alpha})$ we divide this sum into three parts:

$$\begin{aligned} &q(t^{\alpha/2}v, st^{\alpha}) = \\ &e^{-st^{\alpha}}\Big\{ \sum_{n=1}^{st^{\alpha}-(st^{\alpha})^{3/4}} + \sum_{n=st^{\alpha}-(st^{\alpha})^{3/4}}^{st^{\alpha}+(st^{\alpha})^{3/4}} + \sum_{n=st^{\alpha}+(st^{\alpha})^{3/4}}^{\infty} \Big\} \frac{(st^{\alpha})^n}{n!} a^{*n}(t^{\alpha/2}v) \\ &= e^{-st^{\alpha}} \sum_{n=st^{\alpha}-(st^{\alpha})^{3/4}}^{st^{\alpha}+(st^{\alpha})^{3/4}} \frac{(st^{\alpha})^n}{n!} a^{*n}(t^{\alpha/2}v) + O(e^{-ct^{\alpha/4}}); \end{aligned} \tag{3.8}$$

the second relation here is a consequence of the Stirling formula. Let us estimate $t^{\frac{d\alpha}{2}} a^{*n}(t^{\alpha/2}v)$ for all $n \in \big(st^{\alpha} - (st^{\alpha})^{3/4}, st^{\alpha} + (st^{\alpha})^{3/4}\big)$. Notice that $n \to \infty$ as $t \to \infty$ uniformly in $s \in J_3$. Using (3.2) we have

$$t^{\frac{d\alpha}{2}}\, a^{*n}(t^{\alpha/2}v) = \frac{(st^{\alpha})^{d/2}}{s^{d/2}}\, a^{*n}\big((st^{\alpha})^{1/2}\frac{v}{\sqrt{s}}\big)$$

$$= \frac{1}{s^{d/2}} n^{d/2}(1 + o(1))\, a^{*n}\big(\sqrt{n}\frac{v(1+o(1))}{\sqrt{s}}\big) - \frac{1}{s^{d/2}}\Psi\big(\frac{v(1+o(1))}{\sqrt{s}}, 1\big)$$

$$+\frac{1}{s^{d/2}}\Psi\Big(\frac{v(1+o(1))}{\sqrt{s}},1\Big)\to\frac{1}{s^{d/2}}\Psi\Big(\frac{v}{\sqrt{s}},1\Big)=\Psi(v,s).$$

Since the function $\Psi(v,s)$ is uniformly bounded for all $s\in(0,\infty)$, then we get

$$a^{*n}(t^{\alpha/2}v)\le B(v)t^{-\frac{d\alpha}{2}}\quad\text{as}\ \ t\to\infty \tag{3.9}$$

for all $n\in\big(st^\alpha-(st^\alpha)^{3/4},st^\alpha+(st^\alpha)^{3/4}\big)$. Consequently (3.8) together with (3.9) imply (3.7).

As an immediate consequence of (3.7) we obtain

$$\int\limits_{J_2} q(t^{\alpha/2}v,st^\alpha)W_\alpha(s)\,ds\le C_1\delta t^{-\frac{\alpha d}{2}}. \tag{3.10}$$

This yields the required statement.

Step 3. It remains to estimate the contribution of the interval J_1. Again we divide the sum in representation (1.5) into two parts:

$$q(t^{\frac{\alpha}{2}}v,st^\alpha)=e^{-st^\alpha}\sum_{n=1}^{t^{\alpha/2}}\frac{(st^\alpha)^n}{n!}a^{*n}(t^{\frac{\alpha}{2}}v)+e^{-st^\alpha}\sum_{n>t^{\alpha/2}}\frac{(st^\alpha)^n}{n!}a^{*n}(t^{\frac{\alpha}{2}}v)$$

$$=\Sigma_4+\Sigma_5.$$

If $n\ge t^{\alpha/2}$ and $s\le\frac14 t^{-\alpha/2}$, then after a simple computation we obtain

$$\exp\big(-st^\alpha\big)\frac{(st^\alpha)^n}{n!}\le\exp\big(-\kappa_5t^{\alpha/2}\big)$$

with some constant $\kappa_5>0$. Then Σ_5 admits the following upper bound

$$\Sigma_5\le C_5\exp\big(-\kappa_5t^{\alpha/2}\big) \tag{3.11}$$

with a positive constant C_5.

We turn to estimating Σ_4. Observe that we sum up over all integer n from the interval $(0,t^{\frac{\alpha}{2}})$. In particular, n need not tend to infinity as $t\to\infty$.

LEMMA 3.1. *For any $v\in\mathbb{R}^d\backslash\{0\}$ there exist $c(v)>0$ and $C(v)>0$ such that for all $n<t^{\alpha/2}$ we have*

$$a^{*n}(t^{\alpha/2}v)\le C(v)\exp\big(-c(v)t^{\alpha/2}\big). \tag{3.12}$$

P r o o f. The proof of the lemma is based on the Markov inequality. Denote by S_n the sum of n i.i.d. random vectors with a common distribution density $a(x)$. The distribution density of S_n is a^{*n}. The notation S_n^j is used for the j-th coordinate of S_n. For $n<t^{\alpha/2}$ and any $r>0$ we have

$$\int\limits_{|x|>rt^{\alpha/2}} a^{*n}(x)\,dx=\mathbf{P}\{|S_n|\ge rt^{\alpha/2}\}=\mathbf{P}\Big\{|S_n|\ge n\frac{rt^{\alpha/2}}{n}\Big\}$$

$$\leq \sum_{j=1}^{d} \mathbf{P}\big\{|S_n^j| \geq \frac{n}{d}\,\frac{rt^{\alpha/2}}{n}\big\}.$$

According to the Markov's inequality for the terms on the right-hand side of the last estimate the following upper bound holds:

$$\mathbf{P}\big\{|S_n^j| \geq n\frac{rt^{\alpha/2}}{dn}\big\} \leq \exp\Big(-\max_{\gamma\in\mathbb{R}}\big(\gamma\frac{rt^{\alpha/2}}{dn} - L^j(\gamma)\big)\,n\Big),$$

where $L^j(\gamma)$ is the cumulant of S_1^j. Under our assumptions on $a(\cdot)$ there is a positive constants c_0 such that

$$L^j(\gamma) \leq c_0\gamma^2$$

for all γ such that $|\gamma| \leq 1$. Since $\frac{t^{\alpha/2}}{dn} > \frac{4}{d}$, the latter inequality implies the following estimate

$$\max_{\gamma\in\mathbb{R}}\big(\gamma\frac{rt^{\alpha/2}}{dn} - L^j(\gamma)\big) \geq \max_{|\gamma|\leq 1}\big(\gamma\frac{rt^{\alpha/2}}{dn} - L^j(\gamma)\big) \geq c_{d,r}\frac{t^{\alpha/2}}{n}$$

with a positive constant $c_{d,r}$. Hence, for any $r > 0$,

$$\int\limits_{|x|>rt^{\alpha/2}} a^{*n}(x)\,dx \leq \exp\big(-c_{d,r}t^{\alpha/2}\big). \tag{3.13}$$

Combining this estimate with the estimate $a(x) \leq Me^{-b|x|}$ that is granted by our assumptions on a, and writing $a^{*(n+1)} = a^{*n} * a$, one can show in the standard way that

$$a^{*(n+1)}(t^{\alpha/2}v) \leq C(v)\exp\big(-c(v)t^{\alpha/2}\big).$$

Indeed, by (1.2) and (3.13)

$$a^{*(n+1)}(t^{\alpha/2}v) = \int\limits_{\mathbb{R}^d} a^{*n}(y)\,a(t^{\alpha/2}v - y)\,dy$$

$$\leq \int\limits_{|y|\geq\frac{1}{2}t^{\alpha/2}|v|} a^{*n}(y)\,a(t^{\alpha/2}v-y)\,dy + \int\limits_{|y|\geq\frac{1}{2}t^{\alpha/2}|v|} a^{*n}(t^{\alpha/2}v-y)\,a(y)\,dy$$

$$\leq \|a\|_{L^\infty}\big(e^{-c(v)t^{\frac{\alpha}{2}}} + e^{-bc(v)t^{\frac{\alpha p}{2}}}\big).$$

□

Inequality (3.12) immediately implies the following upper bound

$$\Sigma_4 \leq \exp(-ct^{\alpha/2})$$

Combining it with (3.11) yields

$$\int\limits_{J_1} q(t^{\alpha/2}v, st^\alpha)\,ds \leq \exp(-ct^{\alpha/2}). \tag{3.14}$$

Finally, from (3.5), (3.6), (3.10) and (3.14) we deduce that

$$p(t^{\alpha/2}v,t)=t^{-\frac{d\alpha}{2}}\int\limits_0^\infty W_\alpha(s)\Psi(v,s)\,ds\,\big(1+o(1)\big),$$

where $o(1)$ tends to zero as $t\to\infty$. □

4. Moderate deviations region

In this section we consider the region $t^{\frac{\alpha}{2}}\ll|x|\ll t$. The name "moderate deviations region" is related to the fact that studying the large time behaviour of $p(x,t)$ in this region relies on the asymptotic formulae for $q(x,\cdot)$ in the region of moderate deviations. For presentation simplicity we assume that

$$x=vt^\beta(1+o(1))\qquad\text{with }\beta\in\big(\tfrac{\alpha}{2},1\big),\tag{4.1}$$

here $o(1)$ tends to zero as $t\to\infty$.

Theorem 4.1. *Let relation* (4.1) *hold with* $\beta\in\big(\frac{\alpha}{2},1\big)$. *Then, as* $t\to\infty$,

$$p(x,t)=\exp\Big\{-K_vt^{\frac{2\beta-\alpha}{2-\alpha}}(1+o(1))\Big\}\tag{4.2}$$

with $K_v=c_3(\alpha)K$, $c_3(\alpha)=(2-\alpha)\alpha^{\frac{\alpha}{2-\alpha}}$, $K=\big(\frac12(\sigma^{-1}v,v)\big)^{\frac{1}{2-\alpha}}$.

P r o o f. We first prove a lower bound. To this end we let

$$\xi_0=\alpha^{-\frac{\alpha}{2-\alpha}}\big(\frac12(\sigma^{-1}v,v)\big)^{\frac{1-\alpha}{2-\alpha}}t^{(2\beta-\alpha)\frac{1-\alpha}{2-\alpha}}.\tag{4.3}$$

According to [6, Theorem 3.1], for all $\xi\in[\xi_0-1,\xi_0+1]$ we have

$$q(x,t^\alpha\xi)=\exp\Big(-\frac{(\sigma^{-1}x,x)}{2t^\alpha\xi_0}(1+o(1))\Big),$$

where $o(1)$ tends to zero, as $t\to\infty$, uniformly in $\xi\in[\xi_0-1,\xi_0+1]$. Combining this relation with (4.3) and the first formula in (1.20), after straightforward computations we obtain

$$W_\alpha(\xi)q(x,t^\alpha\xi)=\exp\big\{-c_3(\alpha)\big(\frac12(\sigma^{-1}v,v)\big)^{\frac{1}{2-\alpha}}t^{\frac{2\beta-\alpha}{2-\alpha}}(1+o(1))\big\}$$

uniformly in $\xi\in[\xi_0-1,\xi_0+1]$. Integrating the last relation yields the desired lower bound.

To prove the upper bound for $p(x,t)$ we divide the integration domain into three parts:

$$J_1=(0,t^{\beta-\alpha}),\quad J_2=(t^{\beta-\alpha},t^{2\beta-\alpha}),\quad J_3=(t^{2\beta-\alpha},\infty),$$

and show that the second interval J_2 provides the main contribution to the integral in (1.19). We have

$$p(x,t) = \int\limits_{J_1} W_\alpha(s) q\big(x, st^\alpha\big)\, ds + \int\limits_{J_2} W_\alpha(s) q\big(x, st^\alpha\big)\, ds + \int\limits_{J_3} W_\alpha(s) q\big(x, st^\alpha\big)\, ds. \tag{4.4}$$

Our first aim is to calculate the second integral on the right-hand side in (4.4). To this end we split interval J_2 into three parts:

$$J_2^1 = (t^{\beta-\alpha}, t^{\gamma_1}), \quad J_2^2 = (t^{\gamma_1}, t^{2\beta-\alpha-\gamma_2}), \quad J_2^3 = (t^{2\beta-\alpha-\gamma_2}, t^{2\beta-\alpha}),$$

if $\beta \le \alpha$, and

$$J_2^1 = (t^{\beta-\alpha}, t^{\beta-\alpha+\gamma_1}), \quad J_2^2 = (t^{\beta-\alpha+\gamma_1}, t^{2\beta-\alpha-\gamma_2}), \quad J_2^3 = (t^{2\beta-\alpha-\gamma_2}, t^{2\beta-\alpha}),$$

if $\beta > \alpha$. We then show that for sufficiently small $\gamma_1, \gamma_2 > 0$ the contribution of the corresponding integrals over J_2^1 and J_2^3 do not exceed $o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big)$ as $t \to \infty$. Indeed, considering the asymptotics of $W_\alpha(s)$ in (1.20) we conclude that on interval J_2^3 the following upper bound holds:

$$W_\alpha(s) \le C_1 t^{m(\alpha,\beta)} e^{-c_2(\alpha) t^{\frac{2\beta-\alpha-\gamma_2}{1-\alpha}}}, \quad s \in J_2^3,$$

with some $m(\alpha,\beta) > 0$. For $0 < \gamma_2 < \frac{2\beta-\alpha}{2-\alpha}$ this yields

$$\int\limits_{J_2^3} W_\alpha(s) q\big(x, st^\alpha\big)\, ds = o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big). \tag{4.5}$$

We turn to the interval J_2^1. If $\frac{\alpha}{2} < \beta \le \alpha$, then letting

$$0 < \gamma_1 < \frac{(2\beta-\alpha)(1-\alpha)}{2-\alpha} \tag{4.6}$$

we obtain

$$W_\alpha(s) \le C_2, \quad q\big(rt^\beta, st^\alpha\big) \le e^{-c(r) t^{2\beta-\alpha-\gamma_1}} = o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big)$$

for all $s \in J_2^1 = (t^{\beta-\alpha}, t^{\gamma_1})$. Analogously, if $\alpha < \beta < 1$, then we choose γ_1 such that

$$0 < \gamma_1 < \frac{\alpha(1-\beta)}{2-\alpha}. \tag{4.7}$$

In this case

$$W_\alpha(s) q\big(vt^\beta, st^\alpha\big) = o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big),$$

and consequently

$$\int\limits_{J_2^1} W_\alpha(s) q\big(x, st^\alpha\big)\, ds = o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big). \tag{4.8}$$

It remains to compute the asymptotics of the integral over J_2^2. For $\beta > \alpha$ it takes the form

$$\int\limits_{J_2^2} W_\alpha(s) q\big(x, st^\alpha\big)\, ds = \int\limits_{t^{\beta-\alpha+\gamma_1}}^{t^{2\beta-\alpha-\gamma_2}} W_\alpha(s) q\big(x, st^\alpha\big)\, ds \tag{4.9}$$

The case when $J_2^2 = (t^{\gamma_1}, t^{2\beta-\alpha-\gamma_2})$ $(\beta < \alpha)$ can be considered in a similar way.

Since for all $s \in J_2^2$ we have $s > t^{\beta-\alpha+\gamma_1}$, the function $W_\alpha(s)$ meets the first asymptotics in (1.20) as $s \in J_2^2$. Recalling that $x = vt^\beta(1 + o(1))$, we represent $q\big(x, st^\alpha\big)$ as a sum

$$q(vt^\beta, st^\alpha) = e^{-st^\alpha} \left\{ \sum_{k=1}^{t^{(\beta+\gamma_1)}} + \sum_{k=t^{(\beta+\gamma_1)}+1}^{(1+\delta)st^\alpha} + \sum_{k>(1+\delta)st^\alpha} \right\} \frac{(st^\alpha)^k}{k!} a^{*k}(vt^\beta), \tag{4.10}$$

where $\delta > 0$ is a sufficiently small positive constant. Notice that the upper summation limit in the second sum on the right-hand side and the lower summation limit in the third sum depend on s that belongs to the interval $J_2^2 = (t^{\beta-\alpha+\gamma_1},\ t^{2\beta-\alpha-\gamma_2})$.

We start by estimating the first sum in (4.10). Using the Markov inequality in the same was as in the proof of Lemma 3.1 above we obtain

$$\int\limits_{|x|>vt^\beta} a^{*k}(x)\, dx \le C_d \exp\Big\{ - \max_{\gamma\in\mathbb{R}} \big(\gamma \frac{vt^\beta}{dk} - L^j(\gamma)\big)\, k \Big\}. \tag{4.11}$$

The maximum on the right-hand side here admits the lower bound

$$\max_{\gamma\in\mathbb{R}} \big(\gamma \frac{vt^\beta}{dk} - L^j(\gamma)\big) \ge c_{d,v} \big(\frac{t^\beta}{k}\big)^2 \ge \frac{t^{\beta-\gamma_1}}{k}$$

with a constant $c_{d,v} > 0$. This yields the following estimate

$$\int\limits_{|x|>vt^\beta} a^{*k}(x)\, dx \le \exp\big\{ - c_{d,v} t^{\beta-\gamma_1} \big\},$$

which is valid for any $k \le t^{\beta+\gamma_1}$. Combining this estimate with the estimate $a(x) \le Me^{-b|x|}$ and (4.7) we conclude that

$$a^{*(k+1)}(vt^\beta) \le e^{-c_1 t^{\beta-\gamma_1}} = o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big) \quad \text{for all } \ k \le t^{\beta+\gamma_1}. \tag{4.12}$$

The inequality (4.12) combined with a trivial inequality

$$e^{-st^\alpha} \sum_{k=1}^{t^{\beta+\gamma_1}} \frac{(st^\alpha)^k}{k!} \ < \ 1,$$

implies the upper bound for the first sum in (4.10):

$$e^{-st^\alpha}\sum_{k=1}^{t^{\beta+\gamma_1}}\frac{(st^\alpha)^k}{k!}a^{*k}(rt^\beta)\le C_1e^{-c_1t^{\beta-\gamma_1}}=o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big);\tag{4.13}$$

here we assume that γ_1 satisfies (4.7).

The estimation of the third sum in (4.10) is based on the following upper bound

$$e^{-st^\alpha}\frac{(st^\alpha)^k}{k!}\le e^{-\frac14\delta^2st^\alpha}\le e^{-\frac14\delta^2t^{\beta+\gamma_1}}=o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big),$$

which is an immediate consequence of the Stirling formula and valid for any $s\ge t^{\beta-\alpha+\gamma_1}$, $k\ge(1+\delta)st^\alpha$ and $\delta\in(0,1)$. We have also used here an evident inequality $\beta>\frac{2\beta-\alpha}{2-\alpha}$. Since

$$\frac{(st^\alpha)^{k+1}/(k+1)!}{(st^\alpha)^k/k!}=\frac{st^\alpha}{(k+1)}<\frac{st^\alpha}{(1+\delta)st^\alpha}=\frac{1}{1+\delta}$$

for $k>(1+\delta)st^\alpha$, the third sum on the right-hand side of (4.10) can be estimated by the corresponding geometrical progression, and we finally obtain

$$e^{-st^\alpha}\sum_{k>(1+\delta)st^\alpha}\frac{(st^\alpha)^k}{k!}a^{*k}(vt^\beta)=o\big(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}\big).\tag{4.14}$$

The estimation of the second sum in (4.10) with $k\in(t^{\beta+\gamma_1},\,(1+\delta)st^\alpha)$ is based on the statement of Lemma 3.14 from [6], where the following asymptotic formula for $a^{*k}(x)$ has been justified:

$$a^{*k}(x)\;=\;e^{-\frac12\frac{(\sigma^{-1}x,x)}{k}(1+\varphi(\frac{x}{k}))},\quad\text{where }\;\varphi(\xi)\to0\text{ as }\xi\to0,\tag{4.15}$$

provided

$$\frac{|x|}{k}\to0\quad\text{ and }\quad\frac{|x|^2}{k}\to\infty.\tag{4.16}$$

It is easy to see that for all $k\in(t^{\beta+\gamma_1},\,(1+\delta)st^\alpha)$ and $s\in J_2^2$ conditions (4.16) are fulfilled:

$$\frac{|x|}{k}\le C_1\frac{t^\beta}{t^{\beta+\gamma_1}}\to0,\quad\frac{|x|^2}{k}\ge C_2t^{2\beta-\alpha-(2\beta-\alpha-\gamma_2)}\to\infty.$$

Therefore, the relation

$$a^{*k}(x)\;=\;e^{-\frac12\frac{(\sigma^{-1}x,x)}{k}(1+o(1))}\;=\;e^{-\frac12(\sigma^{-1}v,v)\frac{t^{2\beta-\alpha}}{s}(1+o(1))}\tag{4.17}$$

holds uniformly for all $k\in(t^{\beta+\gamma_1},\,(1+\delta)st^\alpha)$ as $t\to\infty$.

Combining (4.17) with the asymptotic formulae in (1.20) and taking into account estimates (4.13) and (4.14) for the first and the third sums on the right-hand side of (4.10), we obtain an asymptotic upper bound for the integral in (4.9):

$$\int\limits_{J_2^2} W_\alpha(s) q\big(x, st^\alpha\big)\, ds \le e^{-K_v\, t^{\frac{2\beta-\alpha}{2-\alpha}}(1+o(1))}, \tag{4.18}$$

which is valid for all sufficiently large t. Here

$$K_v = c_3(\alpha) K = (2-\alpha)\alpha^{\frac{\alpha}{2-\alpha}} \big(\frac{1}{2}(\sigma^{-1}v, v)\big)^{\frac{1}{2-\alpha}}.$$

It is straightforward to check that

$$K_v\, t^{\frac{2\beta-\alpha}{2-\alpha}} \;=\; \min_s f(s,t),$$

where

$$f(s,t) = \frac{1}{2}(\sigma^{-1}v, v)\frac{t^{2\beta-\alpha}}{s} + c_2(\alpha)\, s^{\frac{1}{1-\alpha}}, \quad c_2(\alpha) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}.$$

Notice that $\mathrm{argmin} f(\cdot, t) \in J_2^2$.

From (4.5), (4.8) and (4.18) one can easily deduce that

$$\int\limits_{J_2} W_\alpha(s) q\big(x, st^\alpha\big)\, ds \le e^{-K_v\, t^{\frac{2\beta-\alpha}{2-\alpha}}(1+o(1))} \tag{4.19}$$

with the constant $K_v = c_3(\alpha)K$ defined above.

Now we turn to the remaining integrals on the right-hand side in (4.4). It will be shown that

$$\int\limits_{J_1} W_\alpha(s) q\big(x, st^\alpha\big)\, ds \le O(e^{-c_1 t^\beta}) = o(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}), \tag{4.20}$$

and

$$\int\limits_{J_3} W_\alpha(s) q\big(x, st^\alpha\big)\, ds \le e^{-c_3\, t^{\frac{2\beta-\alpha}{1-\alpha}}} = o(e^{-t^{\frac{2\beta-\alpha}{2-\alpha}}}). \tag{4.21}$$

This means in particular that these two integrals do not contribute to the principal term of the asymptotics of $p(x,t)$.

For $s \ge t^{2\beta-\alpha}$ the asymptotic formula (1.20) implies that

$$W_\alpha(s) \le C e^{-c_2(\alpha) t^{\frac{2\beta-\alpha}{1-\alpha}}}.$$

Since $q(t^\beta v, , st^\alpha)$ is bounded for all $t \ge 1$, we obtain (4.21) with $c_3 = \frac{1}{2}c_2(\alpha)$.

To estimate the integral in (4.20), we represent $q\big(vt^\beta, st^\alpha\big)$ as a sum

$$q(vt^\beta, st^\alpha) = e^{-st^\alpha} \left\{ \sum_{k=1}^{3t^\beta} + \sum_{k>3t^\beta} \right\} \frac{(st^\alpha)^k}{k!} a^{*k}(vt^\beta) \tag{4.22}$$

For all $k \le 3t^\beta$ by the Markov inequality in the same way as in the proof of Lemma 3.1 we have:

$$\int\limits_{|x|>vt^\beta} a^{*k}(x)\,dx \le C_d \exp\Big\{ -\max_{\gamma>0}\big(\gamma\frac{vt^\beta}{dk} - L^j(\gamma)\big)\,k\Big\} \le e^{-c_{d,v}t^\beta}.$$

This yields

$$a^{*(k+1)}(vt^\beta) = \int_{\mathbb{R}} a^{*k}(vt^\beta - z)a(z)dz \le \tilde{C}_1 e^{-c_1 t^\beta}.$$

The second sum in (4.22) can be estimated from above by an appropriate geometric progression. Indeed, since for $k > 3t^\beta$ and $s \le t^{\beta-\alpha}$ we have

$$\frac{st^\alpha}{(k+1)} < \frac{t^\beta}{3t^\beta} = \frac{1}{3},$$

then the second sum admits the following upper bound:

$$e^{-st^\alpha}\sum_{k>3t^\beta}\frac{(st^\alpha)^k}{k!}a^{*k}(vt^\beta) \le \tilde{C}_2\frac{(st^\alpha)^{3t^\beta}}{(3t^\beta)!} \le \tilde{C}_2 e^{-c_2 t^\beta}$$

with $c_2 = 3(\ln 3 - 1)$.

The relations in (4.18) and (4.20) - (4.21) yield the desired estimate from above. □

5. **Large deviations region**

In this section we consider the region of large deviations. Namely, we suppose here that $x = vt(1+o(1))$, where $v \in \mathbb{R}^d \setminus \{0\}$.

For the reader convenience we recall here some definitions and statements from [6]. The notation $I(v)$, $v \in \mathbb{R}^d$, is used for the Legendre transform of $L(\cdot)$, $I(v) = \max\limits_{\gamma\in\mathbb{R}^d}\big(\gamma\cdot v - L(\gamma)\big)$. Under our assumptions on $a(\cdot)$ the function I is smooth and strictly convex in $\mathbb{R}^d$. Moreover, $I(0) = 0$, $I(v) > 0$ for all $v \in \mathbb{R}^d \setminus \{0\}$, and

$$\lim_{|v|\to\infty}\frac{I(v)}{|v|} = +\infty. \tag{5.1}$$

The equation

$$\log\xi = I(\xi v) - \xi v\cdot\nabla I(\xi v), \xi \in \mathbb{R}^+,$$

has a unique solution. It is denoted by ξ_v. A function $\Phi(v)$, $v \in \mathbb{R}^d$, is defined by

$$\Phi(v) = 1 - \frac{1}{\xi_v}\big(1 + \log\xi_v - I(\xi_v v)\big).$$

Then Φ is a smooth convex function such that $\Phi(0) = 0$, $\Phi(v) > 0$ if $v \ne 0$, and

$$\lim_{|v|\to\infty} \frac{\Phi(v)}{|v|} = +\infty. \tag{5.2}$$

In order to formulate our results, we introduce a function

$$F_v(\eta) = c_2(\alpha)\eta^{\frac{1}{1-\alpha}} + \Phi\Big(\frac{v}{\eta}\Big)\eta \tag{5.3}$$

and define

$$\eta(v) = \mathrm{argmin} F_v(\eta), \qquad \eta \ge 0. \tag{5.4}$$

Since $\Phi(\cdot)$ is a convex function, $F_v(\cdot)$ is a strictly convex function on $(0,+\infty)$. Due to (5.2) we have

$$\lim_{\eta\to 0} F_v(\eta) = +\infty, \qquad \lim_{\eta\to+\infty} F_v(\eta) = +\infty$$

for each $v \in \mathbb{R}^d \setminus \{0\}$. Consequently, $\eta(v)$ is a well defined function on $\mathbb{R}^d \setminus \{0\}$.

Denote

$$\mathtt{F}(v) = \min_{\eta>0} F_v(\eta) = F_v(\eta(v)). \tag{5.5}$$

THEOREM 5.1. *Assume that $x = vt(1+o(1))$ as $t\to\infty$ for some $v \in \mathbb{R}^d \setminus \{0\}$. Then, as $t\to\infty$,*

$$p(x,t) = \exp\big(- \mathtt{F}(v)t(1+o(1))\big). \tag{5.6}$$

P r o o f. We begin by proving the lower bound. For all $s \in \big(\eta(v)t^{1-\alpha} - 1, \eta(v)t^{1-\alpha}+1\big)$ we have

$$st^\alpha = \eta(v)t(1+o(1)),$$

where $o(1)$ tends to zero as $t\to\infty$ uniformly in $s \in \big(\eta(v)t^{1-\alpha}-1, \eta(v)t^{1-\alpha}+1\big)$. According to [6, Theorem 3.8] for such s the following relation holds

$$q(x,st^\alpha) = \exp\Big(- \Phi\big(\frac{v}{\eta(v)}\big)\eta(v)t(1+o(1))\Big).$$

Therefore,

$$W_\alpha(s)q(x,st^\alpha) = \exp\Big(- \big[c_2(\alpha)(\eta(v))^{\frac{1}{1-\alpha}} + \Phi\big(\frac{v}{\eta(v)}\big)\eta(v)\big]t(1+o(1))\Big),$$

as $t\to\infty$, uniformly in $s \in \big(\eta(v)t^{1-\alpha}-1, \eta(v)t^{1-\alpha}+1\big)$. Considering (1.19) and the definition of $\mathtt{F}$ in (5.5) we conclude that

$$p(x,t) \ge \exp\big(- \mathtt{F}(v)t(1+o(1))\big).$$

This yields the lower bound in (5.6).

We turn to the upper bound. Our first aim is to estimate the contribution of small s. According to (5.1) under our standing assumptions there exists $\gamma_1 = \gamma_1(v) > 0$ such that for any $\gamma \le \gamma_1$ the following inequality holds

$$I\Big(\frac{v}{\gamma}\Big)\gamma > \mathtt{F}(v).$$

With the help of the Stirling approximation formula, it is straightforward to show that there exists $\gamma_0 = \gamma_0(v) > 0$ such that $\gamma_0 < \gamma_1$ and for any $\gamma \le \gamma_0$ we have

$$\sum_{k\ge\gamma_1 t}\frac{(\gamma t)^k}{k!}e^{-\gamma t} \le \exp\big(\big[\gamma_1 t(\log\gamma - \log\gamma_1) + (\gamma_1-\gamma)t\big](1+o(1))\big) < \exp(-\mathtt{F}(v)t),$$

where $o(1)$ tends to zero as $t\to\infty$. Therefore, for any $s\in(0,\gamma_0 t^{1-\alpha})$,

$$q\big(x,st^\alpha\big) < \max_{k\le\gamma_1 t} a^{*k}(x) + C\exp(-\mathtt{F}(v)t)$$
$$\le \max_{k\le\gamma_1 t}\Big\{\exp\Big(-I\Big(\frac{vt}{k}\Big)k\Big)\Big\} + C\exp(-\mathtt{F}(v)t) \le \exp\big\{-\mathtt{F}(v)t(1+o(1))\big\}.$$

Considering the fact that W_α is a bounded function, we obtain

$$\int\limits_0^{\gamma_0 t^{1-\alpha}} W_\alpha(s)q\big(x,st^\alpha\big)\,ds \le C\int\limits_0^{\gamma_0 t^{1-\alpha}} q\big(x,st^\alpha\big)\,ds$$
$$\le \gamma_0 t^{1-\alpha}\exp\big\{-\mathtt{F}(v)t(1+o(1))\big\} \le \exp\big(-\mathtt{F}(v)t(1+o(1))\big). \qquad (5.7)$$

Due to the first relation in (1.20) and the fact that $q(x,t)$ is bounded, there exists $\gamma_2 > \gamma_1$ such that

$$\int\limits_{\gamma_2 t^{1-\alpha}}^{\infty} W_\alpha(s)q\big(x,st^\alpha\big)\,ds \le \exp\big(-\mathtt{F}(v)t\big) \qquad (5.8)$$

It remains to estimate the contribution of the interval $s\in(\gamma_0 t^{1-\alpha},\gamma_2 t^{1-\alpha})$. Denote $st^\alpha = \gamma t$. Notice that $\gamma\in(\gamma_0,\gamma_2)$. Then by [6, Theorem 3.4]

$$q(x,st^\alpha) \le \exp\Big\{-\Phi\Big(\frac{x}{st^\alpha}\Big)st^\alpha(1+o(1))\Big\} = Ct\exp\Big\{-\Phi\Big(\frac{v}{\gamma}\Big)\gamma t(1+o(1))\Big\},$$

where $o(1)$ tends to zero as $t\to\infty$ uniformly in $\gamma\in(\gamma_0,\gamma_2)$. Combining this relation with (1.20), (5.3), (5.4) and (5.5), we conclude that

$$W_\alpha(s)q(x,st^\alpha) \le \exp\big\{-\mathtt{F}(v)t(1+o(1))\big\}$$

and therefore

$$\int\limits_{\gamma_0 t^{1-\alpha}}^{\gamma_2 t^{1-\alpha}} W_\alpha(s)q\big(x,st^\alpha\big)\,ds \le \exp\big\{-\mathtt{F}(v)t(1+o(1))\big\}.$$

Estimates (5.7), (5.8) and the latter relation yield the desired upper bound. □

6. **Extra large deviations region**

In the region of extra large deviation $|x| \gg t$ our asymptotic estimates are not as sharp as in the other regions. The following statement holds.

THEOREM 6.1. *Assume that $|x| \gg t$. Then there exists a positive constant $\mathsf{c}_+ > 0$ such that*

$$p(x,t) \le \exp\Big\{ -\mathsf{c}_+ |x| \big(\log\big|\tfrac{x}{t}\big|\big)^{\frac{p-1}{p}}\Big\} \qquad (6.1)$$

for all sufficiently large t.

P r o o f. Our analysis relies again on formula (1.19). We consider separately three intervals: $(0,\infty) = (0,1)\cup\big(1, t^{1-\alpha}\big(\frac{|x|}{t}\big)^{1-\frac{\alpha}{2}}\big)\cup\big(t^{1-\alpha}\big(\frac{|x|}{t}\big)^{1-\frac{\alpha}{2}},\infty\big) = I_1 \cup I_2 \cup I_3$. The fact that the contribution of $s \in (0,1)$ does not exceed the right-hand side of (6.1) is a consequence of Proposition 7.1 in Appendix. Indeed, since $st^\alpha \ll |x|$ for $s\in(0,1)$, then by Proposition 7.1 we obtain

$$q(x, st^\alpha) \le \exp\Big\{ -\mathsf{c}_+ |x| \big(\log\big|\tfrac{x}{t^\alpha}\big|\big)^{\frac{p-1}{p}}\Big\}$$

for all $s \in (0,1)$.

For $s\in I_2$ we have $t^\alpha < st^\alpha < \big(\frac{|x|}{t}\big)^{-\frac{\alpha}{2}}|x| \ll |x|$. According to Proposition 7.1, the following estimate holds:

$$q(x,st^\alpha) \le \exp\Big\{-c_4|x|\,\Big[\log\big(\tfrac{|x|}{st^\alpha}\big)\Big]^{\frac{p-1}{p}}\Big\} \le \exp\Big\{-c_4|x|\,\Big[\log\Big(\tfrac{|x|}{|x|}\big(\tfrac{|x|}{t}\big)^{\frac{\alpha}{2}}\Big)\Big]^{\frac{p-1}{p}}\Big\}$$

$$\le \exp\Big\{ -c_5|x|\,\Big[\log\big(\tfrac{|x|}{t}\big)\Big]^{\frac{p-1}{p}}\Big\}$$

for some $c_5>0$ and for all sufficiently large t uniformly in $s\in I_2$. Then

$$\int_{I_2} W_\alpha(s) q(x,t^\alpha s)\,ds \le \exp\Big\{ -c_6|x|\,\Big[\log\big(\tfrac{|x|}{t}\big)\Big]^{\frac{p-1}{p}}\Big\}. \qquad (6.2)$$

In order to estimate the contribution of the interval I_3 we first obtain an upper bound for $W_\alpha(s)$ with $s\in I_3$:

$$W_\alpha(s) \le \exp\Big\{ -c_2(\alpha) t^{-\frac{\alpha}{1-\alpha}} |x|^{\frac{1}{1-\alpha}} \big(\tfrac{|x|}{t}\big)^{-\frac{\alpha}{2(1-\alpha)}}\Big\}$$

$$= \exp\Big\{ -c_2(\alpha)|x|\big(\tfrac{|x|}{t}\big)^{\frac{\alpha}{1-\alpha}-\frac{\alpha}{2(1-\alpha)}}\Big\} \le \exp\Big\{ -c_7|x|\,\Big[\log\big(\tfrac{|x|}{t}\big)\Big]^{\frac{p-1}{p}}\Big\}.$$

Therefore, for sufficiently large t we have

$$\int_{I_3} W_\alpha(s) q(x,t^\alpha s)\,ds \le \exp\Big\{ -c_8|x|\,\Big[\log\big(\tfrac{|x|}{t}\big)\Big]^{\frac{p-1}{p}}\Big\}. \qquad (6.3)$$

To conclude, under a proper choice of a constant c_+ the contribution of each of the intervals I_1, I_2 and I_3 does not exceed the right-hand side in (6.1). This yields (6.1). □

Appendix

Here we prove several inequalities for the fundamental solution $q(x,t)$.

P r o o f o f P r o p o s i t i o n 2.1. We begin with the upper bound. In the region $\{|x| \le t^{\frac{1}{2}} \log t\}$ we can use the technique based on the properties of the Fourier transform $\widehat{a}(\cdot)$ of $a(\cdot)$. We have (see, for instance, formula (2.6) in [6])

$$q(x,t) = \int_{\mathbb{R}^d} e^{ix\cdot p}\big(e^{-t(1-\widehat{a}(p))} - e^t\big)\, dp.$$

From this formula, considering our assumptions on $a(\cdot)$, one can easily derive the desired upper bound. We leave the details to the reader.

If $|x| \ge t^{\frac{1}{2}} \log t$ then for any $\delta > 0$ and all sufficiently large t we have

$$\exp\Big\{ -\delta\frac{|x|^2}{2}\Big\} \le t^{-\frac{d}{2}}.$$

Proof of Proposition 2.1 It was shown in the proof of [6, Lemma 3.18] that for all $k \ge |x|$ the following inequality holds:

$$a^{*k}(x) \le \exp\Big\{ - I(\frac{x}{k})k\Big\} \le \exp\Big\{ - \mathsf{c}\frac{|x|^2}{k}\Big\}.$$

Proof of Proposition 2.1 Combining this inequality with the Stirling formula we conclude that for some constant $c > 0$ and for all sufficiently large t the following estimate holds:

$$q(x,t) \le \exp\Big\{ - c\frac{|x|^2}{t}\Big\} \le t^{-\frac{d}{2}} \exp\Big\{ - (c-\delta)\frac{|x|^2}{t}\Big\}.$$

Proof of Proposition 2.1 This yields the desired upper bound. □

Proposition 7.1. *Under our standing assumptions on $a(\cdot)$ there exists a constant $\mathsf{c} > 0$ such that in the region $\{(x,t) : t > 0, \ \frac{|x|}{t} \gg 1\}$ the following upper bound holds:*

$$q(x,t) \le \exp\Big\{ - \mathsf{c}|x| \left(\log\left|\tfrac{x}{t}\right|\right)^{\frac{p-1}{p}}\Big\}. \tag{7.1}$$

P r o o f. We use representation (1.5). According to estimate (3.61) in [6], there exist constants $\alpha_p > 0$ and $\varkappa > 0$ such that for all sufficiently large x and for all k with $1 \le k \le \alpha_p\,|x|$ we have

$$a^{*k}(x) \le \exp\Big\{ - \varkappa\frac{|x|^p}{k^{p-1}}\Big\}.$$

If k satisfies the estimate $1 \le k \le |x|\big(\log\big(\frac{|x|}{t}\big)\big)^{-\frac{1}{p}}$, then

$$a^{*k}(x) \le \exp\Big\{ - \varkappa|x|\Big[\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{\frac{1}{p}}\Big]^{p-1}\Big\} = \exp\Big\{ - \varkappa|x|\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{\frac{p-1}{p}}\Big\}.$$

We also have

$$\sum_{k\le |x|\left(\log\left(\frac{|x|}{t}\right)\right)^{-\frac1p}} \frac{t^k\, e^{-t}}{k!} \le 1.$$

Notice that the relation $|x| \gg t$ implies $|x|\big(\log\big(\frac{|x|}{t}\big)\big)^{-\frac1p} \gg t$. If $k \ge |x|\big(\log\big(\frac{|x|}{t}\big)\big)^{-\frac1p}$, then, by the Stirling formula,

$$\frac{t^k}{k!} \le \exp\Big\{-k\log\Big(\frac{k}{t}\Big)+k\Big\} \le \exp\Big\{-\frac12|x|\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{-\frac1p}\log\Big(\frac{|x|}{t}\Big)\Big\}$$
$$\le \exp\Big\{-\frac12|x|\Big(\log\Big(\frac{|x|}{t}\Big)\Big)^{\frac{p-1}{p}}\Big\}.$$

Combining the last three estimates yields the desired inequality in (7.1). □

REMARK 7.1. It should be noted that in the formulation of Proposition 7.1 the value of t might be arbitrarily small. The only relation that matters is $\frac{|x|}{t} \gg 1$.

Next we prove (3.3).

PROPOSITION 7.2. *For any* $\delta > 0$

$$\lim_{t\to\infty}\ \sup_{s\ge\delta,\, v\in\mathbb{R}^d} \Big|t^{\frac{d\alpha}{2}} q\big(t^{\frac{\alpha}{2}}v, st^\alpha\big) - \Psi(v,s)\Big| = 0.$$

P r o o f. We divide the sum in formula (1.5) into three parts as follows:

$$q(x,t) = e^{-t}\sum_{n=1}^{\infty}\frac{t^n}{n!}a^{*n}(x) = e^{-t}\Big\{\sum_{n=1}^{t-t^{3/4}} + \sum_{n=t-t^{3/4}}^{t+t^{3/4}} + \sum_{n=t+t^{3/4}}^{\infty}\Big\}\frac{t^n}{n!}a^{*n}(x).$$

With the help of the Stirling formula one can easily check that the first and the last sums here are of order $O\big(e^{-c\sqrt{t}}\big)$ as $t\to\infty$. Therefore,

$$q(x,t) = e^{-t}\sum_{n=t-t^{3/4}}^{t+t^{3/4}}\frac{t^n}{n!}a^{*n}(x) + O(e^{-c\sqrt{t}}). \tag{7.2}$$

We need to estimate the quantity $t^{\frac{d\alpha}{2}}a^{*n}(t^{\frac{\alpha}{2}}v)$ with $n \in \big(st^\alpha - (st^\alpha)^{\frac34}, st^\alpha + (st^\alpha)^{\frac34}\big)$. Observe that for the function $\Psi(v,s)$ defined by (1.21) the following relation holds:

$$\Psi(v,s) = \frac{1}{s^{d/2}}\Psi\Big(\frac{v}{\sqrt{s}},1\Big). \tag{7.3}$$

Then from the uniform in v estimate (3.2) we deduce

$$\frac{(st^\alpha)^{d/2}}{s^{d/2}} a^{*n}\big((st^\alpha)^{1/2}\frac{v}{\sqrt{s}}\big) = \frac{1}{s^{d/2}} n^{d/2}(1+o(1)) a^{*n}\big(\sqrt{n}\frac{v}{\sqrt{s}}(1+o(1))\big) \to$$

$$\to \frac{1}{s^{d/2}}\Psi(\frac{v}{\sqrt{s}},1) = \Psi(v,s);$$

here the inequality $n \geq \frac{\delta}{2}t^\alpha$ has been used. Thus, for any $v \in \mathbb{R}^d$,

$$\max_{n\in(st^\alpha-(st^\alpha)^{\frac{3}{4}},st^\alpha+(st^\alpha)^{\frac{3}{4}})} \left| t^{\frac{d\alpha}{2}} a^{*n}(t^{\alpha/2}v) - \Psi(v,s)\right| \to 0, \tag{7.4}$$

as $t \to \infty$. Moreover, the convergence is uniform with respect to $s \geq \delta$. Finally from (7.2) and (7.4) we obtain

$$t^{\frac{d\alpha}{2}} q(t^{\alpha/2}v, st^\alpha) - \Psi(v,s)$$

$$= e^{-st^\alpha}\sum_{n=1}^{\infty}\frac{(st^\alpha)^n}{n!} a^{*n}(t^{\alpha/2}v)\, t^{\frac{d\alpha}{2}} - e^{-st^\alpha}\sum_{n=1}^{\infty}\frac{(st^\alpha)^n}{n!}\Psi(v,s)$$

$$= e^{-st^\alpha}\sum_{n=st^\alpha-(st^\alpha)^{3/4}}^{st^\alpha+(st^\alpha)^{3/4}}\frac{(st^\alpha)^n}{n!}\Big[a^{*n}(t^{\alpha/2}v)\, t^{\frac{d\alpha}{2}} - \Psi(v,s)\Big] \; + \; O(e^{-c\sqrt{t}}) \to 0.$$

This yields (3.3). □

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