

Estimation of Ambiguity Functions With Limited Spread

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Abstract

This paper proposes a new estimation procedure for the ambiguity function of a non-stationary time series. The stochastic properties of the empirical ambiguity function calculated from a single sample in time are derived. Different thresholding procedures are introduced for the estimation of the ambiguity function. Such estimation methods are suitable if the ambiguity function is only non-negligible in a limited region of the ambiguity plane. The thresholds of the procedures are formally derived for each point in the plane, and methods for the estimation of nuisance parameters that the thresholds depend on are proposed. The estimation method is tested on several signals, and reductions in mean square error when estimating the ambiguity function by factors of over a hundred are obtained. An estimator of the spread of the ambiguity function is proposed.

Index Terms

Ambiguity function, estimation, harmonizable process, non-stationary process, thresholding, underspread process.

I. INTRODUCTION

We propose a new estimation procedure for the Ambiguity Function (AF) of a non-stationary Gaussian process. The suitable definition of a ‘locally stationary process’, and the development of inference methods for a given sample from a non-stationary process is still an open question, see [1]. The AF forms an essential component to this task, as it provides a characterisation of dependency between a given time series and its translates in time and frequency. The AF of a non-stationary process can often be assumed to be mainly limited in support to a small region of the ambiguity plane, and this corresponds to correlation in the process of interest being limited in support. If the support is additionally centred at time and frequency lags of zero, then the process is underspread, see Matz and Hlawatsch [2]. An important class of non-stationary processes, namely semi-stationary processes [3], exhibit limited essential spread in the ambiguity plane. Non-stationary processes can also be constructed from the viewpoint of time-variant linear filtering. Pfander and Walnut [4] have shown that if the spreading of a filtering operator is sufficiently limited then the operator may be identified from its measurements. It is not necessary in this case to assume that the spread is centered at the origin [4]. The key to the estimation or characterization of the generating mechanism of non-stationary processes, in all of these cases, is the compression of the AF of the process.

The AF has also been popularly used in radar and sonar applications [5], where the echo of a known signal is recorded. The delay and frequency shift of the echo can be determined from the AF of the signal [6]. In such applications the emitted signal is deterministic, even if the echo has been contaminated by noise. By using the compression of the emitted signal in the AF domain, the properties of the process of reflection can be determined, see Ma and Goh [7]. The estimation of the AF of a non-zero mean signal immersed in noise is of interest. Unlike Ma and Goh, we shall not assume the compression of the AF is known, but propose an estimator of the AF, suitable if the AF satisfies some (unknown) compression constraints.

Given the importance of the AF it is surprising to find that existing methods for its estimation are quite naive. Methods are usually based on calculating the Empirical AF (EAF), or some smoothed version of the EAF. It is clearly unpalatable to implement a smoothing procedure on the EAF, as the compression of the EAF should be preserved, or even preferably increased, by any proposed estimation procedure. In this spirit Jachan *et al.* [8] have used shrinkage methods to estimate the EAF with an assumed knowledge of its support, using a multiplier that attenuates larger local time and frequency lags. Often the support of the AF is not known, and arbitrary shrinkage at larger lags will not be an admissible estimation procedure.

Given the compression or sparsity of the AF, shrinkage or threshold estimators are a natural choice. In this paper we propose a set of threshold estimators for the AF. Thresholding has been implemented before in the global time-frequency plane for wavelet spectra, or evolutionary spectra, see Fryzlewicz and Nason [9], or von Sachs and Schneider [10]. The estimation

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problem in this case is quite different to estimating the AF, as we cannot assume the raw wavelet/evolutionary spectra exhibit compression. Hedges and Suter [11] calculated the spread of the AF, based on exceedances of the magnitude of the EAF over a (fixed and user tuned) threshold in a given direction in the local time-frequency plane. The stochastic properties of the EAF were not treated by Hedges and Suter, and instead their investigation focused on some very interesting effects from different boundary treatments.

To be able to select a suitable threshold procedure at each local time-frequency point we must establish the distribution of the EAF, and this requires modelling the observed process. We discuss the classes of non-stationary processes that we intend to treat in Section II-A, and initial method of moments estimators of second order structure in Section II-B. In Section III we determine the first and second order properties of the EAF. For strictly underspread processes satisfying constraints on their degree of uniformity we show that the EAF is asymptotically Gaussian proper, if calculated for local frequency and time lags outside the support of the AF.

We introduce the proposed estimation procedure in Section IV for deterministic signals immersed in noise and stochastic processes. In the first estimation procedure we threshold the EAF based on a variance estimated from the entire plane, this yielding the Thresholded EAF (TEAF). For stochastic processes, we also introduce a Local TEAF (LTEAF), where we estimate the variance locally in regions of the ambiguity plane. The EAF of deterministic signals immersed in white noise will be biased. We propose a method to remove this bias prior to local thresholding, thus obtaining the Bias-corrected LTEAF (BLTEAF). We define an estimator of the total spread of a signal in the ambiguity plane in Section IV-A, to numerically measure the support of the signal in the ambiguity plane. The proposed TEAFs will produce sparse approximations to processes that are not strictly underspread. Whenever the EAF is small in comparison to the variance of the EAF, the AF will be estimated as zero. The AF can provide essential information as to the natural bandwidth of a process via the spread of the TEAF.

In Section V we provide simulation studies of the proposed estimators. The Mean Square Errors (MSE) of our procedures reduce in many cases to around a hundredth of the MSE of the EAF. Plots of single realisations show the accuracy of the threshold estimators, based on a single sample. The proposed methodology thus introduces a new method of characterising the features of structured non-stationary processes with high accuracy.

II. SECOND ORDER STRUCTURE DESCRIPTIONS

A. Modelling the Observations

We assume that the observed signal $\{U[t]\}_{t=0}^{N-1}$ is an aggregation of three components, a non-zero and non-stationary mean function $\{\mu_X[t]\} \in \ell_2$, a harmonizable zero-mean stochastic component $\{\check{X}[t]\}$ [12], and additional white noise $\{\varepsilon_X[t]\}$ with variance σ_ε^2 , or

$$U[t] = \mu_X[t] + \check{X}[t] + \varepsilon_X[t] = X[t] + \varepsilon_X[t], \quad t \in \mathbb{Z}. \quad (1)$$

Both $\{\check{X}[t]\}$ and $\{\varepsilon_X[t]\}$ are assumed to be Gaussian processes, and assumed to be mutually independent. Note that $[\cdot]$ is used to indicate a discrete argument, and (\cdot) is used for a continuous argument. As $\{\check{X}[t]\}$ is harmonizable it admits the spectral representation [13] of

$$\check{X}[t] = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi t f} dZ_X(f), \quad t \in \mathbb{Z}, \quad (2)$$

where $\{dZ_X(f)\}$ is an increment process. We note that $\{dZ_X(f)\}$ has a correlation structure specified by the dual-frequency spectrum $S_{XX^*}(\nu, f) d\nu df = E\{dZ_X(\nu + f)dZ_X^*(f)\}$. If $\mu_X[t]$ is zero the AF of $\{X[t]\}$ is defined as a Fourier Transform (FT) pair with the dual-frequency spectrum in variable f . The AF forms an FT pair with the dual-time second moment of the process, or $M_{XX^*}[t, \tau] = E\{X[t]X^*[t - \tau]\}$, for t and $\tau \in \mathbb{Z}$, now in variable t . We refer to [14], [15] for a more detailed exposition of the AF and forms of AFs of common processes.

Estimation of time-frequency descriptions from a real-valued sampled signal is known to display artifacts from aliasing and interference from negative frequencies [16]. For this reason we calculate the EAF of the discrete analytic signal [17], calculated using a discrete Hilbert transform from a periodic length N -filter, where the operation is represented by $\mathcal{Q}_N\{\cdot\}$. Note that since we are working with the analytic signal of a non-stationary process, it may be improper [18] and thus may exhibit complementary correlation [19]. We leave the estimation of the complimentary AF outside the scope of this paper, noting that our proposed methods can in a straight forward fashion be extended to include the estimation of such objects. Furthermore, we will in this paper assume that we are working with proper processes, and any terms that will only be non-zero for improper processes will be omitted.

B. Initial Estimation

We observe a finite sample of $U[t]$ at $t = 0, \dots, N - 1$, which is extended into the discrete analytic signal

$$W_N[t] = U[t] + j\mathcal{Q}_N\{U\}[t] = X[t] + j\mathfrak{S}\{\mathcal{Q}_N\{X\}[t]\} + \varepsilon_X[t] + j\mathfrak{S}\{\mathcal{Q}_N\{\varepsilon_X\}[t]\} = Z_N[t] + \varepsilon_N[t]. \quad (3)$$

The Sample AF (SAF) of the sampled mean, $\{\mu_N[t]\}_{t=0}^{N-1}$, is given by

$$A_{\mu_N \mu_N^*}(\nu, \tau) = \sum_{t=\max(0, \tau)}^{N-1+\min(0, \tau)} \mu_N[t] \mu_N^*[t-\tau] e^{-j2\pi\nu t}, \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tau = [-(N-1), \dots, (N-1)] \quad (4)$$

The properties of $A_{\mu_N \mu_N^*}(\nu, \tau)$ follow directly from properties of AFs of deterministic sequences, see [20], [21]. The effects of discretization on deterministic structure, i.e. trying to infer properties of $\mu_X(t)$ from $\mu_X[t] \equiv \mu_X(t\Delta t)$ for some appropriate sampling period $\Delta t > 0$, will also be left outside the scope of this article [22], [23]. The Empirical Second Moment (ESM) of $\{Z_N[0], \dots, Z_N[N-1]\}$ is given by $\widehat{M}_{ZZ^*}[t, \tau] = W_N[t]W_N^*[t-\tau]$, which has an expectation $E\{\widehat{M}_{ZZ^*}[t, \tau]\} = M_{Z_N Z_N^*}[t, \tau] + 2\sigma_\varepsilon^2$. Thus this estimator is biased, and $\text{var}\{\widehat{M}_{ZZ^*}[t, \tau]\} = M_{Z_N Z_N^*}(t, 0)M_{Z_N Z_N^*}^*(t-\tau, 0) + 2\sigma_\varepsilon^2 M_{Z_N Z_N^*}(t, 0) + 2\sigma_\varepsilon^2 M_{Z_N Z_N^*}(t-\tau, 0) + 4\sigma_\varepsilon^4$. Note that the ESM, its expected value and variance is defined as above for $0 \leq t \leq N-1$ and $t-(N-1) \leq \tau \leq t$, and they are zero otherwise. Clearly the variance of $\{\widehat{M}_{ZZ^*}[t, \tau]\}$ is very large and the estimator should not be put to direct use. We form the EAF of $X[t]$ by

$$\widehat{\mathcal{A}}_{XX^*}(\nu, \tau) = \sum_{t=\max(0, \tau)}^{N-1+\min(0, \tau)} \widehat{M}_{ZZ^*}[t, \tau] e^{-j2\pi\nu t} = \sum_{t=\max(0, \tau)}^{N-1+\min(0, \tau)} W_N[t]W_N^*[t-\tau] e^{-j2\pi\nu t} = A_{W_N W_N^*}(\nu, \tau). \quad (5)$$

Here, $A_{W_N W_N^*}(\nu, \tau)$ is equivalent to the SAF of $\{W_N[t]\}$, and corresponds to an *estimator* of the AF of $\{X[t]\}$. The quantity $\widehat{\mathcal{A}}_{XX^*}(\nu, \tau)$ is also *not* a suitable estimator of $\mathcal{A}_{XX^*}(\nu, \tau)$, as its variance will be extremely large. When implementing (5) we have chosen zero-padding as our choice of extending the estimated covariance matrix. We found that periodic (circular) extension created unpalatable mixing effects of a fundamentally non-stationary object. Edge-effects are discussed more carefully in [11]. The best choice of edge treatment will depend on the statistical and deterministic properties of the chosen EAF.

III. PROPERTIES OF THE EMPIRICAL AMBIGUITY FUNCTION

A. Representation and Moments of the EAF

We shall construct estimators of the AF, based on the EAF. The EAF of a zero-mean harmonizable process $\{X[t]\}$ admits the representation of

$$\begin{aligned} \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) &= 4 \int_0^{1/2} \int_0^{1/2} e^{j\pi(f_1 - f_2 - \nu)(N + \tau - 1)} dZ_X(f_1) dZ_X^*(f_2) \\ &\quad \times e^{j2\pi f_2 \tau} D_{N-|\tau|}(\pi(f_1 - f_2 - \nu)) + O\left(\frac{N-|\tau|}{N}\right), \end{aligned} \quad (6)$$

where $D_N(\pi f) = \sin(\pi f N) / \sin(\pi f)$, and the error term is due to usage of the discrete analytic signal. We respectively denote $\mu_{A,N}(\nu, \tau)$, $\sigma_{A,N}^2(\nu, \tau)$ and $r_{A,N}(\nu, \tau)$ as the mean, variance and relation of the EAF.

Proposition 3.1: Moments of the EAF for noisy Deterministic Signals

For an observed signal taking the form of (1) with the assumption that $\check{X}[t] \equiv 0$ the expected value, variance and relation of the EAF take the form

$$\mu_{A,N}(\nu, \tau) = \mathcal{A}_{\mu_N \mu_N^*}(\nu, \tau) + 2\sigma_\varepsilon^2 e^{-j\pi\nu(N+\tau-1)} D_{N-|\tau|}(\pi\nu) e^{j\pi\tau/2} \text{sinc}(\pi\tau/2) + O\left(\frac{N-|\tau|}{N}\right) \quad (7)$$

$$\sigma_{A,N}^2(\nu, \tau) = 4\sigma_\varepsilon^2 v_{ZZ^*}(\nu, \tau) + 4\sigma_\varepsilon^2 v_{ZZ^*}(-\nu, -\tau) + 16\sigma_\varepsilon^4 [N-|\tau|][1/2-|\nu|] + O(1) \quad (8)$$

$$\begin{aligned} r_{A,N}(\nu, \tau) &= -16[N-|\tau|]\sigma_\varepsilon^4 L(N-|\tau|, \nu) e^{-2j\pi\nu(N-1)} \left[\frac{\sin(2\pi|\nu|\tau)}{2\pi\tau} I(\tau \neq 0) - \frac{1}{2} I(\tau = 0) \right] \\ &\quad + 8\sigma_\varepsilon^2 \Re\{c_{ZZ^*}(\nu, \tau)\} + O(1) \end{aligned} \quad (9)$$

$$L(N-\tau, \nu) = \int_{-\infty}^{\infty} \frac{\sin(\pi f) \sin(\pi(f+2\xi(\nu)))}{\pi f \pi(f+2\xi(\nu))} df, \quad v_{ZZ^*}(\nu, \tau) = \int_{\max(-\nu, 0)}^{1/2-\max(0, \nu)} |\mathcal{M}_Z(f|N, \tau)|^2 df$$

$$c_{ZZ^*}(\nu, \tau) = \int_{\max(0, -\nu)}^{1/2+\min(0, -\nu)} e^{j2\pi f \tau} \mathcal{M}_Z^*(f|N, \tau) \mathcal{M}_Z(f+2\nu|N, -\tau) df$$

where $\xi(\nu) = [N-|\tau|]\nu$ and $\mathcal{M}_Z(f|N, \tau) = \sum_{t=0}^{N-1-\tau} \mu_N[t\tau] e^{-j2\pi f t}$ for $\tau \geq 0$, whilst $\mathcal{M}_Z(f|N, \tau) = \sum_{t=|\tau|}^{N-1} \mu_N[t\tau] e^{-j2\pi f t} e^{-j2\pi f \tau}$ for $\tau < 0$.

Proof: See appendix I. ■

In order to calculate the MSE of any estimator, we need an ideal value to compare with. To that end, for a finite value of N , we define the N -AF for a deterministic sequence to be $\mathcal{A}_{XX^*}^{(N)}(\nu, \tau) = \mathcal{A}_{Z_N Z_N^*}(\nu, \tau)$ if $|\mathcal{A}_{XX^*}(\nu, \tau)| \neq 0$, and zero otherwise.

$\mathcal{A}_{XX^*}(\nu, \tau)$ is based on the *infinite* sequence $\{M_{XX^*}[t, \tau]\}_{t, \tau \in \mathbb{Z}}$ and does *not* exhibit any finite sample issues with spreading in local frequency and time lag due to finite sample effects. We can never observe such values in a finite digital sample, because the maximum concentration of energy will behave like N (our number of sample points), and thus will be finite for finite N . Thus we ideally insist on the concentration of the AF to where $\mathcal{A}_{XX^*}(\nu, \tau)$ is supported, but only letting the N -AF taking finite sample length realizable values.

Proposition 3.2: Moments of the EAF for a Stationary Process

For a zero-mean real-valued stationary stochastic process $X[t]$ the expected value, variance and relation of the EAF take the form

$$\mu_{A,N}(\nu, \tau) = 2D_{N-|\tau|}(\pi\nu)e^{-j\pi\nu(N+\tau-1)} \left(\widetilde{M}_{XX^*}[\tau] + j\mathcal{Q}_N \left\{ \widetilde{M}_{XX^*} \right\} [\tau] \right) + O\left(\frac{N-|\tau|}{N}\right) \quad (10)$$

$$\sigma_{A,N}^2(\nu, \tau) = 16[N-|\tau|][1/2-|\nu|]\overline{S}_{XX^*}(\nu) + O(1) \quad (11)$$

$$\begin{aligned} r_{A,N}(\nu, \tau) &= 16e^{-2j\pi\nu(N+\tau-1)}[N-|\tau|]L(N-|\tau|, \nu) \\ &\quad \times \int_{\max(0, \nu)}^{1/2+\min(0, \nu)} \widetilde{S}_{XX^*}(f)\widetilde{S}_{XX^*}(f-\nu)e^{j4\pi f\tau} df + O(1), \end{aligned} \quad (12)$$

where $\widetilde{M}_{XX^*}[\tau]$ is the autocorrelation sequence of the process, $\widetilde{S}_{XX^*}(f)$ is the spectral density of the process, and

$$\overline{S}_{XX^*}(\nu) = \frac{1}{1/2-|\nu|} \int_{\max(0, -\nu)}^{1/2+\min(0, -\nu)} \widetilde{S}_{XX^*}(g_2 + \nu)\widetilde{S}_{XX^*}(g_2) dg_2. \quad (13)$$

Proof: See appendix II. ■

The AF of a stationary process will be nonzero only on the stationary manifold $\nu = 0$, and takes the form $\mathcal{A}_{XX^*}(\nu, \tau) = \widetilde{M}_{XX^*}[\tau]\delta(\nu)$. For a *fixed* value of N , $D_{N-|\tau|}(\pi\nu)$ does *not* correspond to a delta function in ν . Ideally however the estimated EAF should not exhibit spreading in ν . For a finite value of N we define the N -AF for a stationary process to be $\mathcal{A}_{XX^*}^{(N)}(\nu, \tau) = 2[N-|\tau|] \left(\widetilde{M}_{XX^*}[\tau] + j\mathcal{Q}_N \left\{ \widetilde{M}_{XX^*} \right\} [\tau] \right)$ if $|\mathcal{A}_{XX^*}(\nu, \tau)| \neq 0$, and zero otherwise.

Proposition 3.3: Moments of the EAF for a Uniformly Modulated White Noise Process

For a real-valued zero-mean uniformly modulated white noise process $\{X[t]\}$ the expected value, variance and relation of the EAF take the form

$$\mu_{A,N}(\nu, \tau) = 4\Sigma_{XX^*}(\nu)e^{j\pi(1/2-|\nu|)\tau} \frac{\sin(\pi(1/2-|\nu|)\tau)}{\pi\tau} + O(1) \quad (14)$$

$$\sigma_{A,N}^2(\nu, \tau) = 16[N-|\tau|][1/2-|\nu|]\varsigma_{XX^*}^4(\nu, \tau) + O(1) \quad (15)$$

$$\begin{aligned} r_{A,N}(\nu, \tau) &= 16e^{-4j\pi\nu\tau} \int_{\max(0, \nu)}^{1/2+\min(0, \nu)} \int_{\max(0, \nu)}^{1/2+\min(0, \nu)} \Sigma_{XX^*}(f-\alpha+\nu) \\ &\quad \times \Sigma_{XX^*}^*(f-\nu-\alpha)e^{j2\pi(f+\alpha)\tau} df d\alpha + O(1). \end{aligned} \quad (16)$$

$$\varsigma_{XX^*}^4(\nu, \tau) = \frac{1}{[N-|\tau|][1/2-|\nu|]} \int_{-1/2+|\nu|}^{1/2-|\nu|} (1/2-|\nu|) |\Sigma_{XX^*}(f)|^2 e^{j2\pi f\tau} df,$$

where $\Sigma_{XX^*}(\nu)$ is the DFT of the variance of $\{X[t]\}$, $\sigma_X^2[t]$.

Proof: See appendix III. ■

We note that $\varsigma_{XX^*}^2(\nu, \tau)$ decreases with $|\nu|$ and also with $|\tau|$. We define the N -AF for a uniformly modulated white noise process to be $\mathcal{A}_{XX^*}^{(N)}(\nu, \tau) = 4\Sigma_{XX^*}(\nu)[1/2-|\nu|]$ if $|\mathcal{A}_{XX^*}(\nu, \tau)| \neq 0$, and zero otherwise. Since the process is uncorrelated in time, the true AF is only nonzero for $\tau = 0$, and takes the form $\mathcal{A}_{XX^*}(\nu, \tau) = \Sigma_{XX^*}(\nu)\delta[\tau]$.

B. Distribution of the EAF of an Underspread Process

To determine suitable estimation procedures for the AF we need to derive the distribution of the EAF.

Theorem 3.1: Distribution of the EAF of an underspread process

Assume $\{X[t]\}$ corresponds to a harmonizable real-valued zero-mean Gaussian process whose EAF is strictly underspread, that is $\mathcal{A}_{XX^*}(\nu, \tau)$ is only non-zero for $(\nu, \tau) \in \mathcal{D}$, where there exists some finite non-negative T and Ω such that $\mathcal{D} \subset [-\Omega, \Omega] \times [-T, T]$. Then the EAF evaluated at (ν, τ) has a mean, variance and relation of

$$\mu_{A,N}(\nu, \tau) = \mathcal{A}_{ZZ^*}(\nu, \tau) + O(1) \quad (17)$$

$$\sigma_{A,N}^2(\nu, \tau) = \sum_{t=\max(0, \tau)}^{N-1+\min(0, \tau)} \sum_{\tau'=-T}^{T-1} e^{-j2\pi\nu\tau'} M_{ZZ^*}[t, \tau'] M_{ZZ^*}^*[t-\tau, \tau'] + O\left(\log\left[\frac{N}{T}\right]\right) \quad (18)$$

$$r_{A,N}(\nu, \tau) = O\left(\log\left[\frac{N}{T}\right]\right) \quad \text{if } |\tau| > T. \quad (19)$$

Fix $(\nu, \tau) \notin \mathcal{D}$. Let $\tau \geq 0$ and take

$$R_N[t] = e^{-j2\pi\nu(t+\tau)} (Z_N[t+\tau]Z_N^*[t] - M_{Z_N Z_N^*}[t+\tau, \tau]), \quad t = 0, \dots, N-1-\tau. \quad (20)$$

Assume that the triangular array $\{R_N[t], \quad t = 0, \dots, N-1-\tau\}$ is strongly mixing and satisfies the Lindeberg condition (see (2.2) of theorem 2.1 in [24]), as well as the condition:

$$\sup_N \frac{1}{\sigma_{A,N}^2(\nu, \tau)} \sum_{t=0}^{N-\tau-1} M_{Z_N Z_N^*}[t+\tau, 0] M_{Z_N Z_N^*}[t, 0] < \infty, \quad (21)$$

then with $\xrightarrow{\mathcal{L}}$ denoting convergence in law [25]

$$\frac{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - \mu_{A,N}(\nu, \tau)}{\sigma_{A,N}(\nu, \tau)} \xrightarrow{\mathcal{L}} \mathcal{N}^C(0, 1, 0), \quad (\nu, \tau) \notin \mathcal{D}, \quad t \geq 0. \quad (22)$$

The result follows *mutatis mutandis* for $\tau < 0$.

Proof: See appendix IV. ■

It is tempting to attempt to deduce the distributional results directly from the first and second order structure established in the first part of the theorem. In fact for a stationary process, where T and τ are both even, the result follows directly. For more general classes of processes the result is slightly more involved, and a condition on the variance needs to be combined with a suitable mixing assumption, as above. Note that the CLT theorem that we use does not provide rates of convergence. Also, in some degenerate cases the joint distribution of $\left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\}_{(\nu, \tau)}$ may not be asymptotically multivariate Gaussian. For ease of exposition, such cases are not treated here. Finally, whilst (asymptotically) retrieving the Gaussian distribution for the EAF by Theorem 3.1, we fail to obtain simple interpretable forms for $\sigma_{A,N}^2(\nu, \tau)$ and $r_{A,N}(\nu, \tau)$. For this reason, it is convenient to derive the first and second order structure directly from the modelling assumptions of some commonly used processes, as done in Section III. We define the N -AF of an underspread process as $\mathcal{A}_{XX^*}^{(N)}(\nu, \tau) = \mathcal{A}_{Z_N Z_N^*}(\nu, \tau)$ if $(\nu, \tau) \in \mathcal{D}$, and zero otherwise.

IV. ESTIMATION PROCEDURE

To determine the approximate distribution of the EAF of a deterministic signal immersed in white noise, we need to note the distributions of the four components of (A-1). Here, $A_{Z_N Z_N^*}(\nu, \tau)$ is constant, whilst $A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)$ is asymptotically Gaussian, and its form is determined by Theorem 3.1. We note that as $\varepsilon_X[t]$ is Gaussian white noise, and so from (A-3) and (A-6)

$$A_{\varepsilon_N Z_N^*}(\nu, \tau) + A_{Z_N \varepsilon_N^*}(\nu, \tau) \stackrel{d}{=} \mathcal{N}^C \left(0, 4\sigma_\varepsilon^2 [v_{ZZ^*}(\nu, \tau) + v_{ZZ^*}(-\nu, -\tau)] + O \left(\frac{\|\mathcal{M}_Z\|}{N} \right), \right. \\ \left. 8\sigma_\varepsilon^2 \Re \{c_{ZZ^*}(\nu, \tau)\} + O \left(\frac{\|\mathcal{M}_Z\|}{N} \right) \right). \quad (23)$$

Note that $\|\mathcal{M}_Z\|$ is defined in appendix I. From (A-2) and (A-5) we can note that $A_{\varepsilon_N Z_N^*}(\nu, \tau) + A_{Z_N \varepsilon_N^*}(\nu, \tau)$ is independent of $A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)$. Combining (A-1) and Theorem 3.1, as well as using proposition 3.1, it follows for $(\nu, \tau) \neq (0, 0)$

$$\frac{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - \mu_{A,N}(\nu, \tau)}{\sigma_{A,N}(\nu, \tau)} \xrightarrow{\mathcal{L}} \mathcal{N}^C \left(0, 1, r_{A,N}(\nu, \tau) / \sigma_{A,N}^2(\nu, \tau) \right) \quad (24)$$

$$\mu_{A,N}(\nu, \tau) = \mathcal{A}_{\mu_N \mu_N^*}(\nu, \tau) + 2\sigma_\varepsilon^2 e^{-j\pi\nu(N+\tau-1)} D_{N-|\tau|}(\pi\nu) e^{j\pi\tau/2} \text{sinc}(\pi\tau/2) + O \left(\frac{N-|\tau|}{N} \right). \quad (25)$$

Hence we may note $|\mu_{A,N}(\nu, \tau)|^2 \ll \sigma_{A,N}^2(\nu, \tau)$ in most of the ambiguity plane. Furthermore it follows that

$$\left| \frac{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - \mu_{A,N}(\nu, \tau)}{\sigma_{A,N}(\nu, \tau)} \right|^2 \stackrel{d}{=} \frac{1}{2} \left[1 + \frac{|r_{A,N}(\nu, \tau)|}{\sigma_{A,N}^2(\nu, \tau)} \right] U_1^2 + \frac{1}{2} \left[1 - \frac{|r_{A,N}(\nu, \tau)|}{\sigma_{A,N}^2(\nu, \tau)} \right] U_2^2 + o(1), \quad (26)$$

where U_1 and U_2 are independent standard Gaussian random variables. Thus for such points in the ambiguity plane it follows that $\left| \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - \mu_{A,N}(\nu, \tau) \right|^2$ is a weighted sum of χ_1^2 's. These are equally weighted if $r_{A,N}(\nu, \tau) = 0$. For most points in the ambiguity plane this statement will be roughly valid as apparent from proposition 3.1. Then if $(\nu, \tau) \notin \mathcal{D}$, where \mathcal{D} denotes the support region of $\mathcal{A}_{XX^*}(\nu, \tau)$, we may note that

$$\left| \frac{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - \mu_{A,N}(\nu, \tau)}{\sigma_{A,N}(\nu, \tau)} \right|^2 \stackrel{d}{=} \left[\frac{1}{2} \chi_2^2 + O \left(\frac{|\log(N)|^2}{N-|\tau|} \right) \right] + o(1). \quad (27)$$

A suitable estimation procedure for such random variables with $\sigma_{A,N}^2(\nu, \tau)$ known a priori, is then the following thresholding procedure, for some given threshold λ^2 ,

$$\widehat{\mathcal{A}}_{XX^*}^{(\text{ht})}(\nu, \tau) = \begin{cases} \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) & \text{if } \left| \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right|^2 > \lambda^2 \sigma_{A,N}^2(\nu, \tau) \\ 0 & \text{if } \left| \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right|^2 \leq \lambda^2 \sigma_{A,N}^2(\nu, \tau) \end{cases}. \quad (28)$$

If we normalize $\left\{ \left| \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right|^2 \right\}$ via dividing the sequence, element by element, by $\sigma_{A,N}^2(\nu, \tau)$, then we retrieve a set of *correlated* positive random variables. For any such collection, we may note from [26], that using a threshold $\lambda_{N_X}^2(C) = 2 \log(N_X [\log(N_X)]^C)$ for $C \geq 1$ is suitable. In our example N_X is chosen to be twice the number of observations, as we threshold the ambiguity function frequency by frequency, across the total collection of all time lags. Olhede [26][p. 1529] has calculated the risk of this non-linear estimator for sums of unequally weighted χ_1^2 's.

Of course $\sigma_{A,N}^2(\nu, \tau)$ is *not* known. We standardize the EAF by

$$\widehat{\mathcal{A}}_{XX^*}^{(S)}(\nu, \tau) = \frac{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau)}{\sqrt{[N - |\tau|][1/2 - |\nu|]}}, \quad (29)$$

and note that

$$\text{var} \left\{ \widehat{\mathcal{A}}_{XX^*}^{(S)}(\nu, \tau) \right\} = 16\sigma_\varepsilon^4 + 4\sigma_\varepsilon^2 \frac{v_{ZZ^*}(\nu, \tau) + v_{ZZ^*}(-\nu, -\tau)}{[N - |\tau|][1/2 - |\nu|]} + O\left(\frac{1}{N - |\tau|}\right). \quad (30)$$

We have assumed that $\{\mu_X[t]\} \in \ell_2$, and so $|v_{ZZ^*}(\nu, \tau)| = O(1)$. Thus $4\sigma_\varepsilon^2 \frac{v_{ZZ^*}(\nu, \tau) + v_{ZZ^*}(-\nu, -\tau)}{[N - |\tau|][1/2 - |\nu|]} = o(1)$, where the form of $v_{ZZ^*}(\nu, \tau)$ at high values of $|\nu|$ and $|\tau|$ ensure that the statement is still true for such values. We shall use the Median Absolute Deviation (MAD) estimator of the variance. MAD has been used for estimating the scale of correlated data before, see [27]. The MAD estimator will need an adjustment factor that is different for $\frac{1}{2}\chi_2^2$ random variables compared to χ_1^2 random variables. We note the median of a $\frac{1}{2}\chi_2^2$ is $\ln[2]$. We define an estimator of the white noise variance for any region $\{(\nu, \tau)\} = \mathcal{M} \subset [-1/2, 1/2] \times \{-(N-1), \dots, (N-1)\}$ by

$$\widehat{\sigma}_\varepsilon^4(\mathcal{M}) = \frac{1}{16} \frac{\text{median} \left\{ \left| \widehat{\mathcal{A}}_{XX^*}^{(S)}(\nu, \tau) \right|^2 \right\}_{(\nu, \tau) \in \mathcal{M}}}{\ln[2]}. \quad (31)$$

The imprecision of this procedure will depend on the lack of compression of the representation of $\{X[t]\}$ in the ambiguity domain. We note that MAD has a breakdown point of 50 %, and so with quite severe contamination the estimator will still be useful, if somewhat inefficient. We then take

$$\widehat{\sigma}_{A,N}^2(\nu, \tau | \mathcal{M}) = 16\widehat{\sigma}_\varepsilon^4(\mathcal{M})[N - |\tau|][1/2 - |\nu|]. \quad (32)$$

A suitable threshold procedure (now with an unknown $\sigma_{A,N}^2(\nu, \tau)$) simply corresponds to using (28) with the variance replaced by its estimated value from (32). If the variance is estimated from the entire plane, i.e., $\mathcal{M} = \{-N/(2N), \dots, N/(2N)\} \times \{-(N-1), \dots, (N-1)\}$, we denote $\widehat{\mathcal{A}}_{XX^*}^{(\text{ht})}(\nu, \tau)$ the Thresholded EAF (TEAF).

The EAF of a deterministic signal immersed in white noise is biased, see proposition 3.1. Given an estimator of σ_ε^2 , we can remove this bias prior to thresholding. We define the bias-corrected EAF by

$$\widehat{\mathcal{A}}_{XX^*}^{(B)}(\nu, \tau) = \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - 2\widehat{\sigma}_\varepsilon^2(\mathcal{M}_1)e^{-j\pi\nu(N+\tau-1)}D_{N-|\tau|}(\pi\nu)e^{j\pi\tau/2}\text{sinc}(\pi\tau/2). \quad (33)$$

The noise variance is estimated from some given region \mathcal{M}_1 , which is chosen to only include the rims of the ambiguity plane. We normalize $\widehat{\mathcal{A}}_{XX^*}^{(B)}(\nu, \tau)$ as in (29). For smaller sample sizes, it may be unreasonable to assume that the contributions of $v_{ZZ^*}(\nu, \tau)$ are negligible compared to $\sqrt{N - |\tau|}$. We define $\overline{\sigma}_X^4(\nu, \tau) = \sigma_{A,N}^2(\nu, \tau)/[(N - |\tau|)(1/2 - |\nu|)]$. This is explicitly subscripted by X , as it is only non-constant from contributions due to $v_{ZZ^*}(\nu, \tau)$. We propose to segment the plane into regions, and implement the proposed procedure after taking a separate estimate of $\overline{\sigma}_X^4(\nu, \tau)$ in each region. The optimal choice of region is given by (30). Since $v_{ZZ^*}(\nu, \tau)$ is unknown, we propose to use a centre square around $(0, 0)$, and square annuli, see Fig. 1(a), to estimate the local variance. This is based on assuming $\sigma_{A,N}^2(\nu, \tau)$ smooth and decaying from $(0, 0)$. In general we separate the ambiguity plane into K regions $\{\mathcal{M}_k\}$, so that $\overline{\sigma}_X^4(\nu, \tau) \approx \overline{\sigma}_X^4(\mathcal{M}_k)$, a constant for all $(\nu, \tau) \in \mathcal{M}_k$. We estimate the variance in each region as

$$\widehat{\sigma}_X^4(\mathcal{M}_k) = \frac{1}{16} \frac{\text{median} \left\{ \left| \widehat{\mathcal{A}}_{XX^*}^{(S)}(\nu, \tau) \right|^2 \right\}_{(\nu, \tau) \in \mathcal{M}_k}}{\ln[2]}. \quad (34)$$

We threshold $\widehat{\mathcal{A}}_{XX^*}^{(B)}(\nu, \tau)$ according to (28), but now using $\widehat{\sigma}_X^4(\mathcal{M}_k)$ in each defined region instead of $\widehat{\sigma}_\varepsilon^4(\mathcal{M})$ in (32), this yielding $\widehat{\mathcal{A}}_{XX^*}^{(\text{lbht})}(\nu, \tau)$, or the Local Bias-corrected TEAF (LBTEAF).

Next, we derive results for estimating the EAF of an underspread zero-mean stochastic process. This is strictly speaking a correct treatment for processes satisfying the constraints of Theorem 3.1, but we expect that the distributional result is valid under less constrained scenarios. The conditions are sufficient, but by no means necessary, for the distributional result to hold. We note from Theorem 3.1 that for most points in the ambiguity plane

$$\left| \frac{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - \mu_{A,N}(\nu, \tau)}{\sigma_{A,N}^2(\nu, \tau)} \right|^2 \stackrel{d}{=} \frac{1}{2} \chi_2^2 + O\left(\frac{|\log(N)|^2}{N - |\tau|}\right) + o(1). \quad (35)$$

The question then arises of yet again estimating $\sigma_{A,N}^2(\nu, \tau)$ appropriately. We note from Theorem 3.1 that $\sigma_{A,N}^2(\nu, \tau)$ takes different forms depending on the spreading of the process. The more $\{X[t]\}$ is like a white noise process, the less variation will $\sigma_{A,N}^2(\nu, \tau)$ exhibit from the form of the variance of the EAF of a white noise process. We define $\overline{\sigma}_X^4(\nu, \tau) = \sigma_{A,N}^2(\nu, \tau)/(16[N - |\tau|][1/2 - |\nu|])$. Then

$$\left| \frac{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau) - \mu_{A,N}(\nu, \tau)}{16[N - |\tau|][1/2 - |\nu|]} \right|^2 \stackrel{d}{=} \overline{\sigma}_X^4(\nu, \tau) \left[\frac{1}{2} \chi_2^2 + O\left(\frac{|\log(N)|^2}{N - |\tau|}\right) + o(1) \right]. \quad (36)$$

Using inverse FTs we note from Theorem 3.1 that the variance of the EAF of a strictly underspread process can in fact be rewritten for $\tau > 0$

$$\begin{aligned} \sigma_{A,N}^2(\nu, \tau) &= \sum_{\tau' = -(T-1)}^{T-1} \int_{-\Omega}^{\Omega} \int_{-\Omega}^{\Omega} e^{-j2\pi(\nu\tau' - \nu'\tau)} \mathcal{A}_{ZZ^*}(\nu', \tau') \\ &\quad \mathcal{A}_{ZZ^*}^*(\nu'', \tau') e^{j\pi(N-\tau-1)(\nu' - \nu'')} D_{N-\tau}(\pi(\nu' - \nu'')) d\nu' d\nu'' + O\left(\log\left[\frac{N}{T}\right]\right) \\ &= \sum_{\tau' = -(T-1)}^{T-1} \int_{-\Omega}^{\Omega} e^{-j2\pi(\nu\tau' - \nu'\tau)} |\mathcal{A}_{ZZ^*}(\nu', \tau')|^2 d\nu' + O\left(\log\left[\frac{N}{T}\right]\right) + O(1). \end{aligned} \quad (37)$$

Deriving this requires that $\mathcal{A}_{ZZ^*}(\nu, \tau)$ is sufficiently smooth in ν . If this is not the case, use (18). The variance of the EAF corresponds to aggregating the total magnitude squared of the AF over the plane and implementing phase shifts. If the function is smooth in ν with a stationary phase approximation to the integral we mainly pick up contributions on the line $\nu' = \nu\tau'/\tau$, which we later sum over τ' . This is exemplified for some choices of ν and τ in Figure 1(b), where we plot the lines summed over. Despite this geometrical intuition, the form in (37) is not extremely informative. Equation (37) does indicate that $\sigma_{A,N}^2(\nu, \tau)$ should be a smooth function of ν and τ , as we only integrate over \mathcal{D} . We rely on the forms of propositions 3.2 and 3.3 to give more precise understanding of the variance of the EAF in these special cases. We note from these set of results that the variance is smooth in ν and τ , and often exhibits a decay in (ν, τ) that resembles that of the variance of white noise.

If $\overline{\sigma}_X^4(\nu, \tau)$ is exactly constant across the ambiguity plane we can propose an estimator of $\overline{\sigma}_X^4(\nu, \tau)$ given by equating $\widehat{\sigma}_X^4(\nu, \tau)$ to the estimator from (31). Propositions 3.2 and 3.3 argue that in regions of the ambiguity plane $\overline{\sigma}_X^4(\nu, \tau)$ is smooth, and can be approximated as constant over a given region, again using (31). Noting that as the process is underspread $|\mu_{A,N}(\nu, \tau)| \ll \sigma_{A,N}(\nu, \tau)$ for most (ν, τ) , and so thresholding is an admissible estimation procedure. We therefore propose a Local TEAF (LTEAF), by estimating the variance in given regions, just like the LBTEAF, but without removing the bias. We will divide the ambiguity plane into regions according to Fig. 1(a) to obtain both the LTEAF and LBTEAF. As a final note the proposed estimators may not correspond to valid AFs. Only AFs that are the FTs of valid autocovariance sequences are valid. We do not think this corresponds to a major flaw of the procedure, as we are mainly concerned with functions whose support will inform us of the resolution of the autocovariance.

A. Estimated Spread

Using thresholding methods, we have produced an estimate of the AF in the entire ambiguity plane. We also propose an estimator of the spread of the estimated AF, see [2], [11]. Note that for processes that are not strictly underspread, our thresholding procedure will identify regions where the mean of the EAF is sufficiently distinct in magnitude from the variance of the EAF. This in essence corresponds to comparing the magnitude of the AF at a point, with the magnitude of the AF at other points, see (37). We additionally note from (37) that if the AF is mainly centered at other (ν', τ') then the variance of the EAF will be large compared to the mean of the EAF. In this instance the contribution at (ν, τ) is not significantly contributing to the structure of $\{X[t]\}$. We define the ambiguity indicator cell to be

$$\zeta_X(\nu, \tau) = I(|\mathcal{A}_{XX^*}(\nu, \tau)| \neq 0), \quad \nu \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \tau \in \mathbb{Z}. \quad (38)$$

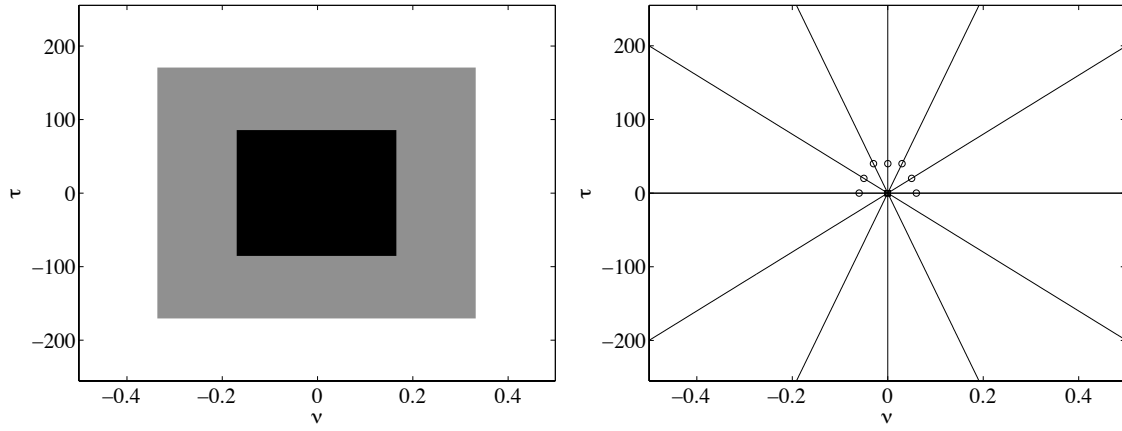


Fig. 1. (a) Partitioning of the ambiguity plane. (b) Radial lines over which the integrand in (37) exhibits stationary phase.

The total spread of the AF of $\{X[t]\}$ over a time-frequency region \mathcal{S} is given by

$$0 \leq \xi_X(\mathcal{S}) = \frac{\int \sum_{\mathcal{S}} \zeta_Z(\nu, \tau) d\nu}{\int \sum_{\mathcal{S}} d\nu} \leq 1. \quad (39)$$

A process $\{X[t]\}$ is strictly AF Compressible if $\xi_X(\mathcal{S}) \ll 1$. We define

$$\hat{\zeta}_X(\nu, \tau) = I\left(\left|\hat{\mathcal{A}}_{XX^*}^{(ht)}(\nu, \tau)\right| \neq 0\right), \quad \nu \in \left(-\frac{1}{2}, \frac{1}{2}\right), \tau \in [-(N-1), (N-1)], \quad (40)$$

and thus

$$\hat{\xi}_X(\mathcal{S}) = \frac{\sum \sum_{\mathcal{S}} \hat{\zeta}_X(\nu_k, \tau)}{\sum \sum_{\mathcal{S}} 1} \quad (41)$$

is an estimator of the total spread of the process. Matz and Hlawatsch define extended underspread processes, as those whose spread is essentially limited. Such processes will be approximated by the TEAF to the region of their essential support, i.e., where their magnitude is non-negligible in comparison to the rest of the ambiguity plane. This permits the quantification of the degree of variability of the time series at each lag, and the band of variability supported at a given lag τ_0 , can be determined by $\hat{\xi}_X(\{(\nu, \tau) : \tau = \tau_0\})$.

V. EXAMPLES

A. Linear chirp immersed in white noise.

We first consider the case of a deterministic linear chirp, $\mu_X[t] = \cos[\pi(2\alpha t + \beta t^2)]$, immersed in white noise. Chirps are commonly characterised in the ambiguity plane [7]. The chirp has a starting frequency $\alpha = 0.1$, with chirp rate $\beta = 9.0196 \cdot 10^{-4}$, and the noise has variance $\sigma_\varepsilon^2 = 0.3$. We generate data sets of length $N = 256$ from this process. We approximate the analytic signal corresponding to the chirp as $\mu_N[t] = \exp[j\pi(2\alpha t + \beta t^2)]$ and find the N -AF of the chirp

$$\hat{\mathcal{A}}_{XX^*}^{(N)}(\nu, \tau) = \exp\{j\pi[2\alpha\tau - \beta\tau^2 + (\beta\tau - \nu)(N + |\tau| - 1)]\} D_{N-|\tau|}(\pi(\beta\tau - \nu)) \quad (42)$$

for $\beta\tau = \nu$ and zero otherwise. We have not inserted $\beta\tau = \nu$ in this equation, since we are dealing with discrete values and $\beta\tau$ will not be identically equal to ν anywhere. Instead, we will for each τ_i find $\nu_i = \min|\beta\tau_i - \nu|$. Thus, the N -AF will be zero except for at the points (ν_i, τ_i) , where it is defined by (42). To quantify the performance of the estimators, we will estimate the MSE by Monte Carlo simulation, where we compare with the N -AF. We generate $K = 5,000$ realisations of the process. The estimated MSE is shown in Fig. 2(a)–(c). Thresholding has reduced the MSE a great deal. To observe differences between the two thresholding procedures, note that the TEAF suffers from estimating a single noise variance, in that the regions where $v_{ZZ^*}(\nu, \tau)$ inflates the variance, noise remains in the estimation (see for example the band around the chirp in Fig. 2(b)). The LBTEAF looks like a considerable improvement, as it removes more noise, but does suffer from difficulties in estimating the noise around the rim of the domain. This leaves a “badly cleaned window” effect.

To quantify our visual impression of these procedures, we look at the total MSE of each estimator, which is obtained by summing the MSE over the ν - τ -plane, averaged over the five thousand realisations. The resulting total MSEs and their standard deviations are shown in Table I. We see that the local bias corrected thresholding yields the lowest MSE, but both thresholding methods have quite significantly reduced the MSE of the EAF. The reduction, is as mentioned, not quite as large as might have

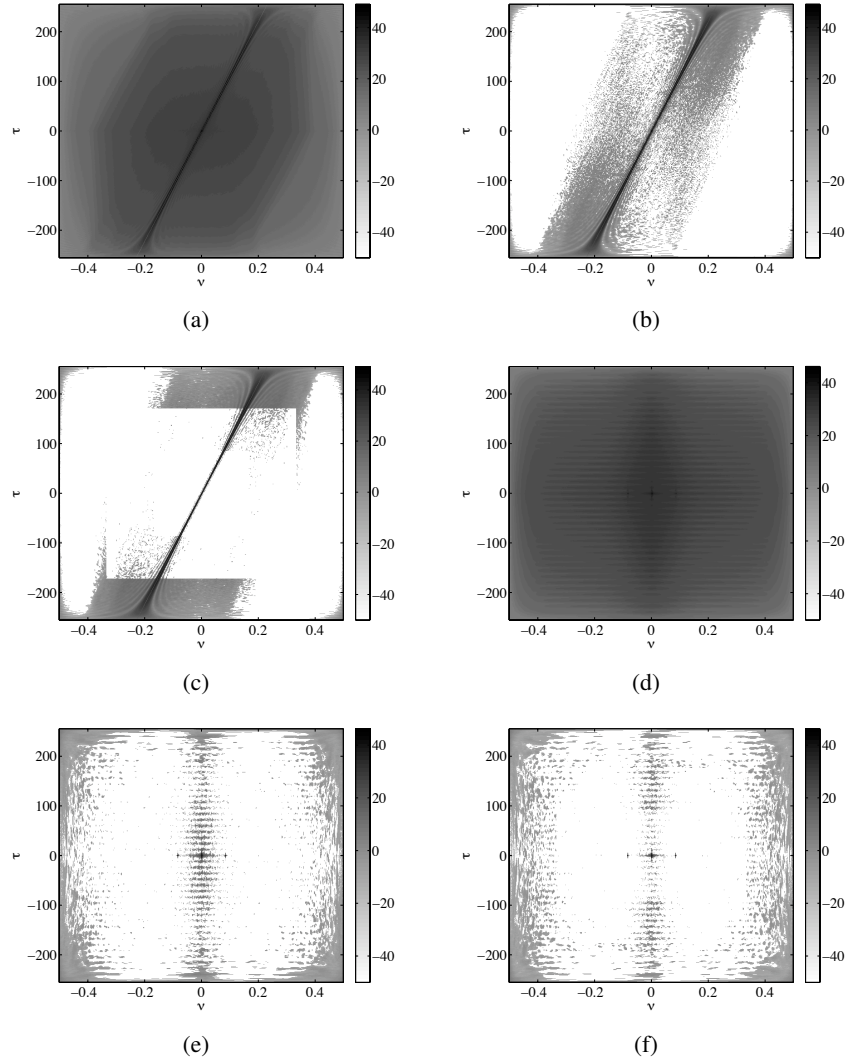


Fig. 2. Chirm immersed in white noise. Estimation of the MSE for (a) EAF (b) TEAF (c) LBTEAF. TVMA process. Estimation of the MSE for (d) EAF (e) TEAF (f) LTEAF.

been anticipated, as the chirm has been broadened near its true support. But the reduction in MSE is respectable corresponding to a factor of almost four.

We estimate the total spread for the chirm immersed in white noise by Monte Carlo simulation. The estimated total spread and the standard deviation of the total spread is given in Table II. The EAF will not be zero anywhere, so we do not need to estimate the total spread for this estimate, but equate this to 1. Theoretically, the AF of the chirm should be nonzero for only 511 of 512×511 cells, which gives us a spread of 0.002. Our estimated spread is larger than this number, but reflects the sparsity of the TEAF.

B. Stationary process

We consider the stationary Moving Average (MA) process $X[t] = \sum_{i=0}^L w_i \xi[t-i]$, where $\xi \sim \mathcal{N}(0, \sigma_\xi^2)$ and $E[\xi[t]\xi[t-\tau]] = \sigma_\xi^2 \delta[\tau]$. The autocorrelation of this process is

$$\widetilde{M}_{XX^*}[\tau] = E[X[t]X^*[t-\tau]] = \sigma_\xi^2 \sum_{i=0}^{L-|\tau|} w_i w_{i+|\tau|} \quad \text{for } |\tau| \leq L. \quad (43)$$

We generate realisations of the process with $\mathbf{w} = [1 \ 0.33 \ 0.266 \ 0.2 \ 0.133 \ 0.066]^T$, length $N = 256$, $L = 5$, and $\sigma_\xi^2 = 1$. Fig. 3(a) shows the EAF and the thresholded EAF for one realisation with the N -AF for $\nu = 0$ and $\tau \in [-10, 10]$. We see that the thresholded EAF and the N -AF are very similar, and only when the variance of the EAF become comparable to

	Chirp	MA	UM	TVMA
Total MSE of EAF	$7.5382 \cdot 10^7$	$2.0337 \cdot 10^8$	$3.3225 \cdot 10^7$	$5.0842 \cdot 10^7$
Total MSE of TEAF	$1.9622 \cdot 10^7$	$9.7961 \cdot 10^5$	$1.7409 \cdot 10^5$	$3.1209 \cdot 10^5$
Total MSE of LTEAF	-	$9.5598 \cdot 10^5$	$1.7076 \cdot 10^5$	$2.951 \cdot 10^5$
Total MSE of LBTEAF	$1.8398 \cdot 10^7$	-	-	-
STD of total MSE of EAF	$9.9790 \cdot 10^6$	$4.6983 \cdot 10^7$	$7.3221 \cdot 10^6$	$1.3809 \cdot 10^7$
STD of total MSE of TEAF	$4.0799 \cdot 10^6$	$4.9168 \cdot 10^5$	$5.1692 \cdot 10^4$	$1.7156 \cdot 10^5$
STD of total MSE of LTEAF	-	$4.1841 \cdot 10^5$	$4.6430 \cdot 10^4$	$1.1522 \cdot 10^5$
STD of total MSE of LBTEAF	$4.0407 \cdot 10^6$	-	-	-

TABLE I

AVERAGE TOTAL MSE AND THE STANDARD DEVIATION (STD) OF TOTAL MSE.

	Chirp	MA	UM	TVMA
Total spread of TEAF	0.013	0.0013	$6.5513 \cdot 10^{-4}$	0.0012
Total spread of LTEAF	-	0.001	$6.3735 \cdot 10^{-4}$	0.001
Total spread of LBTEAF	0.0156	-	-	-
STD of total spread of TEAF	0.0028	0.0016	$8.9984 \cdot 10^{-4}$	0.0018
STD of total spread of LTEAF	-	0.0012	$8.1374 \cdot 10^{-4}$	0.0014
STD of total spread of LBTEAF	0.0039	-	-	-

TABLE II

AVERAGE TOTAL SPREAD AND STD OF TOTAL SPREAD.

the modulus square of the AF is the process thresholded. This exactly corresponds to the support of the real-valued process' autocorrelation.

We see from Table I that the MSE is actually reduced by more than a factor of 100, which is a substantial reduction. Note also that the LTEAF has lower MSE than the TEAF. We estimate the total spread of this process, see Table II. The theoretical spread is in this case equal to 11 cells out of 512×511 which is 4.2044×10^{-5} . Our estimated spread is larger, due to the analytic signal spreading energy in τ and the points at the ends that have not been successfully thresholded, but still reflects the geometry of the support. Using the discrete analytic signal and problems with edge effects makes the inflated spread inevitable. Whilst propositions 3.1–3.3 give the decay of the variance of the EAF, at the rim of the ambiguity plane such expressions are not accurate as the $O(1)$ error term will be of comparable magnitude.

C. Uniformly modulated process

Next, we consider the non-stationary, Uniformly Modulated (UM) process, $X[t] = \sigma_X[t]\xi[t]$, where $\xi \sim \mathcal{N}(0,1)$ and $E[\xi[t]\xi[t-\tau]] = \delta[\tau]$. The time-varying variance is defined as $\sigma_X^2[t] = \sin^2(2\pi f_0 t)$, which has an FT of $\Sigma_{XX^*}(\nu) = \frac{1}{4}(2\delta(\nu) - \delta(\nu - 2f_0) - \delta(\nu + 2f_0))$. The N -AF is $\mathcal{A}_{XX^*}^{(N)}(\nu, \tau) = N$ for $\nu = 0$ and $\tau = 0$, $\mathcal{A}_{XX^*}^{(N)}(\nu, \tau) = -N(1/2 - |2f_0|)$ for $\nu = \pm 2f_0, \tau = 0$, and zero otherwise. We generate a realisation of the process of length $N = 256$ with $f_0 = 0.09$. The EAF is very noisy, but both the TEAF and the LTEAF are substantially cleaner. In Fig. 3(b) we see the EAF and TEAF for $\tau = 0$, and we observe that the thresholding has kept only the three points specified by the N -AF. From Table I we see that also for this process the MSE is reduced with a factor over one hundred from the EAF to the TEAF and LTEAF, with the LTEAF doing a bit better than the TEAF. The theoretical spread of this process is 1.1466×10^{-5} , and our estimated spread is given in Table II.

D. Time-varying MA

Finally, we will combine the MA with the uniformly modulated process, thus defining a Time-Varying MA (TVMA) as $X[t] = \sigma_X[t] \sum_{i=0}^L w_i \xi[t-i]$, where the w 's and $\xi[t]$ is as defined in Section V-B. We use $\sigma_X[t] = \sin(2\pi f_0 t)$ with $f_0 = 0.042$, and we generate samples of length $N = 256$. We find the N -AF of this process as

$$\widehat{\mathcal{A}}_{XX^*}^{(N)}(\nu, \tau) = \begin{cases} (N - |\tau|) \int_0^{1/2} [\widetilde{S}(f - f_0) + \widetilde{S}(f + f_0)] e^{j2\pi f \tau} df & \text{for } \nu = 0, |\tau| \leq L \\ -(N - |\tau|) \int_0^{1/2 - 2f_0} \widetilde{S}(f + f_0) e^{j2\pi f \tau} df & \text{for } \nu = 2f_0, |\tau| \leq L \\ -(N - |\tau|) \int_{2f_0}^{1/2} \widetilde{S}(f - f_0) e^{j2\pi f \tau} df & \text{for } \nu = -2f_0, |\tau| \leq L \end{cases} \quad (44)$$

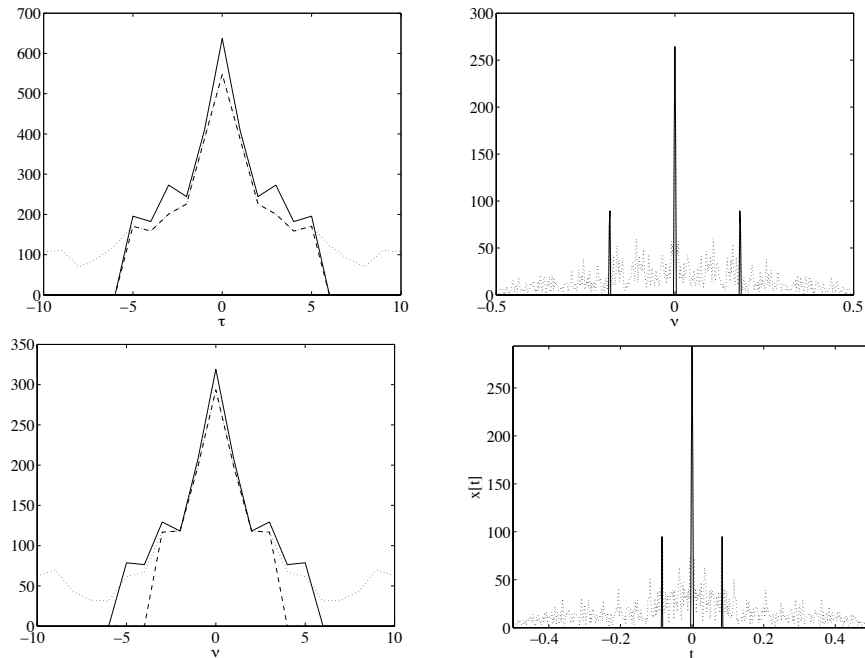


Fig. 3. (a) TEAF (dashed), EAF (dotted) and N -AF (solid) for $\nu = 0$ for one realisation of the MA process. (b) TEAF (solid) and EAF (dotted) for one realisation of the UM process for $\tau = 0$. (c) LTEAF (dashed), EAF (dotted) and N -AF (solid) for $\nu = 0$ for one realisation of the TVMA process. (d) LTEAF (solid) and EAF (dotted) for one realisation of the TVMA process for $\tau = 0$.

where $\tilde{S}(f)$ is the FT of the autocorrelation in (43). Fig. 3(c) shows the N -AF, the EAF and the LTEAF for $\nu = 0$. Likewise, Fig. 3(d) shows the EAF and LTEAF at $\tau = 0$. These two figures demonstrate that the geometry of the non-stationary process is retained by the thresholding estimator, even if the spread of the EAF is cut short when the variance of the EAF becomes too large. The estimated MSEs are shown in 2(d) – (f), and the total MSE in Table I. The MSE for the LTEAF shows a distinct improvement, especially near the region $(0, 0)$. The thresholding methods have again reduced the MSE with a factor over one hundred, and the total estimated spread demonstrates that the AF of this process is extremely sparse.

VI. CONCLUSIONS

In this paper we have introduced a new class of estimators for the AF of a non-stationary process that exhibits sparsity in the ambiguity plane. The AF is a fundamental characterisation of a non-stationary process, and many important properties of a process can be deduced from its AF. The inherent resolution of the process in global time and frequency is an example of such. The AF has been used in estimating the generating mechanism of a non-stationary process [8]. It is also a popular tool in radar and sonar [5]. Despite this fact little efforts have been focused on the estimation of the AF, especially to produce estimators amenable to the determination of spread (size of the support). Characterisation of limited support is vital in designing the best analysis tools for generic second order non-stationary processes, a problem that remains open [1].

Based on the assumption of compression of the AF (small support), we proposed different threshold procedures for estimating the AF. The advantage of using the threshold estimator is that the size of the magnitude of the AF is compared with the magnitude of the AF at other time-frequency cells. Only if the local magnitude is sufficiently large is the value not thresholded. This should enforce a strict support from for example a process whose AF is only extended underspread, see [2][p. 1072]. Unlike Matz and Hlawatsch [2] we do not calculate the spread of the process by fitting the support of the AF into a box centered at $(0, 0)$, but simply count the number of non-zero AF coefficients. Conceptually this can be thought of as determining a finite number of cells that would represent most of the structure of the observed process. If a finite number of cells represent a full process then estimation of its generating mechanism is possible. Hence whilst it clearly is not necessary to constrain the process to be underspread, i.e. be concentrated in support around the origin in the AF domain [4], it is necessary to constrain the possible degree of dependency in the process to ensure that the generating mechanism of the process could be estimated.

Pivotal to thresholding procedures is determining a suitable threshold. We implemented the thresholding with a global variance estimate as well as a local variance estimate, both adjusted for each point in the ambiguity plane. For deterministic signals in white noise we also proposed a procedure which removes bias in the estimator caused by the noise. We demonstrated the superior properties of the threshold estimator in finite sample sizes for a variety of common types of non-stationary and stationary processes. The reductions in MSE compared to a previously used, if naive, estimator for the AF were considerable,

up to factors of over a hundred. Formally our proposed threshold procedure for zero-mean processes is only valid if the process is underspread in which case we determine the asymptotic distribution of the EAF. We conjecture that asymptotic normality can be shown for a larger class of processes, and that in the case of any process with compressed AF, thresholding is an appropriate estimation procedure. The estimated spread does not fully reflect the degree of sparsity of the AF: an inflation is accrued due to the usage of the discrete analytic signal and edge effects. Resolving such effects fully remains an outstanding issue. The definition of spread in this paper has the clear advantage of interpretation, corresponding to the fraction of points where the mean of the EAF dominates the variance of the EAF.

The AF remains the least studied of time-frequency representations of non-stationary signals, perhaps because its arguments lack direct global interpretability. A large class of processes that can be estimated exhibit compression in this domain. We anticipate that further study of the AF of non-stationary processes will yield understanding into what generating mechanisms can be determined from a given sample with fixed sampling rates and sample size.

APPENDIX I PROOF OF PROPOSITION 3.1

The Sample Cross Ambiguity Function for two finite equal length samples of signals $\{Z_1[t]\}_t$ and $\{Z_2[t]\}_t$ of length N is given by $A_{Z_1 Z_2^*}(\nu, \tau) = \sum_{t=\max(0, \tau)}^{N-1+\min(\tau, 0)} Z_1[t] Z_2^*[t - \tau] e^{-j2\pi\nu t}$. Then

$$\begin{aligned} \widehat{A}_{X X^*}(\nu, \tau) &= \sum_{t=\max(0, \tau)}^{N-1+\min(0, \tau)} \widehat{M}_{Z Z^*}[t, \tau] e^{-j2\pi\nu t} = \sum_{t=0}^{N-1-\tau} W_N[t + \tau] W_N^*[t] e^{-j2\pi\nu(t+\tau)} \\ &= A_{Z_N Z_N^*}(\nu, \tau) + A_{\varepsilon_N Z_N^*}(\nu, \tau) + A_{Z_N \varepsilon_N^*}(\nu, \tau) + A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau). \end{aligned} \quad (\text{A-1})$$

From direct calculation, assuming that $\{\mu_X[t]\}$ is a sufficiently finely sampled sequence from a sufficiently smooth signal $\mu_X(t) \in L^2(\mathbb{R})$, it follows $A_{Z_N Z_N^*}(\nu, \tau) = A_{\mu_Z \mu_Z^*}(\nu, \tau) + O\left(\frac{1}{N}\right)$, where $A_{\mu_Z \mu_Z^*}(\nu, \tau)$ is the AF of the continuous analytic signal of $\mu_X(t)$. Secondly, as $Z_N[t] = \mu_N[t]$ is deterministic, and $\varepsilon_X[t]$ is zero-mean, it follows $\mathbb{E}\{A_{\varepsilon_N Z_N^*}(\nu, \tau)\} = 0$, and $\mathbb{E}\{A_{Z_N \varepsilon_N^*}(\nu, \tau)\} = 0$. Using (6) for $\tau \geq 0$ and the fact that $S_{X X^*}(\nu, f) = \sigma_\varepsilon^2 \delta(\nu)$ for a white process, we find

$$\begin{aligned} \mathbb{E}\{A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)\} &= 4 \int_0^{1/2} \int_0^{1/2} e^{j\pi(f_1 - f_2)(N-1-\tau)} \mathbb{E}\{dZ_X(f_1) dZ_X^*(f_2)\} e^{j2\pi f_1 \tau} \\ &\quad \times D_{N-\tau}(\pi(f_1 - f_2 - \nu)) e^{-j\pi\nu(N+\tau-1)} + O\left(\frac{N-\tau}{N}\right) \\ &= 4 \int_0^{1/2} \int_0^{1/2} e^{j\pi(f_1 - f_2)(N-1-\tau)} \sigma_\varepsilon^2 \delta(f_1 - f_2) e^{j2\pi f_1 \tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) \\ &\quad \times e^{-j\pi\nu(N+\tau-1)} df_1 df_2 + O\left(\frac{N-\tau}{N}\right) = 4e^{-j\pi\nu(N+\tau-1)} D_{N-\tau}(\pi\nu) \int_0^{1/2} \sigma_\varepsilon^2 e^{j2\pi f \tau} df \\ &\quad + O\left(\frac{N-\tau}{N}\right) = 2e^{-j\pi\nu(N+\tau-1)} D_{N-\tau}(\pi\nu) \sigma_\varepsilon^2 e^{j\pi\tau/2} \text{sinc}(\pi\tau/2) + O\left(\frac{N-\tau}{N}\right), \end{aligned}$$

where (as usual) $\text{sinc}(x) = \sin(x)/x$. The error terms follow due to the numerical error of the discrete analytic signal as approximating the analytic signal. Thus taking expectations of (A-1) and using the linearity of $\mathbb{E}\{\cdot\}$, the result follows. *Mutatis mutandis* the calculations are implemented for $\tau < 0$.

We start from (A-1) and note that $\text{var}\{\widehat{A}_{X X^*}(\nu, \tau)\}$ can be written in terms of covariances of $A_{Z_N Z_N^*}(\nu, \tau)$, $A_{Z_N \varepsilon_N^*}(\nu, \tau)$ etc. $Z_N[t]$ is deterministic and $\varepsilon_N[t]$ is Gaussian proper and so

$$\begin{aligned} \text{var}\{A_{Z_N Z_N^*}(\nu, \tau)\} &= 0, & \text{cov}\{A_{Z_N Z_N^*}(\nu, \tau), A_{\varepsilon_N Z_N^*}(\nu, \tau)\} &= 0 \\ \text{cov}\{A_{Z_N Z_N^*}(\nu, \tau), A_{Z_N \varepsilon_N^*}(\nu, \tau)\} &= 0, & \text{cov}\{A_{Z_N Z_N^*}(\nu, \tau), A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)\} &= 0 \\ \text{cov}\{A_{\varepsilon_N Z_N^*}(\nu, \tau), A_{Z_N \varepsilon_N^*}(\nu, \tau)\} &= 0, & \text{cov}\{A_{\varepsilon_N Z_N^*}(\nu, \tau), A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)\} &= 0 \\ \text{cov}\{A_{Z_N \varepsilon_N^*}(\nu, \tau), A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)\} &= 0. \end{aligned} \quad (\text{A-2})$$

For $\tau \geq 0$ let $\mathcal{M}_Z(f|N, \tau) = \sum_{t=0}^{N-1-\tau} \mu_N[t] e^{-2j\pi f t}$, where $\{\mu_N[t]\}$ is the discrete analytic signal of $\{\mu_X[t]\}$. For $\tau \geq 0$ with $\mathcal{M}_X(f|N) = \sum_{t=0}^{N-1} \mu_X[t] e^{-2j\pi f t}$ we note that

$$\mathcal{M}_Z(f|N, \tau) = 2 \int_0^{1/2} \mathcal{M}_X(f'|N) e^{-j\pi(f-f')(N-1-\tau)} D_{N-\tau}(\pi(f-f')) df'.$$

Clearly $\mathcal{M}_Z(f|N, \tau)$ is not only supported on positive frequencies, unless $N - \tau \rightarrow \infty$. Then

$$\begin{aligned} \text{var} \{A_{\varepsilon_N Z_N^*}(\nu, \tau)\} &= \text{var} \left\{ \sum_{t=0}^{N-1-\tau} \varepsilon_N[t + \tau] \mu_Z^*[t] e^{-j2\pi\nu(t+\tau)} \right\} \\ &= \text{var} \left\{ 2 \int_0^{1/2} \int_{-1/2}^{1/2} e^{2j\pi(f_1-\nu)\tau} d\mathcal{Z}_X(f_1) \mathcal{M}_Z^*(f_2|N, \tau) \sum_{t=0}^{N-1} e^{j2\pi t(f_1-f_2-\nu)} df_2 \right\} \\ &\quad + O\left(\frac{\|\mathcal{M}_Z\|}{N}\right) = 4\sigma_\varepsilon^2 \int_0^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{j\pi(N-1)(f_4-f_2)} \mathcal{M}_Z^*(f_2|N, \tau) \mathcal{M}_Z(f_4|N, \tau) \\ &\quad \times \frac{\sin(\pi N(f_1-f_2-\nu))}{\sin(\pi(f_1-f_2-\nu))} \frac{\sin(\pi N(f_1-f_4-\nu))}{\sin(\pi(f_1-f_4-\nu))} df_2 df_4 df_1 + O\left(\frac{\|\mathcal{M}_Z\|}{N}\right), \\ \|\mathcal{M}_Z\|^2 &= \int_{-1/2}^{1/2} |\mathcal{M}_Z(f|N, \tau)|^2 df. \end{aligned}$$

The error term follows due to calculating the discrete analytic signal of $\{\varepsilon_N[t]\}_{t=0}^{N-1}$. We assume that $\mathcal{M}_Z(f_2|N, \tau)$ is twice differentiable. We implement changes of variables, and use the asymptotic form of Dirichlet's kernel, in combination with a Taylor expansion of $\mathcal{M}_Z(f_2|N, \tau)$, to deduce that

$$\begin{aligned} \text{var} \{A_{\varepsilon_N Z_N^*}(\nu, \tau)\} &= 4\sigma_\varepsilon^2 \int_{-\nu}^{1/2-\nu} \int_{u_3-1/2}^{u_3+1/2} \int_{u_3-1/2}^{u_3+1/2} e^{j\pi(N-1)(u_1-u_2)} \mathcal{M}_Z^*(u_3-u_1|N, \tau) \\ &\quad \mathcal{M}_Z(u_3-u_2|N, \tau) \times \frac{\sin(\pi N u_1)}{\sin(\pi u_1)} \frac{\sin(\pi N u_2)}{\sin(\pi u_2)} du_1 du_2 du_3 + O\left(\frac{\|\mathcal{M}_Z\|}{N}\right) \quad (\text{A-3}) \\ &= 4\sigma_\varepsilon^2 \int_{\max(0, -\nu)}^{1/2+\min(-\nu, 0)} |\mathcal{M}_Z(\tilde{u}_3|N, \tau)|^2 d\tilde{u}_3 + O\left(\frac{\|\mathcal{M}_Z\|}{N}\right) = 4\sigma_\varepsilon^2 v_{ZZ^*}(\nu, \tau) + O\left(\frac{\|\mathcal{M}_Z\|}{N}\right). \end{aligned}$$

After the change of variables we are given an outer integral of $\int_{-\nu}^{1/2-\nu}$ over u_3 , rather than $\int_0^{1/2}$. For $\nu > 0$ we can rewrite this as $\int_0^{1/2-\nu}$ plus a contribution that will have the inner integrals integrating to negligible contributions. For $\nu < 0$ we can rewrite this as $\int_{-\nu}^{1/2}$ plus a contribution that will have the inner integrals integrating to negligible contributions. For a generic harmonizable process $\{X[t]\}$

$$\begin{aligned} \text{var} \{A_{Z_N Z_N^*}(\nu, \tau)\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} e^{j\pi(f_1-f_2-\alpha_1+\alpha_2)(N-1-\tau)} D_{N-\tau}(\pi(f_1-f_2-\nu)) \\ &\quad e^{j2\pi(f_1-\alpha_1)\tau} D_{N-\tau}(\pi(\alpha_1-\alpha_2-\nu)) \mathbb{E} \{d\mathcal{Z}_X(f_1) d\mathcal{Z}_X^*(f_2) d\mathcal{Z}_X^*(\alpha_1) d\mathcal{Z}_X(\alpha_2)\} \\ &\quad - |\mathbb{E} \{A_{Z_N Z_N^*}(\nu, \tau)\}|^2 + O((N-\tau)/N). \end{aligned}$$

This follows by direct calculation starting from (6), and the order terms follow from the numerical inaccuracy of the discrete analytic signal, and bounding the stochastic terms from considering their variance. Now, if we assume that any zero mean stochastic process $\{X[t]\}$ is Gaussian, then its increment process is also Gaussian. Using Isserlis' theorem [28] we write the variance as

$$\begin{aligned} \text{var} \{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau)\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} S_{XX^*}(f_1-\alpha_1, \alpha_1) S_{XX^*}(\alpha_2-f_2, f_2) \\ &\quad e^{j\pi(f_1-f_2-\alpha_1+\alpha_2)(N-1-\tau)} e^{j2\pi(f_1-\alpha_1)\tau} D_{N-\tau}(\pi(f_1-f_2-\nu)) \\ &\quad D_{N-\tau}(\pi(\alpha_1-\alpha_2-\nu)) df_1 df_2 d\alpha_1 d\alpha_2 + O\left(\frac{N-\tau}{N}\right), \quad (\text{A-4}) \end{aligned}$$

which for white noise reduces to

$$\begin{aligned} \text{var} \{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau)\} &= 16\sigma_\varepsilon^4 \int_0^{1/2} \int_0^{1/2} D_{N-\tau}^2(\pi(f_1-f_2-\nu)) df_1 df_2 + O\left(\frac{N-\tau}{N}\right) \\ &= 16\sigma_\varepsilon^4 [N-\tau] \int_{\max(0, -\nu)}^{1/2+\min(0, -\nu)} \int_{-(1/2-g_2)[N-\tau]}^{g_2[N-\tau]} \frac{\sin^2[\pi\xi]}{\pi^2\xi^2} d\xi_1 dg_2 + O(1) \\ &= 16\sigma_\varepsilon^4 [N-\tau] [1/2 - |\nu|] + O(1). \end{aligned}$$

Implementing the same calculations for $\tau < 0$ derives analogous expressions with τ replaced by $|\tau|$. Putting (A-2), (A-3) and $\text{var}\{A_{Z_N \varepsilon_N^*}(\nu, \tau)\} = \text{var}\{A_{\varepsilon_N Z_N^*}(-\nu, -\tau)\}$ together, and *mutatis mutandis* implementing the calculations for $\tau < 0$, thus yields

$$\text{var}\{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau)\} = 0 + 4\sigma_\varepsilon^2 v_{ZZ^*}(\nu, \tau) + 4\sigma_\varepsilon^2 v_{ZZ^*}(-\nu, -\tau) + 16\sigma_\varepsilon^4 [N - |\tau|][1/2 - |\nu|] + O(1).$$

For $\tau < 0$ $\mathcal{M}_Z(\tilde{u}_3|N, \tau) = \sum_{t=|\tau|}^{N-1} \mu_N[t] e^{-j2\pi f(t+\tau)}$.

We start from (A-1) and note that $\text{rel}\{\widehat{\mathcal{A}}_{XX^*}(\nu, \tau)\}$ can be written as an aggregation of relations. As $Z_N[t]$ is deterministic and $\varepsilon_N[t]$ is zero-mean Gaussian,

$$\begin{aligned} \text{rel}\{A_{Z_N Z_N^*}(\nu, \tau)\} &= 0, & \text{rel}\{A_{Z_N Z_N^*}(\nu, \tau), A_{\varepsilon_N Z_N^*}(\nu, \tau)\} &= 0 \\ \text{rel}\{A_{Z_N Z_N^*}(\nu, \tau), A_{Z_N \varepsilon_N^*}(\nu, \tau)\} &= 0, & \text{rel}\{A_{Z_N Z_N^*}(\nu, \tau), A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)\} &= 0 \\ \text{rel}\{A_{\varepsilon_N Z_N^*}(\nu, \tau), A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)\} &= 0, & \text{rel}\{A_{Z_N \varepsilon_N^*}(\nu, \tau), A_{\varepsilon_N \varepsilon_N^*}(\nu, \tau)\} &= 0. \end{aligned} \quad (\text{A-5})$$

For $\tau \geq 0$ (with = up to $O(\|\mathcal{M}_Z\|/N)$)

$$\begin{aligned} \text{rel}\{A_{\varepsilon_N Z_N^*}(\nu, \tau), A_{Z_N \varepsilon_N^*}(\nu, \tau)\} &= \text{E}\{A_{\varepsilon_N Z_N^*}(\nu, \tau) A_{Z_N \varepsilon_N^*}(\nu, \tau)\} \\ &= \text{E}\left\{4 \int_0^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_0^{1/2} d\mathcal{Z}_X(f_1) d\mathcal{Z}_X^*(f_4) \mathcal{M}_Z^*(f_2|N, \tau) \frac{\sin(\pi N(f_1 - f_2 - \nu))}{\sin(\pi(f_1 - f_2 - \nu))} e^{2j\pi(f_1 - \nu)\tau} \right. \\ &\quad \left. e^{2j\pi(f_3 - \nu)\tau} \mathcal{M}_Z(f_3|N, -\tau) e^{j\pi(N-1)(f_3 - f_4 - \nu)} e^{j\pi(N-1)(f_1 - f_2 - \nu)} \times \frac{\sin(\pi N(f_3 - f_4 - \nu))}{\sin(\pi(f_3 - f_4 - \nu))} df_2 df_3 \right\} \\ &= 4\sigma_\varepsilon^2 \int_0^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{2j\pi(f_1 - \nu)\tau} \mathcal{M}_Z^*(f_2|N, \tau) e^{j\pi(N-1)(f_3 - f_2 - 2\nu)} \frac{\sin(\pi N(f_1 - f_2 - \nu))}{\sin(\pi(f_1 - f_2 - \nu))} \\ &\quad \times df_2 e^{2j\pi(f_3 - \nu)\tau} \mathcal{M}_Z(f_3|N, -\tau) \frac{\sin(\pi N(f_3 - f_1 - \nu))}{\sin(\pi(f_3 - f_1 - \nu))} df_3 df_1. \end{aligned}$$

With an appropriate change of variables, we get

$$\begin{aligned} \text{rel}\{A_{\varepsilon_N Z_N^*}(\nu, \tau), A_{Z_N \varepsilon_N^*}(\nu, \tau)\} &= 4\sigma_\varepsilon^2 \int_0^{1/2} \int_{-1/2 - u_3 - \nu}^{1/2 - u_3 - \nu} \int_{u_3 - 1/2 - \nu}^{u_3 - \nu + 1/2} e^{2j\pi(u_3 - \nu)\tau} \mathcal{M}_Z^*(u_3 - u_1 - \nu|N, \tau) \\ &\quad \times e^{j\pi(N-1)u_1} \frac{\sin(\pi N u_1)}{\sin(\pi u_1)} e^{2j\pi(u_2 + u_3)\tau} \mathcal{M}_Z(u_2 + u_3 + \nu|N, -\tau) e^{j\pi(N-1)u_2} \frac{\sin(\pi N u_2)}{\sin(\pi u_2)} du_1 du_2 du_3 \\ &\quad + O\left(\frac{\|\mathcal{M}_Z\|}{N}\right). \end{aligned}$$

We now let $\tilde{u}_1/N = u_1$ and $\tilde{u}_2/N = u_2$ and get (with equality up to order $\|\mathcal{M}_Z\|/N$)

$$\begin{aligned} \text{rel}\{A_{\varepsilon_N Z_N^*}(\nu, \tau), A_{Z_N \varepsilon_N^*}(\nu, \tau)\} &= 4\sigma_\varepsilon^2 \int_0^{1/2} \int_{[-u_3 - \nu - 1/2]N}^{[1/2 - u_3 - \nu]N} \int_{[u_3 - 1/2 - \nu]N}^{[u_3 - \nu + 1/2]N} \mathcal{M}_Z^*\left(u_3 - \frac{u_1}{N} - \nu|N, \tau\right) \\ &\quad \frac{e^{2j\pi(u_3 - \nu)\tau} e^{j\pi(N-1+2\tau)u_1/N} \sin(\pi u_1)}{\sin(\pi u_1/N)} \mathcal{M}_Z\left(\frac{u_2}{N} + u_3 + \nu|N, -\tau\right) e^{j\pi(N-1+2\tau)u_2/N} \frac{\sin(\pi u_2)}{\sin(\pi u_2/N)} \\ &\quad \times \frac{du_1 du_2 du_3}{N^2} = 4\sigma_\varepsilon^2 \int_{\max(0, -\nu)}^{1/2 + \min(0, -\nu)} e^{2j\pi u_3 \tau} \mathcal{M}_Z^*(u_3|N, \tau) \mathcal{M}_Z(u_3 + 2\nu|N, -\tau) du_3 = 4\sigma_\varepsilon^2 c_{ZZ^*}(\nu, \tau). \end{aligned} \quad (\text{A-6})$$

If $\mathcal{M}_X(u_3|N, \tau)$ is mainly supported only over a range of frequencies we would assume with increasing ν that $c_{ZZ^*}(\nu, \tau) \rightarrow 0$. Furthermore we note that $\text{rel}\{A_{\varepsilon_N Z_N^*}(\nu, \tau)\} = 0$, and $\text{rel}\{A_{Z_N \varepsilon_N^*}(\nu, \tau)\} = 0$. We can note that for a generic harmonizable process $\{X[t]\}$

$$\begin{aligned} \text{rel}\{A_{Z_N Z_N^*}(\nu, \tau)\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} e^{j\pi(f_1 - f_2 + \alpha_1 - \alpha_2)(N-1-\tau)} e^{j2\pi(f_1 + \alpha_1)\tau} \\ &\quad D_{N-\tau}(\pi(f_1 - f_2 - \nu)) D_{N-\tau}(\pi(\alpha_1 - \alpha_2 - \nu)) \text{E}\{d\mathcal{Z}_X(f_1) d\mathcal{Z}_X^*(f_2) d\mathcal{Z}_X(\alpha_1) d\mathcal{Z}_X^*(\alpha_2)\} e^{-2j\pi\nu(N+\tau-1)} \\ &\quad + O\left(\frac{N-\tau}{N}\right) - \text{E}\{A_{Z_N Z_N^*}(\nu, \tau)\}^2. \end{aligned}$$

This follows by direct calculation starting from (6). Using Isserlis' theorem and duplicating the calculations for the variance, we write the relation as

$$\begin{aligned} \text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} S_{XX^*}(f_1 - \alpha_2, \alpha_2) S_{XX^*}(\alpha_1 - f_2, f_2) \\ &e^{j\pi(f_1 - f_2 + \alpha_1 - \alpha_2)(N-1-\tau)} e^{j2\pi(f_1 + \alpha_1)\tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) \\ &\times D_{N-\tau}(\pi(\alpha_1 - \alpha_2 - \nu)) e^{-2j\pi\nu(N+\tau-1)} df_1 df_2 d\alpha_1 d\alpha_2 + O\left(\frac{N-\tau}{N}\right). \end{aligned} \quad (\text{A-7})$$

For a white-noise process this corresponds to

$$\begin{aligned} \text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \sigma_\varepsilon^2 \delta(f_1 - \alpha_2) \sigma_\varepsilon^2 \delta(\alpha_1 - f_2) e^{j\pi(f_1 - f_2 + \alpha_1 - \alpha_2)(N-1-\tau)} \\ &e^{j2\pi(f_1 + \alpha_1)\tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) D_{N-\tau}(\pi(\alpha_1 - \alpha_2 - \nu)) e^{-2j\pi\nu(N+\tau-1)} df_1 df_2 d\alpha_1 d\alpha_2 + O((N-\tau)/N) \\ &= 16 \int_0^{1/2} \int_0^{1/2} \sigma_\varepsilon^4 e^{j\pi(f_1 - f_2 + f_2 - f_1)(N-1-\tau)} e^{j2\pi(f_1 + f_2)\tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) \\ &D_{N-\tau}(\pi(f_2 - f_1 - \nu)) e^{-2j\pi\nu(N+\tau-1)} df_1 df_2 + O((N-\tau)/N). \end{aligned}$$

If $\nu = O(1/[N-\tau])$ then this corresponds to (up to order $(N-\tau)/N$)

$$\begin{aligned} \text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16 \int_0^{1/2} \int_{[f-1/2][N-\tau]}^{f[N-\tau]} \sigma_\varepsilon^4 e^{j4\pi f\tau} D_{N-\tau} \left(\pi \left(\frac{f'}{[N-\tau]} - \nu \right) \right) D_{N-\tau} \left(\pi \left(-\frac{f'}{[N-\tau]} - \nu \right) \right) \\ &\frac{e^{-2j\pi\nu(N+\tau-1)} df df'}{N-\tau} = 16 \int_0^{1/2} \int_{[f-1/2-\nu][N-\tau]}^{[f-\nu][N-\tau]} \sigma_\varepsilon^4 \frac{e^{j4\pi f\tau} D_{N-\tau}(\pi(\frac{f'}{[N-\tau]}) D_{N-\tau}(\pi(\frac{-f'}{[N-\tau]} - 2\nu)) e^{-2j\pi\nu(N+\tau-1)} df df'}{N-\tau}. \end{aligned}$$

If ν is not $O(1/[N-\tau])$, then $\text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\}$ is negligible. If $\nu = O(1/[N-\tau])$ then with $\xi(\nu) = [N-\tau]\nu = O(1)$ we obtain that

$$\begin{aligned} \text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16[N-\tau] \left(\sigma_\varepsilon^4 \int_{-\infty}^{\infty} \frac{\sin(\pi(f')) \sin(\pi(f' + 2\xi(\nu))) e^{-2j\pi\nu(N+\tau-1)}}{\pi(f') \pi(f' + 2\xi(\nu))} df' \int_{\max(0,\nu)}^{1/2+\min(0,\nu)} e^{j4\pi f\tau} df \right. \\ &+ O\left(\frac{1}{N-\tau}\right) \left. \right) = 16[N-\tau] \sigma_\varepsilon^4 L(N-\tau, \nu) e^{-2j\pi\nu(N+\tau-1)} \frac{e^{j4\pi(1/2+\min(0,\nu))\tau} - e^{j4\pi \max(0,\nu)\tau}}{j4\pi\tau} + O(1) \\ &= -16[N-\tau] \sigma_\varepsilon^4 L(N-\tau, \nu) e^{-2j\pi\nu(N-1)} \left[\frac{\sin(2\pi|\nu|\tau)}{2\pi\tau} I(\tau \neq 0) - \frac{1}{2} I(\tau = 0) \right] + O(1). \end{aligned}$$

This defines the quantity $L(N-\tau, \nu)$, which decays like $O\left(\frac{1}{\xi(\nu)^2}\right)$. Thus if $\tau \neq 0$

$$\text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = -16[N-\tau] \sigma_\varepsilon^4 L(N-\tau, \nu) e^{-2j\pi\nu(N-1)} \frac{\sin(2\pi|\nu|\tau)}{2\pi\tau} + 8\sigma_\varepsilon^2 \Re \{ c_{ZZ^*}(\nu, \tau) \} + O(1).$$

Mutatis mutandis calculations can be replicated for $\tau < 0$.

APPENDIX II PROOF OF PROPOSITION 3.2

For $\tau \geq 0$ we note that

$$\begin{aligned} \text{E} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 4 \int_0^{1/2} \int_0^{1/2} e^{j\pi(f_1 - f_2)(N-1-\tau)} \text{E} \{ dZ_X(f_1) dZ_X^*(f_2) \} e^{j2\pi f_1 \tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) \\ &\times e^{-j\pi\nu(N+\tau-1)} + O\left(\frac{N-\tau}{N}\right) \\ &= 2D_{N-\tau}(\pi\nu) e^{-j\pi\nu(N+\tau-1)} \left[\widetilde{M}_{XX^*}[\tau] + j\mathcal{Q}_N \left\{ \widetilde{M}_{XX^*} \right\} [\tau] \right] + O\left(\frac{N-\tau}{N}\right). \end{aligned}$$

Note that $D_{N-\tau}(\pi\nu)$ is an even function, and the expression is valid for $\nu < 0$. Calculations can *mutatis mutandis* be replicated for negative τ . We note that with $\tau \geq 0$

$$\begin{aligned} \text{var} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16 \int_0^{1/2} \int_0^{1/2} \widetilde{S}_{XX^*}(f_1) \widetilde{S}_{XX^*}(f_2) D_{N-\tau}^2(\pi(f_1 - f_2 - \nu)) df_1 df_2 + O\left(\frac{N-\tau}{N}\right) \\ &= 16[N-\tau][1/2 - |\nu|] \overline{S}_{XX^*}(\nu) + O(1), \end{aligned}$$

with $\bar{S}_{XX^*}(\nu) = \int_{\max(0, -\nu)}^{1/2 + \min(-\nu, 0)} \tilde{S}_{XX^*}(g_2 + \nu) \tilde{S}_{XX^*}(g_2) dg_2 / (1/2 - |\nu|)$. *Mutatis mutandis* the calculations can be replicated for $\tau < 0$. Finally we start from (A-7) and note that for stationary processes

$$\begin{aligned}
& \text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = 16 \int_0^{1/2} \int_0^{1/2} \tilde{S}_{XX^*}(f_1) \tilde{S}_{XX^*}(f_2) e^{j2\pi(f_1+f_2)\tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) \\
& e^{-2j\pi\nu(N+\tau-1)} D_{N-\tau}(\pi(f_2 - f_1 - \nu)) df_1 df_2 + O\left(\frac{N-\tau}{N}\right) \\
& = 16e^{-2j\pi\nu(N+\tau-1)} \int_0^{1/2} \int_{[N-\tau](g_2-1/2-\nu)}^{[N-\tau](g_2-\nu)} \tilde{S}_{XX^*}(g_2) \tilde{S}_{XX^*}^{(L)}(g_2 - g_1/[N-\tau] - \nu) \\
& e^{j2\pi(2g_2 - \frac{g_1}{N-\tau} - \nu)\tau} D_{N-\tau}\left(\pi\frac{g_1}{N-\tau}\right) D_{N-\tau}\left(\pi\frac{-g_1 - 2\xi(\nu)}{N-\tau}\right) dg_1 dg_2 / [N-\tau] + O\left(\frac{N-\tau}{N}\right) \\
& = 16e^{-2j\pi\nu(N+\tau-1)} [N-\tau] L(N-\tau, \nu) \int_{\max(0, \nu)}^{1/2 + \min(0, \nu)} \tilde{S}_{XX^*}(g_2) \tilde{S}_{XX^*}(g_2 - \nu) e^{j4\pi g_2 \tau} dg_2 + O(1).
\end{aligned}$$

This again is equivalent to the variance for $\tau = 0$ and $\nu = 0$ as expected, and *mutatis mutandis* results may be derived for $\tau < 0$.

APPENDIX III PROOF OF PROPOSITION 3.3

Using (6) and by direct calculation we note that

$$\begin{aligned}
& \text{E} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = 4 \int_0^{1/2} \int_0^{1/2} e^{j\pi(f_1-f_2)(N-1-\tau)} \text{E} \{ dZ_X(f_1) dZ_X^*(f_2) \} \\
& \times e^{j2\pi f_1 \tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) e^{-j\pi\nu(N+\tau-1)} + O\left(\frac{N-\tau}{N}\right) \\
& = 4 \int_0^{1/2} \int_{f-1/2}^f e^{j\pi f'(N-1-\tau)} \sum_{\tau'} \mathcal{A}_{XX^*}(f', \tau') e^{-j2\pi(f-f')\tau'} e^{j2\pi f \tau} D_{N-\tau}(\pi(f' - \nu)) e^{-j\pi\nu(N+\tau-1)} df' df \\
& + O\left(\frac{N-\tau}{N}\right) = 4 \int_{\max(0, -\nu)}^{1/2 - \max(\nu, 0)} e^{j2\pi f \tau} \int_{(f-1/2)(N-\tau)}^{f(N-\tau)} e^{j\pi f'(N-1-\tau)/(N-\tau)} \Sigma_{XX^*}(f'/(N-\tau) + \nu) \\
& \times D_{N-\tau}(\pi f'/(N-\tau)) df'/(N-\tau) df + O\left(\frac{N-\tau}{N}\right) \\
& = 4 \int_{\max(0, -\nu)}^{1/2 - \max(\nu, 0)} e^{j2\pi f \tau} \Sigma_{XX^*}(\nu) \int_{-\infty}^{\infty} e^{j\pi f'(N-1-\tau)/(N-\tau)} \frac{\sin(\pi f')}{\pi f'} df' df + O(1) \\
& = 4 \Sigma_{XX^*}(\nu) \frac{e^{j2\pi(1/2 - \max(\nu, 0))\tau} - e^{j2\pi \max(-\nu, 0)\tau}}{j2\pi\tau} + O(1) \\
& = 4 \Sigma_{XX^*}(\nu) e^{j\pi(1/2 - \nu)\tau} \frac{\sin(\pi(1/2 - |\nu|)\tau)}{\pi\tau} + O(1).
\end{aligned}$$

Mutatis mutandis we may derive the results for $\tau < 0$. We start from (A-4), and again by direct calculation

$$\begin{aligned}
\text{var} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} S_{XX^*}(f_1 - \alpha_1, \alpha_1) S_{XX^*}(f_2 - \alpha_2, \alpha_2) e^{j2\pi(f_1 - \alpha_1)\tau} \\
&\quad e^{j\pi(f_1 - f_2 - \alpha_1 + \alpha_2)(N-1-\tau)} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) D_{N-\tau}(\pi(\alpha_1 - \alpha_2 - \nu)) df_1 df_2 d\alpha_1 d\alpha_2 + O\left(\frac{N-\tau}{N}\right) \\
&= 16 \int_0^{1/2} \int_0^{1/2} \int_{f-1/2}^f \int_{\alpha-1/2}^\alpha \Sigma_{XX^*}(f - \alpha) \Sigma_{XX^*}^*(f - f' - \alpha + \alpha') e^{j\pi(f' - \alpha')(N-1-\tau)} \\
&\quad e^{j2\pi(f - \alpha)\tau} D_{N-\tau}(\pi(f' - \nu)) D_{N-\tau}(\pi(\alpha' - \nu)) df' d\alpha' df d\alpha + O\left(\frac{N-\tau}{N}\right) \\
&= 16 \int_{-\nu}^{1/2-\nu} \int_{-\nu}^{1/2-\nu} \int_{[f-1/2][N-\tau]}^{f[N-\tau]} \int_{[\alpha-1/2][N-\tau]}^{\alpha[N-\tau]} \Sigma_{XX^*}(f - \alpha) \Sigma_{XX^*}^*\left(f - \frac{f' - \alpha'}{N - \tau} - \alpha\right) \\
&\quad e^{j\pi(f' - \alpha')(N-1-\tau)/(N-\tau)} e^{j2\pi(f - \alpha)\tau} D_{N-\tau}(\pi f'/(N - \tau)) D_{N-\tau}(\pi \alpha'/(N - \tau)) df' d\alpha' df d\alpha / (N - \tau)^2 \\
&\quad + O\left(\frac{N-\tau}{N}\right) \\
&= 16 \int_{\max(0, -\nu)}^{1/2 - \max(\nu, 0)} \int_{\max(0, -\nu)}^{1/2 - \max(\nu, 0)} \Sigma_{XX^*}(f - \alpha) \Sigma_{XX^*}^*(f - \alpha) e^{j2\pi(f - \alpha)\tau} d\alpha df + O(1) \\
&= 16 \int_{-1/2+|\nu|}^{1/2-|\nu|} \left(\frac{1}{2} - |\nu|\right) |\Sigma_{XX^*}(y)|^2 e^{j2\pi y \tau} dy + O(1)
\end{aligned}$$

We start from (A-7)

$$\begin{aligned}
\text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} S_{XX^*}^{(L)}(f_1, \alpha_2) S_{XX^*}^{(L)}(\alpha_1, f_2) e^{j\pi(f_1 - f_2 + \alpha_1 - \alpha_2)(N-1-\tau)} \\
&\quad \times e^{j2\pi(f_1 + \alpha_1)\tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) D_{N-\tau}(\pi(\alpha_1 - \alpha_2 - \nu)) e^{-2j\pi\nu(N+\tau-1)} df_1 df_2 d\alpha_1 d\alpha_2 + O\left(\frac{N-\tau}{N}\right) \\
&= 16 \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \Sigma_{XX^*}(f_1 - \alpha_2) \Sigma_{XX^*}^*(f_2 - \alpha_1) e^{j\pi(f_1 - f_2 + \alpha_1 - \alpha_2)(N-1-\tau)} \\
&\quad e^{j2\pi(f_1 + \alpha_1)\tau} D_{N-\tau}(\pi(f_1 - f_2 - \nu)) D_{N-\tau}(\pi(\alpha_1 - \alpha_2 - \nu)) e^{-2j\pi\nu(N+\tau-1)} df_1 df_2 d\alpha_1 d\alpha_2 + O\left(\frac{N-\tau}{N}\right)
\end{aligned}$$

We implement the change of variables $f'' = [f' - \nu](N - \tau)$ and $\alpha'' = [\alpha' - \nu](N - \tau)$.

$$\begin{aligned}
\text{rel} \left\{ \widehat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} &= 16 \int_0^{1/2} \int_0^{1/2} \int_{[f-1/2-\nu][N-\tau]}^{[f-\nu][N-\tau]} \int_{[\alpha-1/2-\nu][N-\tau]}^{[\alpha-\nu][N-\tau]} \Sigma_{XX^*}\left(f - \alpha + \frac{\alpha''}{N - \tau} + \nu\right) \\
&\quad \times \Sigma_{XX^*}^*\left(f - \frac{f''}{N - \tau} - \nu - \alpha\right) e^{j\pi(f'' + \alpha'')(N-1-\tau)/[N-\tau] + j\pi(N-1-\tau)2\nu} e^{j2\pi(f + \alpha)\tau} D_{N-\tau}\left(\frac{\pi f''}{N - \tau}\right) \\
&\quad D_{N-\tau}(\pi \alpha''/[N - \tau]) e^{-2j\pi\nu(N+\tau-1)} \frac{1}{[N - \tau]^2} d\alpha'' df'' df d\alpha + O\left(\frac{N - \tau}{N}\right) = 16e^{-4j\pi\nu\tau} \int_{\max(0, \nu)}^{1/2 + \min(0, \nu)} \\
&\quad \int_{\max(0, \nu)}^{1/2 + \min(0, \nu)} \Sigma_{XX^*}(f - \alpha + \nu) \Sigma_{XX^*}^*(f - \nu - \alpha) e^{j2\pi(f + \alpha)\tau} df d\alpha + O(1).
\end{aligned}$$

Mutatis mutandis the expressions may be derived for $\tau < 0$.

APPENDIX IV PROOF OF THEOREM 3.1

We outline the proof for $\tau \geq 0$, the same arguments hold *mutatis mutandis* for $\tau < 0$. As $\{X[t]\}$ is an underspread process there exists a constant $T = O(1)$ such that $\forall |\tau| \geq T$, $\text{cov}\{X[t], X[t + \tau]\} = 0$. This implies that $\forall |\tau| \geq T$ $\text{cov}\{X[t], Y[t + \tau]\} = O(1/(\tau - T))$, where we recall that $\{Y[t]\}$ is the DHT of $\{X[t]\}$. For convenience assume $N - \tau = Tn$. This is of course not a necessity but simplifies the proofs. Otherwise for Tm such that $Tm < N - \tau$ and $T(m + 1) > N - \tau$ we can split the sum in the EAF into two parts, and show the latter part has negligible contributions to the total sum. The first part, summing from zero over t up to Tm , can be treated as if $N - \tau = Tm$ with $m = n$. We now take a full length observation $\mathbf{X} = [X_0, \dots, X_{N-1}]$ and construct n subvectors by the following mechanism

$$\mathbf{X}_1 = [X[0], X[T], \dots, X[(n-1)T]], \quad \mathbf{X}_2 = [X[1], X[T+1], \dots, X[(n-1)T+1]], \quad \dots$$

The vectors are constructed so that the elements of \mathbf{X}_k are uncorrelated and if we define \mathbf{Y}_j and $\mathbf{Z}_{N:j}$, as the obvious extensions to \mathbf{X}_j , then $\text{cov}(X_i[t], Y_j[s]) = O(1/|i-j+T(t-s)|)$, $\text{cov}(Z_{N:i}[t], Z_{N:j}[s]) = O(1/|i-j+T(t-s)|)$ if $j \neq i$. Before constructing the CLT type of argument let us study how we shall represent $\hat{\mathcal{A}}_{XX^*}(\nu, \tau)$. We write

$$\hat{\mathcal{A}}_{XX^*}(\nu, \tau) = \sum_{u=0}^{T-1} \sum_{v=0}^{n-1} Z_N[vT+u+\tau] Z_N^*[vT+u] e^{-j2\pi\nu(vT+u)} e^{-j2\pi\nu\tau}.$$

The expectation of $\hat{\mathcal{A}}_{XX^*}(\nu, \tau)$ follows directly from this expression, and we additionally note that $M_{Z_N Z_N^*}[t, \tau] = M_{ZZ^*}[t, \tau] + O(1/N)$.

Using Isserlis' formula [28] and the assumption that the process is proper, we find the variance of $\hat{\mathcal{A}}_{XX^*}(\nu, \tau)$ as

$$\text{var} \left\{ \hat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = \sum_{v_1=0}^{n-1} \sum_{v_2=0}^{n-1} \sum_{u_1=0}^{T-1} \sum_{u_2=0}^{T-1} c_N(\nu, \tau; u_1, u_2, v_1, v_2),$$

where

$$\begin{aligned} c_N(\nu, \tau; u_1, u_2, v_1, v_2) &= e^{-j2\pi\nu(T(v_1-v_2)+u_1-u_2)} [M_{ZZ^*}[v_1T+u_1+\tau, T(v_1-v_2)+u_1-u_2] \\ &\quad + O\left(\frac{1}{N}\right)] \times \left[M_{ZZ^*}^*[v_1T+u_1, T(v_1-v_2)+u_1-u_2] + O\left(\frac{1}{N}\right) \right]. \end{aligned}$$

Thus if $v_1 \neq v_2$ and $v_1 \neq v_2 \pm 1$ we have that with $t_1 = v_1T + u_1$, $c_N(\nu, \tau; u_1, u_2, v_1, v_2) = O(1/(t_1 - t_2)^2)$. Furthermore, $\sum_{|v_1-v_2|>1} c_N(\nu, \tau; u_1, u_2, v_1, v_2) = O(\log(n))$. Otherwise

$$c_N(\nu, \tau; u_1, u_2, v_1, v_2) = M_{ZZ^*}[v_1T+u_1+\tau, u_1-u_2+(v_1-v_2)T] M_{ZZ^*}^*[v_1T+u_1, u_1-u_2+(v_1-v_2)T] + O(1/N).$$

Thus we find that (up to $O(\log(n))$)

$$\text{var} \left\{ \hat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = \sum_{v=0}^{T-1} \sum_{u_2=u_1-(T-1)}^{u_1+(T-1)} \sum_{u_1=0}^{n-1} e^{-j2\pi\nu[u_1-u_2]} M_{ZZ^*}[vT+u_1+\tau, u_1-u_2] M_{ZZ^*}^*[vT+u_1, u_1-u_2].$$

Let $x = vT + u_1$ and $\tau' = u_1 - u_2$. Then this expression is rewritten for $\tau \geq 0$ as

$$\text{var} \left\{ \hat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = \sum_{x=0}^{N-\tau-1} \sum_{\tau'=-\tau}^{T-1} e^{-j2\pi\nu\tau'} M_{ZZ^*}[x+\tau, \tau'] M_{ZZ^*}^*[x, \tau'] + O(\log(n)).$$

Thus $\text{var} \left\{ \hat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = O(N)$. We note that

$$\text{rel} \left\{ \hat{\mathcal{A}}_{XX^*}(\nu, \tau) \right\} = \sum_{v_1=0}^{n-1} \sum_{v_2=0}^{n-1} \sum_{u_1=0}^{T-1} \sum_{u_2=0}^{T-1} r_N(\nu, \tau; u_1, u_2, v_1, v_2)(\nu, \tau),$$

$$\begin{aligned} r_N(\nu, \tau; u_1, u_2, v_1, v_2) &= e^{-j2\pi\nu(T(v_1+v_2)+u_1+u_2+2\tau)} [M_{ZZ^*}[v_1T+u_1+\tau, T(v_1-v_2)+u_1-u_2+\tau] \\ &\quad + O\left(\frac{1}{N}\right)] \times \left[M_{ZZ^*}^*[v_1T+u_1, T(v_1-v_2)+u_1-u_2-\tau] + O\left(\frac{1}{N}\right) \right] \end{aligned}$$

If $|t_1 - t_2 + \tau| > T$ and $|t_1 - t_2 - \tau| > T$

$$r_N(\nu, \tau; u_1, u_2, v_1, v_2) = O\left(\frac{1}{(t_1 - t_2 + \tau)(t_1 - t_2 - \tau)}\right),$$

whilst if $|t_1 - t_2 + \tau| > T$ and $|t_1 - t_2 - \tau| < T$

$$r_N(\nu, \tau; u_1, u_2, v_1, v_2) = O\left(\frac{1}{t_1 - t_2 + \tau}\right).$$

Thus if $|\tau| > T$, $\sum_{v_1, v_2} r_N(\nu, \tau; u_1, u_2, v_1, v_2) = O(\log(n))$. If $|\tau| < T$ $\sum_{|v_1-v_2|>1} r_N(\nu, \tau; u_1, u_2, v_1, v_2) = O(\log(n))$. If $\tau < T$ and $|v_1 - v_2| \leq 1$ we have that $r_N(\nu, \tau; u_1, u_2, v_1, v_2) = e^{-j2\pi\nu(T(v_1+v_2)+u_1+u_2+2\tau)} M_{ZZ^*}[v_1T+u_1+\tau, T(v_1-v_2)+\tau]$

$u_1 - u_2 + \tau] M_{ZZ^*}^*[v_1 T + u_1, T(v_1 - v_2) + u_1 - u_2 - \tau] + O(1/N)$. We deduce that with $x = vT + u_1$ and $\tau' = u_1 - u_2$ that the relation of $\widehat{A}_{XX^*}(\nu, \tau]$ is

$$= \begin{cases} O(\log(n)) & \text{if } |\tau| > T \\ \sum_{x=0}^{N-\tau-1} \sum_{\tau'=-T}^{T-1} e^{-j2\pi\nu[2x+2\tau-\tau']} M_{ZZ^*}[x+\tau, \tau'+\tau] M_{ZZ^*}^*[x, \tau'-\tau] + O(1) & \text{if } |\tau| < T \end{cases} .$$

This completes the proof of the order structure of the EAF of an underspread process.

To prove a CLT for the EAF we add the extra assumptions stated in the theorem. We need to use results for random variables that are both non-stationary and correlated to determine large same results. We note that for $\tau > 0$

$$\widehat{A}_{XX^*}(\nu, \tau] = \sum_{t=0}^{N-\tau-1} R_N[t] + \sum_{t=0}^{N-\tau-1} e^{-j2\pi\nu(t+\tau)} M_{Z_N Z_N^*}[t+\tau, \tau].$$

We note that $\{R_N[t], t = 0, \dots, N - \tau - 1\}$ is a triangular array of centred random variables, that these by assumption are strongly mixing, and have finite second moments. The constraint Peligrad [24] makes on the correlation is satisfied because $X[t]$ is strictly underspread, and the Hilbert transform only induces suitably decaying correlation from such a process. We have assumed the Lindeberg condition holds, and note (21) from our assumptions. Hence from Theorem 2.1 of Peligrad [24] the theorem follows. The condition in (21) does not have to be stated for the real and imaginary part separately as the relation divided by the variance goes to zero for $(\nu, \tau) \notin \mathcal{D}$. We can deduce the exact form of the mean, variance and correlation from the previous part of the theorem.

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