Abelian equations and rank problems for planar webs

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Abstract

We find an invariant characterization of planar webs of maximum rank. For 4-webs, we prove that a planar 4-web is of maximum rank three if and only if it is linearizable and its curvature vanishes. This result leads to the direct web-theoretical proof of the Poincaré's theorem: a planar 4-web of maximum rank is linearizable. We also find an invariant intrinsic characterization of planar 4-webs of rank two and one and prove that in general such webs are not linearizable. This solves the Blaschke problem "to find invariant conditions for a planar 4-web to be of rank 1 or 2 or 3". Finally, we find invariant characterization of planar 5-webs of maximum rank and prove than in general such webs are not linearizable.

1 Introduction

Bol in [6] (see also [4] and [3]) proved that the rank of a planar d-web does not exceed $(d-1)(d-2)/2$. Chern in [7] posed the problem: "Determine all d-webs of curves in the plane having maximum rank $(d-1)(d-2)/2$, $d \geqq 5$."

In the current paper, we find an invariant characterization of planar d-webs of maximum rank and provide a detailed description for the cases $d = 4, 5$. This is the first step for solution of Chern's problem formulated above.

For 4-webs, it is well known that the geometry of a 4-web is determined by the curvature K of one of its 3-subwebs, the basic invariant a and their (covariant) derivatives.

We present the characterization of 4-webs of maximum rank in two forms, an invariant analytic form: *A planar 4-web is of maximum rank three if and only if the curvature K of one of its 3-subwebs and the covariant derivatives K_3 and K_4 of K are expressed in terms of the 4-web basic invariant a and the covariant derivatives of a up to the third order as indicated in formulas of Theorem* 9, and in a pure geometric form: *A planar 4-web is of maximum rank three if and only if it is linearizable and its curvature vanishes* (*Theorem* 12). Note that the curvature of a 4-web is a weighted sum of curvatures of its four 3-subwebs.

As far as we know, these characterizations are the first intrinsic descriptions of 4-webs of maximum rank expressing maximum rank property in terms of the web invariants.

Note that Dou (see [10] and [11]) studied the rank problems for planar 4-webs. The conditions which he found were neither invariant nor effective. This was a reason that Blaschke (who was familiar with Dou's results) in his book [3] (see §48, problem A_2) listed as open the following problem: "Find invariant conditions for a planar 4-web to be of rank 1 or 2 or 3."

Our characterizations of planar 4-webs of maximum rank indicated above along with characterizations of planar 4-webs of rank two and one give a complete solution of the Blaschke problem. The conditions we found are both invariant and effective, and we applied them to several examples.

Pantazi [22] found some necessary and sufficient conditions for a planar web to be of maximum rank. The paper [22] was followed by the papers [23] and [21]. Recently Hénaut [16] (who apparently was not familiar with the paper [22]) associated a connection with the space of abelian equations admitted by a planar web and proved that this connection is integrable (i.e., its curvature form vanishes) if and only if the web is of maximum rank (see also [27] and [28]). Pirio in [25] presented a more detailed exposition of results of Pantazi in [22] and [23] and Mihăileanu in [21]. Both characterizations (of Pantazi in Pirio's interpretation and Hénaut) are not given in terms of the web invariants.

Note also that although a geometric description of planar 4-webs of maximum rank was known (they are algebraizable, i.e., they are equivalent to 4-webs formed by the tangents to an algebraic curve of degree four; see [4], §27), their invariant characterization was not known.

Theorem 12 leads to some interesting results in web geometry. In particular, Theorem 12 implies immediately the Poincaré's theorem (see Corollary 15). The classical (Poincaré's) theorem states: *A planar 4-web of maximum rank three is linearizable.* In our exposition, this theorem becomes obvious because the linearizability conditions are a part of maximum rank conditions (Theorem 12). The Poincaré theorem was noted in the books [4] (§27, p. 239) and [3] (§44) . Note that this theorem is called the Poincaré theorem because it is related to Poincaré's mapping (see [26]) which is widely used in rank problems for webs (see, for example, [8] and [9]). It is worth to note that this theorem can be considered as a reformulation of Sophus Lie's result on surfaces of double translation (see [19]) in the web terms. In fact, in web terms, Lie's result in [19] means that a planar 4-web of maximum rank three is algebraizable (i.e., it is formed by the tangents to a plane algebraic curve of degree four). This implies that any 4-web of maximum rank is linearizable (cf., for example, [3], §44).

Remark also that our proof of the Poincaré theorem uses essentially the linearizability conditions found recently in [2].

In this paper, we also find invariant descriptions of 4-webs of rank two or one (Theorems 24 and 28) and prove that in general such webs are not linearizable (Propositions 27 and 30). Using theorem 12, we prove also that for linearizable 4-webs the vanishing of its curvature is not only necessary but also sufficient for being of maximum rank and that parallelizable 4-webs as well as Mayrhoffer's webs are of maximum rank three.

We also consider concrete examples of 4-webs (Examples 21, 22, 25, 26 and 29) and applying Theorem 12, establish that two of them (Examples 21 and Examples 22) are of maximum rank, two others (Examples 25 and 26) are of rank two and the last one (29) is of rank one. Because 4-webs of Examples 25, 26 and 29 are not linearizable, in general, 4-webs of ranks two and one are not linearizable. We also study rank problems for planar 4- and 5-webs with constant basic invariants.

2 Basic constructions for planar webs

2.1 Planar *d*-webs

A d-web W_d, $d \geq 3$, on a domain $\mathbb{D} \subset \mathbb{R}^2$ is defined by d one-dimensional foliations in general position (i.e., leaves of any pair of foliations are transversal to each other). Such foliations can be defined by d functions (1-st integrals of the foliations) $\langle f_1, ..., f_d \rangle$ such that any pair of functions f_i, f_j, $i \neq j$, are independent, or by d differential 1-forms $< \omega_1, \omega_2, \omega_3, \omega_4, ..., \omega_d >$ such that any two of them are linearly independent.

We fix a co-basis $\langle \omega_1, \omega_2 \rangle$ and a 3-subweb $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$. The forms ω_1, ω_2, and ω_3 can be normalized in such a way that

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

One can easily prove that in this case there is a unique differential 1-form γ such that the so-called *structure equations*

$$d\omega_i = \omega_i \wedge \gamma$$

hold for all $i = 1, 2, 3$ (see [2]).

The form γ determines the Chern connection Γ in the cotangent bundle T^*M with the following covariant differential:

$$d_\Gamma : \omega_i \longmapsto -\omega_i \otimes \gamma.$$

The curvature of this connection is equal to

$$R_\Gamma : \omega_i \longmapsto -\omega_i \otimes d\gamma.$$

If we write

$$d\gamma = K \omega_1 \wedge \omega_2,$$

then the function K is called the *curvature function* of the 3-web W_3.

Note that the curvature form $d\gamma$ is an invariant of the 3-web W_3 while the curvature function K is a relative invariant of the web.

The scale transformation $\langle \omega_1, \omega_2, \omega_3 \rangle \longmapsto \langle \omega_1^s, \omega_2^s, \omega_3^s \rangle$, where s is a nonvanishing smooth function and $\omega_i^s = s^{-1}\omega_i$, preserves the 3-web in the sense that triples $\langle\omega_1,\omega_2,\omega_3\rangle$ and $\langle\omega_1^s,\omega_2^s,\omega_3^s\rangle$ determine the same web. The structure equations for $\langle\omega_1^s,\omega_2^s,\omega_3^s\rangle$ have the form

$$d\omega_i^s = \omega_i^s \wedge \gamma^s$$

with $\gamma^s = \gamma + d\ln|s|$, and therefore $d\gamma = d\gamma^s$.

If one defines the curvature function K^s by the equation

$$d\gamma^s = K^s\omega_1^s \wedge \omega_2^s,$$

then

$$K^s = s^2 K.$$

We emphasize this by saying that K is a relative invariant of weight two.

Let $\langle\partial_1,\partial_2\rangle$ be the basis dual to $\langle\omega_1,\omega_2\rangle$. We put $\partial_3 = \partial_2 - \partial_1$. Then leaves of the 3-web W_3 are trajectories of the vector fields ∂_2,∂_1,and ∂_3.

We denote by δ_i the covariant derivatives in the direction ∂_i with respect to the Chern connection.

Let

$$\gamma = g_1\omega_1 + g_2\omega_2.$$

Then

$$K = \partial_1(g_2) - \partial_2(g_1),$$

and the action of the covariant derivatives δ_i on functions u of weight w is:

$$\delta_i^{(w)}(u) = \partial_i(u) - wg_iu.$$

In what follows, we shall skip the superscript when the weight of u is known. Remark that the covariant derivatives satisfy the Leibnitz rule and

$$\delta_2^{(w+1)}\delta_1^{(w)} - \delta_1^{(w+1)}\delta_2^{(w)} = wK$$

(see [14]).

For general d-web $W_d = \langle\omega_1,\omega_2,\omega_3,....,\omega_d\rangle$, we choose ω_i for $i \geq 4$ in such a way that the normalizations

$$a_i\omega_1 + \omega_2 + \omega_{i+2} = 0$$

hold for $i = 1,...,d-2$, and $a_1 = 1$.

Note that $a_i \neq 0,1$ for $i \geq 2$. Moreover, for the fixed i, the value $a_i(x)$, $x \in \mathbb{D}$, of the function a_i is the cross-ratio of the four straight lines in $T_x^*(\mathbb{D})$ generated by the covectors $\omega_{1,x},\omega_{2,x},\omega_{3,x}$, and $\omega_{i+2,x}$, and therefore it is an invariant. The functions a_i are called the *basic invariants* (cf. [13] or [12], pp. 302–303).

2.2 Web functions

We choose (local) coordinates x, y in $\mathbb{D}$ in such a way that $\omega_1 \wedge dx = 0$ and $\omega_2 \wedge dy = 0$. Let $\omega_3 \wedge df = 0$, $\omega_{i+3} \wedge dg_i = 0$, $i = 1, ..., d-3$, for some functions $f(x, y), g_i(x, y)$.

Using the scale transformation, we assume that $\omega_3 = df$. Then $\omega_1 = -f_x dx$ and $\omega_2 = -f_y dy$.

The dual basis $\{\partial_1, \partial_2\}$ has the form

$$\partial_1 = -f_x^{-1}\partial_x, \quad \partial_2 = -f_y^{-1}\partial_y.$$

The connection form is

$$\gamma = -H\omega_3,$$

where

$$g_1 = g_2 = H = \frac{f_{xy}}{f_x\ f_y}$$

(see [14]). The curvature function has the following expression:

$$K = -f_x^{-1} f_y^{-1} \left(\log \left(f_x f_y^{-1}\right)\right)_{xy}.$$

In terms of the web functions, the basic invariants have the form

$$a_i = \frac{f_y g_{i+1,x}}{f_x g_{i+1,y}}$$

for $i = 2, ..., d-2$.

Definition 1 *A planar d-web W_d is said to be (locally)* ***parallelizable*** *if it is (locally) equivalent to a d-web of parallel straight lines in a domain of the affine plane $\mathbb{A}^2$.*

It is well known (see, for example, [3], §8) that *a planar 3-web is locally parallelizable if and only if $K = 0$.*

For planar d-webs, $d \geq 4$, the following statement holds (cf. [13] or [12], Section 7.2.1 for $d = 4$).

Theorem 2 *A planar d-web $W_d = \langle \omega_1, \omega_2, \omega_3, \omega_4, ..., \omega_d \rangle$ is locally parallelizable if and only if its 3-subweb $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$ is locally parallelizable (i.e., $K = 0$), and all basic invariants a_i are constants.*

Proof. Let $K = 0$ and $a_i = \text{const}$. Then $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$ is locally parallelizable, and we can choose local coordinates x, y in such a way that $\omega_1 = -dx$, $\omega_2 = -dy$, $\omega_3 = d(x + y)$. Since $a_i = \text{const}$, then $\omega_{i+2} = d(a_i x + y)$.

Conversely, suppose that $W_d = \langle \omega_1, \omega_2, \omega_3, \omega_4, ..., \omega_d \rangle$ is locally parallelizable. We choose local coordinates x, y in such a way that leaves of the foliations are parallel straight lines in these coordinates. Then

$$\frac{f_x}{f_y} \text{ and } \frac{g_{i+1,x}}{g_{i+1,y}}$$

are constants.

Therefore $K = 0$, and $a_i = \text{const}$ due to the above formulae for K and a_i. ■

3 Abelian equations

3.1 Classical abelian relations

We begin with an interpretation of the classical Abel addition theorem (see [1]) in terms of planar webs (cf. [3]). A straight line on the affine plane is defined by a pair (r, s): $rx + sy = 1$. Assume that (r, s) satisfies a cubic equation, say, $s^2 - 4r^3 - g_2 r - g_3 = 0$. Given (x, y), one gets the cubic equation for r of the form $r^3 + ar^2 + br + c = 0$ with

$$a = -\frac{x^2}{4y^2},\ b = \frac{1}{4}\left(g_2 + \frac{2x}{y}\right), c = \frac{1}{4}\left(g_3 - \frac{1}{y^2}\right).$$

Then in the domain, where

$$x^4 - 24xy^2 - 12g_2 y^4 > 0,\ \ y \neq 0,$$

the cubic equation has three distinct real roots and consequently three pairwise independent straight lines $(r_1(x, y), s_1(x, y)), (r_2(x, y), s_2(x, y))$ and $(r_3(x, y), s_3(x, y))$ passing through the point (x, y). They generate a 3-web W_3 in the domain.

Let $g_2^3 - 27g_3^2 \neq 0$. Then the solutions of the equation $s^2 - 4r^3 - g_2 r - g_3 = 0$ can be parametrized by the Weierstrass function $\wp:\ r = \wp(t),\ s = \wp'(t)$. As a result, the roots $(r_1(x, y), s_1(x, y)), (r_2(x, y), s_2(x, y))$ and $(r_3(x, y), s_3(x, y))$ correspond to three solutions $(t_1(x, y), t_2(x, y), t_3(x, y))$ of the equation

$$f(t) = \wp(t)\, x + \wp'(t)\, y - 1 = 0.$$

Computing the integral

$$\int t \frac{f'(t)}{f(t)} dt$$

along the boundary of the period parallelogram, one finds the Abel relation

$$t_1(x, y) + t_2(x, y) + t_3(x, y) = \text{const}\,.$$

By the construction, the functions $t_1(x, y), t_2(x, y)$, and $t_3(x, y)$ are constant on the corresponding leaves of W_3.

Consider now an arbitrary planar d-web defined by d functions $W_d = \langle f_1, ..., f_d \rangle$. Then an *abelian relation* is given by d functions $(F_1, ..., F_d)$ of one variable such that

$$F_1(f_1) + \cdots + F_d(f_d) = \text{const}\,.$$

We say that two abelian relations $(F_1, ..., F_d)$ and $(G_1, ..., G_d)$ are *equivalent* if and only if $F_i = G_i + \text{const}_i$ for all $i = 1, ..., d$.

Obviously the set of equivalence classes of abelian relations admits the vector space structure with respect to addition: $(F_1, ..., F_d) + (G_1, ..., G_d) = (F_1 + G_1, ..., F_d + G_d)$ and multiplication by numbers: $\alpha(F_1, ..., F_d) = (\alpha F_1, ..., \alpha F_d)$. The dimension of this vector space is called the *rank* of the web.

In the case when d-web is defined by differential 1-forms $W_d = \langle \omega_1, ..., \omega_d \rangle$, the differentiation of the abelian relation leads us to the *abelian equation*

$$\lambda_1 \omega_1 + \cdots + \lambda_d \omega_d = 0,$$

for functions $(\lambda_1, ..., \lambda_d)$ under the condition that all differential 1-forms $\lambda_i \omega_i$ are closed. The abelian equation is a system of the first order linear PDEs for the functions $(\lambda_1, ..., \lambda_d)$, and the rank of the web is the dimension of the solution space.

The following example of the 3-web illustrates the above constructions. Consider the 3-web W_3 given by the web function

$$f = \frac{2xy - x + y}{x + y}.$$

Then

$$\omega_1 = -f_x dx, \ \omega_2 = -f_y dy, \ \omega_3 = df.$$

The condition

$$\lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3 = 0$$

implies

$$\lambda_1 = \lambda_2 = \lambda_3,$$

and the condition $d(\lambda_3 \omega_3) = 0$ gives

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda(t)$$

for some function $\lambda(t)$.

The other two conditions $d(\lambda \omega_1) = d(\lambda \omega_2) = 0$ lead to the differential equation on λ:

$$2t\lambda(t) + \left(t^2 - 1\right) \lambda'(t) = 0.$$

Thus

$$\lambda(t) = \frac{1}{t^2 - 1},$$

and the abelian relation

$$F_1(x) + F_2(y) + F_3(f) = 0$$

corresponds to the following functions

$$F_1(x) = \ln \frac{x+1}{x}, \ F_2(y) = \ln \frac{y}{y-1}, \ F_3(f) = \ln \frac{f-1}{f+1}.$$

3.2 Abelian differential equations

In this section we formalize the above constructions. Let $W_d = \langle \omega_1, ..., \omega_d \rangle$ be a planar d-web in a domain $\mathbb{D} \subset \mathbb{R}^2$, and let $\pi \colon E \to \mathbb{D}$ be a subbundle of the trivial bundle $\mathbb{R}^d \times \mathbb{D} \to \mathbb{D}$ consisting of points $(x_1, ..., x_d, a)$, where $(x_1, ..., x_d) \in \mathbb{R}^d$, $a \in \mathbb{D}$, such that $\sum_1^d x_i \omega_{i,a} = 0$.

By the **abelian equation** associated with the d-web W_d we mean a system of first order differential equations for sections $(\lambda_1, ..., \lambda_d)$ (i.e., $\sum_1^d \lambda_i \omega_i = 0$) of the bundle π such that (cf. [15]):

$$d(\lambda_1 \omega_1) = \cdots = d(\lambda_d \omega_d) = 0.$$

Let us write down the abelian equation in the explicit form.

In what follows, we shall choose a 3-subweb, say $\langle \omega_1, \omega_2, \omega_3 \rangle$, and normalize the d-web as it was done earlier:

$$a_1 \omega_1 + \omega_2 + \omega_3 = 0, \quad a_2 \omega_1 + \omega_2 + \omega_4 = 0, ..., \; a_{d-2} \omega_1 + \omega_2 + \omega_d = 0,$$

with $a_1 = 1$ *and* $d\omega_3 = 0$.

We call such a normalization *standard*.

Then

$$\begin{aligned}
d(\lambda_1 \omega_1) &= (-\partial_2(\lambda_1) + H\lambda_1)\, \omega_1 \wedge \omega_2, \\
d(\lambda_2 \omega_1) &= (\partial_1(\lambda_2) - H\lambda_2)\, \omega_1 \wedge \omega_2, \\
d(\lambda_3 \omega_3) &= (\partial_2(\lambda_3) - \partial_1(\lambda_3))\, \omega_1 \wedge \omega_2, \\
d(\lambda_i \omega_i) &= (a_{i-2}\partial_2(\lambda_i) - \partial_1(\lambda_i) + \lambda_i (H + \partial_2(a_{i-2}) - a_{i-2}H))\, \omega_1 \wedge \omega_2,
\end{aligned}$$

for all $i = 4, ..., d$.

We shall assume that λ_i are functions of weight 1 and a_i are of weight 0. Then the above formulae take the form

$$\begin{aligned}
d(\lambda_1 \omega_1) &= -\delta_2(\lambda_1)\, \omega_1 \wedge \omega_2, \\
d(\lambda_2 \omega_1) &= \delta_1(\lambda_2)\; \omega_1 \wedge \omega_2, \\
d(\lambda_3 \omega_3) &= (\delta_2(\lambda_3) - \delta_1(\lambda_3))\, \omega_1 \wedge \omega_2, \\
d(\lambda_i \omega_i) &= (\delta_2(a_{i-2}\lambda_i) - \delta_1(\lambda_i))\, \omega_1 \wedge \omega_2.
\end{aligned}$$

The normalization condition $\sum_1^d \lambda_i \omega_i = 0$ implies that

$$\begin{aligned}
\lambda_1 &= \sum_1^{d-2} a_i u_i, \;\; \lambda_2 = \sum_1^{d-2} u_i, \\
\lambda_{i+2} &= u_i, \;\; i = 1, ..., d-2.
\end{aligned}$$

Therefore the abelian equation is equivalent to the following PDEs system

$$\begin{aligned}
&\Delta_1(u_1) = \cdots = \Delta_{d-2}(u_{d-2}) = 0, \\
&\delta_1(u_1) + \cdots + \delta_1(u_{d-2}) = 0,
\end{aligned}$$

where $\Delta_i = \delta_1 - \delta_2 \circ a_i$.

Let $\mathfrak{A}_1 \subset \mathbf{J}^1(\pi)$ be the subbundle of the 1-jet bundle corresponding to the abelian equation, and $\mathfrak{A}_k \subset \mathbf{J}^k(\pi)$ be the $(k-1)$-prolongation of $\mathfrak{A}_1$. Denote by $\pi_{k,k-1}\colon \mathfrak{A}_k \to \mathfrak{A}_{k-1}$ the restrictions of the natural projections $\mathbf{J}^k(\pi) \to \mathbf{J}^{k-1}(\pi)$.

Proposition 3 *Let $k \leq d-2$. Then $\mathfrak{A}_k$ are vector bundles and the maps $\pi_{k,k-1}\colon \mathfrak{A}_k \leftarrow \mathfrak{A}_{k-1}$ are projections. Moreover,* $\dim\ker \pi_{k,k-1} = d-k-2$.

Proof. Let $u_{i,r_1\ldots r_s}$ be coordinates in the jet space $\mathbf{J}^k(\pi)$ which correspond to the covariant derivatives $\delta_{r_1}\cdots\delta_{r_s}$ (see [14] for more details). In these coordinates, the abelian equation takes the following form:

$$
\begin{aligned}
&u_{1,1} = a_1 u_{1,2} + a_{1,2} u_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&u_{d-2,1} = a_{d-2} u_{d-2,2} + a_{d-2,2} u_{d-2},\\
&u_{1,1} + \cdots + u_{d-2,1} = 0.
\end{aligned}
$$

This means that $u_1, ..., u_{d-2}$ are fiberwise coordinates in the bundle $\mathbb{D} \xleftarrow{\pi} E$, while $u_{1,2}, ..., u_{d-3,2}$ are fiberwise coordinates in the bundle $E \xleftarrow{\pi_{1,0}} \mathfrak{A}_1$. Taking covariant derivatives of the abelian equation, we observe that $u_{1,22}, ..., u_{d-4,22}$ are fiberwise coordinates in the bundle $\mathfrak{A}_1 \xleftarrow{\pi_{2,1}} \mathfrak{A}_2$, etc. This process proves the proposition. ■

Proposition 3 shows that there is the following tower of vector bundles:

$$\mathbb{D} \xleftarrow{\pi} E \xleftarrow{\pi_{1,0}} \mathfrak{A}_1 \xleftarrow{\pi_{2,1}} \mathfrak{A}_2 \xleftarrow{\pi_{3,1}} \cdots \xleftarrow{\pi_{d-3,d-4}} \mathfrak{A}_{d-3} \xleftarrow{\pi_{d-2,d-3}} \mathfrak{A}_{d-2}.$$

The last projection $\mathfrak{A}_{d-2} \xrightarrow{\pi_{d-2,d-3}} \mathfrak{A}_{d-3}$ is an isomorphism, and geometrically it can be viewed as a linear connection in the vector bundle $\pi_{d-3}\colon \mathfrak{A}_{d-3} \to \mathbb{D}$. Remark that *the abelian equation is formally integrable if and only if this linear connection is flat.*

The dimension of this bundle is equal to $(d-2)+(d-3)+\cdots+1 = (d-2)(d-1)/2$. This shows that the solution space $Sol(\mathfrak{A})$ of the abelian equation $\mathfrak{A}$ is finite-dimensional and $\dim Sol(\mathfrak{A}) \leq (d-1)(d-2)/2$.

The dimension $\dim Sol(\mathfrak{A})$ is called the *rank of* the corresponding d-web W_d.

As a consequence, we get the following result which was first established by Bol [6] (see also [4] and [3]).

Theorem 4 *The rank of a planar d-web W_d does not exceed* $(d-1)(d-2)/2$.

Remark also, that a different approach for description of the bundle $\pi_{d-3}\colon \mathfrak{A}_{d-3} \to \mathbb{D}$ in the category of analytical webs was used in [16].

The obstruction for compatibility of the abelian equation is given by the multi-bracket (see [17] and Section 7.1). The matrix of the abelian system is

$$\left\| \begin{matrix} \Delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta_{d-2} \\ \delta_1 & \cdots & \delta_1 \end{matrix} \right\| .$$

Computing the multi-bracket, we get

$$(-1)^d \{(\Delta_1, .., 0), ..., (0, ..., \Delta_{d-2}), (\delta_1, ..., \delta_1)\} = \delta_1 \Delta_2 \cdots \Delta_{d-2} (\Delta_1, .., 0)$$
$$+\Delta_1 \delta_1 \Delta_3 \cdots \Delta_{d-2} (0, \Delta_2, .., 0) + \cdots + \Delta_1 \cdots \Delta_{i-1} \delta_1 \Delta_{i+1} \cdots \Delta_{d-2} (0, ..., \Delta_i, ..., 0)$$
$$+ \cdots + \Delta_1 \cdots \Delta_{d-3} \delta_1 (0, ..., \Delta_{d-2}) - \Delta_1 \cdots \Delta_{d-2} (\delta_1, ..., \delta_1).$$

Therefore, the compatibility condition for the abelian system is

$$\varkappa = \square_1 u_1 + \cdots + \square_{d-2} u_{d-2} = 0,$$

where

$$\square_i = \Delta_1 \cdots \Delta_{d-2} \cdot \delta_1 - \Delta_1 \cdots \Delta_{i-1} \cdot \delta_1 \cdot \Delta_{i+1} \cdots \Delta_{d-2} \cdot \Delta_i$$

are linear differential operators of order not exceeding $d-2$.

Summarizing, we get the following

Theorem 5 *A d-web is of maximum rank $(d-1)(d-2)/2$ if and only if $\varkappa = 0$ on $\mathfrak{A}_{d-2}$.*

Remark that $\varkappa$ can be viewed as a linear function on the vector bundle $\mathfrak{A}_{d-2}$, and therefore the above theorem imposes $(d-1)(d-2)/2$ conditions on the d-web (or on $d-2$ web functions) in order the web has the maximum rank. A calculation of these conditions is pure algebraic, and we shall illustrate this calculation below for planar 3-, 4- and 5-webs. All these calculations are based on expressions for total covariant derivatives given in [14]. Note also that expressions for $\varkappa$ in the case of general d-webs are extremely cumbersome while for concrete d-webs it is not the case.

4 Rank of a planar 3-web

Let $d = 3$. Then the maximum rank of W_3 is 1.The abelian equation takes the form

$$\Delta_1 (u_1) = 0,$$
$$\delta_1 (u_1) = 0.$$

The obstruction $\varkappa$ equals

$$\varkappa = (\delta_1 \Delta_1 - \Delta_1 \delta_1) u_1 = (\delta_2 \delta_1 - \delta_1 \delta_2) u_1 = K u_1.$$

Theorem 6 *A 3-web W_3 is of maximum rank one if and only if it is parallelizable. The only abelian equation admitted by such a 3-web is the equation*

$$\omega_1 + \omega_2 + \omega_3 = 0$$

for the standard normalization.

Note that the above theorem was first proved in [5].

5 Planar 4-webs

5.1 The obstruction

In the standard normalization for 4-webs W_4, we put $a_2 = a$:

$$\omega_1 + \omega_2 + \omega_3 = 0,$$
$$a\ \omega_1 + \omega_2 + \omega_4 = 0$$

and reserve the subscripts for the covariant derivatives of a. Thus $a_2 = \delta_2(a)$ and so on.

In what follows, we use the following form of the abelian relation:

$$(u + av)\,\omega_1 + (u + v)\,\omega_2 + u\omega_3 + v\omega_4 = 0,$$

where $\lambda_1 = u + av, \lambda_2 = u + v, \lambda_3 = u, \lambda_4 = v$, and all summands are closed 1-forms under condition that u and v satisfy the abelian equation

$$\delta_1(u) - \delta_2(u) = 0,\ \delta_1(v) - \delta_2(av) = 0,\ \delta_1(u) + \delta_1(v) = 0.$$

For 4-webs, the tower of prolongations of the abelian equation is

$$\mathbb{D} \overset{\pi}{\longleftarrow} E \overset{\pi_{1,0}}{\longleftarrow} \mathfrak{A}_1 \overset{\pi_{2,1}}{\longleftarrow} \mathfrak{A}_2,$$

where $\pi_{2,1} : \mathfrak{A}_2 \to \mathfrak{A}_1$ defines a linear connection on the 3-dimensional vector bundle $\pi_1 \colon \mathfrak{A}_1 \to \mathbb{D}$.

We shall use canonical fiberwise coordinates u, v and $u_1, ...$, etc. in the jet bundles instead of $u_1, u_2, u_{1,1}, ...$

In these coordinates, the abelian equation takes the form

$$u_1 - u_2 = 0, v_2 - av_2 - a_2 v = 0, u_1 + v_1 = 0,$$

and the obstruction

$$\varkappa = (\Delta_1\Delta_2\delta_1 - \delta_1\Delta_1\Delta_2)u + (\Delta_1\Delta_2\delta_1 - \Delta_1\delta_1\Delta_2)v$$

equals $\varkappa = c_0 v_2 + c_1 v + c_2 u$.

The straightforward computation gives the following result.

Theorem 7 *In the canonical coordinates, the restriction $\varkappa$ on $\mathfrak{A}_2$ has the form*

$$\varkappa = c_0 v_2 + c_1 v + c_2 u, \tag{1}$$

where

$$\begin{aligned}
c_0 &= K + \frac{a_{11} - aa_{22} - 2\,(1-a)\,a_{12}}{4a(1-a)} + \frac{(-1+2a)\,a_1^2 - a^2 a_2^2 + 2\,(1-a)^2\,a_1 a_2}{4\,(1-a)^2\,a^2}, \\
c_1 &= \frac{K_2 - K_1}{4(1-a)} + \frac{(a-4)\,a_1 + \left(11 - 20a + 12a^2\right)a_2}{12\,(1-a)^2\,a}K + \frac{a_{112} - a_{122}}{4a(1-a)} \\
&\quad + \frac{a_1 - aa_2}{4a^2(1-a)}a_{22} + \frac{(2a-1)\,(a_1 - aa_2)}{4\,(1-a)^2\,a^2}a_{12} - \frac{a_2^2\,((1-2a)\,a_1 + aa_2)}{4\,(1-a)^2\,a^2}, \\
c_2 &= \frac{aK_2 - K_1}{4a(1-a)} + \frac{(1-2a)\,a_1 - (a-2)\,aa_2}{4\,(1-a)^2\,a^2}K.
\end{aligned}$$

The coefficient c_0 in the expression of $\varkappa$ has an intrinsic geometric meaning. Namely, let us define a curvature of a 4-web as the arithmetic mean of the curvatures of its 3-subwebs. More precisely, consider the 3-subwebs $[1,2,3]$, $[1,2,4]$, $[1,3,4]$ and $[2,3,4]$ of a 4-web W_4 with the following normalizations given by the 1-forms and the basic invariant:

$[1,2,3]$: $\omega_1, \omega_2, \omega_3, \omega_4$;

$[1,2,4]$: $\rho_1 = a\omega_1, \rho_2 = \omega_2, \rho_3 = \omega_4, \rho_4 = \omega_3$;

$[1,3,4]$: $\sigma_1 = (a-1)\omega_1, \sigma_2 = -\omega_3, \sigma_3 = \omega_4, \sigma_4 = -\omega_2$;

$[2,3,4]$: $\tau_1 = (a-1)\omega_2, \tau_2 = a\omega_3, \tau_3 = -\omega_4, \tau_4 = a\omega_1$.

Let $K[l,m,n]$ be the curvature function of the 3-subweb $[l,m,n]$. Define a *curvature* 2*-form* $L\omega_1 \wedge \omega_2$ *of the* 4*-web* as follows

$$\begin{aligned} 4L\omega_1 \wedge \omega_2 &= K[1,2,3]\omega_1 \wedge \omega_2 + K[1,2,4]\rho_1 \wedge \rho_2 \\ &\quad + K[1,3,4]\sigma_1 \wedge \sigma_2 + K[2,3,4]\tau_1 \wedge \tau_2. \end{aligned}$$

Then (see [13] or Ch. 7 of [12] for details)

$$\begin{aligned} K[1,2,3] &= K, \\ K[1,2,4] &= \frac{1}{a}\left(K - \frac{a_{12}}{a} + \frac{a_1 a_2}{a^2}\right), \\ K[1,3,4] &= \frac{1}{a-1}\left[K + \frac{a_2(a_1-a_2)}{(1-a)^2} + \frac{a_{12}-a_{22}}{1-a}\right], \\ K[2,3,4] &= \frac{1}{a(a-1)}\left[K + \frac{(2a-1)a_1(a_1-a_2)}{a^2(1-a)^2} + \frac{a_{11}-a_{12}}{a(1-a)}\right]. \end{aligned}$$

Computing the *curvature function* L from the above formulae, we obtain the following geometric interpretations of the coefficient c_0.

Theorem 8 *The coefficient c_0 equals the curvature function of the* 4*-web:*

$$c_0 = L.$$

5.2 4-webs of maximum rank

A planar 4-web has the maximum rank three if and only if the obstruction $\varkappa$ identically equals zero, i.e., if and only if $c_0 = c_1 = c_2 = 0$. This leads us to the following result.

Theorem 9 *A planar* 4*-web W_4 is of maximum rank three if and only if its curvature K and the covariant derivatives K_3 and K_4 of K, where $\partial_3 = \partial_2 - \partial_1$*

and $\partial_4 = a\partial_2 - \partial_1$, *are expressed in terms of the* 4*-web basic invariant* a *and its covariant derivatives up to the third order as follows:*

$$\begin{aligned}
K &= \frac{-a_{11} + aa_{22} + 2\,(1-a)\,a_{12}}{4a(1-a)} + \frac{(1-2a)\,a_1^2 + a^2 a_2^2 - 2\,(1-a)^2\,a_1 a_2}{4\,(1-a)^2\,a^2}, \\
K_3 &= \frac{(4-a)\,a_1 - \left(11 - 20a + 12a^2\right)a_2}{3\,(1-a)\,a} K + \frac{a_{122} - a_{112}}{a} + \frac{a_4 a_{22}}{a^2} \\
& \quad + \frac{(2a-1)\,a_4 a_{12}}{(1-a)\,a^2} + \frac{2a_2^2\,(1-a)\,a_1 + a_2^2 a_4}{(1-a)\,a^2}, \\
K_4 &= \frac{aa_4 - (1-a)\,a_1 - 2aa_3}{(1-a)\,a} K.
\end{aligned}$$

Taking the covariant derivatives δ_3 and δ_4 of the first equation in the above theorem, we find the values of K_3 and K_4. Comparing the obtained values with their values in the theorem, we arrive at two relations (see them below in Proposition 10) between the 4-web basic invariant a and its covariant derivatives up to the third order.

Conversely, these relations along with the values of K_3 and K_4 obtained by differentiation of K allow us to reconstruct the second and the third equations of the above theorem.

This proves the following result.

Proposition 10 *A planar* 4*-web is of maximum rank three if and only if its curvature* K *has the form indicated in the first equation of Theorem 9, and the* 4*-web basic invariant* a *and its covariant derivatives up to the third order satisfy the following two relations:*

$$\begin{aligned}
& 6(a-1)^2 a^2[-a_{111} + 2(a+1)a_{112} - 3aa_{122}] \\
& + a(a-1)[a(5(7a-5)a_1 - 3(4a^2+5a-4)a_2)a_{11} \\
& - 2((13a^2+18a-19)a_1 + 3a(3-5a)a_2)a_{12} \\
& + a((19a-17)a_1 + 15aa_2)a_{22} \\
& + (-34a^2+49a-19)a_1^3 + (26a^3+40a^2-89a+38)a_1^2 a_2 \\
& + a(-31a^2+53a-18)a_1 a_2^2 - 15a^3 a_2^3 = 0
\end{aligned}$$

and

$$\begin{aligned}
& 6(a-1)^2 a^2[3a_{112} - 2(a+1)a_{122} + aa_{222}] \\
& + a(a-1)[(-15a_1 + (17-19a)a_2)a_{11} \\
& + 2(3(7-9a)a_1 + (5a^2+18a-11)a_2)a_{12} \\
& + (3(4a^2+5a-4)a_1 + a(1-11a)a_2)a_{22} \\
& + 15(2a-1)a_1^3 + (56a^2-101a+41)a_1^2 a_2 \\
& + (-10a^3-41a^2+58a-22)a_1 a_2^2 + a^2(5a-1)a_2^3 = 0
\end{aligned}$$

This proposition allows us to find a geometric meaning of the last two equations of Theorem 9.

Proposition 11 *If the curvature of a* 4*-web vanishes, then conditions of Proposition 10 are equivalent to linearizability of the* 4*-web.*

Proof. It is easy to check that under condition $L = 0$, the 4-web linearizability conditions given in [2] are equivalent to the conditions in Proposition 10. ■

Now we can formulate Theorem 9 in pure geometric terms.

Theorem 12 *A* 4*-web is of maximum rank three if and only if it is linearizable and its curvature vanishes.*

Remark 13 *As far as we know, the above characterizations of* 4*-webs of maximum rank are the first invariant intrinsic descriptions of such webs in terms of the web invariants. Moreover, conditions for a* 4*-web to be of maximum rank include the web linearizability conditions.*

Thus, we have three different (but equivalent) invariant analytic conditions which are necessary and sufficient for a 4-web to be of maximum rank three:

(i) The conditions of Theorem 9;

(ii) Vanishing of the curvature of the 4-web and the conditions of Proposition 10 ; and

(iii) Vanishing of the curvature of the 4-web and the 4-web linearizability conditions from [2].

Each of these three conditions is effective and can be used as a test for determination whether some concrete 4-web is of maximum rank (see examples at the end of this section).

Theorem 12 leads to some interesting results in web geometry.

For *linearizable* 4-webs, the condition of vanishing of the curvature is necessary and sufficient for a 4-web to be of maximum rank.

Corollary 14 *A linearizable planar* 4*-web is of maximum rank three if and only if the curvature vanishes.*

Remark. The proof of Theorem 12 (and Corollary 14) is heavily based on the 4-web linearizability conditions in [2]. For a *linear* 4-web, the result of Corollary 14 was announced (not proved) in [23] (see also [25], Section 5.1.3). Our result is more general than the result for *linear* 4-webs in [23].

The next corollary gives the direct web-theoretical proof of the Poincaré theorem.

Corollary 15 (**Theorem of Poincaré**) *A planar* 4*-web of maximum rank three is linearizable.*

Proof. The result follows directly from Theorem 12, because the linearizability conditions are a part of conditions of Theorem 12. ■

Corollary 16 *If a planar* 4*-web with a constant basic invariant a has maximum rank three, then it is parallelizable.*

Proof. In fact, if $a = \text{const}.$, then $a_i = a_{ij} = a_{ijk} = 0$. If the 4-web is of maximum rank, then by Theorem 12, $L = 0$. Substituting $a_i = a_{ij} = a_{ijk} = 0$ into $L = 0$, we get $K = 0$. Therefore, the web is parallelizable. ■

Corollary 17 *Parallelizable planar 4-webs have maximum rank three.*

Proof. By Proposition 2, a 4-web is parallelizable if and only if the following conditions are satisfied:

$$K = 0, \quad a = \text{const}.$$

It follows that $L = 0$. Because a parallelizable 4-web is linearizable, by Theorem 12, such a web is of maximum rank three. ■

Definition 18 *4-webs all 3-subwebs of which are parallelizable (hexagonal) are called* ***Mayrhofer 4-webs.***

They were introduced by Mayrhofer (see [20]). The following corollary gives new property of Mayrhofer's 4-webs.

Corollary 19 *The Mayrhofer 4-webs are of maximum rank three.*

Proof. First note that by Definition 18, we have $L = 0$. Second, it is well known (see [4], §10; see also [13]) that the Mayrhofer 4-webs are linearizable. Thus, by Theorem 12, the Mayrhofer 4-webs are of, by maximum rank three. ■

Note that the result of Corollary 19 was also proved in the recent paper [27].

In the same way which we used to define the curvature of a 4-web, by taking alternative sums, we can find three additional second-order invariants which are expressed only in terms of the basic invariant a and its covariant derivatives of the first and second order:

$$\begin{aligned} M &= K[1,2,3] - aK[1,2,4] - (a-1)K[1,3,4] + a(a-1)K[2,3,4], \\ P &= K[1,2,3] + aK[1,2,4] - (a-1)K[1,3,4] - a(a-1)K[2,3,4], \\ Q &= K[1,2,3] - aK[1,2,4] + (a-1)K[1,3,4] - a(a-1)K[2,3,4]. \end{aligned}$$

Then

$$\begin{aligned} M &= \frac{-a_{11} - 2aa_{12} - aa_{22}}{a(a-1)} + \frac{(2a-1)a_1^2 - 2a^2a_1a_2 + a^2a_2^2}{a^2(a-1)^2}, \\ P &= \frac{(a_{11} - aa_{22})}{a(a-1)} + \frac{(1-2a)a_1^2 + a^2a_2^2}{a^2(a-1)^2}, \\ Q &= \frac{a_{11} - 2a_{12} + aa_{22}}{a(a-1)} + \frac{(1-2a)a_1^2 + 2(2a-1)a_1a_2 - a^2a_2^2}{a^2(a-1)^2}. \end{aligned}$$

Using these invariants, we can now establish a new invariant characterization of Mayrhofer's 4-webs.

Proposition 20 *A* 4*-web is Mayrhofer's web if and only if the invariants* M, P, Q *and* L *vanish.*

Proof. Consider the system $L = M = P = Q = 0$ as a linear homogeneous system with respect to $K[1,2,3]$, $K[1,2,4]$, $K[1,3,4]$ and $K[2,3,4]$. The determinant of this system is equal to $-16a^2(a-1)^2$. Because $a \neq 0, 1$, the determinant is different from 0. Thus the system has only the trivial solution $K[1,2,3] = K[1,2,4] = K[1,3,4] = K[2,3,4] = 0$.

Therefore, by Definition 18, the 4-web is a Mayrhofer 4-web. The converse statement is obvious. ■

5.2.1 Examples

Remind that we use the following form of the abelian relation:

$$(u + av)\,\omega_1 + (u + v)\,\omega_2 + u\omega_3 + v\omega_4 = 0,$$

where all summands are closed 1-forms under condition that u and v satisfy the abelian equation

$$\delta_1(u) - \delta_2(u) = 0,\ \delta_1(v) - \delta_2(av) = 0,\ \delta_1(u) + \delta_1(v) = 0.$$

The following two cases are important in applications:

$v = 0$: This will be the case if and only if $K = 0$, and the abelian relation has the form

$$u\omega_1 + u\omega_2 + u\omega_3 = 0.$$

$u = 0$: In this case the abelian equation gives

$$\delta_1(v) = 0, \delta_2(v) = \frac{a_2 v}{a},$$

and the compatibility conditions

$$\begin{aligned}\delta_2\delta_1 v - \delta_1\delta_2 v &= Kv,\\ \delta_2\delta_1 v - \delta_1\delta_2 v &= \delta_1\left(\frac{a_2}{a}\right)v\end{aligned}$$

imply

$$K = \delta_1\left(\frac{a_2}{a}\right) = \frac{aa_{12} - a_1 a_2}{a^2}.$$

The abelian relation becomes

$$av\omega_1 + v\omega_2 + v\omega_4 = 0.$$

There are three cases when both these conditions hold, and therefore

$$u\omega_1 + u\omega_2 + u\omega_3 = 0,$$

and

$$av\omega_1 + v\omega_2 + v\omega_4 = 0$$

are abelian relations:

1. Parallelizable 4-webs;
2. Mayrhofer 4-webs; and
3. 4-webs for which $K[1,2,3] = K[1,2,4] = 0$.

Example 21 *We consider the planar* 4*-web formed by the coordinate lines* $y =$ const., $x =$ const., *and by the level sets of the functions*

$$f(x,y) = \frac{x}{y} \text{ and } g(x,y) = \frac{1-y}{1-x}.$$

Note that this is the 4-subweb of the famous Bol 5-web which has the maximum rank six but not linearizable (see Example 7 in Section 5.2 of [2]). Note also that the third and the fourth foliations of this 4-web are the pencils of straight lines with the centers at points $(0,0)$ and $(1,1)$. This 4-web is linear (and therefore linearizable).

First, note that because the 3-subweb $[1,2,3]$ of this 4-web is parallelizable, the web admits the abelian relation

$$u\omega_1 + u\omega_2 + u\omega_3 = 0.$$

The direct calculation shows that for this 4-web the conditions of Theorem 9 are satisfied. Moreover, the straightforward computations show that this 4-web is a Mayrhofer 4-web, and the latter is of maximum rank three by Corollary 19. The corresponding abelian relations are

$$\begin{aligned} &\ln f_1 - \ln f_2 - \ln f_3 = 0, \\ &\ln(1-f_1) - \ln(1-f_2) + \ln f_4 = 0, \\ &\ln \tfrac{1-f_1}{f_1} - \ln \tfrac{1-f_3}{f_3} - \ln(1-f_4) = 0, \end{aligned}$$

where

$$f_1 = x,\ f_2 = y,\ f_3 = \frac{x}{y},\ f_4 = \frac{1-y}{1-x}.$$

Example 22 *We consider the planar* 4*-web formed by the coordinate lines* $y =$ const., $x =$ const., *and by the level sets of the functions*

$$f(x,y) = \frac{x}{y} \text{ and } g(x,y) = \frac{x-xy}{y-xy}$$

(*see Example* 8 *in Section* 5.2 *of* [2]).

Note that this is another 4-subweb of the famous Bol 5-web. Note also that the third and the fourth foliations of this 4-web are the pencil of straight lines with the center at the point $(0,0)$ and the foliation of conics. It was proved in [2] that this 4-web is linearizable.

By the same reason as in Example 21, we have again $K = 0$ and conditions of Theorem 9 are satisfied. Thus, the planar 4-web in question is of maximum rank three with the following abelian relations:

$$\begin{aligned}
&\ln f_1 - \ln f_2 - \ln f_3 = 0,\\
&\ln\left(\tfrac{1}{f_1} - 1\right) - \ln\left(\tfrac{1}{f_2} - 1\right) + \ln f_4 = 0,\\
&\ln(1 - f_1) - \ln(1 - f_3) + \ln(1 - f_4) = 0,
\end{aligned}$$

where

$$f_1 = x,\ f_2 = y,\ f_3 = \frac{x}{y},\ f_4 = \frac{x(1-y)}{y(1-x)}.$$

Example 23 *We consider the planar* 4*-web formed by the coordinate lines* $y = \text{const}.,\ x = \text{const}.,$ *and by the level sets of the functions*

$$f(x,y) = x + y \text{ and } g(x,y) = x^2 + y^2.$$

By the same reason as in Example 21, we have again $K = 0$, and therefore the web admits the abelian relation

$$u\omega_1 + u\omega_2 + u\omega_3 = 0.$$

One can check that the 4-web linearizability conditions from [2] are not satisfied. Therefore, this 4-web is not linearizable. By Theorem 15, this 4-web is not of maximum rank three. Thus, the rank of the 4-web in question can be 1 or 2.

5.3 4-webs of rank two

As we have seen earlier, a 4-web admits an abelian equation (has a positive rank) if and only if the equation

$$c_0 v_2 + c_1 v + c_2 u = 0 \tag{2}$$

has a nonzero solution.

Suppose that the coefficient c_0 in equation (2) equals 0, $c_0 = 0$. Then if two other coefficients c_1 and c_2 of (2) are also 0, then as we know (see Theorem 9), a 4-web is of maximum rank three. If $c_0 = 0$ but one of the coefficients c_1 or c_2 of (2) is not 0, then $c_1 v + c_2 u = 0$ and, say u, satisfies a 1-st order PDEs system of two equations. Therefore, the 4-web admits not more than one abelian equation (i.e., it is of rank one or zero).

In what follows , we assume that the coefficient c_0 in (2) is different from 0: $c_0 \neq 0$. Then a 4-web cannot be of rank more that two.

In this section we shall consider the case when *a* 4*-web is of rank two.*

Theorem 24 *A planar* 4*-web is of rank two if and only if* $c_0 \neq 0$, *and*

$$G_{ij} = 0, i, j = 1, 2, \tag{3}$$

where

$$\begin{aligned}
G_{11} &= ac_0(c_{2,2} - c_{2,1}) + ac_2(c_{0,1} - c_{0,2}) - a(1-a)c_1c_2 \\
&\quad + (2a_2 - a_1 - aa_2)c_0c_2 - Kc_0^2, \\
G_{12} &= ac_0(c_{1,2} - c_{1,1}) + ac_1(c_{0,1} - c_{0,2}) - a(1-a)c_1^2 \\
&\quad + (2a_2 - a_1 - 2aa_2)c_0c_1 + \left(a_2^2 + a_{12} - a_{22}\right)c_0^2, \\
G_{21} &= c_0(c_{2,1} - ac_{2,2}) + c_2(ac_{0,2} - c_{0,1}) - 2a_2c_0c_2 + a(1-a)c_2^2, \\
G_{22} &= c_0(c_{1,1} - ac_{1,2}) + c_1(ac_{0,2} - c_{0,1}) + a(1-a)c_1c_2 - a_2c_0c_1 \\
&\quad - a_2(1-a)c_0c_2 + (a_{22} - K)c_0^2.
\end{aligned}$$

Proof. Adding the compatibility condition (2) to the abelian equations and solving the resulting system with respect to u_1, u_2, v_1, and v_2, we get the Frobenius type PDEs system:

$$\begin{aligned}
u_1 &= -a_2v + \frac{a}{c_0}(c_2u + c_1v), \\
u_2 &= -a_2v + \frac{a}{c_0}(c_2u + c_1v), \\
v_1 &= a_2v - \frac{a}{c_0}(c_2u + c_1v), \\
v_2 &= -\frac{a}{c_0}(c_2u + c_1v).
\end{aligned}$$

We get the integrability conditions for this system from the commutation relation $\delta_2\delta_1 - \delta_1\delta_2 = K$. Computing the commutators and substituting u_1, u_2, v_1 and v_2, due to the system, we arrive at the integrability conditions in the form

$$\begin{aligned}
G_{11}u + G_{12}v = 0, \\
G_{21}u + G_{22}v = 0,
\end{aligned} \tag{4}$$

where G_{11}, G_{12}, G_{21} and G_{22} are defined by formulas in Theorem 24.

But for a 4-web to be of rank two, one needs two independent solutions u and v. This proves (3). ■

5.3.1 Examples

Example 25 *We consider the planar* 4*-web formed by the coordinate lines* $y =$ const., $x =$ const., *and by the level sets of the functions*

$$f(x, y) = x + y \text{ and } g(x, y) = x^2 + y^2.$$

(*see Example* 22).

We have already established that this 4-web admits the abelian relation

$$u\omega_1 + u\omega_2 + u\omega_3 = 0,$$

and its rank is either 1 or 2.

In this case

$$\begin{aligned}
c_0 &= -\frac{3(x-y)^3(x+y)}{xy^5} \neq 0,\\
c_1 &= \frac{x^2-y^2}{4x^2y^3}, \quad c_2 = 0;\\
c_{0,1} &= -\frac{1}{2x^3}, \quad c_{0,2} = \frac{1}{2y^3};\\
c_{1,1} &= -\frac{1}{2x^3y}, \quad c_{1,2} = \frac{3x^2-y^2}{4x^2y^4};\\
c_{2,1} &= c_{2,2} = 0,
\end{aligned}$$

and

$$G_{11} = G_{12} = G_{21} = G_{22} = 0.$$

It follows that conditions (3) are satisfied for this 4-web. Thus, the 4-web is of rank two.

Two abelian relations for this web are:

$$\begin{aligned}
f_1 + f_2 - f_3 &= 0,\\
f_1^2 + f_2^2 - f_4 &= 0,
\end{aligned}$$

where

$$f_1 = x, \ f_2 = y, \ f_3 = x+y, \ f_4 = x^2+y^2.$$

Example 26 *We consider the planar* 4*-web formed by the coordinate lines* $y =$ const., $x =$ const., *and by the level sets of the functions*

$$f(x,y) = \frac{x}{y} \text{ and } g(x,y) = xy(x+y).$$

We have again $K = 0$, and one can check that the 4-web linearizability conditions [2] are not satisfied. Therefore, our 4-web is not linearizable. By Theorem 15, this 4-web is not of maximum rank three. Thus, the rank of the 4-web in question can be 1 or 2.

We have

$$\begin{aligned}
c_0 &= \frac{3y^3(x^2-y^2)}{2x(2x+y)^2(x+2y)^2} \neq 0;\\
c_1 &= \frac{3y^5(y-x)}{2x(2x+y)^2(x+2y)^2} \neq 0; \ c_2 = 0;\\
c_{0,1} &= c_{0,2} = \frac{3y^4(2x^4-5x^3y-12x^2y^2-5xy^3+2y^4)}{2x(2x+y)^2(x+2y)^2};\\
c_{1,1} &= c_{1,2} = -\frac{3y^6(4x^3-10x^2y-7xy^2+4y^3)}{2x^2(2x+y)^3(x+2y)^4};\\
c_{2,1} &= c_{2,2} = 0,
\end{aligned}$$

and as a result

$$G_{11} = G_{12} = G_{21} = G_{22} = 0.$$

Thus, the 4-web is of rank two.

The two abelian relations are

$$\ln f_1 - \ln f_2 - \ln f_3 = 0,$$
$$\ln f_1 + 2\ln f_2 + \ln(1 + f_3) - \ln f_4 = 0,$$

where

$$f_1 = x,\ f_2 = y,\ f_3 = \frac{x}{y},\ f_4 = xy(x+y).$$

Examples 25 and 26 lead us to the important observation:

Proposition 27 *In general* 4*-webs of rank two are not linearizable.*

5.4 4-webs of rank one

As we have seen before, a 4-web can be of rank one if $c_0 = 0$ but one of the coefficients c_1 and c_2 of (2) is not 0 or if $c_0 \neq 0$. The following theorem outlines the four cases when a 4-web can be of rank one.

Theorem 28 *A planar* 4*-web is of rank one if and only if one of the following conditions holds:*

1. $c_0 = 0,\ J_1 = J_2 = 0,$ *where*

$$\begin{aligned} J_1 &= a_2c_1c_2(c_1 - c_2) + ac_2^2(c_{1,2} - c_{1,1}) \\ &\quad + c_1c_2(c_{1,1} + a(c_{2,1} - c_{1,2} - c_{2,2})) + c_1^2(ac_{2,2} - c_{2,1}), \\ J_2 &= c_1^2(c_1 - c_2)^2 K + (c_{1,11} - c_{1,12})\, c_1c_2\,(c_2 - c_1) \\ &\quad + c_1^2(c_1 - c_2)(c_{2,11} - c_{2,12}) - c_2(2c_1 - c_2)\, c_{1,1}(c_{1,2} - c_{1,1}) \\ &\quad + c_1^2c_{2,1}(c_{1,2} - c_{2,2} + c_{2,1}) + c_1^2c_{1,1}(c_{2,2} - 2c_{2,1}) \end{aligned}$$

   and $c_1 \neq c_2,\ c_1 \neq 0.$

2. $c_0 = 0, c_1 = c_2 \neq 0,$ *and* $J_3 = 0,$ *where*

$$J_3 = (a_{22} - a_{12})(1 - a) + a_2(a_2 - a_1) - (1 - a)^2 K.$$

3. $c_0 = 0, c_1 = 0, c_2 \neq 0,$ *and* $J_4 = 0,$ *where*

$$J_4 = a_{12}a - a_1a_2 - Ka^2.$$

4. $c_0 \neq 0,$ *and* $J_{10} = J_{11} = J_{12} = 0,$ *where*

$$J_{10} = G_{11}G_{22} - G_{21}G_{12},$$

$$
\begin{aligned}
J_{11} &= c_0(G_{21,1}G_{22} - G_{22,1}G_{21}) + (a_2c_0 - ac_1)G_{21}^2 \\
&\quad +(ac_2 - a_2c_0 + ac_1)G_{21}G_{22} - ac_2G_{22}^2,
\end{aligned}
$$

$$
\begin{aligned}
J_{12} &= c_0(G_{21,2}G_{22} - G_{22,2}G_{21}) + (a_2c_0 - ac_1)G_{21}^2 \\
&\quad +a(c_2 - c_1)G_{21}G_{22} - c_2G_{22}^2.
\end{aligned}
$$

Proof. First, we consider the case when $c_0 = 0$, one of the coefficients c_1 and c_2 of (2) is not 0, and 4-web W_4 is of rank one.

Then it follows from equation (2) that

$$
u = c_1 t, \quad v = -c_2 t \tag{5}
$$

for some function t.

Differentiating these equations, we find that

$$
\begin{array}{ll}
u_1 = c_{1,1}t + c_1t_1, & u_2 = c_{1,2}t + c_1t_2, \\
v_1 = -c_{2,1}t - c_2t_1, & v_1 = -c_{2,2}t - c_2t_2.
\end{array}
$$

Substituting these expressions into the abelian equation, we get

$$
\begin{array}{l}
(c_{1,1} - c_{2,1})t + (c_1 - c_2)t_1 = 0, \\
(c_{1,1} - c_{1,2})t + c_1(t_1 - t_2) = 0, \\
(-c_{2,1} + ac_{2,2} + a_2c_2)t - c_2t_1 + ac_2t_2 = 0.
\end{array} \tag{6}
$$

If $c_1 - c_2 \neq 0$ and $c_1 \neq 0$, then solving the first two equations with respect to t_1 and t_2, we obtain

$$
t_1 = \frac{t(c_{2,1} - c_{1,1})}{c_1 - c_2}, \; t_2 = \frac{t(c_{1,1} - c_{2,1})}{c_1} + \frac{t(c_{2,1} - c_{1,1})}{c_1 - c_2}. \tag{7}
$$

Substituting these values into the last equation of the previous system, we arrive at the equation $J_1 = 0$, where J_1 is expressed as in Theorem 28.

Next, differentiating the third equation of (6) in the direction $\{\omega_2 = 0\}$ and using symmetric derivatives, we find that

$$
\begin{aligned}
&\frac{t_1}{c_1}(c_{1,1} - c_{1,2}) - \frac{t}{c_1^2}[(c_{1,11} - c_{1,12} + \frac{3Kc_1}{2})c_1 - (c_{1,1} - c_{1,2})c_{1,1}] \\
&-\frac{t_1}{c_1 - c_2}(c_{1,1} - c_{2,1}) - \frac{t}{(c_1 - c_2)^2}[(c_{1,11} - c_{2,11})(c_1 - c_2) - (c_{1,1} - c_{2,1}) \\
&+\frac{t_2(c_{1,1} - c_{2,1})}{c_1 - c_2} + t\frac{c_{1,12} - c_{2,12}}{(c_1 - c_2)^2} + t\frac{3K}{2(c_1 - c_2)} - t\frac{(c_{1,1} - c_{2,1})(c_{1,2} - c_{2,2})}{(c_1 - c_2)^2} \\
= \;& 0.
\end{aligned}
$$

Substituting the values of t_1 and t_2 from (7) into the above equation, we arrive at the equation $J_2 = 0$ of Theorem 28.

Consider now the case: $c_0 = 0, \; c_1 = c_2 \neq 0$.

Then $u = -v$, and

$$u_1 - u_2 = 0,$$
$$u_1 - au_2 - a_2 u = 0.$$

Solving this system with respect to u_1 and u_2, we find that

$$u_1 = \frac{a_2 u}{1-a}, \quad u_2 = \frac{a_2 u}{1-a}.$$

The compatibility of the above equations gives $J_3 = 0$, where

$$J_3 = \left(\frac{a_2 u}{1-a}\right)_2 - \left(\frac{a_2 u}{1-a}\right)_1 - K = \frac{(a_{12} - a_{22})(a-1) - a_2(a_1 - a_2)}{(a-1)^2} - K.$$

Consider now the second excluded case: $c_0 = 0,\ c_1 = 0,\ c_2 \neq 0.$
Then $u = 0$. This implies

$$v_1 = 0,$$
$$av_2 + a_2 v = 0,$$

and the compatibility condition $J_4 = 0$, where $J_4 = a_{12}a - a_1 a_2 - Ka^2$.

Suppose now that $c_0 \neq 0$, and a 4-web is of rank one. Then the abelian equation together with the compatibility condition $\varkappa = 0$ gives the system

$$\begin{aligned} u_1 &= -v_1, \\ u_2 &= -v_1, \\ u_1 &= -a_2 v + \frac{1}{c_0}(c_1 v + c_2 u), \\ v_2 &= -\frac{1}{c_0}(c_1 v + c_2 u). \end{aligned}$$

Because the 4-web is of rank one, system (4) has a nonzero solution. Thus, its determinant vanishes:

$$J_{10} = G_{11}G_{22} - G_{12}G_{21} = 0.$$

Take, for example, the second equation of system (4) and differentiate it. Adding the resulting equations to the above system, we get the conditions $J_{11} = J_{12} = 0$. ■

5.4.1 Example

Example 29 *We consider the planar* 4*-web formed by the coordinate lines* $y =$ const., $x =$ const *t. and by the level sets of the functions*

$$f(x,y) = \frac{xy^2}{(x-y)^2} \text{ and } g(x,y) = \frac{x^2 y}{(x-y)^2}.$$

In this case, $c_0 = 0$ and $J_1 = J_2 = 0$. Thus, we have the web of type 1 as indicated in Theorem 28, and this 4-web is of rank one.

The only abelian relation is

$$\ln f_1 - \ln f_2 + \ln f_3 - \ln f_4 = 0,$$

where

$$f_1 = x,\ f_2 = y,\ f_3 = \frac{xy^2}{(x-y)^2},\ f_4 = \frac{x^2 y}{(x-y)^2}.$$

This example illustrates the following feature:

Proposition 30 *In general, 4-webs of rank one are not linearizable.*

5.5 4-webs with a constant basic invariant

Consider first 4-webs of maximum rank for which the basic invariant is constant on one of web foliations.

Without loss of generality, we assume that a is constant on the second foliation, i.e.,

$$a_1 = 0.$$

Then, solving the system $c_0 = c_1 = c_2 = 0$, we get

$$\begin{aligned} K &= \frac{a_2^2}{4\left(a-1\right)^2} - \frac{a_{22}}{4\left(a-1\right)}, \\ K_1 &= \frac{a_2 a_{22}}{2\left(a-1\right)^2} - \frac{a_2^3}{2\left(a-1\right)^3}, \\ K_2 &= \frac{a_2 a_{22}}{4\left(a-1\right)^2} - \frac{a_2^3}{4\left(a-1\right)^3}. \end{aligned}$$

Differentiating the first equation and taking into account the remaining two equations, we arrive at the conditions

$$a_{22} = \frac{a_2^2}{a-1},\ a_{222} = \frac{a_2^3}{\left(a-1\right)^2}.$$

The second equation above is the covariant derivative of the first one. The first equation implies $K = 0$, and the curvatures of all other 3-subwebs vanish too. In other words, the 4-web is a Mayrhofer web.

On the other hand, if we assume $a_1 = 0$ and $K = 0$, then the only condition for maximum rank is

$$a_{22} - \frac{a_2^2}{a-1} = 0$$

or

$$\delta_2\left(\frac{a_2}{a-1}\right) = 0.$$

Theorem 31 *1. A planar* 4*-web with the basic invariant constant on one of the web foliations is of maximum rank if one of its* 3*-subwebs is parallelizable. Then all other* 3*-subwebs are parallelizable, and the* 4*-web is a Mayrhofer web.*

2. If, say, $a_1 = 0$ *and* $K[1,2,3] = 0$*, then the* 4*-web is of maximum rank if* $a_2/(a-1)$ *is constant on the first foliation, i.e., if*

$$\delta_2\left(\frac{a_2}{a-1}\right) = 0.$$

We conclude our discussion of 4-webs of different ranks by the following theorem whose proof utilizes all previous results.

Theorem 32 *A planar* 4*-web with a nonparallelizable* 3*-subweb and a constant basic invariant is of rank* 0*.*

Proof. Consider a 4-web with a nonparallelizable 3-subweb $[1,2,3]$ and a constant basic invariant a. Then

$$c_0 = K \neq 0, \;\; c_1 = \frac{K_1 - K_2}{4(a-1)}, \; c_2 = \frac{K_1 - aK_2}{4a(a-1)}.$$

Therefore, the web is not of rank three.

Substituting these expressions into equations of Theorem 24, we find

$$\begin{aligned}
G_{11} &= \frac{5(K_1^2 - (a+1)K_1K_2 + aK_2^2) - 4K(4K^2(a-1) + K_{11} - 2aK_{12} + a^2K_{22})}{16(a-1)}, \\
G_{12} &= \frac{5(K_1 - K_2)^2 - 4K(K_{11} - 2K_{12} + K_{22})}{16(a-1)}, \\
G_{21} &= \frac{-5(K_1 - aK_2)^2 + 4K(K_{11} - a(a-1)K_{12} + a^3K_{22})}{16a(a-1)}, \\
G_{22} &= -\frac{5(K_1^2 - (a+1)K_1K_2 + aK_2^2) + 4K(4K^2(a-1) - K_{11} + (1+a)K_{12} - aK_{22})}{16(a-1)}.
\end{aligned}$$

Thus, in general, $G_{ij} \neq 0$. Therefore, 4-webs with a constant basic invariant are not of rank two.

To check whether our 4-web is of rank 1, we note that $c_0 = K \neq 0$ because the 3-subweb $[1,2,3]$ is nonparallelizable. Therefore, if the web admits one abelian equation, then it would belong to class 4 of Theorem 28. This will be the case if $J_{10} = J_{12} = J_{12} = 0$.

But

$$\begin{aligned}
J_{10} &= G_{11}G_{22} - G_{12}G_{21} \\
&= \frac{1}{64}K[64K^5 - 5K_2^2K_{11} + 16aK^3K_{22} - 5(a+1)K_1^2K_{22} + 4(a+1)KK_{11}K_{22} \\
&\quad +5K_1K_2(K_{12} + aK_{22}) - K_{12}(-5K_1^2 + 4K(4K^2 + K_{11} + aK_{22}))].
\end{aligned}$$

Thus in general, $J_{10} \neq 0$. As a result, the web does not admit even one abelian equation, and it is of rank 0. ■

6 Planar 5-webs

6.1 5-webs of maximum rank

Let us consider a planar 5-web in the standard normalization

$$\begin{aligned} &\omega_1+\omega_2+\omega_3=0,\\ &a\omega_1+\omega_2+\ \omega_4=0,\\ &b\omega_1+\omega_2+\ \omega_5=0, \end{aligned}$$

where a and b are the basic invariants of the web.

The abelian relation for such a web has the form

$$(w+au+bv)\omega_1+(w+u+v)\omega_2+w\omega_3+u\omega_4+v\omega_5=0,$$

where we have $\lambda_1=w+au+bv, \lambda_2=w+u+v, \lambda_3=w, \lambda_4=u,$and $\lambda_5=v$.

The functions $w,u,$ and v satisfy the abelian equation

$$\begin{aligned} &\delta_1(w)-\delta_2(w)=0,\\ &\delta_1(u)-\delta_2(au)=0,\\ &\delta_1(v)-\delta_2(bv)=0,\\ &\delta_1(w)+\delta_1(u)+\delta_1(v)=0, \end{aligned}$$

and their compatibility condition takes the form

$$\begin{aligned} \varkappa &=(\Delta_1\Delta_2\Delta_3\delta_1-\delta_1\Delta_2\Delta_3\Delta_1)(w)+(\Delta_1\Delta_2\Delta_3\delta_1-\Delta_1\delta_1\Delta_3\Delta_2)(u)\\ &+(\Delta_1\Delta_2\Delta_3\delta_1-\Delta_1\Delta_2\delta_1\Delta_3)(v)=0. \end{aligned}$$

In the canonical coordinates in the jet bundles, the abelian equation has the form

$$\begin{aligned} &w_1-w_2=0,\\ &u_1-au_2-a_2u=0,\\ &v_1-bv_2-b_2v=0,\\ &u_1+v_1+w_1=0, \end{aligned}$$

and the obstruction $\varkappa$ equals

$$c_0w_{22}++c_1w_2+c_2v_2+c_3w+c_4u+c_5v=0,$$

where

$$\begin{aligned} c_0 &= K+R[a,b]+R[b,a]\\ &-\frac{a-a^2+b-b^2-4ab+2a^2b+2ab^2}{10\,(-1+a)\,a\,(a-b)^2\,(-1+b)\,b}a_1b_1\\ &+\frac{2a-a^2+2b-b^2-4ab+a^2b+ab^2}{10\,(-1+a)\,(a-b)^2\,(-1+b)}a_2b_2, \end{aligned}$$

and

$$
\begin{aligned}
R[a,b] &= = \frac{(1-3a+b)\,a_{11} + \left(4a-3a^2-3b+4ab\right)a_{12} + \left(a^2-3ab+a^2b\right)a_{22}}{10\,(-1+a)\,a\,(a-b)} \\
&\quad + \frac{2a-6a^2+6a^3-b+4ab-6a^2b-b^2+2ab^2}{10\,(-1+a)^2\,a^2\,(a-b)^2} a_1^2 \\
&\quad + \frac{4a^2-8a^3+3a^4-6ab+14a^2b-8a^3b+3b^2-6ab^2+4a^2b^2}{10\,(-1+a)^2\,a^2\,(a-b)^2} a_1a_2 \\
&\quad + \frac{-a^4-2a^2b+6a^3b-a^4b-2a^2b^2}{10\,(-1+a)^2\,a^2\,(a-b)^2} a_2^2 + \frac{-1+ab}{10\,(-1+a)\,(a-b)^2\,(-1+b)} a_1b_2
\end{aligned}
$$

and the expressions for c_1, c_2, c_3, c_4, and c_5 are given in Section 7.2.

Theorem 33 *A planar* 5*-web is of maximum rank six if and only if the invariants* c_0, c_1, c_2, c_3, c_4 *and* c_5 *vanish.*

Note that c_0 contains the curvature function K, while c_1, c_2 contain K and the linear combinations of covariant derivatives K_1, K_2, and c_3, c_4, c_5 contain K and the linear combinations of the covariant derivatives K_1, K_2 and the second symmetrized covariant derivatives K_{11}, K_{12}, K_{22}.

Corollary 34 *For* 5*-webs of maximum rank, the curvature function* K *of the* 3*-subweb* $[1,2,3]$ *has the following expression in terms of the basic invariants* a *and* b *and their covariant derivatives:*

$$
\begin{aligned}
K &= \frac{a-a^2+b-b^2-4ab+2a^2b+2ab^2}{10\,(-1+a)\,a\,(a-b)^2\,(-1+b)\,b} a_1b_1 \\
&\quad - \frac{2a-a^2+2b-b^2-4ab+a^2b+ab^2}{10\,(-1+a)\,(a-b)^2\,(-1+b)} a_2b_2 - R[a,b] - R[b,a].
\end{aligned}
$$

6.2 Curvature of planar 5-webs

Similar to the case of 4-webs, the coefficient c_0 in the expression of $\varkappa$ for 5-webs has also an intrinsic geometric meaning, Namely, let us define a curvature of a 5-web as the arithmetic mean of the curvatures of its 3-subwebs. A planar 5-web has ten 3-subwebs:

$[1,2,3]$, $[1,2,4]$, $[1,3,4]$, $[2,3,4]$, $[1,2,5]$, $[1,3,5]$, $[2,3,5]$, $[1,4,5]$, $[2,4,5]$, $[3,4,5]$.

Normalizations with the curvature functions for 3-subwebs $[1,2,3], [1,2,4]$, $[1,3,4]$ and $[2,3,4]$ are given in Section 5.1. The similar expressions for 3-subwebs $[1,2,5]$, $[1,3,5]$ and $[2,3,5]$ can be obtained by the substitutions $a \to b$ and $4 \to 5$:

$[1,2,5]$: $\widetilde{\rho}_1 = b\omega_1, \widetilde{\rho}_2 = \omega_2, \widetilde{\rho}_3 = \omega_5, \widetilde{\rho}_4 = \omega_3;$

$[1,3,5]$: $\widetilde{\sigma}_1 = (b-1)\omega_1, \widetilde{\sigma}_2 = -\omega_3, \widetilde{\sigma}_3 = \omega_5, \widetilde{\sigma}_4 = -\omega_2;$

[2, 3, 5]: $\widetilde{\tau}_1 = (b-1)\omega_2, \widetilde{\tau}_2 = b\omega_3, \widetilde{\tau}_3 = -\omega_5, \widetilde{\tau}_4 = b\omega_1$;

For the last three cases, we have

[1, 4, 5]: $\zeta_1 = (a-b)\omega_1,\ \ \zeta_2 = -a\omega_1 - \omega_2,\ \zeta_3 = b\omega_1 + \omega_2$;

[2, 4, 5]: $\eta_1 = \frac{b-a}{b}\omega_2,\ \ \eta_2 = -a\omega_1 - \omega_2,\ \ \ \eta_3 = \frac{a}{b}(b\omega_1 + \omega_2)$;

[3, 4, 5]: $\theta_1 = (a-b)\omega_3,\ \theta_2 = (1-b)(a\omega_1 + \omega_2),\ \ \theta_3 = (a-1)(b\omega_1 + \omega_2)$;

For the curvature functions $K[l, m, n]$, we get the following expressions:

$$
\begin{aligned}
K[1,2,5] &= \frac{1}{b}\left(K - \frac{b_{12}}{b} + \frac{b_1 b_2}{b^2}\right), \\
K[1,3,5] &= \frac{1}{b-1}\left[K + \frac{b_2(b_1 - b_2)}{(1-b)^2} + \frac{b_{12} - b_{22}}{1-b}\right], \\
K[2,3,5] &= \frac{1}{b(b-1)}\left[K + \frac{(2b-1)b_1(b_1 - b_2)}{b^2(1-b)^2} + \frac{b_{11} - b_{12}}{b(1-b)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
K[1,4,5] &= \frac{K - a_{22}}{b-a} + \frac{b_{12} - a_{12} + a(a_{22} - b_{22})}{(a-b)^2} \\
&\quad + \frac{(b_2 - a_2)(a_2 b - a b_2 - a_1 + b_1)}{(a-b)^3},
\end{aligned}
$$

$$
\begin{aligned}
K[2,4,5] &= \frac{bK}{a(b-a)} - \frac{a a_{12} - a_1 a_2}{a^3(b-a)} + \frac{a_{11} b - a b_{11}}{a^2 b(b-a)^2} \\
&\quad - \frac{a_{12} b + a_1 b_2 - a_2 b_1 - a b_{12}}{ab(b-a)^2} + \frac{(a_1 b - a b_1)(2b b_2 - a b_2 - a_2 b)}{a b^2 (b-a)^3} \\
&\quad - \frac{(a_1 b - a b_1)(a_1 b^2 + 2ab(b_1 - a_1) - a^2 b_1)}{a^3 b^2 (b-a)^3},
\end{aligned}
$$

$$
\begin{aligned}
K[3,4,5] &= \frac{K}{(a-1)(b-1)(b-a)} - \frac{a a_2(b_1 - a_1 - b_2 + a_2)}{(a-1)^3(b-1)(b-a)^2} \\
&\quad + \frac{a_2(a b_2 - b_1 + (b-1)a_2)}{(a-1)^3(b-1)^2(b-a)} + \frac{a_1\,(b_1 - a_1 - b_2 + a_2)}{(a-1)^3(b-1)(b-a)^2} \\
&\quad + \frac{a_1\,(a b_2 - b_1 + (b-1)a_2)}{(a-1)^3(b-1)^2(b-a)} - \frac{a(b_{12} - a_{12} - b_{22} + a_{22})}{(a-1)^2(b-1)(b-a)^2} \\
&\quad - \frac{a_2(b_1 - a_1 - b_2 + a_2}{(a-1)^2(b-1)(b-a)^2} + \frac{a(b_1 - a_1 - b_2 + a_2)(b_2 - a_2)}{(a-1)^2(b-1)(b-a)^3} \\
&\quad - \frac{(b_1 - a b_2)b_2}{(a-1)^2(b-1)^3(b-a)} - \frac{a_2 b_2 - b_{12} + a b_{22} + (b-1)a_{22}}{(a-1)^2(b-1)^2(b-a)} \\
&\quad + \frac{(b_1 - a_1 - b_2 + a_2)(b_1 - a_1)}{(a-1)^2(b-1)(b-a)^3} - \frac{b_{11} - a_{11} - b_{12} + a_{12}}{(a-1)^2(b-1)(b-a)^2} \\
&\quad - \frac{-b_{11} + a_1 b_2 + a b_{12} + (b-1)a_{12}}{(a-1)^2(b-1)^2(b-a)} - \frac{(b_1 - a b_2)b_1}{(a-1)^2(b-1)^3(b-a)},
\end{aligned}
$$

Define a *curvature* 2*-form* $L\omega_1 \wedge \omega_2$ *of the* 5*-web* as follows

$$
\begin{aligned}
10L\omega_1 \wedge \omega_2 &= K[1,2,3]\omega_1 \wedge \omega_2 \\
&+K[1,2,4]\rho_1 \wedge \rho_2 + K[1,3,4]\sigma_1 \wedge \sigma_2 + K[2,3,4]\tau_1 \wedge \tau_2 \\
&+K[1,2,5]\widetilde{\rho}_1 \wedge \widetilde{\rho}_2 + K[1,3,5]\widetilde{\sigma}_1 \wedge \widetilde{\sigma}_2 + K[2,3,5]\widetilde{\tau}_1 \wedge \widetilde{\tau}_2 \\
&+K[1,4,5]\zeta_1 \wedge \zeta_2 + K[2,4,5]\eta_1 \wedge \eta_2 + K[3,4,5]\theta_1 \wedge \theta_2
\end{aligned}
$$

The straightforward calculation shows that

$$
\begin{aligned}
10L &= K + aK[1,2,4] + (a-1)K[1,3,4] \\
&+a(a-1)K[2,3,4] + bK[1,2,5] + (b-1)K[1,3,5] + b(b-1)K[2,3,5] \\
&+(b-a)K[1,4,5] + \frac{b}{a(b-a)}K[2,4,5] + (a-1)(b-1)(b-a)K[3,4,5]
\end{aligned}
$$

and

$$L = c_0.$$

Theorem 35 *The curvature of a planar* 5*-web of maximum rank equals zero.*

6.3 5-webs with constant basic invariants

We conclude this section by consideration of 5-webs with constant basic invariants. One can check that in this case c_0 coincides with the curvature function K of the 3-subweb $[1,2,3]$, and the expressions for c_1, c_2, c_3, c_4 and c_5 are linear combinations of K_i, K_{ij} and K^2. In other words, $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$ if and only if $K = 0$.

Theorem 36 *A planar* 5*-web with constant basic invariants is of maximum rank if and only if it is parallelizable.*

6.4 Examples

In this section we consider two examples of 5-webs.

Example 37 *We consider the Bol* 5*-web formed by the coordinate lines* $y =$ const., $x =$ const., *and by the level sets of the functions*

$$f(x,y) = \frac{x}{y},\ \ g_4(x,y) = \frac{1-y}{1-x} \ \text{and}\ g_5(x,y) = \frac{x-xy}{y-xy}$$

(*see Example* 8 *in Section* 5.2 *of* [2]).

First note that because the 3-subweb $[1,2,3]$ of this 5-web is parallelizable, we have $K = 0$.

For this 5-web, we have

$$a = \frac{x(y-1)}{y(x-1)}, b = \frac{y-1}{x-1},$$

and

$$c_i = 0$$

for $i = 0, 1, ..., 5$.

Thus, the Bol 5-web is of maximum rank.

Using the linearizability conditions for planar 5-webs given in [2], we observe that the Bol 5-web is not linearizable. Indeed, one of the invariants, namely, the second-order invariant

$$\mu_{[1,2,3,4]} - \mu_{[1,2,3,5]} = \frac{\partial_1 a - a\partial_2 a}{a - a^2} - \frac{\partial_1 b - b\partial_2 b}{b - b^2}$$

does not vanish.

For this 5-web, we have

$$\begin{aligned} \omega_1 &= -\frac{1}{y}dx, \ \ \omega_2 = \frac{x}{y^2}dy, \ \ \omega_3 = \frac{1}{y}dx - \frac{x}{y^2}dy, \\ \omega_4 &= \frac{x(y-1)}{(x-1)y^2}dx - \frac{x}{y^2}dy, \ \omega_5 = \frac{x(y-1)}{(x-1)y}dx - \frac{x^2}{y^2}dy, \end{aligned}$$

and solving the abelian equation, we get the following abelian relations:

$$\begin{aligned} &\ln f_1 - \ln f_2 - \ln f_3 = 0, \\ &\ln f_3 + \ln f_4 - \ln f_5 = 0, \\ &\ln(1 - f_1) - \ln(1 - f_2) + \ln f_4 = 0, \\ &\ln(1 - f_1) - \ln(1 - f_3) + \ln(1 - f_5) = 0, \\ &\ln \tfrac{1-f_1}{f_1} - \ln \tfrac{1-f_3}{f_3} + \ln(1 - f_4) = 0, \\ &\mathbf{D}_2(f_1) - \mathbf{D}_2(f_2) - \mathbf{D}_2(f_3) - \mathbf{D}_2(f_4) + \mathbf{D}_2(f_5) = 0, \end{aligned}$$

where

$$f_1 = x, \ f_2 = y, \ f_3 = \frac{x}{y}, \ f_4 = \frac{1-y}{1-x}, \ \ f_5 = \frac{x - xy}{y - xy}$$

and

$$\mathbf{D}_2(t) = \mathbf{Li}_2 t + \frac{1}{2}\ln t \ \ln(1-t) - \frac{\pi^2}{6}, \ \ 0 < t < 1; \ \mathbf{Li}_2 t = \sum_{n=1}^{\infty} \frac{t^n}{n^2},$$

or

$$\mathbf{D}_2(t) = -\frac{1}{2}\int_0^t \left(\frac{\ln|1-s|}{s} + \frac{\ln|s|}{1-s} \right) ds - \frac{\pi^2}{6}, \ \ 0 < t < 1,$$

is the version of the original Rogers dilogarithm (see [29] and [18]) normalized so that RHS of the last abelian relation is 0.

This example leads us to the following important observation

Proposition 38 *In general, planar 5-webs of maximum rank are not linearizable (algebraizable).*

Example 39 *We consider the* 5*-web formed by the coordinate lines* $y = \text{const}.,\ x = \text{const}.,$ *and by the level sets of the functions*

$$f(x,y) = \frac{x}{y},\ g_4(x,y) = \frac{x}{(x-1)(y-1)} \text{ and } g_5(x,y) = \frac{y}{(x-1)(y-1)}.$$

This is the 5*-subweb* $[1,2,3,7,8]$ *of the* 29*-web* $K(5)$ *(see [24], Section* 7.2.3*).*

As in Example 37, we have

$$K = 0,\ a = \frac{1-y}{y(x-1)},\ b = \frac{x(1-y)}{x-1},$$

and

$$c_i = 0$$

for $i = 0, 1, ..., 5$.

Thus, the planar 5-web in question is of maximum rank.

One can also check the linearizability conditions (see of [2], p. 445) hold, and this 5-web is linearizable (algebraizable). Therefore, the 5-web in question is algebraizable 5-web of maximum rank.

For this 5-web, we have

$$\begin{aligned}
\omega_1 &= -\frac{1}{y}dx,\ \omega_2 = \frac{x}{y^2}dy,\ \omega_3 = \frac{1}{y}dx - \frac{x}{y^2}dy, \\
\omega_4 &= \frac{1-y}{(x-1)y^2}dx - \frac{x}{y^2}dy,\ \omega_5 = \frac{x(1-y)}{(x-1)y^2}dx - \frac{x}{y^2}dy,
\end{aligned}$$

and solving the abelian equation, we get the following abelian relations

$$\begin{aligned}
&\ln f_1 - \ln f_2 - \ln f_3 = 0, \\
&\ln f_3 - \ln f_4 + \ln f_5 = 0, \\
&\ln \tfrac{f_1}{1-f_1} - \ln(1-f_2) - \ln f_4 = 0, \\
&\ln(1-f_1) - \ln \tfrac{f_2}{1-f_2} + \ln f_5 = 0, \\
&\tfrac{f_1-1}{f_1} + f_2 - f_3 - \tfrac{1}{f_4} = 0, \\
&f_1 - \tfrac{1-f_2}{f_2} + f_3 + \tfrac{1}{f_5} = 0,
\end{aligned}$$

where

$$f_1 = x,\ f_2 = y,\ f_3 = \frac{x}{y},\ f_4 = \frac{x}{(1-x)(1-y)},\quad f_5 = \frac{y}{(1-x)(1-y)}.$$

7 Appendix

7.1 Multi-brackets

Compatibility conditions for linear (as well as nonlinear) PDEs systems can be expressed in terms of multi-brackets introduced in [17]. Here we give the necessary formulae for multi-brackets which we use in this paper.

Consider a linear PDEs system of $n+1$ differential equations for n unknown functions:

$$\begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{n+11} & \cdots & a_{n+1\ n} \end{Vmatrix} \begin{Vmatrix} u_1 \\ \vdots \\ u_n \end{Vmatrix} = 0,$$

where a_{ij} are linear differential operators.

Then the multi-bracket $\{a_1, ..., a_{n+1}\}$ of scalar differential operators $a_i = (a_{i1}, ..., a_{in})$ is given by the formula

$$\{a_1, ..., a_{n+1}\} = \sum_{i=1}^{n+1} (-1)^{i-1} \operatorname{Ndet}(A_i)\, a_i,$$

where A_i is the $n \times n$ matrix obtained from the matrix $\|a_{ij}\|$ by deleting i-th row, and Ndet is a non-commutative version of the determinant function produced by the standard formula of decomposition of the determinant with respect to the first column.

The abelian equation is the equation of this form and it satisfies the conditions of [17]. Thus, the multi-brackets give the compatibility conditions for the abelian equations.

7.2 Coefficients for obstruction

Below we list coefficients c_1, c_2, c_3, c_4 and c_5. They are

$$c_1 = \frac{j_1}{3ad},\ c_2 = \frac{j_2}{3ad}, c_3 = \frac{j_3}{d}, c_4 = \frac{j_4}{6a}, c_5 = \frac{j_5}{6a},$$

where

$$d = -10a^2b^2\,(a-1)^2\,(b-1)^2\,(a-b)^2$$

and

$$\begin{aligned} j_1 = \ & -3(-1+b)b^2(6a^3 - 6a^2(1+b) - b(1+b) + 2a(1+b)^2)a_1^3 \\ & -3ba_1^2(-(-1+b)b(-6a^4 - b^2(5+2b) + a^3(16+13b) \\ & +ab(13+12b+4b^2) - a^2(10+23b+12b^2))a_2 + a((b+a^2(7-12b) \\ & b - 2b^3 + a^3(-2+4b) + a(1-8b+7b^2+4b^3))b_1 + b(-(-2+b) \\ & b^2 + a^3(-6+4b) + a^2(3+7b-5b^2) + a(1-6b-b^2+2b^3))b_2)) \\ & -a_1(3(-1+b)b^2(-6b^3 - 3a^4(2+b) + ab^2(17+8b) \\ & -a^2b(16+23b) + a^3(4+24b+b^2))a_2^2 + 3(-1+a) \\ & aba_2((2(2-3b)b^2 + a^3(-2+4b) + a^2(1+5b-12b^2) \\ & +ab(-8+9b+5b^2))b_1 + b(-2(-2+b)b^2 + a^3(-6+4b) \\ & +a^2(7+3b-4b^2) + ab(-8-b+3b^2))b_2) \\ & +a(-31a^3b^2K + 28a^4b^2K + 3a^5b^2K + 50a^2b^3K + 22a^3b^3K - 69a^4b^3K \\ & -3a^5b^3K - 19ab^4K - 92a^2b^4K + 70a^3b^4K + 41a^4b^4K + 42ab^5K \\ & +19a^2b^5K - 61a^3b^5K - 23ab^6K + 23a^2b^6K - 3(-1+b)b^2(6a^3 \\ & -b(1+2b) - 3a^2(2+3b) + a(1+8b+3b^2))a_{11} + 3(-1+b)b^2(-6a^4 \\ & -b^2(5+2b) + 3a^3(4+5b) + ab(13+12b+4b^2) - a^2(8+22b+13b^2))a_{12} \end{aligned}$$

$$
\begin{aligned}
&+6a^3b^2a_{22}-6a^4b^2a_{22}-24a^2b^3a_{22}+21a^3b^3a_{22}+27ab^4a_{22}-6a^2b^4a_{22}\\
&-18a^3b^4a_{22}+6a^4b^4a_{22}-9b^5a_{22}-18ab^5a_{22}+27a^2b^5a_{22}-9a^3b^5a_{22}\\
&+9b^6a_{22}-9ab^6a_{22}+3a^2b^6a_{22}+3a^2b_1^2-3a^4b_1^2-6abb_1^2-6a^2bb_1^2+6a^3bb_1^2\\
&+6a^4bb_1^2+18ab^2b_1^2-18a^3b^2b_1^2-18ab^3b_1^2+18a^2b^3b_1^2-9a^3b_1b_2+9a^4b_1b_2\\
&+18a^2bb_1b_2-18a^4bb_1b_2-12ab^2b_1b_2-30a^2b^2b_1b_2+30a^3b^2b_1b_2\\
&+12a^4b^2b_1b_2+24ab^3b_1b_2-24a^3b^3b_1b_2-9ab^4b_1b_2+9a^2b^4b_1b_2\\
&+6a^2b^2b_2^2-6a^4b^2b_2^2-18a^2b^3b_2^2+18a^3b^3b_2^2+3ab^4b_2^2-3a^3b^4b_2^2-3a^2bb_{11}\\
&+3a^4bb_{11}+3ab^2b_{11}+12a^2b^2b_{11}-12a^3b^2b_{11}-3a^4b^2b_{11}-12ab^3b_{11}\\
&+12a^3b^3b_{11}+9ab^4b_{11}-9a^2b^4b_{11}+9a^3bb_{12}-9a^4bb_{12}-21a^2b^2b_{12}\\
&+21a^4b^2b_{12}+12ab^3b_{12}+30a^2b^3b_{12}-30a^3b^3b_{12}-12a^4b^3b_{12}-21ab^4b_{12}\\
&+21a^3b^4b_{12}+9ab^5b_{12}-9a^2b^5b_{12}+9a^3b^2b_{22}-9a^4b^2b_{22}-12a^2b^3b_{22}\\
&+12a^4b^3b_{22}+3ab^4b_{22}+12a^2b^4b_{22}-12a^3b^4b_{22}-3a^4b^4b_{22}-3ab^5b_{22}\\
&+3a^3b^5b_{22}))-a(-3a^2(2+a(-3+b))(-1+b)b^3a_2^3+3a^2ba_2^2((a^2(1-2b)\\
&+b+b^2-3b^3+ab(-3+4b+b^2))b_1+(-2+b)b(a^2(-2+b)+b-2b^2\\
&+a(1+b^2))b_2)+.a_2(-3(-1+b)b^2(3a^4-7a^3(1+b)+2b^2(1+b)\\
&-ab(5+9b+2b^2)+a^2(3+14b+6b^2))a_{11}+3(-1+b)b^2(6b^3\\
&+3a^4(2+b)-ab^2(17+8b)+a^2b(15+26b)-a^3(4+24b+3b^2))a_{12}\\
&+a(3a(-1+b)b^3(a^2(-4+b)-3b+a(3+4b-b^2))a_{22}+\\
&(-1+a)(3a(a^2(-1+2b)+a(-1+4b-6b^2)+2b(1-3b+3b^2))b_1^2\\
&+3a(b^2(4-8b+3b^2)-2ab(3-7b+4b^2)+a^2(3-6b+4b^2))b_1b_2\\
&+b(-3ab(2a^2+b^2+a(2-6b+b^2))b_2^2+(a-b)(-1+b)\\
&(-3a(1+a-3b)b_{11}-3a((4-3b)b+a(-3+4b))b_{12}\\
&-b((a^2(23-19b)+3b^2+ab(-38+31b))K+3a(a(-3+b)+b)b_{22}))))))\\
&+3(-1+a)a(a-b)b((a-b)(-1+b)ba_{111}+(-1+b)b(3a^2+2b(1+b)\\
&-a(2+5b))a_{112}+2a^2ba_{122}-5ab^2a_{122}+3b^3a_{122}+3ab^3a_{122}-2a^2b^3a_{122}\\
&-3b^4a_{122}+2ab^4a_{122}-a^2b^2a_{222}+ab^3a_{222}+a^2b^3a_{222}-ab^4a_{222}+4a^2Kb_1\\
&-4a^3Kb_1-8abKb_1+8a^3bKb_1+12ab^2Kb_1-12a^2b^2Kb_1+aa_{11}b_1\\
&-2ba_{11}b_1-2aba_{11}b_1+3b^2a_{11}b_1-2aa_{12}b_1+2a^2a_{12}b_1+4ba_{12}b_1\\
&-4a^2ba_{12}b_1-6b^2a_{12}b_1+6ab^2a_{12}b_1-a^2a_{22}b_1+2aba_{22}b_1+2a^2ba_{22}b_1\\
&-3ab^2a_{22}b_1+12a^2bKb_2-12a^3bKb_2-8ab^2Kb_2+8a^3b^2Kb_2\\
&+4ab^3Kb_2-4a^2b^3Kb_2+3aba_{11}b_2-2b^2a_{11}b_2-2ab^2a_{11}b_2+b^3a_{11}b_2\\
&-6aba_{12}b_2+6a^2ba_{12}b_2+4b^2a_{12}b_2-4a^2b^2a_{12}b_2-2b^3a_{12}b_2+2ab^3a_{12}b_2\\
&-3a^2ba_{22}b_2+2ab^2a_{22}b_2+2a^2b^2a_{22}b_2-ab^3a_{22}b_2-5a^2bK_1+5a^3bK_1\\
&+5ab^2K_1-5a^3b^2K_1-5ab^3K_1+5a^2b^3K_1+5a^2b^2K_2-5a^3b^2K_2\\
&-5ab^3K_2+5a^3b^3K_2+5ab^4K_2-5a^2b^4K_2));
\end{aligned}
$$

$$
\begin{aligned}
j_2 = \;& -3(-1+b)b^3(6a^3-6a^2(1+b)-b(1+b)+2a(1+b)^2)a_1^3\\
&+3b^2a_1^2((-1+b)b(-6a^4-b^2(5+2b)+a^3(16+13b)\\
&+ab(13+12b+4b^2)-a^2(10+23b+12b^2))a_2+a((2a^3(-2+b)+b^3\\
&+a^2(6-7b+6b^2)-a(3-6b+5b^2+2b^3))b_1+(-1+a)b(2a^2b\\
&+b(-1+2b)-a(-3+5b+b^2))b_2))+ba_1(-3(-1+b)b^2(-6b^3\\
&-3a^4(2+b)+ab^2(17+8b)-a^2b(16+23b)+a^3(4+24b+b^2))a_2^2\\
&-3aba_2((a^4(1+b)+b^2(-1+3b)+ab(2-5b-5b^2)+a^3(-5+b-4b^2)\\
&+a^2(3-3b+11b^2+b^3))b_1-b(-a^4(-3+b)+b^2(1+b)+a^3(-5+b-4b^2)
\end{aligned}
$$

$$
\begin{aligned}
&-ab(2+5b+b^2)+a^2(3+b+7b^2+b^3))b_2)+a(7a^3b^2K+8a^4b^2K-3a^5b^2K\\
&-14a^2b^3K-34a^3b^3K-3a^4b^3K+3a^5b^3K+7ab^4K+44a^2b^4K+26a^3b^4K\\
&-5a^4b^4K-18ab^5K-31a^2b^5K+a^3b^5K+11ab^6K+a^2b^6K-3a^2(2+3b)\\
&+3(-1+b)b^2(6a^3-b(1+2b)+a(1+8b+3b^2))a_{11}\\
&+3(-1+b)b^2(6a^4+b^2(5+2b)-3a^3(4+5b)-ab(13+12b+4b^2)\\
&+a^2(8+22b+13b^2))a_{12}-6a^3b^2a_{22}+6a^4b^2a_{22}+24a^2b^3a_{22}-21a^3b^3a_{22}\\
&-27ab^4a_{22}+6a^2b^4a_{22}+18a^3b^4a_{22}-6a^4b^4a_{22}9b^5a_{22}+18ab^5a_{22}-27a^2b^5a_{22}\\
&+9a^3b^5a_{22}-9b^6a_{22}+9ab^6a_{22}-3a^2b^6a_{22}-3a^4b_1^2+9abb_1^2-18a^2bb_1^2\\
&+15a^3bb_1^2+6a^4bb_1^2-18ab^2b_1^2+21a^2b^2b_1^2-18a^3b^2b_1^2+12ab^3b_1^2-6a^2b^3b_1^2\\
&-3a^3b_1b_2+9a^4b_1b_2+6a^2bb_1b_2-15a^3bb_1b_2-15a^4bb_1b_2+9ab^2b_1b_2-9a^2b^2b_1b_2\\
&+33a^3b^2b_1b_2+3a^4b^2b_1b_2-15ab^3b_1b_2+3a^2b^3b_1b_2-12a^3b^3b_1b_2+3ab^4b_1b_2\\
&+3a^2b^4b_1b_2-3a^2b^2b_2^2+3a^3b^2b_2^2-3a^4b^2b_2^2+9a^2b^3b_2^2-6a^3b^3b_2^2+3a^4b^3b_2^2\\
&-6a^2b^4b_2^2+3a^3b^4b_2^2+3a^2bb_{11}-6a^3bb_{11}+6a^4bb_{11}-3ab^2b_{11}-3a^2b^2b_{11}\\
&+3a^3b^2b_{11}-6a^4b^2b_{11}+9ab^3b_{11}-3a^2b^3b_{11}+3a^3b^3b_{11}-6ab^4b_{11}+3a^2b^4b_{11}\\
&+3a^3bb_{12}-9a^4bb_{12}+3a^2b^2b_{12}+6a^3b^2b_{12}+15a^4b^2b_{12}-6ab^3b_{12}-6a^2b^3b_{12}\\
&-18a^3b^3b_{12}-6a^4b^3b_{12}+9ab^4b_{12}+6a^2b^4b_{12}+9a^3b^4b_{12}-3ab^5b_{12}-3a^2b^5b_{12}\\
&-3a^3b^2b_{22}+3a^2b^3b_{22}+9a^3b^3b_{22}-3a^4b^3b_{22}-9a^2b^4b_{22}-3a^3b^4b_{22}+3a^4b^4b_{22}\\
&+6a^2b^5b_{22}-3a^3b^5b_{22}))+a(3a^2(2+a(-3+b))(-1+b)b^4a_2^3\\
&-3a^2b^3a_2^2((-1+a^2(-2+b)+b-b^2+a(3-2b+b^2))b_1+b(-4+5b)\\
&+a^2(3-4b+2b^2)+a(-2+7b-8b^2+b^3))b_2)\\
&-ba_2(-3(-1+b)b^2(3a^4-7a^3(1+b)+2b^2(1+b)-ab(5+9b+2b^2)\\
&+a^2(3+14b+6b^2))a_{11}+3(-1+b)b^2(6b^3+3a^4(2+b)-ab^2(17+8b)\\
&+a^2b(15+26b)-a^3(4+24b+3b^2))a_{12}+a(3a(-1+b)b^3(a^2(-4+b)-3b\\
&+a(3+4b-b^2))a_{22}-3a(-1+b)(a^2(-2+b)-3b+a(1+5b-2b^2))b_1^2\\
&+3a(a^3(1-b+b^2)+b^2(3-5b+3b^2)+a^2(1-5b+7b^2-4b^3)\\
&+ab(-2+b+b^2-b^3))b_1b_2+b(-3ab(a^3b+b(-2+3b)+a(-4+7b-4b^2)\\
&+a^2(5-8b+2b^2))b_2^2+(a-b)(-1+b)(3a(1+a(-2+b))b_{11}\\
&+3a(a+a^2(1-2b)-ab^2+b(-2+3b))b_{12}-b((a^3(11-7b)-3b^2\\
&+2ab(1+4b)+a^2(1-19b+7b^2))K+3a(a+a^2(-2+b)+2b-2ab)b_{22})))))\\
&+(-1+a)a(-3(a-b)^2(-1+b)b^3a_{111}-3(-2+3a-2b)(a-b)^2(-1+b)b^3a_{112}\\
&-6a^3b^3a_{122}+21a^2b^4a_{122}-24ab^5a_{122}-9a^2b^5a_{122}+6a^3b^5a_{122}+9b^6a_{122}\\
&+18ab^6a_{122}-12a^2b^6a_{122}-9b^7a_{122}+6ab^7a_{122}+3a^3b^4a_{222}-6a^2b^5a_{222}\\
&-3a^3b^5a_{222}+3ab^6a_{222}+6a^2b^6a_{222}-3ab^7a_{222}+7a^3b^2Kb_1-11a^4b^2Kb_1\\
&-14a^2b^3Kb_1+30a^3b^3Kb_1-a^4b^3Kb_1+7ab^4Kb_1-27a^2b^4Kb_1-a^3b^4Kb_1\\
&+8ab^5Kb_1+5a^2b^5Kb_1-3ab^6Kb_1-3ab^2a_{11}b_1+6a^2b^2a_{11}b_1+3b^3a_{11}b_1\\
&-3a^2b^3a_{11}b_1-6b^4a_{11}b_1-3ab^4a_{11}b_1+6b^5a_{11}b_1-6a^2b^2a_{12}b_1+3a^3b^2a_{12}b_1\\
&+3ab^3a_{12}b_1-3a^2b^3a_{12}b_1+3a^3b^3a_{12}b_1+3b^4a_{12}b_1+9ab^4a_{12}b_1-9a^2b^4a_{12}b_1\\
&-9b^5a_{12}b_1+6ab^5a_{12}b_1+3a^2b^3a_{22}b_1-6a^3b^3a_{22}b_1-3ab^4a_{22}b_1+6a^2b^4a_{22}b_1\\
&+3a^3b^4a_{22}b_1-3a^2b^5a_{22}b_1-3a^2b_1^3-3a^3b_1^3+6abb_1^3+12a^2bb_1^3+6a^3bb_1^3\\
&-18ab^2b_1^3-18a^2b^2b_1^3+18ab^3b_1^3-3a^4b^2Kb_2+5a^3b^3Kb_2+8a^4b^3Kb_2\\
&-a^2b^4Kb_2-27a^3b^4Kb_2+7a^4b^4Kb_2-ab^5Kb_2+30a^2b^5Kb_2-14a^3b^5Kb_2\\
&-11ab^6Kb_2+7a^2b^6Kb_2-3ab^3a_{11}b_2+3b^4a_{11}b_2+6ab^4a_{11}b_2-3a^2b^4a_{11}b_2\\
&-6b^5a_{11}b_2+3ab^5a_{11}b_2+6a^2b^3a_{12}b_2-9a^3b^3a_{12}b_2-9ab^4a_{12}b_2+9a^2b^4a_{12}b_2\\
&+3a^3b^4a_{12}b_2+3b^5a_{12}b_2-3ab^5a_{12}b_2+3a^2b^5a_{12}b_2+3b^6a_{12}b_2-6ab^6a_{12}b_2
\end{aligned}
$$

$$+6a^3b^3a_{22}b_2 - 3a^2b^4a_{22}b_2 - 6a^3b^4a_{22}b_2 - 3ab^5a_{22}b_2 + 3a^3b^5a_{22}b_2 + 6ab^6a_{22}b_2$$
$$-3a^2b^6a_{22}b_2 + 15a^3b_1^2b_2 + 6a^4b_1^2b_2 - 39a^2bb_1^2b_2 - 36a^3bb_1^2b_2 - 12a^4bb_1^2b_2$$
$$+30ab^2b_1^2b_2 + 69a^2b^2b_1^2b_2 + 36a^3b^2b_1^2b_2 - 48ab^3b_1^2b_2 - 39a^2b^3b_1^2b_2 + 18ab^4b_1^2b_2$$
$$-18a^4b_1b_2^2 + 51a^3bb_1b_2^2 + 24a^4bb_1b_2^2 - 48a^2b^2b_1b_2^2 - 69a^3b^2b_1b_2^2 + 12ab^3b_1b_2^2$$
$$+72a^2b^3b_1b_2^2 + 3a^3b^3b_1b_2^2 - 18ab^4b_1b_2^2 - 9a^2b^4b_1b_2^2 - 6a^2b^3b_2^3 + 9a^2b^4b_2^3$$
$$-3a^3b^4b_2^3 + 3a^2bb_1b_{11} + 6a^3bb_1b_{11} - 3ab^2b_1b_{11} - 24a^2b^2b_1b_{11} - 9a^3b^2b_1b_{11}$$
$$+18ab^3b_1b_{11} + 27a^2b^3b_1b_{11} - 18ab^4b_1b_{11} - 6a^3bb_2b_{11} - 6a^4bb_2b_{11} + 15a^2b^2b_2b_{11}$$
$$+27a^3b^2b_2b_{11} + 6a^4b^2b_2b_{11} - 9ab^3b_2b_{11} - 42a^2b^3b_2b_{11} - 18a^3b^3b_2b_{11}$$
$$+21ab^4b_2b_{11} + 21a^2b^4b_2b_{11} - 9ab^5b_2b_{11} - 15a^3bb_1b_{12} - 6a^4bb_1b_{12} + 39a^2b^2b_1b_{12}$$
$$+36a^3b^2b_1b_{12} + 12a^4b^2b_1b_{12} - 24ab^3b_1b_{12} - 66a^2b^3b_1b_{12} - 39a^3b^3b_1b_{12}$$
$$+36ab^4b_1b_{12} + 45a^2b^4b_1b_{12} - 18ab^5b_1b_{12} + 18a^4bb_2b_{12} - 51a^3b^2b_2b_{12}$$
$$-24a^4b^2b_2b_{12} + 45a^2b^3b_2b_{12} + 78a^3b^3b_2b_{12} - 12ab^4b_2b_{12} - 72a^2b^4b_2b_{12}$$
$$-9a^3b^4b_2b_{12} + 18ab^5b_2b_{12} + 9a^2b^5b_2b_{12} + 9a^4bb_1b_{22} - 27a^3b^2b_1b_{22}$$
$$-9a^4b^2b_1b_{22} + 24a^2b^3b_1b_{22} + 30a^3b^3b_1b_{22} + 3a^4b^3b_1b_{22} - 6ab^4b_1b_{22}$$
$$-27a^2b^4b_1b_{22} - 9a^3b^4b_1b_{22} + 6ab^5b_1b_{22} + 6a^2b^5b_1b_{22} - 9a^3b^3b_2b_{22}$$
$$+9a^2b^4b_2b_{22} + 12a^3b^4b_2b_{22} - 3a^4b^4b_2b_{22} - 12a^2b^5b_2b_{22} + 3a^3b^5b_2b_{22}$$
$$-3a^3b^2b_{111} + 6a^2b^3b_{111} + 3a^3b^3b_{111} - 3ab^4b_{111} - 6a^2b^4b_{111} + 3ab^5b_{111}$$
$$+6a^3b^2b_{112} + 6a^4b^2b_{112} - 12a^2b^3b_{112} - 27a^3b^3b_{112} - 6a^4b^3b_{112} + 6ab^4b_{112}$$
$$+36a^2b^4b_{112} + 21a^3b^4b_{112} - 15ab^5b_{112} - 24a^2b^5b_{112} + 9ab^6b_{112} - 9a^4b^2b_{122}$$
$$+24a^3b^3b_{122} + 15a^4b^3b_{122} - 21a^2b^4b_{122} - 36a^3b^4b_{122} - 6a^4b^4b_{122} + 6ab^5b_{122}$$
$$+27a^2b^5b_{122} + 12a^3b^5b_{122} - 6ab^6b_{122} - 6a^2b^6b_{122} - 3a^4b^3b_{222} + 6a^3b^4b_{222}$$
$$+3a^4b^4b_{222} - 3a^2b^5b_{222} - 6a^3b^5b_{222} + 3a^2b^6b_{222} - 15a^4b^3K_1 + 45a^3b^4K_1$$
$$+15a^4b^4K_1 - 45a^2b^5K_1 - 45a^3b^5K_1 + 15ab^6K_1 + 45a^2b^6K_1 - 15ab^7K_1$$
$$+15a^4b^3K_2 - 45a^3b^4K_2 - 15a^4b^4K_2 + 45a^2b^5K_2 + 45a^3b^5K_2 - 15ab^6K_2$$
$$-45a^2b^6K_2 + 15ab^7K_2));$$

$$j_3 = \quad -(-1+b)b^2(6a^3 - 6a^2(1+b) - b(1+b) + 2a(1+b)^2)Ka_1^2$$
$$+2a^2(-2-5b+5b^2+2b^3))Ka_2 - ba_1(b(3a^4(-1+b) + 3(-1+b)b^2$$
$$-8a^3(-1+b^2) - 6ab(-1+b^2) + a((b + a^2(7-12b)b - 2b^3$$
$$+a^3(-2+4b) + a(1-8b+7b^2+4b^3))Kb_1 + b((-(-2+b)b^2$$
$$+a^3(-6+4b) + a^2(3+7b-5b^2) + a(1-6b-b^2+2b^3))Kb_2$$
$$+(-1+b)(3a^3 - b^2 + ab(3+2b) - a^2(2+5b))(-K_1 + bK_2))))$$
$$-a(-a(-1+b)b^2(-6ab + a^2(1+b) + 2b(1+b))Ka_2^2$$
$$+aba_2((b + 3b^2 - 6b^3 + a^3(-1+2b) + ab(-6+7b+4b^2)$$
$$-a^2(-2+b+5b^2))Kb_1 + b((a^3(-3+2b) + b(-2+7b-3b^2)$$
$$-a^2(-7+b+2b^2) + a(-2-4b-b^2+2b^3))Kb_2$$
$$+(-1+b)(a^3 - 3b^2 + ab(5+2b) - a^2(2+3b))(-K_1 + bK_2)))$$
$$+(-1+a)(-(-1+b)b^2(-a + 3a^2 + b - 4ab + b^2)Ka_{11}$$
$$-(-1+b)b^2(3a^3 - 3b^2 + ab(7+4b) - a^2(4+7b))Ka_{12}$$
$$+a((-1+b)b^2(3b^2 + a^2(1+b) - ab(4+b))Ka_{22}$$
$$+(a^2(-1+2b) + a(-1+4b-6b^2) + 2b(1-3b+3b^2))Kb_1^2$$
$$+b_1((b^2(4-8b+3b^2) - 2ab(3-7b+4b^2) + a^2(3-6b+4b^2))Kb_2$$
$$+b(a(3-5b)b + a^2(-1+2b) + b^2(-2+3b))(-K_1 + aK_2))$$

$$+b(-b(2a^2+b^2+a(2-6b+b^2))Kb_2^2$$
$$+b(a(5-3b)b+(-2+b)b^2+a^2(-3+2b))b_2(-K_1+aK_2)$$
$$+(a-b)(-1+b)(-(1+a-3b)Kb_{11}+(a(3-4b)+b(-4+3b))Kb_{12}$$
$$-b((a(-3+b)+b)Kb_{22}-(a-b)(-10K^2+5aK^2+5bK^2+K_{11}$$
$$-(a+b)K_{12}+abK_{22})))))));$$

$$\begin{aligned}
j_4 = \;& -6(-1+b)b^2(6a^3-6a^2(1+b)-b(1+b)+2a(1+b)^2)a_1^3b_2\\
&-6ba_1^2(-(-1+b)b(-6a^4-b^2(5+2b)+a^3(16+13b)+ab(13+12b+4b^2)\\
&-a^2(10+23b+12b^2))a_2b_2+a((b+a^2(7-12b)b-2b^3+a^3(-2+4b)\\
&+a(1-8b+7b^2+4b^3))b_1b_2+b(-(-1+a)(2a^2b+b(-1+2b)\\
&-a(-3+5b+b^2))b_2^2+(-1+b)(6a^3-6a^2(1+b)-b(1+b)\\
&+2a(1+b)^2)(bK-b_{12}))))+2a_1(-3(-1+b)b^2(-6b^3-3a^4(2+b)\\
&+ab^2(17+8b)-a^2b(16+23b)+a^3(4+24b+b^2))a_2^2b_2\\
&-3aba_2((-1+a)(2(2-3b)b^2+a^3(-2+4b)+a^2(1+5b-12b^2)\\
&+ab(-8+9b+5b^2))b_1b_2+b((a^4(-3+b)-b^2(1+b)+ab(2+5b+b^2)\\
&+a^3(5-b+4b^2)-a^2(3+b+7b^2+b^3))b_2^2+(-1+b)(3a^4+3b^2\\
&-8a^3(1+b)-6ab(1+b)+2a^2(2+7b+2b^2))(bK-b_{12})))\\
&+a(-3a(2a^2(1-3b)b+a^3(-1+2b)-2b(1-3b+3b^2)\\
&+a(1-2b+6b^3))b_1^2b_2+ab_1(3(a^3(-3+4b)+b^2(5-8b+b^2)\\
&+a^2(3-3b^2-4b^3)+ab(-5+2b+7b^2+b^3))b_2^2\\
&+b(b(3a^3(-5+3b)+a^2(14+b^2)-b(3-7b+b^2)\\
&+a(-3+3b-14b^2+2b^3))K+3(b+a^2(7-12b)b-2b^3\\
&+a^3(-2+4b)+a(1-8b+7b^2+4b^3))b_{12}-3(-1+b)(3a^3-b^2+ab(3+2b)\\
&-a^2(2+5b))b_{22}))+b(3a^2(1+a^2+a(-1+b)-2b)(-1+b)bb_2^3\\
&+b_2(9a^3bK+5a^4bK-3a^5bK-20a^2b^2K-34a^3b^2K+9a^4b^2K+3a^5b^2K\\
&+8ab^3K+54a^2b^3K+9a^3b^3K-8a^4b^3K-22ab^4K-23a^2b^4K+a^3b^4K\\
&+11ab^5K+a^2b^5K+3(-1+b)b(6a^3-b(1+2b)-3a^2(2+3b)\\
&+a(1+8b+3b^2))a_{11}+3(-1+b)b(6a^4+b^2(5+2b)-3a^3(4+5b)\\
&-ab(13+12b+4b^2)+a^2(8+22b+13b^2))a_{12}-6a^3ba_{22}+6a^4ba_{22}\\
&+24a^2b^2a_{22}-21a^3b^2a_{22}-27ab^3a_{22}+6a^2b^3a_{22}+18a^3b^3a_{22}-6a^4b^3a_{22}\\
&+9b^4a_{22}+18ab^4a_{22}-27a^2b^4a_{22}+9a^3b^4a_{22}-9b^5a_{22}+9ab^5a_{22}\\
&-3a^2b^5a_{22}+3a^2b_{11}-3a^4b_{11}-3abb_{11}-12a^2bb_{11}+12a^3bb_{11}+3a^4bb_{11}\\
&+12ab^2b_{11}-12a^3b^2b_{11}-9ab^3b_{11}+9a^2b^3b_{11}-9a^3b_{12}+9a^4b_{12}+24a^2bb_{12}\\
&-9a^3bb_{12}-12a^4bb_{12}-12ab^2b_{12}-21a^2b^2b_{12}+24a^3b^2b_{12}-3a^4b^2b_{12}\\
&+18ab^3b_{12}-12a^2b^3b_{12}+9a^3b^3b_{12}-3ab^4b_{12}-3a^2b^4b_{12}-3a^3bb_{22}\\
&+3a^2b^2b_{22}+9a^3b^2b_{22}-3a^4b^2b_{22}-9a^2b^3b_{22}-3a^3b^3b_{22}+3a^4b^3b_{22}\\
&+6a^2b^4b_{22}-3a^3b^4b_{22})+3a(-1+b)b(3a^3-b^2+ab(3+2b)\\
&-a^2(2+5b))(-b_{112}+b_{122}+b(K_1-K_2)))))\\
&-a(-6a^2(2+a(-3+b))(-1+b)b^3a_2^3b_2-6a^2ba_2^2((a^2(-1+2b)\\
&-ab(-3+4b+b^2)+b(-1-b+3b^2))b_1b_2+b(-(b(-4+5b)\\
&+a^2(3-4b+2b^2)+a(-2+7b-8b^2+b^3))b_2^2+(-1+b)(-6ab+a^2(1+b)\\
&+2b(1+b))(bK-b_{12})))+2a_2(3a^2(2a^2(1-3b)b+a^3(-1+2b)\\
&-2b(1-3b+3b^2)+a(1-2b+6b^3))b_1^2b_2-a^2b_1(-3(a^3(3-5b+2b^2)
\end{aligned}$$

$$+b^2(-5+5b+3b^2)-ab(-6+4b+7b^2+b^3)-a^2(3+2b-11b^2+3b^3))b_2^2$$
$$+b(b(a^3(-4+b)+a^2(8+10b-6b^2)-3b(1-4b+b^2)+ab(-17+2b^2))K$$
$$+3(b+3b^2-6b^3+a^3(-1+2b)+ab(-6+7b+4b^2)-a^2(-2+b+5b^2))b_{12}$$
$$-3(-1+b)(a^3-3b^2+ab(5+2b)-a^2(2+3b))b_{22}))$$
$$+b(-3a^2b(a^3b+b(-2+3b)+a(-4+7b-4b^2)+a^2(5-8b+2b^2))b_2^3$$
$$+b_2(-3(-1+b)b(3a^4-7a^3(1+b)+2b^2(1+b)-ab(5+9b+2b^2)$$
$$+a^2(3+14b+6b^2))a_{11}+3(-1+b)b(6b^3+3a^4(2+b)-ab^2(17+8b)$$
$$+a^2b(15+26b)-a^3(4+24b+3b^2))a_{12}+a(-a^3bK+12a^4bK-17a^3b^2K$$
$$-25a^4b^2K-14ab^3K+26a^2b^3K+41a^3b^3K+10a^4b^3K+3b^4K+12ab^4K$$
$$-48a^2b^4K-11a^3b^4K-3b^5K+8ab^5K+7a^2b^5K+3a(-1+b)b^2(a^2(-4+b)$$
$$-3b+a(3+4b-b^2))a_{22}-3(-1+a)a(-1+b)(a+a^2-4ab+b(-1+3b))b_{11}$$
$$+9a^3b_{12}-9a^4b_{12}-15a^2bb_{12}-3a^3bb_{12}+21a^4bb_{12}+18ab^2b_{12}-3a^2b^2b_{12}$$
$$-18a^3b^2b_{12}-9a^4b^2b_{12}-15ab^3b_{12}+30a^2b^3b_{12}-9ab^4b_{12}+3a^2b^4b_{12}+3a^3bb_{22}$$
$$-6a^4bb_{22}+3a^2b^2b_{22}-3a^3b^2b_{22}+9a^4b^2b_{22}-6ab^3b_{22}+3a^2b^3b_{22}-3a^3b^3b_{22}$$
$$-3a^4b^3b_{22}+6ab^4b_{22}-6a^2b^4b_{22}+3a^3b^4b_{22}))-3a^2(-1+b)b(a^3-3b^2$$
$$+ab(5+2b)-a^2(2+3b))(-b_{112}+b_{122}+b(K_1-K_2))))$$
$$+(-1+a)a(30a^3b^3K^2-30a^4b^3K^2-60a^2b^4K^2+30a^3b^4K^2$$
$$+30a^4b^4K^2+30ab^5K^2+30a^2b^5K^2-60a^3b^5K^2-30ab^6K^2+30a^2b^6K^2$$
$$-6a^3b^3Ka_{22}+24a^2b^4Ka_{22}-18ab^5Ka_{22}-18a^2b^5Ka_{22}+6a^3b^5Ka_{22}$$
$$+18ab^6Ka_{22}-6a^2b^6Ka_{22}-6a^2bKb_1^2+8a^3bKb_1^2+12ab^2Kb_1^2-18a^2b^2Kb_1^2$$
$$-16a^3b^2Kb_1^2-8ab^3Kb_1^2+34a^2b^3Kb_1^2-6ab^4Kb_1^2-6a^2b^2a_{111}b_2+12ab^3a_{111}b_2$$
$$+6a^2b^3a_{111}b_2-6b^4a_{111}b_2-12ab^4a_{111}b_2+6b^5a_{111}b_2+12a^2b^2a_{112}b_2$$
$$-18a^3b^2a_{112}b_2-24ab^3a_{112}b_2+36a^2b^3a_{112}b_2+18a^3b^3a_{112}b_2+12b^4a_{112}b_2$$
$$-18ab^4a_{112}b_2-48a^2b^4a_{112}b_2+42ab^5a_{112}b_2-12b^6a_{112}b_2+12a^3b^2a_{122}b_2$$
$$-42a^2b^3a_{122}b_2+48ab^4a_{122}b_2+18a^2b^4a_{122}b_2-12a^3b^4a_{122}b_2-18b^5a_{122}b_2$$
$$-36ab^5a_{122}b_2+24a^2b^5a_{122}b_2+18b^6a_{122}b_2-12ab^6a_{122}b_2-6a^3b^3a_{222}b_2$$
$$+12a^2b^4a_{222}b_2+6a^3b^4a_{222}b_2-6ab^5a_{222}b_2-12a^2b^5a_{222}b_2+6ab^6a_{222}b_2$$
$$+20a^3bKb_1b_2-22a^4bKb_1b_2-42a^2b^2Kb_1b_2+10a^3b^2Kb_1b_2+44a^4b^2Kb_1b_2$$
$$+28ab^3Kb_1b_2+68a^2b^3Kb_1b_2-84a^3b^3Kb_1b_2-56ab^4Kb_1b_2+28a^2b^4Kb_1b_2$$
$$+6ab^5Kb_1b_2-6a^3ba_{22}b_1b_2+18a^2b^2a_{22}b_1b_2+12a^3b^2a_{22}b_1b_2-12ab^3a_{22}b_1b_2$$
$$-30a^2b^3a_{22}b_1b_2+18ab^4a_{22}b_1b_2+4a^3b^2Kb_2^2-8a^4b^2Kb_2^2-18a^2b^3Kb_2^2$$
$$+36a^3b^3Kb_2^2-14a^4b^3Kb_2^2+2ab^4Kb_2^2-32a^2b^4Kb_2^2+22a^3b^4Kb_2^2+22ab^5Kb_2^2$$
$$-14a^2b^5Kb_2^2-12a^3b^2a_{22}b_2^2+6a^2b^3a_{22}b_2^2+12a^3b^3a_{22}b_2^2+6ab^4a_{22}b_2^2$$
$$-6a^3b^4a_{22}b_2^2-12ab^5a_{22}b_2^2+6a^2b^5a_{22}b_2^2+6a^2b_1^2b_2^2+6a^3b_1^2b_2^2-12abb_1^2b_2^2$$
$$-24a^2bb_1^2b_2^2-12a^3bb_1^2b_2^2+36ab^2b_1^2b_2^2+36a^2b^2b_1^2b_2^2-36ab^3b_1^2b_2^2-18a^3b_1b_2^3$$
$$+36a^2bb_1b_2^3+30a^3bb_1b_2^3-24ab^2b_1b_2^3-66a^2b^2b_1b_2^3-12a^3b^2b_1b_2^3+36ab^3b_1b_2^3$$
$$+18a^2b^3b_1b_2^3+12a^2b^2b_2^4-18a^2b^3b_2^4+6a^3b^3b_2^4+6a^2b^2Kb_{11}-18a^3b^2Kb_{11}$$
$$-6ab^3Kb_{11}+18a^2b^3Kb_{11}+18a^3b^3Kb_{11}-24a^2b^4Kb_{11}+6ab^5Kb_{11}-6a^2bb_2^2b_{11}$$
$$-6a^3bb_2^2b_{11}+6ab^2b_2^2b_{11}+30a^2b^2b_2^2b_{11}+6a^3b^2b_2^2b_{11}-24ab^3b_2^2b_{11}-24a^2b^3b_2^2b_{11}$$
$$+18ab^4b_2^2b_{11}-6(a-b)ba_{11}(-(a-2ab+b(-2+3b))b_1b_2$$
$$-b((1+(-2+a)b)b_2^2+(-1+b)(1-3a+b)(bK-b_{12})))$$
$$-6(a-b)ba_{12}(2(a-3ab^2+a^2(-1+2b)+b(-2+3b))b_1b_2$$
$$+b((a^2(-3+b)-b(1+b)+2a(1+b^2))b_2^2$$
$$+(-1+b)(3a^2+3b-4a(1+b))(bK-b_{12})))$$

$-36a^3b^2Kb_{12}+42a^4b^2Kb_{12}+78a^2b^3Kb_{12}-42a^3b^3Kb_{12}-42a^4b^3Kb_{12}$
$-42ab^4Kb_{12}-42a^2b^4Kb_{12}+78a^3b^4Kb_{12}+42ab^5Kb_{12}-36a^2b^5Kb_{12}$
$+6a^3b^2a_{22}b_{12}-24a^2b^3a_{22}b_{12}+18ab^4a_{22}b_{12}+18a^2b^4a_{22}b_{12}-6a^3b^4a_{22}b_{12}$
$-18ab^5a_{22}b_{12}+6a^2b^5a_{22}b_{12}+6a^2b_1^2b_{12}+6a^3b_1^2b_{12}-12abb_1^2b_{12}$
$-24a^2bb_1^2b_{12}-12a^3bb_1^2b_{12}+36ab^2b_1^2b_{12}+36a^2b^2b_1^2b_{12}-36ab^3b_1^2b_{12}$
$-18a^3b_1b_2b_{12}+36a^2bb_1b_2b_{12}+18a^3bb_1b_2b_{12}-24ab^2b_1b_2b_{12}-30a^2b^2b_1b_2b_{12}$
$+12a^3b^2b_1b_2b_{12}+12ab^3b_1b_2b_{12}-42a^2b^3b_1b_2b_{12}+36ab^4b_1b_2b_{12}+18a^3bb_2^2b_{12}$
$-30a^2b^2b_2^2b_{12}-48a^3b^2b_2^2b_{12}+24ab^3b_2^2b_{12}+66a^2b^3b_2^2b_{12}+24a^3b^3b_2^2b_{12}$
$-36ab^4b_2^2b_{12}-18a^2b^4b_2^2b_{12}-6a^2bb_{11}b_{12}-6a^3bb_{11}b_{12}+6ab^2b_{11}b_{12}$
$+30a^2b^2b_{11}b_{12}+6a^3b^2b_{11}b_{12}-24ab^3b_{11}b_{12}-24a^2b^3b_{11}b_{12}+18ab^4b_{11}b_{12}$
$+18a^3bb_{12}^2-42a^2b^2b_{12}^2-24a^3b^2b_{12}^2+24ab^3b_{12}^2+48a^2b^3b_{12}^2+6a^3b^3b_{12}^2$
$-24ab^4b_{12}^2-6a^2b^4b_{12}^2+6a^4b^2Kb_{22}-24a^3b^3Kb_{22}+18a^2b^4Kb_{22}+18a^3b^4Kb_{22}$
$-6a^4b^4Kb_{22}-18a^2b^5Kb_{22}+6a^3b^5Kb_{22}-6a^3b_1^2b_{22}-6a^4b_1^2b_{22}+18a^2bb_1^2b_{22}$
$+18a^3bb_1^2b_{22}+12a^4bb_1^2b_{22}-12ab^2b_1^2b_{22}-24a^2b^2b_1^2b_{22}-24a^3b^2b_1^2b_{22}$
$+12ab^3b_1^2b_{22}+12a^2b^3b_1^2b_{22}+18a^4b_1b_2b_{22}-36a^3bb_1b_2b_{22}-18a^4bb_1b_2b_{22}$
$+6a^2b^2b_1b_2b_{22}+36a^3b^2b_1b_2b_{22}-12a^4b^2b_1b_2b_{22}+12ab^3b_1b_2b_{22}-6a^2b^3b_1b_2b_{22}$
$+24a^3b^3b_1b_2b_{22}-12ab^4b_1b_2b_{22}-12a^2b^4b_1b_2b_{22}+18a^3b^2b_2^2b_{22}-18a^2b^3b_2^2b_{22}$
$-24a^3b^3b_2^2b_{22}+6a^4b^3b_2^2b_{22}+24a^2b^4b_2^2b_{22}-6a^3b^4b_2^2b_{22}+6a^3bb_{11}b_{22}+6a^4bb_{11}b_{22}$
$-12a^2b^2b_{11}b_{22}-18a^3b^2b_{11}b_{22}-6a^4b^2b_{11}b_{22}+6ab^3b_{11}b_{22}+18a^2b^3b_{11}b_{22}$
$+12a^3b^3b_{11}b_{22}-6ab^4b_{11}b_{22}-6a^2b^4b_{11}b_{22}-18a^4bb_{12}b_{22}+48a^3b^2b_{12}b_{22}$
$+12a^4b^2b_{12}b_{22}-30a^2b^3b_{12}b_{22}-42a^3b^3b_{12}b_{22}+6a^4b^3b_{12}b_{22}+30a^2b^4b_{12}b_{22}$
$-6a^3b^4b_{12}b_{22}-6a^3bb_1b_{112}+18a^2b^2b_1b_{112}+12a^3b^2b_1b_{112}-12ab^3b_1b_{112}$
$-30a^2b^3b_1b_{112}+18ab^4b_1b_{112}+6a^3b^2b_2b_{112}-18a^2b^3b_2b_{112}-12a^3b^3b_2b_{112}$
$+12ab^4b_2b_{112}+30a^2b^4b_2b_{112}-18ab^5b_2b_{112}+6a^3bb_1b_{122}+6a^4bb_1b_{122}$
$-18a^2b^2b_1b_{122}-18a^3b^2b_1b_{122}-12a^4b^2b_1b_{122}+12ab^3b_1b_{122}+24a^2b^3b_1b_{122}$
$+24a^3b^3b_1b_{122}-12ab^4b_1b_{122}-12a^2b^4b_1b_{122}-6a^3b^2b_2b_{122}-6a^4b^2b_2b_{122}$
$+18a^2b^3b_2b_{122}+18a^3b^3b_2b_{122}+12a^4b^3b_2b_{122}-12ab^4b_2b_{122}-24a^2b^4b_2b_{122}$
$-24a^3b^4b_2b_{122}+12ab^5b_2b_{122}+12a^2b^5b_2b_{122}-6a^4bb_1b_{222}+12a^3b^2b_1b_{222}$
$+6a^4b^2b_1b_{222}-6a^2b^3b_1b_{222}-12a^3b^3b_1b_{222}+6a^2b^4b_1b_{222}+6a^4b^2b_2b_{222}$
$-12a^3b^3b_2b_{222}-6a^4b^3b_2b_{222}+6a^2b^4b_2b_{222}+12a^3b^4b_2b_{222}-6a^2b^5b_2b_{222}$
$+6a^3b^2b_{1112}-12a^2b^3b_{1112}-6a^3b^3b_{1112}+6ab^4b_{1112}+12a^2b^4b_{1112}$
$-6ab^5b_{1112}-6a^3b^2b_{1122}-6a^4b^2b_{1122}+12a^2b^3b_{1122}+18a^3b^3b_{1122}$
$+6a^4b^3b_{1122}-6ab^4b_{1122}-18a^2b^4b_{1122}-12a^3b^4b_{1122}+6ab^5b_{1122}+6a^2b^5b_{1122}$
$+6a^4b^2b_{1222}-12a^3b^3b_{1222}-6a^4b^3b_{1222}+6a^2b^4b_{1222}+12a^3b^4b_{1222}$
$-6a^2b^5b_{1222}-15a^3b^2b_1K_1+24a^2b^3b_1K_1+9a^3b^3b_1K_1-9ab^4b_1K_1-12a^2b^4b_1K_1$
$+3ab^5b_1K_1-a^3b^2b_2K_1+29a^4b^2b_2K_1+2a^2b^3b_2K_1-75a^3b^3b_2K_1-29a^4b^3b_2K_1$
$-ab^4b_2K_1+69a^2b^4b_2K_1+82a^3b^4b_2K_1-23ab^5b_2K_1-83a^2b^5b_2K_1+30ab^6b_2K_1$
$+7a^3b^2b_1K_2+7a^4b^2b_1K_2-8a^2b^3b_1K_2-21a^3b^3b_1K_2-a^4b^3b_1K_2+ab^4b_1K_2$
$+15a^2b^4b_1K_2+2a^3b^4b_1K_2-ab^5b_1K_2-a^2b^5b_1K_2-27a^4b^2b_2K_2+78a^3b^3b_2K_2$
$+21a^4b^3b_2K_2-81a^2b^4b_2K_2-66a^3b^4b_2K_2+30ab^5b_2K_2+75a^2b^5b_2K_2$
$-30ab^6b_2K_2-6a^3b^3K_{11}+12a^2b^4K_{11}+6a^3b^4K_{11}-6ab^5K_{11}-12a^2b^5K_{11}$
$+6ab^6K_{11}+6a^3b^3K_{12}+6a^4b^3K_{12}-12a^2b^4K_{12}-18a^3b^4K_{12}-6a^4b^4K_{12}$
$+6ab^5K_{12}+18a^2b^5K_{12}+12a^3b^5K_{12}-6ab^6K_{12}-6a^2b^6K_{12}-6a^4b^3K_{22}$
$+12a^3b^4K_{22}+6a^4b^4K_{22}-6a^2b^5K_{22}-12a^3b^5K_{22}+6a^2b^6K_{22}));$

$$
\begin{aligned}
j_5 = \ & -6(-1+b)b^2(6a^3-6a^2(1+b)-b(1+b)+2a(1+b)^2)a_1^3a_2 \\
& -2ba_1^2(-3(-1+b)b(-b^2(5+2b)+a^3(10+7b)+ab(12+11b+4b^2) \\
& -a^2(8+19b+10b^2))a_2^2-a(-1+b)b(a(3a^3+a^2(4-17b) \\
& +(3-4b)b+a(-6+9b+8b^2))K+3(6a^3-6a^2(1+b)-b(1+b) \\
& +2a(1+b)^2)a_{12}-3(2a^3(1+b)-b^2(1+b)-2a^2(1+2b+2b^2) \\
& +ab(3+3b+2b^2))a_{22})+3aa_2((b+a^2(7-12b)b-2b^3+a^3(-2+4b) \\
& +a(1-8b+7b^2+4b^3))b_1+b(-(-2+b)b^2+a^3(-6+4b) \\
& +a^2(3+7b-5b^2)+a(1-6b-b^2+2b^3))b_2)) \\
& -a_1(-6(-1+b)b^3(a^3(-13+b)+6b^2-2ab(7+4b)+2a^2(5+9b))a_2^3 \\
& +6aba_2^2((a^3(3-6b)+2b^2(-2+3b)+ab(7-5b-9b^2) \\
& +a^2(-2-5b+14b^2+b^3))b_1+b(2(-2+b)b^2+ab(8+3b-4b^2) \\
& +a^3(10-8b+b^2)+a^2(-8-5b+4b^2+b^3))b_2) \\
& +2aa_2(-21a^3b^2K+20a^4b^2K+35a^2b^3K+14a^3b^3K-39a^4b^3K \\
& -17ab^4K-70a^2b^4K+50a^3b^4K+19a^4b^4K+39ab^5K+14a^2b^5K \\
& -43a^3b^5K-22ab^6K+21a^2b^6K-3(-1+b)b^2(6a^3-b(1+2b) \\
& -3a^2(2+3b)+a(1+8b+3b^2))a_{11}+3(-1+b)b^2(-2b^2(4+b) \\
& +2a^3(7+4b)+ab(19+15b+4b^2)-a^2(12+27b+11b^2))a_{12}-27a^2b^3a_{22} \\
& +30a^3b^3a_{22}+45ab^4a_{22}-21a^2b^4a_{22}-33a^3b^4a_{22}-18b^5a_{22}-27ab^5a_{22} \\
& +51a^2b^5a_{22}+3a^3b^5a_{22}+18b^6a_{22}-18ab^6a_{22}-3a^2b^6a_{22}+3a^2b_1^2-3a^4b_1^2 \\
& -6abb_1^2-6a^2bb_1^2+6a^3bb_1^2+6a^4bb_1^2+18ab^2b_1^2-18a^3b^2b_1^2-18ab^3b_1^2 \\
& +18a^2b^3b_1^2-9a^3b_1b_2+9a^4b_1b_2+18a^2bb_1b_2-18a^4bb_1b_2-12ab^2b_1b_2 \\
& -30a^2b^2b_1b_2+30a^3b^2b_1b_2+12a^4b^2b_1b_2+24ab^3b_1b_2-24a^3b^3b_1b_2 \\
& -9ab^4b_1b_2+9a^2b^4b_1b_2+6a^2b^2b_2^2-6a^4b^2b_2^2-18a^2b^3b_2^2+18a^3b^3b_2^2 \\
& +3ab^4b_2^2-3a^3b^4b_2^2-3a^2bb_{11}+3a^4bb_{11}+3ab^2b_{11}+12a^2b^2b_{11}-12a^3b^2b_{11} \\
& -3a^4b^2b_{11}-12ab^3b_{11}+12a^3b^3b_{11}+9ab^4b_{11}-9a^2b^4b_{11}+9a^3bb_{12}-9a^4bb_{12} \\
& -21a^2b^2b_{12}+21a^4b^2b_{12}+12ab^3b_{12}+30a^2b^3b_{12}-30a^3b^3b_{12}-12a^4b^3b_{12} \\
& -21ab^4b_{12}+21a^3b^4b_{12}+9ab^5b_{12}-9a^2b^5b_{12}+9a^3b^2b_{22}-9a^4b^2b_{22} \\
& -12a^2b^3b_{22}+12a^4b^3b_{22}+3ab^4b_{22}+12a^2b^4b_{22}-12a^3b^4b_{22}-3a^4b^4b_{22} \\
& -3ab^5b_{22}+3a^3b^5b_{22})+a^2b(6(-1+b)b(3a^3-b^2+ab(3+2b) \\
& -a^2(2+5b))a_{112}-6b(b^2-b^4+2a^3(-1+b^2)+a^2(2+2b-4b^3) \\
& +ab(-3+b^2+2b^3))a_{122}+6a^2b^2a_{222}-6a^3b^2a_{222}-12ab^3a_{222}+6a^2b^3a_{222} \\
& +6a^3b^3a_{222}+6b^4a_{222}+6ab^4a_{222}-12a^2b^4a_{222}-6b^5a_{222}+6ab^5a_{222} \\
& +6a^2Kb_1-14a^3Kb_1+2a^4Kb_1+6abKb_1-6a^2bKb_1+28a^3bKb_1 \\
& -4a^4bKb_1-28ab^2Kb_1-2a^3b^2Kb_1+30ab^3Kb_1-18a^2b^3Kb_1-6aa_{12}b_1 \\
& +12a^3a_{12}b_1-6ba_{12}b_1+48aba_{12}b_1-42a^2ba_{12}b_1-24a^3ba_{12}b_1-42ab^2a_{12}b_1 \\
& +72a^2b^2a_{12}b_1+12b^3a_{12}b_1-24ab^3a_{12}b_1+6a^2a_{22}b_1-6a^3a_{22}b_1-18aba_{22}b_1 \\
& +6a^2ba_{22}b_1+12a^3ba_{22}b_1+12b^2a_{22}b_1+18ab^2a_{22}b_1-30a^2b^2a_{22}b_1 \\
& +34a^2b^2Kb_2-18b^3a_{22}b_1+18ab^3a_{22}b_1+6a^2bKb_2-24a^3bKb_2+6a^4bKb_2 \\
& -4a^4b^2Kb_2-16ab^3Kb_2-20a^2b^3Kb_2+12a^3b^3Kb_2+8ab^4Kb_2-2a^2b^4Kb_2 \\
& -6aba_{12}b_2-18a^2ba_{12}b_2+36a^3ba_{12}b_2+36ab^2a_{12}b_2-42a^2b^2a_{12}b_2 \\
& -24a^3b^2a_{12}b_2-12b^3a_{12}b_2+6ab^3a_{12}b_2+30a^2b^3a_{12}b_2+6b^4a_{12}b_2 \\
& -12ab^4a_{12}b_2+18a^2ba_{22}b_2-18a^3ba_{22}b_2-30ab^2a_{22}b_2+18a^2b^2a_{22}b_2 \\
& +12a^3b^2a_{22}b_2+12b^3a_{22}b_2+6ab^3a_{22}b_2-18a^2b^3a_{22}b_2-6b^4a_{22}b_2
\end{aligned}
$$

$$+6ab^4a_{22}b_2+9a^3bK_1-3a^4bK_1-24a^2b^2K_1+3a^3b^2K_1+3a^4b^2K_1$$
$$+15ab^3K_1+15a^2b^3K_1-12a^3b^3K_1-15ab^4K_1+9a^2b^4K_1-a^3bK_2$$
$$+a^4bK_2+8a^2b^2K_2-14a^3b^2K_2-7ab^3K_2+13a^2b^3K_2+13a^3b^3K_2$$
$$-a^4b^3K_2-20a^2b^4K_2+2a^3b^4K_2+7ab^5K_2-a^2b^5K_2))$$
$$-a(2(-1+b)b^2a_2^2(3(3a^3(1+b)-2b^2(1+b)+2ab(2+4b+b^2)$$
$$-a^2(2+9b+5b^2))a_{11}+b(a(a^3(14-3b)-3b^2+ab(7+4b)$$
$$-a^2(10+9b))K+3(a^3(-13+b)+6b^2-2ab(7+4b)+2a^2(5+9b))a_{12}))$$
$$+aba_2(6(-1+a)(a-b)^2(-1+b)ba_{111}-6(-1+b)b(3a^3(1+b)-2b^2(1+b)$$
$$+2ab(2+4b+b^2)-a^2(2+9b+5b^2))a_{112}+24a^2b^2a_{122}-30a^3b^2a_{122}$$
$$-42ab^3a_{122}+30a^2b^3a_{122}+30a^3b^3a_{122}+18b^4a_{122}+18ab^4a_{122}-54a^2b^4a_{122}$$
$$-18b^5a_{122}+24ab^5a_{122}-2a^3Kb_1+8a^4Kb_1+12a^2bKb_1-20a^3bKb_1$$
$$-16a^4bKb_1-4ab^2Kb_1+34a^3b^2Kb_1+6ab^3Kb_1-24a^2b^3Kb_1+6a^3b^3Kb_1$$
$$-6a^2a_{11}b_1+6a^3a_{11}b_1+18aba_{11}b_1-6a^2ba_{11}b_1-12a^3ba_{11}b_1-12b^2a_{11}b_1$$
$$-18ab^2a_{11}b_1+30a^2b^2a_{11}b_1+18b^3a_{11}b_1-18ab^3a_{11}b_1+12a^2a_{12}b_1$$
$$-18a^3a_{12}b_1-42aba_{12}b_1+30a^2ba_{12}b_1+36a^3ba_{12}b_1+24b^2a_{12}b_1$$
$$+30ab^2a_{12}b_1-84a^2b^2a_{12}b_1-36b^3a_{12}b_1+54ab^3a_{12}b_1-6a^2b^3a_{12}b_1$$
$$-18a^3bKb_2+30a^4bKb_2-2a^2b^2Kb_2-28a^4b^2Kb_2-4ab^3Kb_2+28a^2b^3Kb_2$$
$$-6a^3b^3Kb_2+6a^4b^3Kb_2+2ab^4Kb_2-14a^2b^4Kb_2+6a^3b^4Kb_2-18a^2ba_{11}b_2$$
$$+18a^3ba_{11}b_2+30ab^2a_{11}b_2-18a^2b^2a_{11}b_2-12a^3b^2a_{11}b_2-12b^3a_{11}b_2$$
$$-6ab^3a_{11}b_2+18a^2b^3a_{11}b_2+6b^4a_{11}b_2-6ab^4a_{11}b_2+48a^2ba_{12}b_2-60a^3ba_{12}b_2$$
$$-48ab^2a_{12}b_2+30a^2b^2a_{12}b_2+48a^3b^2a_{12}b_2+24b^3a_{12}b_2-18ab^3a_{12}b_2$$
$$-24a^2b^3a_{12}b_2-6a^3b^3a_{12}b_2-12b^4a_{12}b_2+24ab^4a_{12}b_2-6a^2b^4a_{12}b_2$$
$$+a^3bK_1-7a^4bK_1-2a^2b^2K_1+20a^3b^2K_1+ab^3K_1-13a^2b^3K_1-13a^3b^3K_1$$
$$+7a^4b^3K_1+14a^2b^4K_1-8a^3b^4K_1-ab^5K_1+a^2b^5K_1-9a^3b^2K_2+15a^4b^2K_2$$
$$+12a^2b^3K_2-15a^3b^3K_2-15a^4b^3K_2-3ab^4K_2-3a^2b^4K_2+24a^3b^4K_2$$
$$+3ab^5K_2-9a^2b^5K_2)+6(-1+a)a((-1+b)b^2(-3b^2-a^2(4+b)+ab(7+b))a_{12}^2$$
$$+(a-b)(-1+b)b^2a_{11}(a(1+a-3b)K+(-1+3a-b)a_{12}-(a-b)(1+b)a_{22})$$
$$+a_{12}((-1+b)b^3(-a^2(-5+b)+a(-8+b)b+3b^2)a_{22}+a((a+a^2-2b-4ab$$
$$-2a^2b+6b^2+6ab^2-6b^3)b_1^2+(a^2(-3+6b-4b^2)+b^2(-4+8b-3b^2)$$
$$+2ab(3-7b+4b^2))b_1b_2+b(b(2a^2+b^2+a(2-6b+b^2))b_2^2$$
$$-(a-b)(-1+b)(-(1+a-3b)b_{11}+(a(3-4b)+b(-4+3b))b_{12}$$
$$-b((a(7-6b)+b(-6+7b))K+(a(-3+b)+b)b_{22})))))$$
$$+a(5a^3b^2K^2-10a^2b^3K^2-10a^3b^3K^2+5ab^4K^2+20a^2b^4K^2+5a^3b^4K^2$$
$$-10ab^5K^2-10a^2b^5K^2+5ab^6K^2-(-1+b)b^3(-a^2(-3+b)+a(-4+b)b+b^2)Ka_{22}$$
$$-(a-b)^2(-1+b)b^2a_{1112}-a^2b^2a_{1122}+2ab^3a_{1122}-b^4a_{1122}+a^2b^4a_{1122}-2ab^5a_{1122}$$
$$+b^6a_{1122}+a^2b^3a_{1222}-2ab^4a_{1222}-a^2b^4a_{1222}+b^5a_{1222}+2ab^5a_{1222}-b^6a_{1222}$$
$$-a^2ba_{112}b_1+3ab^2a_{112}b_1+2a^2b^2a_{112}b_1-2b^3a_{112}b_1-5ab^3a_{112}b_1+3b^4a_{112}b_1$$
$$+a^2ba_{122}b_1-3ab^2a_{122}b_1-2a^2b^2a_{122}b_1+2b^3a_{122}b_1+5ab^3a_{122}b_1-3b^4a_{122}b_1$$
$$-a^2Kb_1^2-a^3Kb_1^2+2abKb_1^2+4a^2bKb_1^2+2a^3bKb_1^2-6ab^2Kb_1^2-6a^2b^2Kb_1^2$$
$$+6ab^3Kb_1^2-3a^2b^2a_{112}b_2+5ab^3a_{112}b_2+2a^2b^3a_{112}b_2-2b^4a_{112}b_2-3ab^4a_{112}b_2$$
$$+b^5a_{112}b_2+3a^2b^2a_{122}b_2-5ab^3a_{122}b_2-2a^2b^3a_{122}b_2+2b^4a_{122}b_2+3ab^4a_{122}b_2$$
$$-b^5a_{122}b_2+3a^3Kb_1b_2-6a^2bKb_1b_2-6a^3bKb_1b_2+4ab^2Kb_1b_2+14a^2b^2Kb_1b_2$$
$$+4a^3b^2Kb_1b_2-8ab^3Kb_1b_2-8a^2b^3Kb_1b_2+3ab^4Kb_1b_2-2a^2b^2Kb_2^2$$
$$-2a^3b^2Kb_2^2+6a^2b^3Kb_2^2-ab^4Kb_2^2-a^2b^4Kb_2^2a^2bKb_{11}+a^3bKb_{11}-ab^2Kb_{11}$$

$-5a^2b^2Kb_{11} - a^3b^2Kb_{11} + 4ab^3Kb_{11} + 4a^2b^3Kb_{11} - 3ab^4Kb_{11} - 3a^3bKb_{12}$
$+7a^2b^2Kb_{12} + 7a^3b^2Kb_{12} - 4ab^3Kb_{12} - 14a^2b^3Kb_{12} - 4a^3b^3Kb_{12} + 7ab^4Kb_{12}$
$+7a^2b^4Kb_{12} - 3ab^5Kb_{12} - 3a^3b^2Kb_{22} + 4a^2b^3Kb_{22} + 4a^3b^3Kb_{22} - ab^4Kb_{22}$
$-5a^2b^4Kb_{22} - a^3b^4Kb_{22} + ab^5Kb_{22} + a^2b^5Kb_{22} + a^3bb_1K_1 - 3a^2b^2b_1K_1$
$-2a^3b^2b_1K_1 + 2ab^3b_1K_1 + 5a^2b^3b_1K_1 - 3ab^4b_1K_1 + 3a^3b^2b_2K_1 - 5a^2b^3b_2K_1$
$-2a^3b^3b_2K_1 + 2ab^4b_2K_1 + 3a^2b^4b_2K_1 - ab^5b_2K_1 - a^3bb_1K_2 + 3a^2b^2b_1K_2$
$+2a^3b^2b_1K_2 - 2ab^3b_1K_2 - 5a^2b^3b_1K_2 + 3ab^4b_1K_2 - 3a^3b^2b_2K_2 + 5a^2b^3b_2K_2$
$+2a^3b^3b_2K_2 - 2ab^4b_2K_2 - 3a^2b^4b_2K_2 + ab^5b_2K_2 - a^3b^2K_{11} + 2a^2b^3K_{11}$
$+a^3b^3K_{11} - ab^4K_{11} - 2a^2b^4K_{11} + ab^5K_{11} + a^3b^2K_{12} - 2a^2b^3K_{12} + ab^4K_{12}$
$-a^3b^4K_{12} + 2a^2b^5K_{12} - ab^6K_{12} - a^3b^3K_{22} + 2a^2b^4K_{22} + a^3b^4K_{22} - ab^5K_{22}$
$-2a^2b^5K_{22} + ab^6K_{22})))$

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[1]In the bibliography we will use the following abbreviations for the review journals: JFM for Jahrbuch für die Fortschritte der Mathematik, MR for Mathematical Reviews, and Zbl for Zentralblatt für Mathematik.

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