Spencer δ-cohomology, restrictions, characteristics and involutive symbolic PDEs

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Abstract

We generalize the notion of involutivity to systems of differential equations of different orders and show that the classical results relating involutivity, restrictions, characteristics and characteristicity, known for first order systems, extend to the general context. This involves, in particular, a new definition of strong characteristicity. The proof exploits a spectral sequence relating Spencer δ-cohomology of a symbolic system and its restriction to a non-characteristic subspace. [1]

Introduction

This paper concerns some algebraic aspects of systems of differential equations. We will investigate their systems of symbols $g_k \subset S^k T^* \otimes N$, where T and N are finite dimensional vector spaces, representing the spaces of independent and dependent variables respectively. The collection $g = \{g_k\}_{k\geq 0}$ will be called a symbolic system and we do not require that it is generated in one particular order (more details will be given in §1 below).

Spencer δ-cohomology groups $H^{i,j}(g)$ are algebraic invariants of such structures, important in the study of formal integrability of PDEs [S]. Let $W \subset T$ be a subspace and $V^* = \operatorname{ann}(W)$. From the exact sequence

$$0 \to V^* \hookrightarrow T^* \to W^* \to 0 \qquad (\dagger)$$

we get the restricted symbolic system $\bar{g}_k \subset S^k W^* \otimes N$.

A subspace $V^* \subset T^*$ is called non-characteristic ([S]) if no non-zero element $\omega \in g_k$ restricts to zero on W: $\omega(\xi_1, \dots, \xi_k) = 0 \,\forall \xi_i \in W \Rightarrow \omega = 0$, where $k = r_{\min}(g)$ is the minimal order of the system g (§2). This classical definition leads however to the following confusion for higher-order systems ($k > 1$). Consider a subspace $g_k \subset S^k T^*$ of codimension 1, $k \geq 2$, and let g be the corresponding symbolic system. Then by dimension reasons any one-dimensional subspace V^* is characteristic (see Example 4). Identifying such subspaces with projectivized covectors we would conclude that every covector is characteristic for one scalar PDE (not ODE: $\dim T > 1$) of order $k > 1$, which can't be true.

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Thus we risk changing the standard terminology and call non-characteristic (by Spencer and al) subspaces V^* strongly non-characteristic (with opposite being weakly characteristic; so in the above example all covectors are weakly characteristic, but not characteristic); more motivations for this will come below.

Let us denote

$$\Upsilon^{i,j}=\bigoplus_{r>0}S^rV^*\otimes\delta(S^{i+1-r}W^*\otimes\Lambda^{j-1}W^*)\otimes N,\quad \Theta^{i,j}=\bigoplus_{q>0}\Upsilon^{i,q}\otimes\Lambda^{j-q}V^*,$$

where δ is the Spencer operator. Then we have: $\Upsilon^{i,0}=\Upsilon^{0,j}=0$ and $\Theta^{i,0}=\Theta^{0,j}=0$. The other terms are however non-zero.

Let also $\Pi^{i,j}=\delta(S^{i+1}V^*\otimes N\otimes\Lambda^{j-1}V^*)$. Note that $\Pi^{i,0}=0$ by definition and for $j>0$ we have: $\Pi^{i,j}=S^iV^*\otimes N\otimes\Lambda^jV^*\cap\operatorname{Ker}\delta$. In particular, $\Pi^{0,j}=N\otimes\Lambda^jV^*$ for $j>0$. Finally denote $S^{i,j}=\operatorname{Im}(S^{i+j}V^*\to S^iV^*\otimes S^jV^*)$.

Involutivity of a symbolic system g is equivalent to vanishing of certain δ-cohomology groups, see §3.

Theorem A. *Let V^* be a strongly non-characteristic subspace for a symbolic system g. If g is involutive, then its W-restriction $\bar g$ is also involutive.*

Moreover the Spencer cohomology of g and $\bar g$ are related by the formula:

$$H^{i,j}(g)\simeq\bigoplus_{q>0}H^{i,q}(\bar g)\otimes\Lambda^{j-q}V^*\oplus\delta^{i+1}_{r_{min}(g)}\cdot[\Theta^{i,j}\oplus\Pi^{i,j}]\oplus\delta^i_0\delta^j_0\cdot H^{0,0}(\bar g),$$

where δ^t_s is the Kronecker symbol.

If $\bar g$ is an involutive system of pure order $k=r_{min}(\bar g)=r_{max}(\bar g)$, then g is also an involutive system of pure order k and the above formula holds.

Corollary. *If g is an involutive system and V^* is strongly non-characteristic subspace of T^*, then the δ-differential induces the following exact sequences:*

$$0\to\delta^{i+1}_{r_{min}(g)}\cdot S^{j,i}\otimes N\to S^{j-1}V^*\otimes H^{i,1}(g)\to\dots$$
$$\dots\to V^*\otimes H^{i,j-1}(g)\to H^{i,j}(g)\to H^{i,j}(\bar g)\oplus\delta^{i+1}_{r_{min}(g)}\cdot\Upsilon^{i,j}\to 0,$$

for $i\ge r_{min}(g)-1$.

The implication (g involutive)$\Rightarrow$($\bar g$ involutive) from Theorem A constitutes Guillemin's theorem A obtained for the first order systems in [G]. Our proof is based on the technique, developed for other purposes in [KL$_2$], which allows to generalize the statement to arbitrary symbolic systems. The inverse Guillemin's theorem, i.e. the implication, ($\bar g$ involutive)$\Rightarrow$(g involutive) was not known.

The above Corollary for $k=0$ and system g of the first order is a theorem of Quillen-Guillemin [G, Q] (see §5 for details).

This paper is a generalization of the classical results known for the first order systems ([G, GK, BCG^3]). But it is not straightforward. Indeed, Spencer's reformulation of Guillemin's results for higher order holds true in the stable range $m\ge\mu$ (§1.7-1.8 of [S]), but one inevitably gets into the trouble if adjusts the order $k \geq r_{\min}(g)$ precisely to the place, where involutivity of g starts (see §7). In addition, Spencer's generalization of results due to Quillen and Guillemin (loc.cit.) contains mistakes (see Remark in §2).

More arguably, it was not noticed previously that the definition of characteristic subspace has two meaningful generalizations to the case of higher order: one important for the Cauchy problem (standard one adapted to restrictions as in [S]) and the other one important in the study of characteristics, which we call strong characteristicity. Namely, we call V^* strongly characteristic for g if $\exists\, \omega \in g_k \setminus \{0\}$ such that the directional derivative $\delta_\xi \omega = 0\, \forall \xi \in W$, where $k = r_{\max}(g)$ is the maximal order of the system.

Theorem B. *Let g be an involutive system over $\mathbb{C}$. Then a subspace $V^* \subset T^*$ is strongly characteristic iff it contains a characteristic covector.*

This result is a generalization of Guillemin's theorem B from [G], which concerns the pure order one systems – the only case, where the two introduced notions of characteristicity coincide. Thus both Guillemin's theorems have analogs for higher (and even various) order systems, but for this two different notions of characteristicity should be imposed.

Some other results appear at the end of this paper. We also provide a series of counter-examples showing importance of all our hypotheses.

1. Symbolic systems

Consider the Spencer δ-complex:

$$0 \to S^kT^* \otimes N \xrightarrow{\delta} S^{k-1}T^* \otimes N \otimes T^* \xrightarrow{\delta} \cdots \xrightarrow{\delta} S^{k-n}T^* \otimes N \otimes \Lambda^n T^* \to 0,$$

where $S^iT^* = 0$ for $i < 0$. The *first prolongation* of a subspace $h \subset S^kT^* \otimes N$ is

$$h^{(1)} = \{p \in S^{k+1}T^* \otimes N \mid \delta p \in h \otimes T^*\}$$

Higher prolongations are defined inductively and satisfy $(h^{(l)})^{(m)} = h^{(l+m)}$. An alternative definition is: $h^{(l)} = S^lT^* \otimes h \cap S^{k+l}T^* \otimes N$.

Definition 1. *Symbolic system is a sequence of subspaces $g_k \subset S^kT^* \otimes N$, $k \geq 0$, with $g_0 = N$ and $g_k \subset g_{k-1}^{(1)}$.*

With every such a system we associate its Spencer δ-complex of order k:

$$0 \to g_k \xrightarrow{\delta} g_{k-1} \otimes T^* \xrightarrow{\delta} g_{k-2} \otimes \Lambda^2 T^* \to \cdots \xrightarrow{\delta} g_{k-n} \otimes \Lambda^n T^* \to 0.$$

The cohomology group at the term $g_i \otimes \Lambda^j T^*$ is denoted by $H^{i,j}(g)$ and is called the Spencer δ-cohomology of g.

Note that $g_k = S^kT^* \otimes N$ for $0 \leq k < r$ and the first number $r = r_{\min}(g)$, where the equality is violated is called the minimal order of the system. Actually the system has several orders:

$$\operatorname{ord}(g) = \{k \in \mathbb{Z}_+ \mid g_k \neq g_{k-1}^{(1)}\}.$$

Note that multiplicity of an order $m(r) = \dim g_{r-1}^{(1)}/g_r$ is equal to $\dim H^{r-1,1}(g)$. Hilbert basis theorem implies finiteness of the set of orders:

$$\operatorname{codim}(g) := \dim H^{*,1}(g) = \#\operatorname{ord}(g) < \infty.$$

If $r_{\max}$ is the maximal order of the system, then $g_{k+1} = g_k^{(1)}$ for $k \geq r_{\max}$.

Denote by $\bar{g}$ the image of the restriction map $g \to SW^* \otimes N$.

Proposition 1. *For any subspace $W \subset T$ the restriction $\bar{g}$ is a symbolic system.*

Proof. This follows from naturality of the δ-differential. □

2. Characteristics

A covector $v \in {}^{\mathbb{C}}T^* \setminus \{0\}$ is called (complex) characteristic for g_k if $v^k \otimes w \in g_k^{\mathbb{C}}$ for some $w \in {}^{\mathbb{C}}N \setminus \{0\}$ (in this paper characteristics will be considered only over the field $\mathbb{C}$). Clearly, if v is characteristic for g_k, it is characteristic for its prolongation $g_k^{(1)}$ and vice versa. We call v characteristic for a symbolic system g if it is characteristic on every level g_k (or equivalently only for the level $k = r_{\max}(g)$). The projectivized set of all characteristic covectors forms the characteristic variety $\operatorname{Char}^{\mathbb{C}}(g) \subset P^{\mathbb{C}}T^*$.

Definition 2. *Call a subspace $V^* \subset T^*$ strongly non-characteristic for g_k if $g_k \cap V^* \cdot S^{k-1}T^* \otimes N = 0$. In the opposite case, when the intersection is non-zero, let's call V^* weakly characteristic.*

Call V^* weakly non-characteristic if $g_k \cap S^k V^* \otimes N = 0$. If the intersection is non-zero, the subspace V^* will be called strongly characteristic.

Note that strong characteristicity implies weak characteristicity, as well as strong non-characteristicity implies weak non-characteristicity, but there are spaces g_k, for which certain V^* are (don't be confused!) simultaneously weakly characteristic and weakly non-characteristic. Of course, then they are neither strongly characteristic nor strongly non-characteristic for g_k.

Remark. *Spencer's notion of non-characteristicity (definition 1.8.1 of [S]) formally coincides with our weak non-characteristicity (after translation from D-complex to the symbolic language). But then his Theorem 1.8.1(i) (as well as preceding Theorem 1.7.3) becomes wrong, since weak non-characteristic subspaces don't need to be strong non-characteristic (see §7). However changing tensorial product to symmetric and shifting the index by one, the definition turns into our strong non-characteristicity and the subsequent statements hold. Thus importance of distinction between characteristicities becomes apparent.*

Two introduced notions of weak and strong characteristicity coincide for first order systems, $k = 1$ (the same for non-characteristicity), but not for the case of higher orders. The following property follows directly from the definition:

Proposition 2. *A subspace V^* is strongly non-characteristic for g_k iff the restriction to W map $g_k \to \bar{g}_k$ is an isomorphism. It is weakly non-characteristic iff kernels of the maps $\delta_w : g_k \to g_{k-1}$, $w \in W$, jointly intersect only by zero.*

Proposition 3. *If $g_k \subset S^k T^* \otimes N$ is weakly/strongly non-characteristic, then any subspace of its prolongation $g_{k+1} \subset g_k^{(1)} \subset S^{k+1} T^* \otimes N$ is such as well.*

Proof. For the weak case the statement is obvious.

Let V^* be strongly non-characteristic for g_k. Then the restriction $g_k \to \bar{g}_k$ is an isomorphism. Assume that V^* is weakly characteristic for g_{k+1}. From the commutative diagram

$$\begin{array}{ccc} g_{k+1} & \stackrel{\delta}{\longrightarrow} & g_k \otimes T^* \\ \downarrow & & \downarrow \\ \bar{g}_{k+1} & \stackrel{\delta}{\longrightarrow} & \bar{g}_k \otimes W^* \end{array}$$

we conclude that a non-zero element $p \in g_{k+1}$ belongs to the kernel of the restriction map iff $\delta(p) \in g_k \otimes V^*$, which implies that V^* is strongly characteristic for g_{k+1} and hence strongly characteristic for g_k. But this yields that V^* is weakly characteristic for g_k, contradicting our assumption. □

Thus weak and strong non-characteristicity for a space g_k are inherited by the prolonged spaces $g_k^{(1)}$. Theorem B assures that the same is true for strong characteristicity (over $\mathbb{C}$) in involutive case, but thanks to Example 6 not in general. As will be seen in Example 7 the property of g_k being weakly characteristic is not hereditary upon prolongations too.

Definition 3. *A subspace $V^* \subset T^*$ is called weakly or strongly non-characteristic for a symbolic system g if this requirement holds for $g_{r_{min}(g)}$ and hence for every g_k with $k \geq r_{min}(g)$.*

Call V^* weakly or strongly characteristic for g if $g_k \cap V^* \cdot S^{k-1} T^* \otimes N \neq 0$, resp. $g_k \cap S^k V^* \otimes N \neq 0$, for $k = r_{max}(g)$.

Note that now the notion of strong characteristicity is not opposite to weak non-characteristicity unless the system has a pure order $r_{\min} = r_{\max}$ (and the same for weak characteristicity and strong non-characteristicity).

Relation between characteristic covectors and characteristic subspaces were clarified by Guillemin [G] in the case of first order involutive systems. We extend his result to the case of arbitrary orders in Theorem B.

3. Involutivity

The classical Cartan's definition of involutivity involves quasi-regular sequences. Namely a subspace $g_k \subset S^k T^* \otimes N$ is involutive if for some and hence for any generic basis $v_1, \dots, v_n$ of T the maps

$$\delta_{v_i} : g_k^{(1)} \cap S^{k+1} \operatorname{ann}\langle v_1, \dots, v_{i-1}\rangle \otimes N \to g_k \cap S^k \operatorname{ann}\langle v_1, \dots, v_{i-1}\rangle \otimes N \quad (*)$$

are surjective for all $1 \le i \le n$.

It is well-known (see Serre's letter in [GS], also [BCG^3]) that this is equivalent to the requirement

$$H^{i,j}(g) = 0 \text{ for } i \ge k, \qquad (**)$$

where g is the system generated by g_k, and so (g_k involutive) $\Rightarrow$ ($g_k^{(1)}$ involutive). Basing on this homological characterization a system g of pure order k is called involutive if its generating subspace g_k is such.

But there are several ways of generalizing this for arbitrary symbolic systems:

I_1: $H^{i,1}(g) = 0 \Longrightarrow H^{i,j}(g) = 0\ \forall j > 1$.

I_2: For $k \notin \operatorname{ord}(g)$ and a generic basis $\{v_i\}$ the following maps are surjective: $\delta_{v_i} : g_k \cap S^k \operatorname{ann}\langle v_1, \dots, v_{i-1}\rangle \otimes N \to g_{k-1} \cap S^{k-1} \operatorname{ann}\langle v_1, \dots, v_{i-1}\rangle \otimes N$.

I_3: Denoting $\operatorname{ord}(g) = \{r_1 < \dots < r_s\}$, there is a splitting $T = \oplus_{j=1}^s U_j$ and a basis $\{v_i\}$ subordinated to it such that the above maps δ_{v_i} are surjective unless $k = r_m$ is an order and i corresponds to $v_i \in U_m$.

One easily proves $I_3 \Rightarrow I_2 \Rightarrow I_1$, but the implications are irreversible. Properties I_1, I_2 are too weak for the general definition of involutivity and I_3 seems to be too strong (this property holds for the direct sum $g \subset S(\sum T_i^*) \otimes (\sum N_i)$ of involutive systems $g_i \subset ST_i^* \otimes N_i$). So we give:

Definition 4. *A symbolic system $g \subset ST^* \otimes N$ is called involutive if each subspace $g_k \subset S^k T^* \otimes N$ is involutive. When $\operatorname{ord}(g) = \{r_1, \dots, r_s\}$, this is a condition only for $k = r_i$, $1 \le i \le s$.*

This definition most appropriately reflects the dual picture of quasi-regular sequences in the symbolic module g^* ([GS, BCG^3]) known for pure order systems (it is also interesting to investigate involutivity coupled with property I_3).

Let us denote by $g^{|k\rangle}$ the symbolic system generated by all differential corollaries of the system deduced from the order k:

$$g_i^{|k\rangle} = \begin{cases} S^i T^* \otimes N, & \text{for } i < k; \\ g_k^{(i-k)}, & \text{for } i \ge k. \end{cases}$$

Theorem 4. *A system g is involutive iff $H^{i,j}(g^{|k\rangle}) = 0$ for all $i \ge k$ (this condition is to be checked for $k \in \operatorname{ord}(g)$ only).*

Proof. This follows from the classical equivalence $(*) \Leftrightarrow (**)$ because involutivity of g means involutivity of the pure order systems $g^{|r_1\rangle}, \dots, g^{|r_s\rangle}$. $\square$

In particular, $H^{i,j}(g) = 0$ for $i \notin \operatorname{ord}(g) - 1$, $(i,j) \ne (0,0)$, and properties I_1, I_2 follow from involutivity. This however is not invertible:

Example 1. Consider the system $u_{xx} = 0, u_{yy} = 0, v_{yyy} = 0$. The only non-zero cohomologies are $H^{0,0}(g) = \mathbb{R}^1$, $H^{1,1}(g) = \mathbb{R}^2$, $H^{2,1}(g) = \mathbb{R}^1$, $H^{2,2}(g) = \mathbb{R}^1$. So I_1 holds. The surjectivity requirement I_2 holds as well. But the pure order 2 system $g^{|2\rangle}$ is not involutive because $H^{2,2}(g^{|2\rangle}) = \mathbb{R}^1$. Thus g is not involutive.

Example 2. The system g given by equations $u_{xx} = 0, u_{xy} = 0, u_{yyz} = 0$ is involutive. But it does not satisfy the property I_3.

4. Proof of Theorem A

Our proof is based on a spectral sequence constructed in [KL$_2$] for the need of a reduction theorem (which means that instead of projection of the symbolic system g to $SW^* \otimes N$ we intersect it with $SV^* \otimes N$).

Define a filtration in the l-th Spencer complex, induced by the filtration in ΛT^* via the powers of V^*:

$$F^{p,q} = g_{l-p-q} \otimes \Lambda^p V^* \wedge \Lambda^q T^*.$$

Lemma 5. *The filtration is monotone decreasing, $F^{p+1,q-1} \subset F^{p,q}$, and is preserved by the δ-map, $\delta F^{p,q} \subset F^{p,q+1}$.* □

This filtration determines the spectral sequence of Leray-Serre type with

$$E_0^{p,q} = F^{p,q}/F^{p+1,q-1} = g_{l-p-q} \otimes \Lambda^p V^* \otimes \Lambda^q W^*.$$

The differential $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ acts by W and so

$$E_1^{p,q} = H^{l-p-q,q}(g, \delta') \otimes \Lambda^p V^*,$$

where δ' is the induced differential (along W).

Denote $\Upsilon^{i,j} = V^* \cdot \delta'(S^i T^* \otimes N \otimes \Lambda^{j-1} W^*)$, $\Xi^{i,j} = \delta'(g_{i+1} \otimes \Lambda^{j-1} W^*) \cap \Upsilon^{i,j}$.

If V^* is strongly non-characteristic, then $\Xi^{i,1} = 0$. In fact, suppose that for some $p \in g_{i+1}$ we have: $\delta' p \in \Upsilon^{i,1}$. Then $(\delta' p)|_W = 0$ and so $p\,|_W = 0$. This contradicts injectivity of the projection $g_{i+1} \to \bar{g}_{i+1}$. However for $j > 1$ the term $\Xi^{i,j}$ can be non-zero.

Lemma 6. *Let g be a symbolic system, V^* be strongly non-characteristic and $k = r_{min}(g)$. The cohomology of g with respect to the induced differential δ' are related to the Spencer cohomology of the restricted symbolic system as follows:*

$$H^{i,j}(g, \delta') = \begin{cases} 0, & i < k-1, j > 0; \\ S^i V^* \otimes N, & i \le k-1, j = 0; \\ H^{i,j}(\bar{g}, \bar{\delta}) \oplus \Upsilon^{i,j}/\Xi^{i,j}, & i = k-1, j > 0; \\ H^{i,j}(\bar{g}, \bar{\delta})/\Xi^{i-1,j+1}, & i = k; \\ H^{i,j}(\bar{g}, \bar{\delta}), & i > k. \end{cases}$$

The third and forth lines above represent the cohomology non-canonically. We actually mean here the exact sequences:

$$0 \to \Upsilon^{k-1,j}/\Xi^{k-1,j} \to H^{k-1,j}(g, \delta') \to H^{k-1,j}(\bar{g}, \bar{\delta}) \to 0,$$
$$0 \to H^{k,j-1}(g, \delta') \to H^{k,j-1}(\bar{g}, \bar{\delta}) \to \Xi^{k-1,j} \to 0.$$

Proof. The restriction map induces an isomorphism of the complexes

$$\begin{array}{ccccccccc} 0 & \to & g_l & \stackrel{\delta'}{\longrightarrow} & g_{l-1}\otimes W^* & \stackrel{\delta'}{\longrightarrow} & g_{l-2}\otimes\Lambda^2W^* & \stackrel{\delta'}{\longrightarrow} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bar g_l & \stackrel{\bar\delta}{\longrightarrow} & \bar g_{l-1}\otimes W^* & \stackrel{\bar\delta}{\longrightarrow} & \bar g_{l-2}\otimes\Lambda^2W^* & \stackrel{\bar\delta}{\longrightarrow} & \dots \end{array}$$

at first $l-k+1$ terms and hence an isomorphism of cohomologies. For $i<k$ we have: $g_i=S^iT^*$, $\bar g_i=S^iW^*$. So the boundary cohomologies $H^{k,j-1}$, $H^{k-1,j}$ make the only difference, being found from the commutative diagram:

$$\begin{array}{ccccccc} \dots & \to & g_k\otimes\Lambda^{j-1}W^* & \stackrel{\delta'}{\longrightarrow} & S^{k-1}T^*\otimes N\otimes\Lambda^jW^* & \to & \dots \\ & & \wr\downarrow & & \downarrow & & \\ \dots & \to & \bar g_k\otimes\Lambda^{j-1}W^* & \stackrel{\bar\delta}{\longrightarrow} & S^{k-1}W^*\otimes N\otimes\Lambda^jW^* & \to & \dots \end{array}$$ □

Lemma 7. *If g is involutive, then the differentials $d_r^{p,q}:E_r^{p,q}\to E_r^{p+r,q-r+1}$ are trivial for $r>0$, save for the map $d_1^{p,0}$ with $l-p<k=r_{min}(g)$, which is the δ-differentiation along V.*

Proof. We prove at first this statement for the case of pure order k symbolic system g. Afterwards we deduce the general case.

Let g be involutive. We prove by induction on $l\geq k$ that $H^{i,l-i}(g,\delta')$ vanishes except for $i=k-1$. Let $\dim V=t$.

Then the table $E_1^{p,q}$ consists of $(t+1)$ columns: $E_1^{0,q}=H^{l-q,q}(g,\delta')$, $E_1^{1,q}=H^{l-1-q,q}(g,\delta')\otimes V^*$, $\dots$, $E_1^{t,q}=H^{l-t-q,q}(g,\delta')\otimes\Lambda^tV^*$. By the induction hypothesis for $0<i\leq t$ there is only one non-zero term among $E_1^{i,q}$, $q>0$, corresponding to $q=l-i-k+1$ (if this number is non-negative, otherwise all terms vanish).

Also the row $(E_1^{p,0},d_1^{p,0})$ is exact except for the left boundary position, whence $E_2^{p,0}=0$ for $p\neq l-k+1$. For $p=l-k+1\leq t$ we have: $E_2^{p,0}=H(E_1^{p,0},d_1^{p,0})=\delta(S^kV^*\otimes N\otimes\Lambda^{l-k}V^*)$.

Thus since only the elements of anti-diagonal $p+q=l-k+1$ can survive in E_∞, the single non-zero term in the $E_1^{0,q}$ column except $E_1^{0,l-k+1}=H^{k-1,l-k+1}(g,\delta')$ can be $E_1^{0,l-k}=H^{k,l-k}(g,\delta')$ provided that one of the differentials $d_i^{0,l-k}$, $i=1,\dots,t$, is injective.

But then in the spectral sequence for $(l+t)$-th Spencer complex we find a non-zero term $E_1^{t,l-k}=H^{k,l-k}(g,\delta')\otimes\Lambda^tV^*$ and to kill its contribution to the Spencer group $H^{k,*}(g)=0$ (involutivity) we need to assume a non-zero term $E_1^{i,l+t-k-i-1}=H^{k+1,l+t-k-i-1}(g,\delta')\otimes\Lambda^iV^*$, $0\leq i<t$. Continuing we obtain an infinite sequence of non-zero groups $H^{s,q_s}(g,\delta')=H^{s,q_s}(\bar g)$, $s\to\infty$. But this contradicts Poincaré δ-lemma, according to which $\dim H^{*,*}(\bar g)<\infty$.

Now consider the general case, when the symbolic system has different orders. To prove vanishing of $d_r^{p,q}$ for $r>0$ we use an alternative (but standard) definition of the r-th term of the spectral sequence:

$$E_r^{p,q}=Z_r^{p,q}/(Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}),$$

where $Z_r^{p,q} = \{\omega \in F^{p,q} \,|\, \delta\omega \in F^{p+r,q-r+1}\}$ are r-th order cocycles and $B_r^{p,q} = \{\delta\theta \in F^{p,q} \,|\, \theta \in F^{p-r,q+r-1}\}$ are r-th order coboundaries. Since the differential $d_r^{p,q}$ is induced by δ it is helpful to introduce the following spaces (we stress dependence on the symbolic system for these terms):

$$\hat{E}_r^{p,q}(g) = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + \operatorname{Ker}\delta \cap Z_r^{p,q}), \qquad \check{E}_r^{p,q}(g) = Z_r^{p,q}/B_{r-1}^{p,q}.$$

Actually, the differential $d_r^{p,q}$ factorizes via the natural δ-induced map $\bar{d}_r^{p,q}$: $\hat{E}_r^{p,q}(g) \to \check{E}_r^{p+r,q-r+1}(g)$ and the natural projections $\varsigma_r^{p,q} : \check{E}_r^{p,q}(g) \to E_r^{p,q}(g)$ and $\varrho_r^{p,q} : E_r^{p,q}(g) \to \hat{E}_r^{p,q}(g)$ as follows:

$$d_r^{p,q} = \varsigma_r^{p+r,q-r+1} \circ \bar{d}_r^{p,q} \circ \varrho_r^{p,q}.$$

Note that $Z_r^{p,q}(g)$ as well as $Z_{r-1}^{p+1,q-1}(g)$ depends only on the graded term g_{l-p-q}, if we consider the l-th Spencer complex. Thus $\hat{E}_r^{p,q}(g) = \hat{E}_r^{p,q}(g^{|l-p-q\rangle})$ and consequently obtain the following commutative diagram:

$$\begin{array}{ccccccc}
E_r^{p,q}(g) & \to & E_r^{p,q}(g^{|l-p-q\rangle}) & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1}(g^{|l-p-q\rangle}) & \to & E_r^{p+r,q-r+1}(g) \\
\downarrow \varrho_r & & \downarrow \varrho_r & & \uparrow \varsigma_r & & \uparrow \varsigma_r \\
\hat{E}_r^{p,q}(g) & \stackrel{\sim}{\to} & \hat{E}_r^{p,q}(g^{|l-p-q\rangle}) & \xrightarrow{\bar{d}_r^{p,q}} & \check{E}_r^{p+r,q-r+1}(g^{|l-p-q\rangle}) & \to & \check{E}_r^{p+r,q-r+1}(g).
\end{array}$$

In this diagram all arrows except differentials $d_r, \bar{d}_r$ are projections and thus the differential $d_r^{p,q}(g)$ for the general symbolic system g factorizes via the differential $d_r^{p,q}$ for the pure order involutive system $g^{|l-p-q\rangle}$.

Since $d_r^{p,q} = 0$, $q, r > 0$, for pure order involutive systems as was proved above, we obtain the same conclusion $d_r^{p,q}(g) = 0$ (in the same range) for the general involutive symbolic system g. □

Lemma 8. *Let $\bar{g}$ be an involutive system of pure order k. Then for $r > 0$ the differentials $d_r^{p,q}$ vanish as in Lemma 7 (except $d_1^{p,0}$, $l - p < k$) and g is also involutive of pure order k.*

Proof. If $\bar{g}$ is involutive, then $E_1^{p,q}$ has support in the lines $p+q = l-k+1$ and $q = 0$. The latter is exact when equipped with the differential d_1, except for the term $E_1^{p,0}$, $p = l-k+1 \le t$, and the former survives until E_∞ by the graphical evidence.

Now involutivity of g follows from Lemma 6 and the spectral sequence, since $H^{i,j}(\bar{g}) = 0$ for $i \ne k-1$ implies $H^{i,j}(g) = 0$ for $i \ne k-1$. □

Lemma 9. *Let the system g be involutive. Then the system $\bar{g}$ is also involutive and $\Xi^{i,j} = 0$ for $i \ge r_{min}(g) - 1$.*

Proof. By our definition the first statement suffices to prove for pure order k systems. For $k = 1$ the claim that $\bar{g}$ is involutive is the Guillemin's Theorem A and for $k > 1$ it follows by the equivalence reduction, see the end of §6.

Since $H^{i,j-1}(\bar{g}) = 0$ for $i \ge k$, the homomorphic image of this group $\Xi^{i-1,j}$ vanishes too. The same obviously holds for a general system g and $k = r_{\min}(g)$, since the term $\Xi^{i-1,j}$ depends only on the subspace g_i and hence on the involutive system $g^{|i\rangle}$. □

Now we can finish the proof of Theorem A. By Lemma 7 we have: $E_\infty^{p,q} = E_1^{p,q}$ for $q>0$, $E_\infty^{p,0}=E_2^{p,0}=0$ for $p\neq l-k+1$, $k=r_{\min}(g)$, and $E_2^{l-k+1,0} = \Pi^{k-1,l-k+1}$. Thus we conclude for $l>0$:

$$H^{l-j,j}(g,\delta)\simeq \oplus_{p+q=j}\,E_\infty^{p,q} = \oplus_{q>0}\,H^{l-j,q}(g,\delta')\otimes\Lambda^{j-q}V^*\oplus\delta_{r_{\min}(g)}^{l-j+1}\cdot\Pi^{l-j,j}$$

and the claim follows from Lemmata 6 and 9. □

5. Proof of the Corollary

For $r_{\min}(g)=1$ the sequence of the Corollary is exact for all $i\geq 0$. When $i=0$ it reads:

$$0\to S^jV^*\otimes H^{0,0}(g)\to S^{j-1}V^*\otimes H^{0,1}(g)\to\dots$$
$$\dots\to V^*\otimes H^{0,j-1}(g)\to H^{0,j}(g)\to H^{0,j}(\bar g)\to 0.$$

This follows from the formula of Theorem A, which in the considered case can be rewritten as:

$$H^{0,j}(g)=\oplus_{q\geq 0}H^{0,q}(\bar g)\otimes\Lambda^{j-q}V^*.$$

Substitution of this into the above sequence decomposes it into the sum of the trivial Spencer complexes $(S^{\alpha-t}V^*\otimes\Lambda^tV^*,\delta)$ tensorially multiplied with $H^{0,s}(\bar g)$.

Of course, the decomposition is not natural, so this argument is not justified. But we can filter the cohomology $H^{0,j}(g)$ via the spectral sequence:

$$H^{0,j}(g)=F_\infty^{0,j}\supset F_\infty^{1,j-1}\supset\dots\supset F_\infty^{j,0}\quad\text{with}$$
$$F_\infty^{a,b}/F_\infty^{a+1,b-1}=E_\infty^{a,b}=H^{0,b}(\bar g)\otimes\Lambda^aV^*.$$

The associated graded sum is as in the considered formula. Therefore we can filter the above complex and the consecutive quotients are exact. The required exactness of the whole complex follows.

Remark. *In [G] involutivity for pure order 1 restricted systems was deduced from the exactness of the above sequence. Here we use the opposite idea, concluding exactness from a by-product (or tool) of Theorem A on involutivity.*

For $i=r_{\min}(g)-1>0$ we have the following complex:

$$0\to S^{j,i}\otimes N\to S^{j-1}V^*\otimes H^{i,1}(g)\to\dots$$
$$\dots\to V^*\otimes H^{i,j-1}(g)\to H^{i,j}(g)\to H^{i,j}(\bar g)\oplus\Upsilon^{i,j}\to 0.$$

This complex is again filtered via the filtration of cohomology

$$H^{i,j}(g)=F_\infty^{0,j}\supset F_\infty^{1,j-1}\supset\dots\supset F_\infty^{j,0}\quad\text{with}$$
$$F_\infty^{a,b}/F_\infty^{a+1,b-1}=E_\infty^{a,b}=(H^{i,b}(\bar g)\oplus\Upsilon^{i,b})\otimes\Lambda^aV^*\oplus\delta_0^b\cdot\Pi^{i,a}$$

The consecutive quotient complexes equal $(S^{\alpha-t}V^*\otimes\Lambda^tV^*,\delta)$ tensorially multiplied with $H^{\beta,s}(\bar g)\oplus\Upsilon^{\beta,s}$ (again this summation is not natural, so one should consider in stead the short exact sequence as after Lemma 6 and perform an additional factorization) and so are exact. The last occurring complex is:

$$0\to S^{j,i}\to S^{j-1}V^*\otimes\Pi^{i,1}\to\dots\to V^*\otimes\Pi^{i,j-1}\to\Pi^{i,j}\to 0.$$

Its exactness follows from the following anti-commutative diagram (or bi-complex: the sum of compositions of arrows along the boundary of a square is zero), in which rows and columns are exact, save for the one-term sequences:

$$\begin{array}{ccccccc}
\ddots & & \uparrow & & \ddots & & \\
\dots\to & & S^aV^*\otimes S^{b-1}V^*\otimes\Lambda^{c+1}V^* & \to & S^{a-1}V^*\otimes S^{b-1}V^*\otimes\Lambda^{c+2}V^* & & \to\dots\\
 & & \uparrow & & \uparrow & & \\
\dots\to & & S^aV^*\otimes S^{b}V^*\otimes\Lambda^{c}V^* & \to & S^{a-1}V^*\otimes S^{b}V^*\otimes\Lambda^{c+1}V^* & & \to\dots\\
 & & \uparrow & & \uparrow & & \\
 & & \ddots & \to & S^{a-1}V^*\otimes S^{b+1}V^*\otimes\Lambda^{c}V^* & & \to\dots
\end{array}$$

In the remaining cases $i\ge r_{\min}(g)$ and the complex from the corollary equals:

$$0\to S^{j-1}V^*\otimes H^{i,1}(g)\to\dots\to V^*\otimes H^{i,j-1}(g)\to H^{i,j}(g)\to H^{i,j}(\bar g)\to 0.$$

Again we have a decreasing filtration $F_\infty^{t,j-t}$ of $H^{i,j}$ with

$$F_\infty^{a,b}/F_\infty^{a+1,b-1}=E_\infty^{a,b}=H^{i,b}(\bar g)\otimes\Lambda^aV^*.$$

Thus the considered complex is filtered with all the consecutive quotients being exact and the claim follows.

6. Proof of Theorem B

We give an indirect proof, though a direct approach, similar to the one presented in Appendix B and using the Corollary, is plausible.

Recall that every system of PDEs of higher orders can be equivalently written as a system of first order equations. This is achieved via the map $\mathcal{E}\subset J^k(\pi)\hookrightarrow J^1(J^{k-1}(\pi))$. Let us call this composition map the equivalence reduction (er).

Example 3. The equation $u_{xy}=0$ on the plane $T=\mathbb{R}^2(x,y)$ is equivalent to the following system of the first order: $p_y=0$, $q_x=0$ ($p=u_x,q=u_y$). We will identify T with its tangent spaces and consider the corresponding symbolic system g on T. Let W be a proper subspace of T which equals neither $\mathbb{R}^1(x)$ nor $\mathbb{R}^1(y)$. The corresponding subspace $V^*=\operatorname{ann}(W)\subset T^*$ is weakly characteristic for the second order PDE, but is not for the equivalent first order system.

Thus the notion of weakly characteristic subspace is not invariant under the equivalence reductions. However we will show the strong characteristicity is well-posed.

Similarly, if g is a scalar ($\dim N=1$) symbolic system on T with $\dim T>1$ generated by one higher order PDE, then any 1-dimensional subspace V^* is weakly characteristic, while it is weakly (in this case also strongly) characteristic for the first order reduction of g iff the corresponding covector is characteristic.

On the algebraic level the above equivalence reduction is obtained via the embedding $\delta: S^kT^*\to S^{k-1}T^*\otimes T^*$, which induces the following correspondence:

$$S^kT^*\otimes N\supset g_k \ \rightsquigarrow\ \hat g_1=\operatorname{er}_k(g_k)\subset T^*\otimes(S^{k-1}T^*\otimes N).$$

More generally for $l\geq k$ the coupling map $S^{k-1}T\otimes S^lT^*\to S^{l-k+1}T^*$ yields:

$$S^lT^*\otimes N\supset g_l \ \rightsquigarrow\ \hat g_{l-k+1}=\operatorname{er}_k(g_l)\subset S^{l-k+1}T^*\otimes(S^{k-1}T^*\otimes N).$$

The map er_k acts on elements as follows:

$$\prod_{i=1}^m a_i\otimes\xi\mapsto\sum_{i_1,\dots,i_{k-1}}\frac{l!}{(l-k+1)!}\prod_{j\neq i_s}a_j\otimes(a_{i_1}\cdots a_{i_{k-1}})\otimes\xi$$

We shall show this correspondence is respected by the prolongation procedure, so that it descends to symbolic systems with $r_{\min}(g)=k$.

Lemma 10. *The subspaces* $\operatorname{er}_k(g_l^{(i)}),\operatorname{er}_k(g_l)^{(i)}\subset S^{l+i-k+1}T^*\otimes(S^{k-1}T^*\otimes N)$ *coincide for* $l\geq k$.

Proof. The following diagram commutes:

$$\begin{array}{ccc} S^lT^*\otimes N\otimes\Lambda^{j-1}T^* & \rightsquigarrow & S^{l-k+1}T^*\otimes(S^{k-1}T^*\otimes N)\otimes\Lambda^{j-1}T^*\\ \delta\downarrow & & \delta\downarrow\\ S^{l-1}T^*\otimes N\otimes\Lambda^jT^* & \rightsquigarrow & S^{l-k}T^*\otimes(S^{k-1}T^*\otimes N)\otimes\Lambda^jT^*.\end{array}$$

This suffices to check on generators:

$$\begin{array}{ccc} z_1^l\otimes\xi\otimes z_2\wedge\ldots\wedge z_j & \overset{\operatorname{er}_k}{\mapsto} & \frac{l!}{(l-k+1)!}z_1^{l-k+1}\otimes z_1^{k-1}\otimes\xi\otimes z_2\wedge\ldots\wedge z_j\\ \downarrow & & \downarrow\\ lz_1^{l-1}\otimes\xi\otimes z_1\wedge z_2\wedge\ldots\wedge z_j & \overset{\operatorname{er}_k}{\mapsto} & \frac{l!}{(l-k)!}z_1^{l-k}\otimes z_1^{k-1}\otimes\xi\otimes z_1\wedge z_2\wedge\ldots\wedge z_j.\end{array}$$

Thus the equivalence reduction and the prolongation commute. $\square$

Consequently we can define equivalence reduction of a symbolic system g. Let's denote the reduction $\operatorname{er}_k(g)$ of a symbolic system g by $\hat g$.

Proposition 11. *Let g be a symbolic system of minimal order $r_{min}(g) \geq k$. Then involutivity of the system g is equivalent to involutivity of the system $\hat{g}$.*

Proof. Consider at first a symbolic system g of pure order $\geq k$. We have the following isomorphism of complexes for $l \geq k$:

$$\begin{array}{ccc} \vdots & & \vdots \\ \delta\downarrow & & \delta\downarrow \\ g_{l+1}\otimes\Lambda^{j-1}T^* & \rightsquigarrow & \hat{g}_{l-k+2}\otimes\Lambda^{j-1}T^* \\ \delta\downarrow & & \delta\downarrow \\ g_l\otimes\Lambda^j T^* & \rightsquigarrow & \hat{g}_{l-k+1}\otimes\Lambda^j T^* \end{array}$$

Thus $H^{i,j}(g) = H^{i-k+1,j}(\hat{g})$, if $i > k-1$. But we can also consider the derived systems $g^{|s\rangle}$ instead of g and get the same conclusion. The claim follows. □

Lemma 12. *$V^* \subset T^*$ is strongly characteristic for g iff it is such for $\hat{g}$.*

Proof. If $g_k \cap S^k V^* \otimes N \neq 0$, then clearly $\hat{g}_1 \cap V^* \otimes (S^{k-1}V^* \otimes N) \neq 0$. On the other hand, if $\hat{g}_1 \cap V^* \otimes (S^{k-1}T^* \otimes N) \neq 0$, then the pairing of $S^{k-1}T \otimes N^*$ and some element $p \in g_k$ takes values in V^*, whence $g_k \cap S^k V^* \otimes N \neq 0$. □

Proposition 13. *The characteristic varieties of the systems g and $\hat{g}$ coincide.*

Proof. Actually, the characteristic variety $\mathrm{Char}^{\mathbb{C}}(g)$ is defined by the characteristic ideal $I(g) = \mathrm{ann}(g^*) \subset ST$ ([S]). But from the description of this ideal given in [KL$_2$] we see that $I(g) = I(\hat{g})$. □

Thus we have reduced Theorem B for pure order k systems to its partial case for $k = 1$, i.e. Theorem B of [G] (see Appendix B for the proof). If g is a general involutive system, then $g^{|k\rangle}$, $k \leq r_{\max}(g)$, is a pure order involutive system and already proved part of the statement implies the whole claim. □

Proof that (g involutive) ⇒ ($\bar{g}$ involutive). As noted in §4 this implication suffices to prove for pure order k systems. We do it by reducing to the case of first order.

By Proposition 11 the equivalence reduction $\hat{g}$ is an involutive first order system. Since V^* is strongly non-characteristic for g it is also such for $\hat{g}$ (to this side the claim is true: otherwise the subspace V^* is strongly characteristic for $\hat{g}$ and we apply Lemma 12). Thus by Guillemin's Theorem A the reduction $\bar{\hat{g}} \subset SW^* \otimes (S^{k-1}T^* \otimes N)$ is involutive.

Since V^* is strongly non-characteristic, the coefficients reduction $S^{k-1}T^* \otimes N \to S^{k-1}W^* \otimes N$ maps the system $\bar{\hat{g}}$ isomorphically onto its image, which is the equivalence reduction of the restricted system $\hat{\bar{g}} \subset SW^* \otimes (S^{k-1}W^* \otimes N)$. But the map eq_k does not change involutivity, and therefore the system $\bar{g}$ is involutive. □

7. Examples

Here we demonstrate that all assumptions in our results are essential. At first we consider the statement of Theorem B.

Example 4. Consider a system generated by a subspace $g_k \subset S^kT^*$ of codimension 1. One easily checks the system g is involutive (this follows immediately from the reduction theorem of [KL$_2$]). The set of characteristic covectors forms a hypersurface in $\mathbb{P}^{\mathbb{C}}T^*$ of degree k and so a generic covector is non-characteristic. However all 1-dimensional subspaces V^* are weakly characteristic if $k > 1$ and $\dim T > 2$ (while weakly non-characteristic subspaces are plentiful too).

Example 5. For the Laplace equation on T with $\dim T \geq 2$ all V^* of dimension 2 are strongly characteristic. But there are no real characteristics. Thus working over $\mathbb{C}$ is important.

Let $g \subset ST^*$ be a scalar system of *complete intersection type* ([KL$_2$]), which means that if $\operatorname{codim}(g) = t$ and $\dim T = n$, we have: $t \leq n$ and $\operatorname{codim}\operatorname{Char}^{\mathbb{C}}(g) = t$.

Example 6. Let $\dim T = n > 2$ and a symbolic system $g \subset ST^*$ of order 2 be given by n equations of complete intersection type. There are strongly characteristic subspaces V^* of dimension > 1, but the system is of finite type and hence is free of characteristics. Note that for this system

$$H^{0,0}(g) \simeq \mathbb{R}^1,\ H^{1,1}(g) \simeq \mathbb{R}^n, \dots, H^{i,i}(g) \simeq \mathbb{R}^{\binom{n}{i}}, \dots, H^{n,n}(g) \simeq \mathbb{R}^1. \qquad (\ddagger)$$

and so it is not involutive.

Now we study the situation of Theorem A. Consider a general symbolic complete intersection system g.

Proposition 14. *If a subspace $V^* \subset T^*$ is strongly non-characteristic for $g_{r_{max}(g)}$, then $\dim V^* \leq m(1) + 1$. The equality can be achieved only if the system has two orders $\operatorname{ord}(g) = \{1, k\}$ with multiplicities $m(1), m(k)$, satisfying $n = m(1) + m(k)$ (finite type) and either $k = 2$ or $n - m(1) = m(k) = 1$.*

Notice that if $m(1) \neq 0$, then strongly non-characteristic subspace V^* for g should satisfy the inequality $\dim V^* \leq m(1)$ and every such generic subspace is strongly non-characteristic. However if $m(1) = 0$ and $n > 1$, then the proposition implies that the system has pure order 2 and finite type (provided it is a complete intersection).

Proof. If $\operatorname{ord}(g) = \{1\}$, the claim is obvious. So consider the case with higher order equations $\operatorname{ord}(g) = \{1 < k_1 < \cdots < k_t = k\}$. Note that equations of the first order can be normalized to be $u_{x_{n-m(1)+1}} = 0, \dots, u_{x_n} = 0$ and these variables x_i can be excluded for considerations in higher orders.

Namely, we decompose $T = \tilde{T} \oplus U$, where $\tilde{T} = \mathbb{R}^{n-m(1)}(x_1, \dots, x_{n-m(1)})$ and $U = \mathbb{R}^{m(1)}(x_{n-m(1)+1}, \dots, x_n)$ with $g_k \subset S^k\tilde{T}^*$. Denote $\tilde{V} = V \cap \tilde{T}$. Then $g_k \cap \tilde{V}^* \cdot S^{k-1}\tilde{T}^* = 0$ and so $\sum m(k_i) \geq \operatorname{codim}(g_k \subset S^k\tilde{T}^*) \geq \dim \tilde{V}^* \cdot S^{k-1}\tilde{T}^*$.

We have $\dim \tilde{V} \geq \dim V - m(1)$ and if $\dim \tilde{V} > 0$, we get:

$$n - m(1) \geq \sum_{i=1}^{t} m(k_i) \geq \operatorname{codim} g_k \geq \dim \tilde{V}^* \cdot S^{k-1}\tilde{T}^* \geq \dim \tilde{V}^* \cdot \tilde{T}^* \geq n - m(1).$$

Thus we must have equalities everywhere and the claim follows. $\square$

Let us consider a scalar complete intersection system g, not of the first order (everything is clear), where the equality in Proposition 14 is achieved. Due to a remark before the proof we then restrict to a pure second order finite type system of complete intersection type.

We will use this example to show that Theorem A does not extend generally. The spectral sequence, having $E_1^{p,q}$ in a product form until $p+q \neq l-k+1$ as in §4, may seem to split and stabilize, but we will show the differentials $d_1^{p,q} \not\equiv 0$.

Example 7. Let a scalar symbolic system g be given by $g_0 = \mathbb{R}^1$, $g_1 = T^*$, $g_2 \subset S^2T^*$ of codimension n and complete intersection type, $g_{2+i} = g_2^{(i)}$. Then $\dim g_i = \binom{n}{i}$ and the non-zero cohomologies are listed in (‡).

Let V^* be a generic 1-dimensional subspace of T^* and $W = \operatorname{ann}(V^*)$. Then

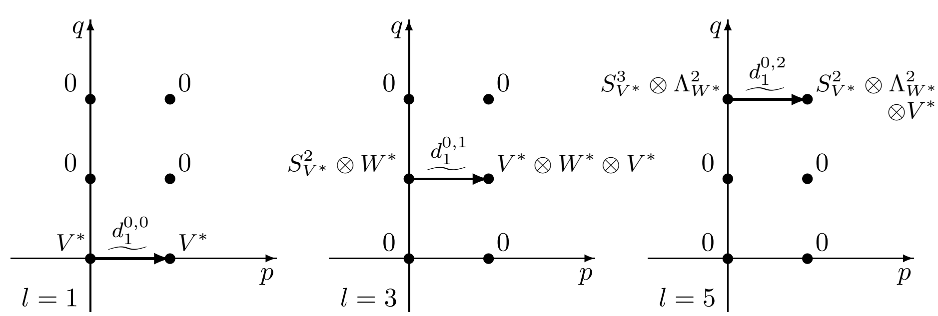
$$\bar{g}_0 = \mathbb{R}^1,\ \bar{g}_1 = W^*,\ \bar{g}_2 = S^2W^*,\ \bar{g}_{2+i} \subset S^{2+i}W^* \text{ has codimension } \tbinom{n+i}{i+2} - \tbinom{n}{i+2}.$$

The first non-trivial case from our point of view is $n = 3$ ($\dim W = 2$), which we consider in details (the case of arbitrary n is absolutely similar). In this case the only non-zero Spencer δ-cohomologies are:

$$H^{0,0}(g,\delta') = \mathbb{R}^1,\ H^{1,0}(g,\delta') = V^* \simeq \mathbb{R}^1,\ H^{1,1}(g,\delta') = V^* \otimes W^* \simeq \mathbb{R}^2,$$
$$H^{2,1}(g,\delta') = S^2V^* \otimes W^* \simeq \mathbb{R}^2,\ H^{2,2}(g,\delta') = S^2V^* \otimes \Lambda^2W^* \simeq \mathbb{R}^1,$$
$$H^{3,2}(g,\delta') = S^3V^* \otimes \Lambda^2W^* \simeq \mathbb{R}^1.$$

Thus the spectral sequences for l-th Spencer complex stabilize by graphical reasons at E_1 for even l $(= 2, 4, 6)$, converging to $H^{i,l-i}(g)$.

For odd l the following table describes the term $E_1^{p,q}$ (we compactify the notations $S^iV^* = S^i_{V^*}$ etc):



Thus $E_2^{p,q} = 0$ $(p+q = l)$ and we get stabilization at E_2.

This example shows that Lemmata 7, 9 are wrong in the non-involutive case (namely: $d_1^{0,s} \neq 0$ for $l = 2s+1$ and $\Xi^{1,2} \neq 0$).

Example 8. Let $T = N = \mathbb{R}^2 = \langle \partial_x, \partial_y \rangle$. Consider the symbolic system: $g_0 = N$, $g_1 = \mathrm{so}(2) = \langle \partial_y \otimes dx - \partial_x \otimes dy \rangle$, $g_{1+i} = g_1^{(i)} = 0$ for $i > 0$. The only non-zero cohomology are: $H^{0,0}(g) = \mathbb{R}^2$, $H^{0,1}(g) = \mathbb{R}^3$, $H^{1,2}(g) = \mathbb{R}$. Thus g does not satisfy I_1 and so is not involutive.

Let $W = \langle \partial_y \rangle$. Its annulator $V^* = \langle dx \rangle$ is a non-characteristic subspace for the system g (in this case weakly and strongly) . The restriction to W is equal to: $\bar{g}_0 = N$, $\bar{g}_1 = \langle \partial_x \otimes dy \rangle$, $\bar{g}_{1+i} = 0$. Thus $\bar{g}$ is not involutive in the pure order sense, but it is involutive with our definition of involutivity for multiple-order. The Spencer cohomology equal: $H^{0,0}(\bar{g}) = \mathbb{R}^2$, $H^{0,1}(\bar{g}) = \mathbb{R}^1$, $H^{1,1}(\bar{g}) = \mathbb{R}^1$. So $\bar{g}$ satisfies the properties I_1, I_2, but does not satisfy I_3 (though it satisfies a modified I_3', where we split not only the base T, but also the fiber N).

Therefore, we see that the second part of Theorem A does not hold for involutive system $\bar{g}$ of arbitrary orders. Also the formula of Theorem A does not hold for the systems g and $\bar{g}$ (for instance for $i = j = 1$).

8. Other results and a discussion

We deduce one more result from the spectral sequence of §4. A symbolic system g is called m-acyclic if $H^{i,j}(g) = 0$ for $i \notin \mathrm{ord}(g) - 1$ and $0 \leq j \leq m$. Involutivity corresponds to the case $m = \dim T$.

Theorem 15. *Let V^* be strongly non-characteristic for the symbolic system g of pure order k. Then g is m-acyclic iff $\bar{g}$ is m-acyclic.*

In a weak form this also generalizes to general symbolic systems.

Proof. The reasoning for the direct implication is the same as in the proof of Theorem A (see Lemma 7): We prove by induction on $l \geq k$ that $H^{l-j,j}(g, \delta') = 0$, $j \leq \min\{m, l-k\}$. The base of induction is obvious. Let us study at first the case $\dim V^* = 1$.

Consider the spectral sequence $E_r^{p,q}$ of the l-th Spencer complex. By induction hypothesis $E_1^{i,j} = 0$ for all $i \geq 1, j \leq m, l-k-1$. Thus all the differentials $d_r^{0,j}$ for $j < m, l-k$ vanish and since $H^{l-j,j}(g) = 0$ for $j \leq m, l-k$, we conclude that the only non-zero group among $E_1^{0,j}$ for $j \leq m, l-k$ can occur when $j = j_0 = \min\{m, l-k\}$ and only when the differential d_1^{0,j_0} is injective.

So suppose $E_1^{0,j_0} = H^{l-j_0,j_0}(g, \delta') \neq 0$. Then for the spectral sequence of $(l+1)$-st Spencer complex the group $E_1^{1,j_0} = H^{l-j_0,j_0}(g, \delta') \otimes V^* \neq 0$ and in order to have $H^{l-j_0,j_0}(g) = 0$ the group E_1^{1,j_0} should be killed by $d_1^{0,j_0} : E_1^{0,j_0} \to E_1^{1,j_0}$. Thus $E_1^{0,j_0} = H^{l+1-j_0,j_0}(g, \delta') \neq 0$. Continuing this process we obtain a sequence $H^{s-j_0,j_0}(g, \delta') \neq 0$, $s \to \infty$, which cannot happen by the δ-lemma.

Thus the claim is proved for $\dim V = 1$. When $\dim V = t > 1$ we can find a complete flag $\{0\} \subset V_1^* \subset \cdots \subset V_t^* = V^*$ of strongly non-characteristic subspaces and apply the previous arguments successively. Since again the corresponding terms Ξ vanish, we have $H^{l-j,j}(\bar{g}) = H^{l-j,j}(g, \delta') = 0$ for $j \leq m, l-k$.

The reverse statement follows directly from the spectral sequence of §4. □

An alternative approach to the direct implication is via the equivalence reduction and Proposition 2 of [GK], which is equivalent to our statement for the pure first order systems (or via the long diagram chase as in [G], p. 275).

In particular, 2-acyclicity is very important since obstructions for formal integrability belong to the groups $H^{i,2}(g)$. After some number of prolongations the system becomes m-acyclic, even involutive. The above result states that the places, where this stabilization happens, is the same for the systems g and $\bar{g}$.

Remark. *It is possible to consider vanishing of δ-cohomology on the other part of the spectrum: $H^{i,j}(g) = 0$ for $n - j \leq m$. This m-coacyclicity is not so wide-spread as m-acyclicity, but is closer in spirit to the notion of involutivity (because it implies existence of a quasi-regular sequence of length m, see also Appendix A). Then by similar methods one proves for $i \geq k$:*

$$H^{i,j}(g) = 0 \ \forall j \geq n - m \Leftrightarrow H^{i,j}(\bar{g}) = 0 \ \forall j \geq n - m.$$

Involutive systems became one of the most important classes of PDEs during the profound investigation of differential equations compatibility and integrability problem at the beginning of the last century ([C, J, V]). However Cohen-Macaulay systems, being very important in the commutative algebra, were introduced into differential equations context quite recently in [KL_2]. Our frequent example (§7) of complete intersections is a partial case.

Recall that projection to the r.h.s. in the formula (†) determines the restriction $\bar{g}$ of a symbolic system g, while intersection with the l.h.s. yields the reduction $\tilde{g}$. Recall also ([KL_2]) that a subspace V^* is called transversal if its complexification is transversal to the characteristic variety $\text{Char}^{\mathbb{C}}(g)$.

Now we wish to compare these two classes of symbolic systems. The following table shows an apparent duality between them:

Involutive systems	Cohen-Macaulay systems
Restriction $\bar{g}$ of an involutive g to a strictly non-characteristic subspace $W \subset T$ is involutive	Reduction $\tilde{g}$ of a Cohen-Macaulay g to a transversal subspace $V^* \subset T^*$ is Cohen-Macaulay
Restriction induces an isomorphism of symbolic systems $g \simeq \bar{g}$, but not δ-cohomologies $H^{i,j}(g) \neq H^{i,j}(\bar{g})$	Reduction yields an isomorphism of δ-cohomologies $H^{i,j}(g) \simeq H^{i,j}(\tilde{g})$, but not symbolic systems $g \neq \tilde{g}$
The Spencer cohomology $H^{*,*}(g)$ is a free ΛV^*-module, and change of coefficients $SW \subset ST$ induces an isomorphism of the module g^*	The symbolic module g^* is a free SW-module, and coefficients change $\Lambda V^* \subset \Lambda T^*$ induces an isomorphism of the cohomology $H^{*,*}(g)$
Provides a canonical Koszul resolution of the symbolic module g^*	Provides an effective calculation of the Spencer groups $H^{i,j}(g)$

The statements gathered here are combinations of results from [G], [KL$_2$] and the present papers.

A. Descended symbolic systems

Here we describe a new property of involutive systems, not related to the main topic of the paper, but still important in our discussion.

Given a subspace $g_k \subset S^kT^* \otimes N$ we can consider its *descended* subspace $\partial g_k = \langle \delta_v p \,|\, v \in T, p \in g_k \rangle \subset S^{k-1}T^* \otimes N$.

If $g = \{g_k\}$ is a symbolic system, then $\partial g_k \subset g_{k-1}$.

Proposition 16. *Let $n = \dim T$. Then $H^{i,n}(g) = 0$ iff $g_i = \partial g_{i+1}$.*

Proof. This follows from exactness of the sequence:

$$g_{i+1} \otimes \Lambda^{n-1}T^* \longrightarrow \partial g_{i+1} \otimes \Lambda^n V^* \to 0. \qquad \square$$

We define the *descended* system ∂g by the rule: $(\partial g)_k = \partial g_{k+1}$.

Lemma 17. *The descender ∂g is a symbolic system.*

Proof. Let $v \in T$ and $p \in \partial g_{k+1}$. Then $p = \sum \delta_{w_i} q_i$ for some $q_i \in g_{k+1}$, $w_i \in T$. We have: $\delta_v p = \sum \delta_v \delta_{w_i} q_i = \sum \delta_{w_i}(\delta_v q_i) \in \partial g_k$. $\square$

Proposition 18. *$H^{i,n}(g) = 0$ for $i \geq k$ implies $H^{i-1,n}(\partial g) = 0$ for $i \geq k$.*

Proof. By a Bourbaki's lemma for the symbolic module g^* (see Serre's letter in [GS]) the first property is equivalent to the existence of $v \in T$ such that $\delta_v : g_{i+1} \to g_i$ is epimorphic for $i \geq k$. But then $\delta_v : \partial g_{i+1} \to \partial g_i$ is epimorphic too and the claim follows. $\square$

Thus the descender ∂g characterizes vanishing of certain cohomologies. Recall that this vanishing is closely related to the existence of quasi-regular elements ([GS]) and so to involutivity. In fact, the descended system behave nicely w.r.t. involutivity property:

Theorem 19. *If g is an involutive system, so is it descender ∂g.*

Proof. The statement suffices to check for the systems of pure order k. Consider the commutative diagram of δ-sequences:

$$\begin{array}{ccccc}
\ddots & \vdots & & \vdots & \\
\cdots \to & g_{k+2} \otimes \Lambda^i T^* \otimes \Lambda^{n-2} T^* & \longrightarrow & g_{k+1} \otimes \Lambda^{i+1} T^* \otimes \Lambda^{n-2} T^* & \to \dots \\
 & \downarrow & & \downarrow & \\
\cdots \to & g_{k+1} \otimes \Lambda^i T^* \otimes \Lambda^{n-1} T^* & \longrightarrow & g_k \otimes \Lambda^{i+1} T^* \otimes \Lambda^{n-1} T^* & \to \dots \\
 & \downarrow & & \downarrow & \\
\cdots \to & \partial g_{k+1} \otimes \Lambda^i T^* \otimes \Lambda^n T^* & \longrightarrow & \partial g_k \otimes \Lambda^{i+1} T^* \otimes \Lambda^n T^* & \to \dots \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

The columns are exact (in the symbolic grading $\geq k$). The rows are also exact, save for the bottom one. So this latter is exact too (in the symbolic grading $\geq k-1$) by the standard diagram chase. $\square$

Though $\partial g \subset g$, the equality does not hold even for involutive systems. For instance, the Frobenius type system $g_{k-1} = S^{k-1}T^* \otimes N$, $g_k = 0$, exhibits the strict inclusion. However, it is the only finite type involutive system of pure order and one can expect that modulo sums with such systems other involutive systems enjoy the considered property.

In any case, either $\partial g = g$ or we have a canonical sub-system, which can be considered as an intermediate integral of the system. Such integrals are known to facilitate the integration ([KL$_2$]).

We can iterate the procedure and construct further descended systems: $g \supset \partial g \supset \partial^2 g \supset \dots$. This systems of strict inclusions necessary terminates and in a finite number of steps we get the least descender $\partial^\infty g$.

B. Guillemin's theorem on characteristics

For completeness we provide here a proof of Theorem B from [G], following the author's ideas, but with a different exposition and the coordinate-free approach. Namely we prove that a characteristic subspace contains a characteristic covector (over $\mathbb{C}$), the inverse statement being obvious.

Let g be a first order symbolic system and V^* a characteristic subspace, which in this case is the same as a strongly characteristic subspace. Let V_0^* be its maximally non-characteristic subspace. Shrinking V^* if necessary we can assume that codimension of V_0^* in V^* is one.

Let $\omega \in V^* \setminus V_0^*$. We assume also that ω is not a characteristic covector, for otherwise we are done. Denote $N' = [\omega \otimes N \cap (V_0^* \otimes N + g)]/\omega$. Since $V_0^* \otimes N \cap g = 0$, we get a well-defined map:

$$\lambda : N' \to V_0^* \otimes N, \qquad \xi \mapsto \omega \otimes \xi \mod g.$$

Proposition 20. (i) $\mathrm{Im}(\lambda) \subset V_0^* \otimes N'$.
(ii) *The following λ-generated sequence is a complex:*

$$0 \to N' \xrightarrow{\lambda} V_0^* \otimes N' \xrightarrow{\lambda} \Lambda^2 V_0^* \otimes N' \to \dots$$

Proof. (i) Let $\lambda_v(\xi) = \langle \lambda(\xi), v \rangle$, $v \in V_0 = \mathrm{ann}(\omega) \subset V$, be the map $\lambda_v : N' \to N$. We have:

$$\begin{array}{ccccc} & & V_0^* \otimes T^* \otimes N & & \\ & & \rho \downarrow & & \\ 0 \to S^2 V_0^* \otimes N & \xrightarrow{\varphi_0} & V_0^* \otimes H^{0,1}(g) & \xrightarrow{\varphi_1} & H^{0,2}(g) \end{array}$$

The vertical map is epimorphic and for $\omega \otimes \lambda(\xi) \in V_0^* \otimes T^* \otimes N$ we have: $\varphi_1 \circ \rho(\omega \otimes \lambda(\xi)) = 0$. From exactness of the horizontal complex (Quillen's Theorem, see our Corollary) we get: $\omega \otimes \lambda_v(\xi) \in i_v S^2 V_0^* \otimes N = V_0^* \otimes N \mod g$,

i.e. $\lambda_v(\xi) \in N' \; \forall v \in V_0$.
(ii) This claim follows from the identity $\omega \wedge \omega = 0$. □

Lemma 21. *There exists a non-zero vector $\xi_0 \in N'$ such that $\lambda(\xi_0) \in V_0^* \otimes \xi_0$.*

Proof. This equivalently means that the set of commuting (by (ii) above) linear operators has a common eigenvector (over $\mathbb{C}$). □

Now $\omega \otimes \xi_0 = p \otimes \xi_0 \mod g$ for some covector $p \in V_0^*$, i.e. the covector $\bar{p} = \omega - p \in V^*$ is characteristic: $\bar{p} \in \operatorname{Char}^{\mathbb{C}}(g)$.

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