Linearizability of d-webs, $d \geq 4$, on two-dimensional manifolds

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February 1, 2008

Abstract

We find $d-2$ relative differential invariants for a d-web, $d \geq 4$, on a two-dimensional manifold and prove that their vanishing is necessary and sufficient for a d-web to be linearizable. If one writes the above invariants in terms of web functions $f(x,y)$ and $g_4(x,y), ..., g_d(x,y)$, then necessary and sufficient conditions for the linearizabilty of a d-web are two PDEs of the fourth order with respect to f and g_4, and $d-4$ PDEs of the second order with respect to f and $g_4, ..., g_d$. For $d=4$, this result confirms Blaschke's conjecture on the nature of conditions for the linearizabilty of a 4-web. We also give Mathematica codes for testing 4- and d-webs ($d>4$) for linearizability and examples of their usage.

0 Introduction

Let W_d be a d-web given by d one-parameter foliations of curves on a two-dimensional manifold M^2. The web W_d is linearizable (rectifiable) if it is equivalent to a linear d-web, i.e., to a d-web formed by d one-parameter foliations of straight lines on a projective plane.

The problem of the linearizability of webs was posed by Blaschke in the 1920s (see, for example, his book [2], §17 and §42) who claimed that it is hopeless to find such a criterion because of the complexity of calculations involving high order jets. Blaschke in [2] (§ 42) formulated the problems of finding conditions for the linearizability of 3-webs (§ 17) and 4-webs (§ 42) given on M^2. Comparing the numbers of absolute invariants for a general 3-web W_3 (a general 4-web W_4) and a linear 3-web (a linear 4-web), Blaschke made the conjectures that conditions of linearizability for a 3-web W_3 should consist of four relations for the 9th order web invariants (4 PDE of 9th order) and those for a 4-web W_4 should consist of two relations for the 4th order web invariants (2 PDE of 4th order) .

A criterion for linearizability is very important in web geometry and in its applications. It is also important in applications to nomography (see [2], §17 and [3], §18).

A new approach for finding conditions of linearizability for webs on the plane has been proposed by Akivis (1973) in his talk at the Seminar on Classical Differential Geometry in Moscow State University. Goldberg [6] implemented this approach for 3-webs. The goals of the authors of the paper [8] were to find linearizability conditions for a 3-web W_3 and to improve Bol's and Boruvka's result related to the Gronwall conjecture. For the formulation of the Gronwall conjecture, the statement of the results of Bol and Boruvka and references to their works see [2], §17.

In this paper we use Akivis' approach to establish a criterion of linearizability for d-webs, $d \geq 4$. The results of the present paper do not rely on the results or methods of the paper [8] mentioned above. We prove that the Blaschke conjecture was correct: a 4-web W_4 is linearizable if and only its two 4th order invariants vanish. In terms of the invariants defining the geometry of a 4-web W_4, the vanishing of these two invariants means that the covariant derivatives K_1 and K_2 of the web curvature K are expressed in terms of the curvature K itself, the basic web invariant a and its covariant derivatives up to the 3rd order. We find explicit expressions for these invariants in terms of symmetrized covariant derivatives. Note that expressions for these invariants in terms of web functions contain 262 terms each. After this paper was submitted, one of the authors used the conditions of linearizability described above to check whether numerous known classes of 4-webs are linearizable (see [7]).

The results obtained in this paper give a complete solution of the linearizability problem for d-webs, $d \geq 4$, and provide tests for establishing linearizability of such webs. In particular, for 4-webs W_4, our results provide a complete solution of the longstanding problem posed by Blaschke (see, for example [2], §42).

We also investigate the linearizability of d-webs W_d for $d \geq 5$. In this case the linearizability conditions involve $d-2$ differential invariants. Two of them have order 4 and the rest are of order 2.

All computations in this paper were done manually, and the more routine ones (for example, equations (13), (14), 15) and the formulas for K_1 and K_2 in Section 2.3.4) were checked by Mathematica package. At the end of the paper, we provide the Mathematica codes for testing 4- and d-webs, $d > 4$, for linearizability and examples of their usage. The material in Section 4 (tests and examples) essentially relies on using Mathematica.

Note that a different approach to the linearizability problem for webs W_d for $d \geq 4$ was used by Hènaut in [9]. However, Hènaut did not find conditions in the form suggested by Blaschke. His conditions do not contain web invariants.

1 Basics Constructions

We recall main constructions for 3-webs on 2-dimensional manifolds (see, for example, [3] or [2] , or [6]) in a form suitable for us.

Let M^2 be a 2-dimensional manifold, and suppose that a 3-web W_3 is given by three differential 1-forms ω_1, ω_2, and ω_3 such that any two of them are linearly independent.

Proposition 1 *The forms $\omega_1, \omega_2,$ and ω_3 can be normalized in such a way that the normalization condition*

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{1}$$

holds.

Proof. In fact, if we take the forms ω_1 and ω_2 as cobasis forms of M^2, then the form ω_3 is a linear combination of the forms ω_1 and ω_2 :

$$\omega_3 = \alpha\omega_1 + \beta\omega_2 \,,$$

where $\alpha, \beta \neq 0$.

After the substitution

$$\omega_1 \to \frac{1}{\alpha}\omega_1, \; \omega_2 \to \frac{1}{\beta}\omega_2, \; \omega_3 \to -\omega_3$$

the above equation becomes (1). ■

It is easy to see that any two of such normalized triplets $\omega_1, \omega_2, \omega_3$ and $\omega_1^s, \omega_2^s, \omega_3^s$ determine the same 3-web W_3 if and only if

$$\omega_1^s = s\omega_1, \; \omega_2^s = s\omega_2, \; \omega_3^s = s\omega_3 \tag{2}$$

for a non-zero smooth function s.

We will investigate local properties of W_3. Thus we can assume that M^2 is a simply connected domain of $\mathbb{R}^2$, and therefore there exists a smooth function f such that ω_3 is proportional to df, that is, $\omega_3 \wedge df = 0$. The function f is called a *web function.* Note that this function is defined up to renormalization $f \longmapsto F(f)$.

We choose such a representation of W that

$$\omega_3 = df. \tag{3}$$

Similarly we find smooth functions x and y for forms ω_1 and ω_2 such that

$$\omega_1 = adx, \; \omega_2 = bdy$$

for some smooth functions a and b.

Moreover, functions x and y are independent and therefore can be viewed as (local) coordinates. In these coordinates the normalization condition gives

$$\omega_1 = -f_x dx, \; \omega_2 = -f_y dy, \; \omega_3 = df.$$

Let the vector fields ∂_1 and ∂_2 form the basis dual to the cobasis ω_1, ω_2, i.e., $\omega_i(\partial_j) = \delta_{ij}$ for $i, j = 1, 2$.

Then

$$\partial_1 = -\frac{1}{f_x}\frac{\partial}{\partial x}, \quad \partial_2 = -\frac{1}{f_y}\frac{\partial}{\partial y}$$

and

$$dv = \partial_1(v)\;\omega_1 + \partial_2(v)\;\omega_2 \tag{4}$$

for any smooth function v.

1.1 Structure Equations

From now on we shall assume that a 3-web W_3 is given by differential 1-forms ω_1, ω_2, and ω_3 normalized by conditions (1) and (3).

Since on a two-dimensional manifold the exterior differentials $d\omega_1$ and $d\omega_2$ as 2-forms differ from the 2-form $\omega_1 \wedge \omega_2$ only by factors, we get $d\omega_1 = h_1\ \omega_1 \wedge \omega_2$ and $d\omega_2 = h_2\ \omega_2 \wedge \omega_1$ for some functions h_1 and h_2.

By $d\omega_3 = 0$, one gets $h_1 = h_2$. Denote this function by H. Then $d\omega_1 = H\omega_1 \wedge \omega_2$ and $d\omega_2 = H\omega_2 \wedge \omega_1$ or

$$d\omega_1 = \omega_1 \wedge \gamma, \quad d\omega_2 = \omega_2 \wedge \gamma, \tag{5}$$

where

$$\gamma = -H\omega_3. \tag{6}$$

We call relations (5) the *first structure equations* of the 3-web W_3. In terms of the web function f, one has

$$\gamma = -\frac{f_{xy}}{f_x f_y}\omega_3$$

and

$$H = \frac{f_{xy}}{f_x f_y}.$$

If we change the representative according to (2), then the first structure equations take the form

$$d\omega_p^s = \omega_p^s \wedge \gamma^s,\ p = 1, 2, 3,$$

where

$$\gamma^s = \gamma - d\log(s)$$

It follows that $d\gamma^s = d\gamma$.

One has

$$d\gamma = K\omega_1 \wedge \omega_2. \tag{7}$$

This equation is called *the second structure equation of the web,* and the function K is called the *web curvature.*

If we put $d\gamma^s = K^s\omega_1^s \wedge \omega_2^s$, then $K^s = s^{-2}K$. Therefore the curvature function K is a relative invariant of weight 2.

In terms of the web function f, one has

$$K = -\frac{1}{f_x f_y}\left(\log\left(\frac{f_x}{f_y}\right)\right)_{xy} \tag{8}$$

(cf.[2], § 9, or [1], p. 43).

For the basis vector fields ∂_1 and ∂_2, the structure equations take the form

$$[\partial_1, \partial_2] = H\ (\partial_2 - \partial_1). \tag{9}$$

where [,] is the commutator of vector fields.

Substituting (6) into (7), one gets $d\gamma = dH \wedge \omega_1 + \omega_2)$,and from (4) it follows that

$$K = \partial_1(H) - \partial_2(H). \tag{10}$$

1.2 The Chern Connection

Let us use the differential 1-form γ to define a connection in the cotangent bundle $\tau^* : T^*M \rightarrow M$ by the following covariant differential:

$$d_\gamma : \Lambda^1(M) \rightarrow \Lambda^1(M) \otimes \Lambda^1(M),$$

where

$$\begin{aligned} d_\gamma(\omega_1) &= -\omega_1 \otimes \gamma, \\ d_\gamma(\omega_2) &= -\omega_2 \otimes \gamma; \end{aligned}$$

and $\otimes$ denotes the tensor product.

In what follows we shall denote by $\Lambda^p(M)$, $p = 1, 2$, the modules of smooth differential p-forms on M.

It is easy to check that the curvature form of the above connection is equal to $-d\gamma$, that is, $d_\gamma^2 : \Lambda^1(M) \rightarrow \Lambda^1(M) \otimes \Lambda^2(M)$ is the multiplication by $-d\gamma$:

$$d_\gamma^2(\omega) = -\omega \otimes d\gamma$$

for any differential form $\omega \in \Lambda^1(M)$.This connection is called the *Chern connection* of the web.

It is also easy to check that the Chern connection satisfies the relations

$$d_\gamma(\omega_i^s) = -\omega_i^s \otimes \gamma^s$$

for $i = 1, 2$, and any non-zero smooth function s. The straightforward computation shows also that d_γ is a torsion-free connection.

Recall (see, for example, [12], p. 128) that for the covariant differential $d_\nabla : \Lambda^1(M) \rightarrow \Lambda^1(M) \otimes \Lambda^1(M)$ of any torsion-free connection ∇, one has $d_\nabla = d_\gamma - T$, where

$$T : \Lambda^1(M) \rightarrow S^2(M) \subset \Lambda^1(M) \otimes \Lambda^1(M)$$

is the *deformation tensor* of the connection, and $S^2(M)$ is the module of the symmetric 2-tensors on M.

Below we shall use the notation $\nabla_X(\theta) \stackrel{\text{def}}{=} (d_\nabla\theta)(X)$ for the covariant derivative of a differential 1-form θ along vector field X with respect to connection ∇.

Proposition 2 *Let $d_\nabla : \Lambda^1(M) \rightarrow \Lambda^1(M) \otimes \Lambda^1(M)$ be the covariant differential of a connection ∇ in the cotangent bundle of M. Then a foliation $\{\theta = 0\}$ on M given by the differential 1-form $\theta \in \Lambda^1(M)$ consists of geodesics of ∇ if and only if*

$$d_\nabla(\theta) = \alpha \otimes \theta + \theta \otimes \beta$$

for some differential 1-forms $\alpha, \beta \in \Omega^1(M)$.

Proof. Let θ' be a differential 1-form such that θ and θ' are linearly independent.

Then

$$d_\nabla(\theta) = \alpha \otimes \theta + \theta \otimes \beta + h\theta' \otimes \theta'.$$

Assume that X is a geodesic vector field on M such that $\theta(X) = 0$. Then $\nabla_X(\theta)$ must be equal to zero on X. But

$$d_\nabla\theta(X) = \beta(X)\,\theta + h\theta'(X)\,\theta'.$$

Therefore, $h = 0$. ■

Corollary 3 *Foliations* $\{\omega_1 = 0\}, \{\omega_2 = 0\},$ *and* $\{\omega_3 = 0\}$ *are geodesic with respect to the Chern connection.*

1.3 Akivis–Goldberg Equations

The problem of linearization of webs can be reformulated as follows: *find a torsion-free flat connection such that the foliations of the web are geodesic with respect to this connection.*

Proposition 4 *Let* $d_\nabla = d_\gamma - T : \Lambda^1(M) \to \Lambda^1(M) \otimes \Lambda^1(M)$ *be the covariant differential of a torsion-free connection* ∇ *such that the foliations* $\{\omega_p = 0\},\ p = 1, 2, 3,$ *are geodesic. Then*

$$\begin{array}{l} T(\omega_1) = T^1_{11}\omega_1 \otimes \omega_1 + T^1_{12}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1), \\ T(\omega_2) = T^2_{22}\omega_2 \otimes \omega_2 + T^2_{12}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1), \end{array} \tag{11}$$

where the components of the deformation tensor have the form

$$T^2_{12} = \lambda_1, \quad T^1_{12} = \lambda_2,\ T^1_{11} = 2\lambda_1 + \mu,\ T^2_{22} = 2\lambda_2 - \mu \tag{12}$$

for some smooth functions $\lambda_1, \lambda_2,$ *and* μ.

Proof. Due to (2) and the requirement that the foliations $\{\omega_1 = 0\}$ and $\{\omega_2 = 0\}$ are geodesic, one gets (11). The same requirement for the foliation $\{\omega_3 = 0\}$ gives the following relation for the components of the deformation tensor T :

$$T^1_{11} + T^2_{22} = 2(T^1_{12} + T^2_{12}),$$

and this implies (12). ■

Therefore, in order to linearize the 3*-web, one should find functions* λ_1, λ_2 *and* μ *in such a way that the connection corresponding to* $d_\gamma - T$*, where the deformation tensor* T *has form* (12)*, is flat.*

Let us denote by ∇_i the covariant derivatives along $\partial_i, i = 1, 2,$ with respect to the connection ∇ and by

$$R : \Lambda^1(M) \to \Lambda^1(M)$$

the curvature tensor .

From the standard formula for the curvature $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (see, for example, [11], p. 133) and (9) we find that

$$R(\omega) = [\nabla_1, \nabla_2](\omega) + H\ (\nabla_1 - \nabla_2)(\omega)$$

for any $\omega \in \Lambda^1(M)$.

It follows from the above proposition that for the connection corresponding to $d_\gamma - T$ we get

$$\begin{aligned}
\nabla_1(\omega_1) &= -(2\lambda_1 + \mu + H)\ \omega_1 - \lambda_2\ \omega_2,\\
\nabla_1(\omega_2) &= -(\lambda_1 + H)\ \omega_2,\\
\nabla_2(\omega_1) &= -(\lambda_2 + H)\ \omega_1,\\
\nabla_2(\omega_2) &= -\lambda_1\ \omega_1 - (2\lambda_2 - \mu + H)\ \omega_2.
\end{aligned}$$

and

$$\begin{aligned}
R(\omega_1) &= (2\partial_2(\lambda_1) - \partial_1(\lambda_2) + \partial_2(\mu) - H(2\lambda_1 - \lambda_2 + \mu) - \lambda_1\lambda_2 - K)\ \omega_1 +\\
&\quad (\partial_2(\lambda_2) + \lambda_2(-H - \lambda_2 + \mu))\ \omega_2,\\
R(\omega_2) &= (-\partial_1(\lambda_1) + \lambda_1(H + \lambda_1 + \mu))\ \omega_1 +\\
&\quad (\partial_2(\lambda_1) - 2\partial_1(\lambda_2) + \partial_1(\mu) - H(\lambda_1 - 2\lambda_2 + \mu) + \lambda_1\lambda_2 - K)\ \omega_2
\end{aligned}$$

Therefore, in order to obtain a flat torsion-free connection, components of the deformation tensor must satisfy the following *Akivis-Goldberg equations*

$$R(\omega_1) = 0,\ R(\omega_2) = 0. \tag{13}$$

Since ω_1 and ω_2 are linearly independent, equations (13) imply that

$$\begin{aligned}
2\partial_2(\lambda_1) - \partial_1(\lambda_2) + \partial_2(\mu) - H(2\lambda_1 - \lambda_2 + \mu) - \lambda_1\lambda_2 - K &= 0,\\
\partial_2(\lambda_2) + \lambda_2(-H - \lambda_2 + \mu) &= 0,\\
-\partial_1(\lambda_1) + \lambda_1(H + \lambda_1 + \mu) &= 0,\\
\partial_2(\lambda_1) - 2\partial_1(\lambda_2) + \partial_1(\mu) - H(\lambda_1 - 2\lambda_2 + \mu) + \lambda_1\lambda_2 - K &= 0.
\end{aligned}$$

Resolving the system with respect to the derivatives of λ_1 and λ_2, we obtain the following system of PDEs:

$$\begin{aligned}
\partial_1(\lambda_1) &= \lambda_1(H + \lambda_1 + \mu),\\
\partial_2(\lambda_1) &= \frac{K}{3} + H\left(\lambda_1 + \frac{\mu}{3}\right) + \lambda_1\lambda_2 + \frac{1}{3}\partial_1(\mu) - \frac{2}{3}\partial_2(\mu),\\
\partial_1(\lambda_2) &= -\frac{K}{3} + H\left(\lambda_2 - \frac{\mu}{3}\right) + \lambda_1\lambda_2 + \frac{2}{3}\partial_1(\mu) - \frac{1}{3}\partial_2(\mu),\\
\partial_2(\lambda_2) &= \lambda_2(H + \lambda_2 - \mu).
\end{aligned}$$

We shall look at the above system as a system of partial differential equations with respect to the functions λ_1 and λ_2 provided that μ is given.

We get the compatibility conditions for this system from structure equations (9) for λ_1 and λ_2 presented in the form

$$\partial_1(\partial_2(\lambda_i)) - \partial_2(\partial_1(\lambda_i)) + H(\partial_1(\lambda_i) - \partial_2(\lambda_i)) = 0,$$

where $i = 1, 2$.

After a series of long and straightforward computations, we obtain the following two compatibility equations:

$$I_1(\mu) = 0, \; I_2(\mu) = 0, \tag{14}$$

where $I_1(\mu)$ and $I_2(\mu)$ have the form

$$I_1(\mu) = -\partial_1^2(\mu) + 2\partial_1\partial_2(\mu) + (\mu + H)\partial_1(\mu) - 2(2H + \mu)\partial_2(\mu)$$
$$+H\mu^2 + (2H^2 - \partial_2(H))\mu - \partial_1(K) + 2HK$$

and

$$I_2(\mu) = -\partial_2^2(\mu) + 2\partial_1\partial_2(\mu) + 2(\mu - H)\partial_1(\mu) - (H + \mu)\partial_2(\mu) - H\mu^2$$
$$+\left(2H^2 - \partial_1(H)\right)\mu - \partial_2(K) + 2HK.$$

We sum up these results in the following

Theorem 5 *The Akivis-Goldberg equations as differential equations with respect to the components $T^1_{12} = \lambda_2$ and $T^2_{12} = \lambda_1$ of the deformation tensor T are compatible if and only if the component μ satisfies the following differential equations:*

$$I_1(\mu) = 0, \; I_2(\mu) = 0.$$

If the above conditions (14) are valid, then the system (13) of PDEs is the Frobenius-type system, and for given values $\lambda_1(x_0)$ and $\lambda_2(x_0)$ at a point $x_0 \in M$, there is (a unique) smooth solution of the system in some neighborhood of x_0.

It is worthwhile to note the peculiarity of the Akivis-Goldberg system of differential equations and our presentation of components of the deformation tensor. This is a non-linear overdetermined system with respect to components $\lambda_1, \lambda_2, \mu$ of the deformation tensor, but the compatibility conditions in our case depend on μ only while for general systems they depend on all components of the deformation tensor. This gives us a method to find the linearizability conditions in a constructive way.

2 Linearizability of 4-Webs

2.1 The Basic Invariant of a 4-Web

A 4-web W_4 on M^2 can be defined by 4 differential 1-forms $\omega_1, \omega_2, \omega_3$, and ω_4 such that any two of them are linearly independent.

We prove the following proposition:

Proposition 6 *The forms $\omega_1, \omega_2, \omega_3$, and ω_4 can be normalized in such a way that the normalization condition* (1) *holds for the first three of them, and in addition, the following condition holds for the forms ω_1, ω_2, and ω_4:*

$$\omega_4 + a\omega_1 + \omega_2 = 0, \tag{15}$$

where a is a nonzero function.

Proof. In fact, if we take the forms ω_1 and ω_2 as cobasis forms of M^2, then the forms ω_3 and ω_4 are linearly expressed in terms of ω_1 and ω_2 :

$$\omega_3 = \alpha\omega_1 + \beta\omega_2\,,$$
$$\omega_4 = \alpha'\omega_1 + \beta'\omega_2\,,$$

where $\alpha, \beta, \alpha', \beta' \neq 0$, $\alpha \neq \alpha'$, $\alpha\beta' - \alpha'\beta \neq 0$.

Making the substitution

$$\omega_1 \to -\frac{1}{\alpha}\omega_1,\ \omega_2 \to \frac{1}{\beta}\omega_2,\ \omega_3 \to -\omega_3,\ \omega_4 \to -\frac{\beta'}{\beta}\omega_4,$$

we get (1) and (15) with $a = \frac{\alpha'\beta}{\beta'\alpha}$. ■

Note that $a \neq 0, 1$. Moreover, the value $a\,(x)$, $x \in M$, of the function a is the cross-ratio of the four tangents to the lines in $T_x^*(M^2)$ generated by the covectors $\omega_{1,x}, \omega_{2,x}, \omega_{3,x}$, and $\omega_{4,x}$, and therefore is an invariant of the 4-web. The function a is called the *basic invariant* of the 4-web (see [4] and [5], pp. 302–303).

2.2 The Expression for μ

We shall consider a 4-web $\langle\omega_1, \omega_2, \omega_3, \omega_4\rangle$ as the 3-web $\langle\omega_1, \omega_2, \omega_3\rangle$ and an extra foliation given by form ω_4 which satisfies (15). Moreover, by the Chern connection, the curvature, etc. that we discussed above for a 3-web we shall mean the corresponding constructions for the 3-web $\langle\omega_1, \omega_2, \omega_3\rangle$.

Theorem 7 *Let ∇ be a torsion-free connection in the cotangent bundle $\tau^* : T^*M \to M$ such that the foliations $\{\omega_1 = 0\}, \{\omega_2 = 0\}, \{\omega_3 = 0\}$, and $\{\omega_4 = 0\}$ are geodesic for ∇. Then the components of the deformation tensor T have the form* (12) *and*

$$\mu = \frac{\partial_1\,(a) - a\partial_2\,(a)}{a - a^2}. \tag{16}$$

Proof. Let $d_\nabla = d_\gamma - T$ be the covariant differential of the connection ∇. Then (15) gives

$$-d_\nabla\,(\omega_4) = \omega_1 \otimes da - \omega_4 \otimes \gamma - aT\,(\omega_1) - T\,(\omega_2)\,.$$

If $\omega_4 = 0$, then $\omega_2 = -a\omega_1$, and the right-hand side takes the form

$$\left(\partial_1(a) - a\partial_2(a) + \mu\left(a^2 - a\right)\right)\omega_1 \otimes \omega_1.$$

Therefore, this tensor equals zero if and only if equation (16) holds. ■

Formula (16) shows that the quantity μ occurring in expressions (12) of the components of the deformation tensor, is expressed in terms of the basic invariant a and its derivatives. Namely this fact made it possible to express the linearizability conditions for 4-webs in terms of 4th order jets and solve the linearizability problem for 4-webs without use of computers.

3 Differential Invariants of 4-Webs

For the values of the operators $I_1(\mu)$ and $I_2(\mu)$ on the function $\mu = (\partial_1(a) - a\partial_2(a))/(a - a^2)$, we introduce the following operators:

$$I_1^0(f, a) = I_1\left(\frac{\partial_1(a) - a\partial_2(a)}{a - a^2}\right)$$

and

$$I_2^0(f, a) = I_2\left(\frac{\partial_1(a) - a\partial_2(a)}{a - a^2}\right).$$

These are differential operators of order three in the basic invariant a and of order four in the web function f. If they are equal to zero, then μ satisfies the conditions $I_1(\mu) = I_2(\mu) = 0$, and therefore the Akivis–Goldberg equations for the 3-web generated by ω_1, ω_2, and ω_3 are compatible. They can be solved with respect to the functions λ_1 and λ_2, and we get finally the deformation tensor and such a flat connection in which the leaves of $\omega_p = 0$ for all $p = 1, 2, 3, 4$ are geodesics.

Summarizing we get the following theorem.

Theorem 8 *The 4-web W_4 is linearizable if and only if the conditions $I_1^0(f, a) = 0$ and $I_2^0(f, a) = 0$ hold.*

We call the quantities $I_1^0(f, a)$ and $I_2^0(f, a)$ the *basic differential invariants* of the 4-web W_4.

In order to make the expressions for these invariants more symmetric, we introduce a second web function for a 4-web W_4. Namely, locally one can find a function $g(x, y)$ such that $\omega_4 \wedge dg = 0$, or

$$\omega_4 = u\, dg$$

for some function u. Note that the function $f(x, y)$ defines the 3-subweb of the 4-web W_4 formed by the foliations $\{\omega_1 = 0\}, \{\omega_2 = 0\}$, and $\{\omega_3 = 0\}$,and the function $g(x, y)$ defines the 3-subweb of the 4-web W_4 formed by the foliations $\{\omega_1 = 0\}, \{\omega_2 = 0\}$, and $\{\omega_4 = 0\}$.

It follows from (15) that

$$ug_x = -af_x, \ ug_y = -f_y.$$

These two equations imply that

$$a = \frac{f_y g_x}{f_x g_y}$$

and

$$a = \frac{\partial_1 (g)}{\partial_2 (g)}. \tag{17}$$

Substituting this expression into (16) and the result obtained into (14), one gets two differential invariants $I_1 (f, g)$ and $I_2 (f, g)$ each of which is of order three in f and g.

3.1 Computation of the Differential Invariants

3.1.1 Calculus of Covariant Derivatives

Let $d_\gamma : \Lambda^1(M) \to \Lambda^1 (M) \otimes \Lambda^1 (M)$ be the covariant differential with respect to the Chern connection.

Denote by $\Theta^k (M) = \left(\Lambda^1 (M)\right)^{\otimes k}$ the module of covariant tensors of order k. Then the Chern connection induces a covariant differential

$$d_\gamma^{(k)} : \Theta^k (M) \to \Theta^{k+1} (M),$$

where

$$d_\gamma^{(k)} : h\theta \longmapsto hd_\nabla^{(k)} (\theta) + \theta \otimes dh$$

and $h \in C^\infty (M)$ and $\theta \in \Theta^k (M)$.

If θ has the form $\theta = u\omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_k}$ in the basis $\{\omega_1, \omega_2\}$, where $i_1, i_2, ..., i_k = 1, 2$, and $u \in C^\infty (M)$, then

$$d_\gamma^{(k)} (\theta) = \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_k} \otimes (du - ku\gamma).$$

We say that u is of weight k and call the form

$$\delta^{(k)} (u) = du - ku\gamma \tag{18}$$

the *covariant differential* of u. Decomposing the form $\delta^{(k)} (u)$ in the basis $\{\omega_1, \omega_2\}$, we obtain

$$\delta^{(k)} (u) = \delta_1^{(k)} (u) \ \omega_1 + \delta_2^{(k)} (u) \ \omega_2,$$

where

$$\begin{aligned} \delta_1^{(k)} (u) &= \partial_1 (u) - kHu, \\ \delta_2^{(k)} (u) &= \partial_2 (u) - kHu \end{aligned} \tag{19}$$

are the covariant derivatives of u with respect to the Chern connection. Note that $\delta_1^{(k)} (u)$ and $\delta_2^{(k)} (u)$ are of weight $k + 1$.

Lemma 9 *For any $s = 0, 1, ...,$ the relation*

$$\delta_2^{(s+1)} \circ \delta_1^{(s)} - \delta_1^{(s+1)} \circ \delta_2^{(s)} = sK \tag{20}$$

holds for the commutator.

Proof. We have

$$\delta_2^{(s+1)} \circ \delta_1^{(s)} = \partial_2\partial_1 - sH\partial_2 - (s+1)\,H\partial_1 + \left(s\,(s+1)\,H^2 - s\partial_2 H\right)$$

and

$$\delta_1^{(s+1)} \circ \delta_2^{(s)} = \partial_1\partial_2 - sH\partial_1 - (s+1)\,H\partial_2 + \left(s\,(s+1)\,H^2 - s\partial_1 H\right).$$

The statement follows now from (10). ■

3.1.2 Prolongations of the Curvature and the Basic Invariant

As we have seen, the geometry of a 4-web is determined by the curvature K, the basic invariant a and their (covariant) derivatives. In order to express the invariants I_1 and I_2 in terms of K, a and their covariant derivatives, we need the first covariant derivatives of K and covariant derivatives of a up to the third order.

We apply (19) to K and a.

The curvature function K is of weight two. Hence

$$K_1 = \delta_1^{(2)}\,(K) = \partial_1\,(K) - 2HK,$$
$$K_2 = \delta_1^{(2)}\,(K) = \partial_2\,(K) - 2HK.$$

The basic invariant is of weight zero. Hence

$$a_1 = \delta_1^{(0)}\,(a) = \partial_1 a,$$
$$a_2 = \delta_2^{(0)}\,(a) = \partial_2 a.$$

Note that (20) for $s = 0$ implies that $\delta_2^{(1)} \circ \delta_1^{(0)} = \delta_1^{(1)} \circ \delta_2^{(0)}$.

Thus, we have the following expressions for the second covariant derivatives of a :

$$a_{11} = \delta_1^{(1)} \circ \delta_1^{(0)}(a) = \partial_1^2 a - H\partial_1 a,$$
$$a_{12} = a_{21} := \delta_2^{(1)} \circ \delta_1^{(0)}(a) = \partial_1\partial_2 a - H\partial_2 a,$$
$$a_{22} = \delta_2^{(1)} \circ \delta_2^{(0)}(a) = \partial_2^2 a - H\partial_2 a.$$

Formula (20) for $s = 1$ gives $\delta_2^{(2)} \circ \delta_1^{(1)} - \delta_1^{(2)} \circ \delta_2^{(1)} = K$.

Define the third covariant derivatives as follows:

$$\widetilde{a}_{ijk} = \delta_k^{(2)} \circ \delta_j^{(1)} \circ \delta_i^{(0)}\,(a)\,.$$

Note that these expressions are symmetric in (i, j). In order to get symmetry in (i, j, k) for *all* third covariant derivatives, we define the *symmetrized third covariant derivatives* a_{ijk} as follows:

$$\begin{aligned} a_{111} &= \widetilde{a}_{111}, a_{222} = \widetilde{a}_{222}, \\ a_{112} &= \frac{1}{3}\left(\widetilde{a}_{112} + \widetilde{a}_{121} + \widetilde{a}_{211}\right), \\ a_{122} &= \frac{1}{3}\left(\widetilde{a}_{122} + \widetilde{a}_{212} + \widetilde{a}_{221}\right). \end{aligned}$$

For them we have the following expressions:

$$\begin{aligned} a_{111} &= \partial_1^3 a - 2H\partial_1^2 a + (H^2 - \partial_1 H)\partial_1 a, \\ a_{112} &= \partial_1\partial_2\partial_1 a - H\partial_1^2 a - 2H\partial_2\partial_1 a + \left(2H^2 - \frac{2\partial_1 H + \partial_2 H}{3}\right)\partial_1 a, \\ a_{122} &= \partial_2\partial_1\partial_2 a - H\partial_2^2 a - 2H\partial_1\partial_2 a + \left(2H^2 - \frac{\partial_1 H + 2\partial_2 H}{3}\right)\partial_2 a, \\ a_{222} &= \partial_2^3 a - 2H\partial_2^2 a + (H^2 - \partial_2 H)\partial_2 a. \end{aligned}$$

3.1.3 Cartan's Prolongations

In this section we show the relationship of the above calculus to Cartan's prolongations of the curvature K and the basic invariant a of a 4-web W_4.

Since K is a relative invariant of weight two, it satisfies the following Pfaffian equation:

$$\delta K = K_1\omega_1 + K_2\omega_2,$$

where $\delta K = \delta^{(2)} K = dK - 2K\gamma$.

Since a is an absolute invariant, we have

$$\delta a = a_1\omega_1 + a_2\omega_2,$$

where $\delta a = \delta^{(0)} a = da$.

Applying (18) to a_1 and a_2, we obtain

$$\begin{aligned} \delta a_1 &= a_{11}\omega_1 + a_{12}\omega_2, \\ \delta a_2 &= a_{12}\omega_1 + a_{22}\omega_2 \end{aligned}$$

because $a_{12} = a_{21}$.

Here $\delta a_i = \delta^{(1)} a_i = da_i - a_i\gamma$, $i = 1, 2$.

For the covariant differentials of a_{ij}, we have

$$\begin{aligned} \delta a_{11} &= \widetilde{a}_{111}\omega_1 + \widetilde{a}_{112}\omega_2, \\ \delta a_{12} &= \widetilde{a}_{121}\omega_1 + \widetilde{a}_{122}\omega_2, \\ \delta a_{22} &= \widetilde{a}_{221}\omega_1 + \widetilde{a}_{222}\omega_2, \end{aligned} \tag{21}$$

where $\delta a_{ij} = \delta^{(2)} a_{ij} = da_{ij} - 2a_{ij}\gamma$.

Passing to the symmetrized derivatives and using (20) , we find that

$$\frac{\widetilde{a}_{112}+2\widetilde{a}_{121}}{3} = a_{112},$$
$$\frac{\widetilde{a}_{112}-\widetilde{a}_{121}}{2} = \frac{K}{2}a_1.$$

Therefore,

$$\widetilde{a}_{112} = a_{112} + \frac{2K}{3}a_1,$$

and the first equation in (21) takes the following form:

$$\delta a_{11} = a_{111}\omega_1 + (a_{112} + \frac{2}{3}a_1 K)\omega_2.$$

For the second equation of (21), we have

$$\widetilde{a}_{121} = a_{112} - \frac{K}{3}a_1$$

and

$$\delta a_{12} = (a_{112} - \frac{1}{3}a_1 K)\omega_1 + \widetilde{a}_{122}\omega_2.$$

For the third equation of (21), we have $\widetilde{a}_{122} = \widetilde{a}_{212}$ and

$$\frac{\widetilde{a}_{221}+2\widetilde{a}_{122}}{3} = a_{122},$$
$$\frac{\widetilde{a}_{221}-\widetilde{a}_{122}}{2} = -\frac{K}{2}a_2.$$

and

$$\widetilde{a}_{221} = a_{122} - \frac{2}{3}Ka_2,$$
$$\widetilde{a}_{122} = a_{122} + \frac{1}{3}Ka_2.$$

Therefore,

$$\delta a_{12} = (a_{112} - \frac{1}{3}a_1 K)\omega_1 + (a_{122} + \frac{1}{3}Ka_2)\omega_2,$$
$$\delta a_{22} = (a_{122} - \frac{2}{3}a_2 K)\omega_1 + a_{222}\omega_2.$$

3.1.4 Differential Invariants in Terms of Covariant Derivatives

Here we express invariants $I_1^0(f,a)$ and $I_2^0(f,a)$ in terms of the curvature function K , basic invariant a and their covariant derivatives. To do this, we express the ordinary derivatives in terms of the covariant derivatines according to the above formulae. After long computations, we get that the linearizability conditions $I_1^0(f,a) = I_2^0(f,a) = 0$ are equivalent to the following two equations:

$$
\begin{aligned}
K_1 &= \frac{1}{a-a^2}\Big[\frac{1}{3}((1-a)a_1 + aa_2)K - a_{111} + (2+a)a_{112} - 2aa_{122}\Big] \\
&+\frac{1}{(a-a^2)^2}\{[(4-6a)a_1 + (a^2+3a-2)a_2]a_{11} \\
&+[(2a^2+7a-6)a_1 + (2a-3a^2)a_2]a_{12} + [2(a-a^2)a_1 - 2a^2a_2]\}a_{22} \\
&+\frac{1}{(a-a^2)^3}[(-6a^2+8a-3)(a_1)^3 - 2a^3(a_2)^3 \\
&+(2a^3+9a^2-15a+6)(a_1)^2a_2 + (-3a^3+6a^2-2a)a_1(a_2)^2]
\end{aligned}
$$

and

$$
\begin{aligned}
K_2 &= \frac{1}{a-a^2}\Big[\frac{1}{3}(a_1 + (a-1)a_2)K + 2a_{112} - (2a+1)a_{122} + aa_{222}\Big] \\
&+\frac{1}{(a-a^2)^2}\{[2a_1 + (2a-2)a_2]a_{11} \\
&+[(6a-5)a_1 + (-2a^2-3a+2)a_2]a_{12} + [(1-a-2a^2)a_1 + 2a^2a_2]\}a_{22} \\
&+\frac{1}{(a-a^2)^3}[(4a-2)(a_1)^3 + a^3(a_2)^3 \\
&+(6a^2-12a+5)(a_1)^2a_2 + (-2a^3-3a^2+5a-2)a_1(a_2)^2].
\end{aligned}
$$

4 Linearizability of d-Webs

A d-web W_d on M^2 is defined by d differential 1-forms $\omega_1, \omega_2, \omega_3, ..., \omega_d$ such that any two of them are linearly independent. We shall fix the 3-subweb $\langle\omega_1, \omega_2, \omega_3\rangle$ and by the Chern connection, curvature, etc. we shall mean the corresponding constructions for this 3-web.

For any $4 \le \alpha \le d$, we shall consider a 4-subweb W_4^α defined by the forms $\omega_1, \omega_2, \omega_3, \omega_\alpha$. We denote the basic invariant of this subweb by a_α and continue use the notation a for a_4. Then

$$\omega_\alpha + a_\alpha \omega_1 + \omega_2 = 0.$$

In the same way we used above, we prove the following theorem:

Theorem 10 *Let ∇ be a torsion-free connection in the cotangent bundle $\tau^* : T^*M \to M$ such that the foliations $\{\omega_1 = 0\}, \{\omega_2 = 0\}, \{\omega_3 = 0\}$, and $\{\omega_\alpha = 0\}$ are ∇-geodesic for all $\alpha \ge 4$. Then the components of the deformation tensor T have form* (12) *and*

$$\mu = \frac{\partial_1(a_\alpha) - a_\alpha \partial_2(a_\alpha)}{a_\alpha - a_\alpha^2} \tag{22}$$

for all $\alpha = 4, ..., d$.

Comparing the expressions for μ, we get the following $d-4$ new relative invariants of the d-web W_d :

$$I_\alpha = \frac{\partial_1(a_\alpha) - a_\alpha \partial_2(a_\alpha)}{a_\alpha - a_\alpha^2} - \frac{\partial_1(a) - a\partial_2(a)}{a - a^2},$$

where $\alpha = 5, ..., d$.

The web W_d can be defined by the functions $f, g_4 = g, ..., g_d$ and

$$a_\alpha = \frac{\partial_1(g_\alpha)}{\partial_2(g_\alpha)}.$$

This gives the following expressions for the invariants I_α :

$$I(f, g, g_\alpha) = I(f, g_\alpha) - I(f, g),$$

where $\alpha = 5, ..., d$, and

$$I(f, p) = \frac{(\partial_1 p)^2 \partial_2^2 p - 2\partial_1 p\, \partial_2 p\, \partial_1 \partial_2 p + (\partial_2 p)^2 \partial_1^2 p}{\partial_1 p\, \partial_2 p\, (\partial_2 p - \partial_1 p)}.$$

Summarizing we get the following theorem:

Theorem 11 *The d-web W_d is linearizable if and only if the conditions $I_1(f, g) = 0$, $I_2(f, g) = 0$ and $I(f, g, g_5) = 0, ..., I(f, g, g_d) = 0$ hold.*

4.1 Method of *d*-Web Linearization

4.1.1 4-Webs

We define a 4-web W_4 by two web functions f and g. Then the procedure for the linearization of such a web can be outlined as follows:

Step 1 Check the linearizability conditions $I_1(f, g) = 0,\ I_2(f, g) = 0$.

Step 2 Find the function μ from (16). Solve the Akivis-Goldberg equations (13) with respect to the functions λ_1 and λ_2. This is the Frobenius-type PDEs system due to Step 1. Find the components of the deformation tensor T from (12).

Step 3 The connection $\delta_0 - T$ is flat. Find local coordinates x_1 and x_2 in which the connection coincides with the standard one on M^2. In these coordinates, the leaves of W_4 are straight lines.

Remark 12 *Step* 2 *and Step* 3 *can be performed in a constructive way (in quadratures) if the web under consideration admits a nontrivial symmetry group. In this case one can find the first integrals for the system of Akivis-Goldberg equations and hence the deformation tensor. If this deformation tensor also possesses nontrivial symmetries, then the local coordinates in Step* 3 *can be found.*

4.1.2 d-Webs, $d > 4$

We define a d-web W_d by $d-2$ web functions f and $g = g_4, ..., g_d$. Then the procedure for linearization can be outlined as follows:

Step 1 Check the linearizability conditions $I_1(f,g) = 0,\ I_2(f,g) = 0, I(f,g,g_5) = 0, ..., I(f,g,g_d) = 0$.

Step 2 Find the function μ from (16). Solve the Akivis-Goldberg equations (13) with respect to the functions λ_1 and λ_2. This is the Frobenius-type PDEs system due to Step 1. Find the components of the deformation tensor T from (12).

Step 3 The connection $\delta_0 - T$ is flat. Find local coordinates x_1 and x_2 in which the connection coincides with the standard one on M^2. In these coordinates, the leaves of W_d are straight lines.

5 Tests and Examples

5.1 Test Notebooks

Below we give Mathematica codes for testing 4- and 5-webs for linearizability.

The following program computes differential invariants of d-webs for $d \geq 4$:

$\textbf{webInvariants}[fTab\_] := [\{f, g, X, Y, h, A, I1, I2, J, a, \mu, d, ans\},$

$f = fTab[[1]];\ d = Length[fTab];\ g[i\_] = fTab[[i]];$

$X[A\_] := -\dfrac{D[A,x]}{D[f,x]};\ Y[A\_] := -\dfrac{D[A,y]}{D[f,y]};\ h = \dfrac{D[f,x,y]}{D[f,x]*D[f,y]};$

$a[i\_] = \dfrac{D[f,y]*D[g[i],x]}{D[f,x]*D[g[i],y]};\quad \nu[i\_] := \dfrac{X[a[i]] - a[i]*Y[a[i]]}{a[i]^2 - a[i]};\ \mu = \nu[2];$

$I1 = X[X[\mu]] - 2*X[Y[\mu]] + (\mu - h)*X[h] + (4*h - 2*\mu)*Y[\mu] +$

$h*\mu^2 - (2*h^2 - Y[h])*\mu - X[X[h]] + X[Y[h]] + 2*h*X[h]$

$-2*h*Y[h]//\textbf{Simplify};$

$I2 = X[Y[\mu]] - 2*X[Y[\mu]] + (2*\mu + 2*h)*X[h] + (h - \mu)*Y[\mu] -$

$h*\mu^2 - (2*h^2 - X[h])*\mu + Y[Y[h]] - Y[X[h]] + 2*h*X[h]$

$-2*h*Y[h]//\textbf{Simplify};$

$J[i\_] := (\mu - \nu[i])//\textbf{Simplify};$

$ans = \{I1, I2, Table[J[i], \{i, 3, d\}]\}\ \ ]$

The following program tests 4-webs for the linearizability:

$$\mathbf{LinTest4Web}[f\_, g\_] := \mathbf{Module}[$$
$$\{X, Y, h, A, I1, I2, a, \mu, Z, ans\},$$
$$X[A\_] := -\frac{D[A, x]}{D[f, x]};\ Y[A\_] := -\frac{D[A, y]}{D[f, y]};\ h =$$
$$\frac{D[f, x, y]}{D[f, x] * D[f, y]};$$
$$a = \frac{D[f, y] * D[g, x]}{D[f, x] * D[g, y]};\ \mu = \frac{X[a] - a * Y[a]}{a^2 - a};$$
$$I1 = X[X[\mu]] - 2 * X[Y[\mu]] + (\mu - h) * X[h] + (4 * h - 2 * \mu) * Y[\mu] +$$
$$h * \mu^2 - (2 * h^2 - Y[h]) * \mu - X[X[h]] + X[Y[h]] + 2 * h * X[h]$$
$$-2 * h * Y[h]//\mathbf{Simplify};$$
$$I2 = X[Y[\mu]] - 2 * X[Y[\mu]] + (2 * \mu + 2 * h) * X[h] + (h - \mu) * Y[\mu] -$$
$$h * \mu^2 - (2 * h^2 - X[h]) * \mu + Y[Y[h]] - Y[X[h]] + 2 * h * X[h]$$
$$-2 * h * Y[h]//\mathbf{Simplify};$$
$$Z = If[I1 === 0\&\&I2 === 0, "YES", "NO"];$$
$$ans = Z\quad]$$

Finally we give the code which tests d-webs.

$$\mathbf{LindTestdWeb}[fun\_] := \mathbf{Module}[\{f, g, X, Y, h, d, I1, I2, J, a, \nu, \mu, Z, ans\},$$
$$f = fun[[1]]; d = Length[fun]; g[i\_] := fun[[i]];$$
$$X[A\_] := -\frac{D[A, x]}{D[f, x]};\quad Y[A\_] := -\frac{D[A, y]}{D[f, y]};\quad h = \frac{D[f, x, y]}{D[f, x] * D[f, y]};$$
$$a[i\_] := \frac{D[f, y] * D[g[i], x]}{D[f, x] * D[g[i], y]};\quad \nu[i\_] := \frac{X[a[i]] - a[i] * Y[a[i]]}{a[i]\hat{}2 - a[i]};\ \mu = \nu[2];$$
$$I1 = X[X[\mu]] - 2 * X[Y[\mu]] + (\mu - h) * X[h] + (4 * h - 2 * \mu) * Y[\mu] +$$
$$h * \mu^2 - (2 * h^2 - Y[h]) * \mu - X[X[h]] + X[Y[h]] + 2 * h * X[h] -$$
$$2 * h * Y[h]//\mathbf{Simplify};$$
$$I2 = X[Y[\mu]] - 2 * X[Y[\mu]] + (2 * \mu + 2 * h) * X[h] + (h - \mu) * Y[\mu] -$$
$$h * \mu^2 - (2 * h^2 - X[h]) * \mu + Y[Y[h]] - Y[X[h]] + 2 * h * X[h] -$$
$$2 * h * Y[h]//\mathbf{Simplify};$$
$$J[i\_] := (\mu - \nu[i])//\mathbf{Simplify};$$
$$Z = \mathbf{If}[I1 === 0\&\&I2 === 0\&\&$$
$$\mathbf{Table}[J[i], \{i, 3, d\}] === \mathbf{Table}[0, \{i, 3, d\}], "YES", "NO"];$$
$$ans = Z\quad]$$

In the last test fun is a collection $\{f_1, ..., f_{d-2}\}$ of functions determining the d-web.

Results of the tests are "YES" or "NO" depending on the linearizability of the web. Note that the computer testing gives the same results if in each example we replace the functions $f(x, y)$ and $g(x, y)$ by the functions $f(p(x), q(y))$ and $g(p(x), q(y))$, where $p(x)$ and $q(y)$ are arbitrary smooth functions of x and y, respectively (i.e., if we consider equivalent webs).

5.2 Examples

1. **LinTest4Web**$[x/y, x + y] =$ "YES"

   This is the **4**-web whose **3**rd foliation consists of straight lines of the pencil with center at the origin, and the **4**th foliation consists of parallel straight lines forming the angle **135** degrees with positive direction of the axis Ox, i.e., this **4**-web is linear, and the test is just for demonstration that it is working.

2. **LinTest4Web**$[x/y, (1 - y)/(1 - x)] =$ "YES"

   In this case the **3**rd and **4**th foliations are straight lines of two pencils with their vertices at $(0, 0)$ and $(1, 1)$. This **4**-web is also linear, and the test is just for demonstration that it is working.

3. **LinTest4Web**$[x + \sqrt{x^2 - y}, x + y] =$ "YES"

   In this case the curves of the **3**rd foliation are tangent to the parabola $y = x^2$, and the **4**th foliation consists of parallel straight lines forming the angle **135** degrees with positive direction of the axis Ox, i.e., this **4**-web is linear. But here it is not obvious, that the **3**rd foliation consists of straight lines.

4. **LinTest4Web**$[x + \sqrt{x^2 - y}, y + \sqrt{y^2 - x}] =$ "YES"

   Here the curves of the **3**rd foliation are tangent to the parabola $y = x^2$, and the curves of the **4**th foliation are tangent to the parabola $x = y^2$, i.e., this **4**-web is linear.

5. **LinTest4Web**$[x/y, (x + y) * Exp[-x]] =$ "NO"

   This is the **4**-web whose **3**rd foliation consists of straight lines of the pencil with center at the origin, and the **4**-subweb defined by the **4**th foliation and the coordinate lines is parallelizable. The **4**-web in this example is not linearizable, although two of its **3**-subwebs are linearizable.

6. **LinTest4Web**$[x/y, x^n + y^n] =$ "YES"

   This web is equivalent to the **4**-web of the **1**st example. This web is not linear but it is linearizable.

7. **LinTestdWeb**$[\{y/x, (1 - y)/(1 - x), (x - xy)/(y - xy)\}] =$ "NO"

   This is the famous **5**-web constructed by Bol (see [2], § 46 and [3], §12 and §31). The web consists of **4** pencils of straight lines (the first two are the pencils of parallel coordinate lines, and the **3**rd and the **4**th are the pencils with centers at $(0,0)$ and $(1,1)$), and a foliation of conics passing through **4** centers of the **4** pencils. Bol constructed this example to show that there exists a **5**-web of maximum rank **6** which is not linearizable. Bol gave an indirect proof that this **5**-web is not linearizable. Our test gives the direct proof of this fact.

8. **LinTest4Web**$[y/x,(x-xy)/(y-xy)] = "YES"$

   This is a **4**-subweb of the Bol **5**-web considered in the previous example. It is formed by **3** pencils of straight lines and the same foliation of conics. It appeared that this **4**-web is linearizable while the Bol **5**-web is not linearizable. Note that we can prove the linearizability of this **4**-web using the quadratic transformation $x = 1/x^*, y = 1/y^*$ suggested by Blaschke in [2], §46.

9. **LinTestdWeb**$[\{x/y,(1-y)/(1-x),(x-xy)/(y-xy),xy,(x-xy)/(x-1),(1-y)/(xy-y),x(1-y)^2/y(1-x)^2\}] = "NO"$

   This is the Spence–Kummer **9**-web constructed by Pirio and Robert (see [13], [14] and [15]). This web consists of **4** pencils of straight lines described in Example 7, **4** foliations of conics and a foliation of cubics passing through **4** centers of the **4** pencils. Pirio and Robert constructed this example and other examples of d-webs, $d = 6,7,8$, to show that there exist nonlinearizable webs of maximum rank different from the Bol **5**-web considered in Example 7. They proved that all their d-webs are not linearizable. Our test gives the direct proof of this fact for the Spence–Kummer **9**-web (and all other d-webs constructed in [13], [14] and [15]).

References

[1] Akivis, M. A. and A. M. Shelekhov, *Geometry and algebra of multidimensional three-webs*, translated from the Russian by V. V. Goldberg, Kluwer Academic Publishers, Dordrecht, 1992, xvii+358 pp. (MR 93k:53021; Zbl 771:53001.)

[2] Blaschke, W., *Einführung in die Geometrie der Waben*, Birkhäuser-Verlag, Basel-Stutgart, 1955, 108 pp. (MR **17**, p. 780; Zbl **68**, p. 365.)

[3] Blaschke, W. and G. Bol, *Geometrie der Gewebe*, Springer-Verlag, Berlin, 1938, viii+339 pp. (MR **6**, p. 19; Zbl **20**, p. 67.)

[4] Goldberg, V. V., *On the theory of four-webs on a differentiable manifold* X^{2r}, Serdica **6** (1980), no. 2, 105–119. (MR 82f:53023; Zbl 453:53010.)

[5] Goldberg, V. V., *Theory of multicodimensional* $(n+1)$*-webs*, Kluwer Academic Publishers, Dordrecht, 1988, xxii+466 pp. (MR 89h:53021; Zbl 668:53001.)

[6] Goldberg, V. V., *On a linearizability condition for a three-web on a two-dimensional manifold*, Differential Geometry, Peniscola 1988, 223–239, Lecture Notes in Math. **1410**, Springer, Berlin–New York, 1989. (MR 91a:53032; Zbl 689:53008.)

[7] Goldberg, V. V., *Four-webs in the plane and their linearizability*, Acta Appl. Math. **80** (2004), no. 1, 35–55.

[8] Grifone, J., Z. Muzsnay and J. Saab, *On the linearizability of 3-webs*, Proceedings of the Third World Congress of Nonlinear Analysts, Part 4 (Catania, 2000), Nonlinear Anal. **47** (2001), no. 4, 2643–2654. (MR 1972389.)

[9] Hènaut, A., *Sur la linéarisation des tissus de* C^2, Topology **32** (1993), 531–542. (MR 94g:53013; Zbl 799:32010.)

[10] Hènaut, A., *Caractérisation des tissus de* C^2 *dont le rang est maximal et qui sont linéarisables,* Composito Math. **94** (1994), no. 3, 247–268. (MR 96a:32057; Zbl 877:53013.)

[11] Kobayashi, S. and K. Nomizu, *Foundations of differential geometry*, vol. 1, Wiley–Interscience, New York–London–Sydney, 1963, xi+329 pp. (MR **27** ♯2945; Zbl. **119**, p. 375.)

[12] Norden, A. P., *Affinely connected spaces* (Russian), Nauka, Moscow, 1976, 432 pp. (MR **57** ♯7421.)

[13] Pirio, L., *Study of a functional equation associated to the Kummer's equation of the trilogarithm,* arXiv: math.CV/0206172, 11 pp.

[14] Pirio, L., *Abelian functional equations, planar web geometry and polylogarithms,* arXiv: math.CV/0212146, 30 pp.

[15] Robert, G. F., *Relations fonctionelles polylogarithmiques et tissus plans,* Prépublication n° **146** , Univ. Bordeaux 1, 23 pp.

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