

# Higher weight spectra of codes from Veronese threefolds

Trygve Johnsen<sup>\*,a,1,2</sup>, Hugues Verdure<sup>a,2</sup>

<sup>a</sup>*Department of Mathematics and Statistics, UiT-The Arctic University of Norway  
9037 Tromsø, Norway*

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## Abstract

We study binary linear codes  $C$  obtained from the quadric Veronese embedding of  $\mathbb{P}^3$  in  $\mathbb{P}^9$  over  $\mathbb{F}_2$ . We show how one can find the higher weight spectra of these codes. Our method will be a study of the Stanley-Reisner rings of a series of matroids associated to each code  $C$ .

*Key words:* Binary Veronese code, higher weight spectra

*2000 MSC:* 05E45, 94B05, 05B35, 13F55

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## 1. Introduction

Projective Reed-Müller codes is a class of error-correcting codes that has attracted much attention over the last decades. To find the code parameters, including the generalized Hamming weights, has been a difficult task. In [16] one found these parameters for projective Reed-Müller codes of order two. For projective Reed-Müller codes of higher order important contributions have appeared quite recently. See [2] and [4], where one gives results that have wide applications in determining such generalized Hamming weights for these codes. For affine Reed-Müller codes there are recent nice results in [1]. To find the

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\*Corresponding author

*Email addresses:* `Trygve.Johnsen@uit.no` (Trygve Johnsen), `Hugues.Verdure@uit.no` (Hugues Verdure)

<sup>1</sup>T. Johnsen is grateful for the generous hospitality of IIT-Bombay during parts of the work with this paper

<sup>2</sup>This work was partially supported with a joint grant from the Research Council of Norway, and the Department of Science and Technology, India (grant 280731 from RCN) This work was also partially supported by the project Pure Mathematics in Norway, funded by Bergen Research Foundation and Tromsø Research Foundation

more refined information lying in higher weight spectra of projective Reed-Müller codes is more difficult, when their orders are higher than one, and to our knowledge there are few results about the higher weight spectra of these codes.

Let  $C(a, b)_q$  be the (projective Reed-Müller) code with a generator matrix  $G_q$  with  $n = q^b + \dots + q + 1$  columns that are coordinate vectors of the  $n$  points of the image of  $\mathbb{P}_q^b$  by the  $a$ -uple embedding of  $\mathbb{P}_q^b$  into  $\mathbb{P}_q^{\frac{b(b+3)}{2}}$ . Such a matrix is only defined up to permutations of the columns, and multiplications of each column by a non-zero constant, but such operations are irrelevant for the properties that we will study.

In an earlier paper [9] one studied the simplest projective Reed-Müller codes of order at least 2, namely the codes  $C(2, 2)_q$  over  $\mathbb{F}_q$ . In the present paper we study the code  $C(2, 3)_2$ , defined by the Veronese embedding of  $\mathbb{P}_q^3$  into  $\mathbb{P}_q^9$ . Here the 15 columns of the generator matrix  $G$  correspond to the points of  $\mathbb{P}^3$ , and each row is obtained by taking an element of a basis for the vector space of all homogeneous polynomials of degree 2 in 4 variables, and evaluating it at the points of  $\mathbb{P}^3$ . We will compute the higher weight spectra of this code, that is the quantities  $A_w^{(r)}$  that give the number of subcodes of  $C(2, 3)_2$  of dimension  $r$  and support weight  $w$ . In addition we are interested in the equivalent data of the generalized weight polynomials  $P_j(Z)$ : for each  $j = 0, \dots, n$  the value  $P_j(2^m)$  is the number of codewords of weight  $j$  in the extension code  $C(2, 3)_2 \otimes_{\mathbb{F}_q} \mathbb{F}_{2^m}$ . A key result is Corollary 17 which tells exactly what information it is essential to extract from these resolutions.

Our method will consist of finding the  $\mathbb{N}$ -graded resolutions of the Stanley-Reisner rings of a series of 10 matroids derived from the parity check matroid  $M$  of the code. We will compute them by identifying some sets of points in  $\mathbb{P}_2^3$  with some geometric properties (how many quadrics pass through these points). The  $\mathbb{N}$ -graded Betti numbers of these resolutions will give us the generalized weight polynomials  $P_j(Z)$ , which are equivalent to the higher weight spectra.

The purpose of this article is mainly to reveal the power of the sketched method, and demonstrate how it can be used, through this example  $C(2, 3)_2$ . The method summarized above is exemplified through the concrete calculations

in Sections 3-6, giving rise to Theorems 21 and 22. The binary code in the present paper has only  $2^{10} = 1024$  elements, and we have been able to verify Theorem 21 using more straightforward methods. We hope nevertheless that, together with a better general understanding of the geometry of quadrics in  $\mathbb{P}^3$ , our method will give the higher weight spectra of  $C(2, 3)_q$  for general  $q$ .

## 2. Definitions and notation

Let  $q$  be a prime power and let  $\nu_q$  be the Veronese map that maps  $\mathbb{P}^3$  into  $\mathbb{P}^9$  over  $\mathbb{F}_q$ , i.e.  $(x, y, z, w)$  is mapped to  $(x^2, xy, xz, xw, y^2, yz, yw, z^2, zw, w^2)$ , and let  $V = V_q$  be the image, a non-degenerate smooth threefold of degree 8. The cardinality  $|V|$  of  $V$  is  $|\mathbb{P}^2| = q^3 + q^2 + q + 1$ . Fix some order for the points of  $V$ , and for each such point, fix a coordinate 10-tuple that represents it. Let  $G_q$  be the  $(10 \times (q^3 + q^2 + q + 1))$ -matrix, whose columns are the coordinate 10-tuples of the points of  $V$ , taken in the order fixed.

**Definition 1.** *The Veronese code  $C(2, 3)_q$  is the linear  $[q^3 + q^2 + q + 1, 10]_q$ -code with generator matrix  $G_q$ .*

For  $q = 2$  we thus get a  $[15, 10]$ -code  $C(2, 3)_2$ . This is the code which will be the main object of interest in this paper.

### 2.1. Hamming weights, spectra and generalized weight polynomials

**Definition 2.** *Let  $C$  be a  $[n, k]$  linear code over  $\mathbb{F}_q$ . Let  $\mathbf{c} = (c_1, \dots, c_n) \in C$ . The support of  $\mathbf{c}$  is the set*

$$\text{Supp}(\mathbf{c}) = \{i \in \{1, \dots, n\} : c_i \neq 0\}.$$

*Its weight is*

$$\text{wt}(\mathbf{c}) = |\text{Supp}(\mathbf{c})|.$$

*Similarly, if  $T \subset C$ , then its support and weight are*

$$\text{Supp}(T) = \bigcup_{\mathbf{c} \in T} \text{Supp}(\mathbf{c}) \text{ and } \text{wt}(T) = |\text{Supp}(T)|.$$

Important invariants of a code are the generalized Hamming weights, introduced by Wei in [15]:

**Definition 3.** Let  $C$  be a  $[n, k]$  linear code over  $\mathbb{F}_q$ . Its generalized Hamming weights are

$$d_i = \min\{wt(D) : D \subset C \text{ is a subcode of dimension } i\}$$

for  $1 \leq i \leq k$ .

We also have

**Definition 4.** Let  $C$  be a  $[n, k]$  linear code over  $\mathbb{F}_q$ . For  $1 \leq w \leq n$  and  $1 \leq r \leq k$ , the higher weight spectra of  $C$  are

$$A_w^{(r)} = |\{D : D \text{ subcode of } C \text{ of dimension } r \text{ and weight } w\}|.$$

In particular, we have

$$d_r = \min\{w : A_w^{(r)} \neq 0\}.$$

In [12], one shows that the number of codewords of a given code extended to a field extension of a given weight can be expressed by polynomials (the generalized weight polynomials). More precisely, if  $C$  is a  $[n, k]$ -code over  $\mathbb{F}_q$ , then the code  $C^{(i)} = C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^i}$  for  $i \geq 1$  is a  $[n, k]$  code over  $\mathbb{F}_{q^i}$ . Then

**Theorem 5.** Let  $C$  be a  $[n, k]$ -code over  $\mathbb{F}_q$ . Then, there exists polynomials  $P_w \in \mathbb{Z}[Z]$  for  $0 \leq w \leq n$  such that

$$\forall i \geq 1, P_w(q^i) = \left| \left\{ \mathbf{c} \in C^{(i)} : wt(\mathbf{c}) = w \right\} \right|.$$

In [11], one gives a relation between the higher weight spectra and the polynomials defined above, namely

**Theorem 6.** Let  $C$  be a  $[n, k]$  code over  $\mathbb{F}_q$ . Let  $0 \leq w \leq n$ . Then

$$P_w(q^m) = \sum_{r=0}^m A_w^{(r)} \prod_{i=0}^{r-1} (q^m - q^i).$$

## 2.2. Matroids, resolutions and elongations

There are many equivalent definitions of a matroid. We refer to [13] for a deeper study of the theory of matroids.

**Definition 7.** A matroid is a pair  $(E, \mathcal{I})$  where  $E$  is a finite set and  $\mathcal{I}$  is a set of subsets of  $E$  satisfying

$$(R_1) \quad \emptyset \in \mathcal{I}$$

$$(R_2) \quad \text{If } I \in \mathcal{I} \text{ and } J \subset I, \text{ then } J \in \mathcal{I}$$

$$(R_3) \quad \text{If } I, J \in \mathcal{I} \text{ and } |I| < |J|, \text{ then } \exists j \in J - I \text{ such that } I \cup \{j\} \in \mathcal{I}.$$

The elements of  $\mathcal{I}$  are called independent sets. The subsets of  $E$  that are not independent are called dependent sets, and inclusion minimal dependent sets are called circuits.

For any  $X \subset E$ , its rank is

$$r(X) = \max\{|I| : I \in \mathcal{I}, I \subset X\}$$

and its nullity is  $n(X) = |X| - r(X)$ . The rank of the matroid is  $r(M) = r(E)$ . Finally, for any  $0 \leq i \leq |E| - r(M)$ ,

$$N_i = \{X \subset E : n(X) = i\}.$$

If  $C$  is a  $[n, k]$ -linear code given by a  $(n - k) \times n$  parity check matrix  $H$ , then we can associate to it a matroid  $M_C = (E, \mathcal{I})$ , where  $E = \{1, \dots, n\}$  and  $X \in \mathcal{I}$  if and only if the columns of  $H$  indexed by  $X$  are linearly independent over  $\mathbb{F}_q$ . It can be shown that this matroid is independent of the choice of the parity check matrix of the code, and we may thus call it the parity check matroid of  $C$ .

Let  $\mathbb{K}$  be a field. We can associate to  $M$  a monomial ideal  $I_M$  in  $S = K[\{X_e\}_{e \in E}]$  defined by

$$I_M = \langle \mathbf{X}^\sigma : \sigma \notin \mathcal{I} \rangle$$

where  $\mathbf{X}^\sigma$  is the monomial product of all  $X_e$  for  $e \in \sigma$ . This ideal is called the Stanley-Reisner ideal of  $M$  and the quotient  $S_M = S/I_M$  the Stanley-Reisner

ring associated to  $M$ . We refer to [6] for the study of such objects. As described in [8] the Stanley-Reisner ring has minimal  $\mathbb{N}$  and  $\mathbb{N}^n$ -graded free resolutions

$$0 \leftarrow S_M \leftarrow S \leftarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \leftarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{2,j}} \leftarrow \dots \leftarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{|E|-r(M),j}} \leftarrow 0 \quad (1)$$

and

$$0 \leftarrow S_M \leftarrow S \leftarrow \bigoplus_{\alpha \in \mathbb{N}^n} S(-\alpha)^{\beta_{1,\alpha}} \leftarrow \dots \leftarrow \bigoplus_{\alpha \in \mathbb{N}^n} S(-\alpha)^{\beta_{|E|-r(M),\alpha}} \leftarrow 0.$$

In particular the numbers  $\beta_{i,j}$  and  $\beta_{i,\alpha}$  are independent of the minimal free resolution and of the field  $\mathbb{K}$ , and are called respectively the  $\mathbb{N}$ -graded and  $\mathbb{N}^n$ -graded Betti numbers of the matroid. We have

$$\beta_{i,j} = \sum_{wt(\alpha)=j} \beta_{i,\alpha}.$$

We also note that  $\beta_{0,0} = 1$ .

As a consequence of a more general result by Peskine and Szpiro ([14], Lemma on p. 1422]):

**Theorem 8.** *Let  $M$  be a matroid of rank  $r = n - k$  on a set of cardinality  $n$ . Then the  $\mathbb{N}$ -graded Betti numbers of  $R_M$  satisfy the equations*

$$\sum_{i=0}^k \sum_{j=0}^n (-1)^i j^s \beta_{i,j} = 0, \quad (2)$$

for  $0 \leq s \leq k - 1$ , where by convention,  $0^0 = 1$ .

The  $k$  equations (2) from Theorem 8 are often called the Herzog-Kühl equations, and have a particularly nice form and solution when the resolution is pure. See also [3] for more on this topic.

**Definition 9.** *For a matroid  $M$  we define  $\phi_j(M) = \sum_{i=0}^k (-1)^i \beta_{i,j}$ .*

**Remark 10.** *The equations (2) can be written:*

$$\sum_{j=0}^n j^s \phi_j(M) = 0, \quad (3)$$

and it is clear that these equations are independent in the variables  $\phi_j(M)$  with a Vandermonde coefficient matrix.

Also, as explained in [8, Theorem 1], we can compute the  $\mathbb{N}^n$ -graded Betti number  $\beta_{i,\alpha}$  as the Euler characteristic of a certain matroid. If  $M$  is a matroid and  $\sigma$  is a subset of the ground set  $E$ , then  $M_\sigma$  is the matroid with independent sets

$$\mathcal{I}(M_\sigma) = \{\tau \in \mathcal{I}(M) : \tau \subset \sigma\}.$$

Moreover, the Euler characteristic of  $M$  is

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{|E|} (-1)^{i-1} |\{\tau \subset E : |\tau| = i \text{ and } \tau \notin \mathcal{I}\}| \\ &= \sum_{i=0}^{|E|} (-1)^i |\{\tau \subset E : |\tau| = i \text{ and } \tau \in \mathcal{I}\}| \end{aligned}$$

The following result ([8, Theorem 1], last part) will be very useful:

**Theorem 11.** *Let  $M$  be a matroid on the ground set  $E$ . Let  $\sigma \subset E$ . Then*

$$\beta_{n(\sigma),\sigma} = (-1)^{r(\sigma)-1} \chi(M_\sigma).$$

*In particular, for any circuit  $\sigma$ ,  $\beta_{1,\sigma} = 1$ .*

We will also frequently use ([8, Theorem 1], first part):

**Theorem 12.** *Let  $C$  be a  $[n, k]$ -code over  $\mathbb{F}_q$ . The  $\mathbb{N}$ -graded Betti numbers of the parity check matroid  $M_C$  satisfy:  $\beta_{i,j} \neq 0$  if and only if there exists an inclusion minimal set in  $N_i$  of cardinality  $j$ . In particular,  $\beta_{i,X} = 0$  if and only if  $X$  is not inclusion minimal in  $N_i$ . Furthermore  $d_i = \min\{j : \beta_{i,j} \neq 0\}$ .*

**Remark 13.** *Because of this result we will call the term  $\bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{i,j}}$  in (1) "the nullity  $i$  part of the (graded) resolution (1)".*

**Definition 14.** *Let  $M = (E, \mathcal{I})$  be a matroid, with  $|E| = n$ , and let  $l \geq 0$ . Then, the  $l$ -th elongation of  $M$  is the matroid  $M^{(l)} = (E, \mathcal{I}^{(l)})$  with*

$$\mathcal{I}^{(l)} = \{I \cup X : I \in \mathcal{I}, X \subset E, |X| \leq l\}.$$

The  $l$ -th elongation of  $M$  is a matroid of rank  $\min\{n, r(M) + l\}$ .

We denote by  $N_i^{(l)}$  be the set of subsets  $X$  of  $E$  with  $n^{(l)}(X) = i$ . The following result is trivial, but useful:

**Proposition 15.**  $N_i^{(l)} = N_{i+l}$ , for  $i = 0, \dots, n - r(M) - l$ . In particular the inclusion minimal elements of  $N_i^{(l)}$  are the same as the inclusion minimal elements of  $N_{i+l}$ .

The main theorem of [10] gives an expression of the generalized weight polynomials of a code to the Betti numbers of its associated matroid and its elongations, namely:

**Theorem 16.** Let  $C$  be a  $[n, k]$  code over  $\mathbb{F}_q$ .

$$P_w(Z) = \sum_{l \geq 0} (\phi_w(M^{(l)}) - \phi_w(M^{(l-1)})) Z^l, \quad (4)$$

where  $\phi_w(M^{(-1)}) = 0$ .

**Corollary 17.** For a linear code  $C$  the following 3 pieces of information are equivalent:

- (a) The knowledge of the generalized weight polynomials  $P_w(Z)$ , for all  $w$ .
- (b) The knowledge of the  $A_w^{(r)}$  for all  $r$  and  $w$ .
- (c) The knowledge of the  $\phi_j^{(l)}$  for  $M_C$  for all  $l$  and  $j$ .

PROOF. That (a) and (b) are equivalent follows directly from Theorem 6. Theorem 16 gives the equivalence between (a) and (c).

**Remark 18.** The knowledge of the  $\phi_j(M^{(l)})$  for all  $j, l$  is then enough to find the higher weight spectra. We will nevertheless have to compute some of the individual  $\beta_{i,j}^{(l)}$  in order to find all the  $\phi_j(M^{(l)})$ .

Let us now focus on  $C(2, 3)_q$ . There is a one-to-one correspondence between words of this  $C(2, 3)_q$  and affine equations for quadrics in  $\mathbb{P}_q^3$ , and under this correspondence, the support of a codeword correspond to points of  $\mathbb{P}_q^3$  that

are not on the quadric. Thus, the circuits of  $M_C$  correspond to quadrics with inclusion maximal point sets over  $\mathbb{F}_q$ .

From [7, Tables 15.4 and 15.9], we see that the set of quadrics in  $\mathbb{P}_q^3$  (where by a quadric we will here mean a non-zero quadric polynomial up to a multiplicative constant) is:

**Proposition 19.** *In  $\mathbb{P}_q^3$  the  $\frac{q^{10}-1}{q-1}$  quadrics are as follows.*

- *There are  $(q^2 + 1)(q + 1)$  double planes, each with  $q^2 + q + 1$  points,*
- *There are  $\frac{1}{2}q(q^2 + q + 1)(q^2 + 1)(q + 1)$  pairs of two distinct planes, each pair with  $2q^2 + q + 1$  points*
- *There are  $\frac{1}{2}q^4(q^3 - 1)(q^2 + 1)$  hyperbolic quadrics, each with  $q^2 + 2q + 1$  points.*
- *There are  $\frac{1}{2}q^4(q^3 - 1)(q^2 - 1)$  elliptic quadrics, each with  $q^2 + 1$  points.*
- *There are  $q^2(q^3 - 1)(q^2 + 1)(q + 1)$  cones, each with  $q^2 + q + 1$  points.*
- *There are  $\frac{1}{2}q(q^3 - 1)(q^2 + 1)$  lines, each with  $q + 1$  points.*

In order to compute the Betti numbers of the parity check matroid of  $C(2, 3)_q$ , we will need the following lemma. An analogous result was proved in [9, Lemma 1] for the codes  $C(2, 2)_q$  and the proof for our codes  $C(2, 3)_q$  is identical:

**Lemma 20.** *For any  $X \subset E = \{1, \dots, q^2 + q + 1\}$  the nullity  $n(X)$  is equal to the dimension over  $\mathbb{F}_q$  of the affine set of polynomial expressions that define quadrics that pass through all the points of  $E - X$ .*

### 3. The resolution of the Stanley-Reisner ring of the parity check matroid

From now on, we set  $q = 2$ ,  $C = C(2, 3)_2$ , and  $M = M_C$ . The rest of the paper will mainly consist of finding the right input in order to be able to utilize the formulas (4). In this section we will simply find the  $\phi_j(M^{(0)}) = \phi_j(M)$ .

Later, in Section 7, these values will be used (via (4)), in combination with other input, to find the  $P_w(Z)$ .

By Proposition 19 the codewords of  $C$  have weights 4, 6, 8, 8, 10, 12, depending on whether the complements of their supports are plane pairs, hyperbolic quadrics, cones, double planes, elliptic quadrics or lines, respectively. Furthermore, by [16] we have:

$$d_1 = 4, d_2 = 6, d_3 = 7, d_4 = 8, d_5 = 10,$$

$$d_6 = 11, d_7 = 12, d_8 = 13, d_9 = 14, d_{10} = 15.$$

We then have  $rk(M) = 15 - 10 = 5$ .

We will identify minimal elements of  $N_i$  (the sets of nullity  $i$ ) for  $i = 1, \dots, 10$ , and then use Theorem 12 to obtain the structure of the minimal resolutions of the Stanley-Reisner ring of  $M$ . Since  $d_1 = 4$ , all  $X$  with  $|X| \leq 3$  are independent, so  $|X| \geq 4$  for all (minimal)  $X$  in  $N_1$ . Moreover  $n(X) \geq 7 - 5 = 2$  for all  $X$  with  $|X| \geq 7$ , so the only candidates for minimal elements in  $N_1$  are those  $X$  with  $|X| = 4, 5, 6$ . For  $|X| = 4$ , the only elements of  $N_1$  are the 105 supports of codewords that are complements of plane pairs. There are no minimal elements of  $N_1$  of cardinality 5 since their complements are necessarily contained in a pair of planes. Hence, by Theorems 12 and 11, the nullity 1 part of the resolution (1) is

$$S(-4)^{105} \oplus S(-6)^{\beta_{1,6}},$$

where the  $S(-6)$ -part is due to codewords of weight 6 with supports that are complements of hyperbolic quadrics that are not contained in plane pairs. Let us compute this number that we will need for the first elongation. From [7, Table 15.8], any plane intersects a hyperbolic quadric in either 3 or 5 points (in the latter case, the plane is a tangent plane at one point, and the intersection between the quadric and the plane is 2 lines intersecting at the tangent point). So, if the hyperbolic quadric (with 9 points) is contained in a pair of planes, these planes have to be tangent planes, say  $\Pi_1$  and  $\Pi_2$ , at distinct points  $P_1$  and  $P_2$  respectively. Let  $L_1^{(1)}, L_2^{(1)}, L_1^{(2)}, L_2^{(2)}$  be the lines  $\Pi_1 \cap H$  and  $\Pi_2 \cap H$

respectively. Then  $L_1^{(i)} \cap L_2^{(i)} = P_i$ . Let  $L = \Pi_1 \cap \Pi_2$ . If  $L$  is one of the lines  $L_j^{(i)}$ , then  $H \cap (\Pi_1 \cup \Pi_2)$  has 7 points. If not,  $L$  intersects the lines  $L_j^{(i)}$  in at least 2 distinct points (otherwise  $P_1 = P_2$ ), so that  $H \cap (\Pi_1 \cup \Pi_2)$  has at most 8 points. In any case, this shows that  $H$  is not included in  $\Pi_1 \cup \Pi_2$ , and  $b$  is thus equal to the number of hyperbolic quadrics, that is  $\beta_{1,6} = 280$ .

Let us analyze the nullity 2 term. Since  $d_2 = 6$ , all  $X$  with  $|X| \leq 5$  are out of the question. Since  $rk(M) = 5$ , all  $X$  with  $|X| \geq 8$  have  $n(X) \geq 8 - 5 = 3$ , so they are also out of the question. Hence we are left with  $X$  with  $|X| = 6, 7$ , and by Theorem 12 again the nullity 2 term is of type  $S(-6)^{c_1} \oplus S(-7)^{c_2}$  (here  $c_1, c_2$  could be zero a priori).

In a perfectly analogous manner we argue for the higher nullity terms. Hence the entire resolution looks like:

$$\begin{aligned} 0 \leftarrow S/I \leftarrow S \leftarrow S(-4)^{105} \oplus S(-6)^{280} \leftarrow S(-6)^{c_1} \oplus S(-7)^{c_2} \leftarrow S(-7)^{c_3} \oplus S(-8)^{c_4} \\ \leftarrow S(-8)^{c_5} \oplus S(-9)^{c_6} \leftarrow S(-10)^{c_7} \leftarrow S(-11)^{c_8} \leftarrow S(-12)^{c_9} \\ \leftarrow S(-13)^{c_{10}} \leftarrow S(-14)^{c_{11}} \leftarrow S(-15)^{c_{12}} \leftarrow 0. \end{aligned}$$

Remark 10 gives 10 independent equations in 10 variables (the  $\phi_j(M)$ ), and we get

$$\phi_j(M) = -770, 3960, -31185, 86240, -135828, 136080, -88935, 36960, -8910, 952$$

for  $6 \leq j \leq 15$ . Moreover, from  $\phi_6(M) = -770$ , we get that

$$c_1 = \beta_{2,6} = 280 + 770 = 1050. \quad (5)$$

#### 4. Resolution for the first elongation matroid

From Proposition 15 and Theorem 12, using the shape of the resolution of  $M$ , we see that the resolution associated to  $M^{(1)}$  looks like:

$$\begin{aligned} 0 \leftarrow S/I^{(1)} \leftarrow S \leftarrow S(-6)^{e_1} \oplus S(-7)^{e_2} \leftarrow S(-7)^{e_3} \oplus S(-8)^{e_4} \\ \leftarrow S(-8)^{e_5} \oplus S(-9)^{e_6} \leftarrow S(-10)^{e_7} \leftarrow S(-11)^{e_8} \leftarrow S(-12)^{e_9} \end{aligned}$$

$$\leftarrow S(-13)^{e_{10}} \leftarrow S(-14)^{e_{11}} \leftarrow S(-15)^{e_{12}} \leftarrow 0.$$

Remark 10 gives 9 independent equations in 10 variables (the  $\phi_j(M^{(1)})$ ). In order to compute them all, we need to compute one, say  $e_1$ . This corresponds to minimal elements of  $N_1^{(1)} = N_2$  of cardinality 6. We will now see that these elements corresponds to (complements) of either a plane and a line outside or four lines through a point, not three of them in the same plane. The way to do it is that we show that these two configurations are minimal in  $N_2$ , compute the number of such configurations, their local Betti numbers, and show that there can't be any other, since we already know  $\beta_{2,6}$ .

If  $X$  is minimal in  $N_2$ , this means that  $E - X$  is a set of points, maximal (for inclusion) with the properties that there are exactly 2 independent conics passing through them by Lemma 20. Since  $d_2 = 6$ , we already know that any set of cardinality 6 has nullity at most 2, so that we just need to show that these configurations have at least 2 independent conics passing through them. This will also show that  $X$  is minimal in  $N_2$ . So consider first a plane  $\Pi$  together with a line  $L$  not contained in  $\Pi$ . The union has clearly 9 points. And if  $\Pi_1$  and  $\Pi_2$  are two distinct planes passing through  $L$ , then the conics  $\Pi \cup \Pi_1$  and  $\Pi \cup \Pi_2$  are clearly independent, and containing  $\Pi \cup L$ . Then, in the case 4 lines  $L_1, L_2, L_3, L_4$  passing through a point, not 3 of them in the same plane, it has also clearly 9 points. If  $n(X) = 1$ , this means that the unordered pair of planes  $((L_1, L_2), (L_3, L_4))$  and  $((L_1, L_3), (L_2, L_4))$  are the same, which shows that either  $L_1, L_2, L_3$  or  $L_1, L_2, L_4$  lie on the same plane. Now, we compute the "local" contribution  $\beta_{2,X}$  for these 2 configurations, using Euler characteristic (Theorem 11). In both cases, since  $n(X) = 2$ , all subsets of cardinality 6 and 5 are dependent, and there are exactly 1 and 6 of them. All subsets of cardinality 3 or less are independent, so we just have to see how many subsets of cardinality 4 are dependent. In the first case, there are exactly 3 (corresponding to the 3 planes passing through the line  $L$ ). In the second case, also 3, corresponding to the plane pairs  $((L_1, L_2), (L_3, L_4)), ((L_1, L_3), (L_2, L_4))$  and  $((L_1, L_4), (L_2, L_3))$ .

In both cases then, we have

$$\beta_{2,X} = -1 + 6 - 3 = 2.$$

Finally, it is not difficult to see that there are  $15 \cdot (35 - 7) = 420$  plane-line configurations, and  $15 \cdot \frac{7 \cdot 6 \cdot 4 \cdot 1}{4!} = 105$  configurations of 4 lines through a point, not 3 of them in the same plane. Since

$$2 \cdot 420 + 2 \cdot 105 = 1050 = \beta_{2,6},$$

this shows us that the minimal elements of  $N_2$  are exactly the configurations listed above, and that there are 525 of them. Thus  $-\phi_6(M^{(1)}) = \beta_{1,6}^{(1)} = 525$ , and we get that

$$\phi_j(M^{(1)}) = -1710, 26145, -96040, 186102, -220500, 166110, -78120, 21015, -2478$$

for  $7 \leq j \leq 15$ .

## 5. Resolutions for the second and third elongation matroids

In a similar way as in the previous section, the resolutions for the second and third elongation matroids look like

$$\begin{aligned} 0 \leftarrow S/I^{(2)} \leftarrow S \leftarrow S(-7)^{f_1} \oplus S(-8)^{f_2} \leftarrow S(-8)^{f_3} \oplus S(-9)^{f_4} \leftarrow S(-10)^{f_5} \\ \leftarrow S(-11)^{f_6} \leftarrow S(-12)^{f_7} \leftarrow S(-13)^{f_8} \leftarrow S(-14)^{f_9} \leftarrow S(-15)^{f_{10}} \leftarrow 0. \end{aligned}$$

and

$$\begin{aligned} 0 \leftarrow S/I^{(3)} \leftarrow S \leftarrow S(-8)^{g_1} \oplus S(-9)^{g_2} \leftarrow S(-10)^{g_3} \leftarrow S(-11)^{g_4} \\ \leftarrow S(-12)^{g_5} \leftarrow S(-13)^{g_6} \leftarrow S(-14)^{g_7} \leftarrow S(-15)^{g_8} \leftarrow 0. \end{aligned}$$

respectively. In both cases, Remark 10 gives one independent equation less than the number of variables (the  $\phi_w(M^{(i)})$  for  $i = 2, 3$ ). So, as for the first elongation, we will find the number of minimal elements of  $N_1^{(2)} = N_3$  of cardinality 7 and  $N_1^{(3)} = N_4$  of cardinality 8 respectively.

Elements of  $N_2$  of cardinality 7 are necessarily complements of intersections of 2 quadrics. The only way to achieve that is to intersect 2 pair of planes, a plane and a hyperbolic quadric, or 2 hyperbolic quadrics, all of them with 8 points in the intersection. As before, in order for an hyperbolic quadric to intersect with a pair of planes in 8 points, the planes have to be tangent planes, and the intersection is 4 lines. From [5, E.1], the intersection of 2 hyperbolic quadrics with 8 points in the intersection is also 4 lines. Both intersections can thus be seen also as the intersection of 2 pairs of planes (with no points common to all the planes). It is not difficult to see that these configurations are minimal in  $N_2$ , and that they are  $\frac{105 \cdot 12 \cdot 8}{2 \cdot 2} = 2520$  such. Thus,  $\beta_{1,7}^{(1)} = 2520$ , and  $\beta_{2,7}^{(1)} = 2520 - 1710 = 810$ .

As we did in the previous section, we identify 2 types of sets of points, that together with the contribution of their local Betti numbers, fills  $\beta_{2,7}^{(1)}$ , so that we can compute  $\beta_{1,7}^{(2)}$ . These sets are on the one hand a plane with a point outside, and on the other hand 8 points so that no 3 of them are collinear. There are 120 sets of the first type, and 15 of the second. It is also easy to compute that  $\beta_{2,X}^{(1)} = 6$  for each of these sets. This leads us to

$$\phi_7(M^{(2)}) = -\beta_{7,1}^{(2)} = -(120 + 15) = -135,$$

and together with Remark 10,

$$\phi_j(M^{(2)}) = -5355, 36260, -100548, 154350, -142590, 79380, -24660, -3297$$

for  $8 \leq j \leq 15$ .

For the second resolution of this section, it is not hard to find that the minimal elements of  $N_4$  of cardinality are the complements of planes. This says that

$$\phi_8(M^{(3)}) = -\beta_{1,8}^{(3)} = -\#\{\text{planes of } \mathbb{P}_2^3\} = -15,$$

which, together with Remark 10 gives

$$\phi_j(M^{(3)}) = -4900, 26712, -60900, 74550, -51660, 19200, -2988$$

for  $9 \leq j \leq 15$ .

## 6. Resolutions for the remaining elongation matroids

The remaining elongation matroids are easier to deal with, since they have pure (and actually linear) resolutions. This means that the equations from Remark 10 are independent, with the same number of unknowns. So that we get

$$0 \leftarrow S/I^{(4)} \leftarrow S \leftarrow S(-10)^{3003} \leftarrow S(-11)^{13650} \leftarrow S(-12)^{25025} \\ \leftarrow S(-13)^{23100} \leftarrow S(-14)^{10725} \leftarrow S(-15)^{2002} \leftarrow 0,$$

$$0 \leftarrow S/I^{(5)} \leftarrow S \leftarrow S(-11)^{1365} \leftarrow S(-12)^{5005} \\ \leftarrow S(-13)^{6930} \leftarrow S(-14)^{4290} \leftarrow S(-15)^{1001} \leftarrow 0,$$

$$0 \leftarrow S/I^{(6)} \leftarrow S \leftarrow S(-12)^{455} \leftarrow S(-13)^{1260} \leftarrow S(-14)^{1170} \leftarrow S(-15)^{364} \leftarrow 0,$$

$$0 \leftarrow S/I^{(7)} \leftarrow S \leftarrow S(-13)^{105} \leftarrow S(-14)^{195} \leftarrow S(-15)^{91} \leftarrow 0,$$

$$0 \leftarrow S/I^{(8)} \leftarrow S \leftarrow S(-14)^{15} \leftarrow S(-15)^{14} \leftarrow 0,$$

$$0 \leftarrow S/I^{(9)} \leftarrow S \leftarrow S(-15) \leftarrow 0,$$

and

$$0 \leftarrow S/I^{(10)} \leftarrow S \leftarrow 0.$$

## 7. The generalized weight polynomials and higher weight spectra

In this section, we are finally able to compute the generalized weight polynomials and the higher weight spectra of  $C$ :

**Theorem 21.** *The code  $C(2, 3)_2$  has the following non-zero generalized weight polynomials*

$$\begin{aligned}
P_0(Z) &= 1, \\
P_4(Z) &= 105Z - 105, \\
P_6(Z) &= 525Z^2 - 1295Z + 770, \\
P_7(Z) &= 135Z^3 + 1575Z^2 - 5670Z + 3960, \\
P_8(Z) &= 15Z^4 + 5340Z^3 - 31500Z^2 + 57330Z - 31185, \\
P_9(Z) &= 4900Z^4 - 41160Z^3 + 132300Z^2 - 182280Z + 86240, \\
P_{10}(Z) &= 3003Z^5 - 29715Z^4 + 127260Z^3 - 286650Z^2 + 321930Z - 135828, \\
P_{11}(Z) &= 1365Z^6 - 15015Z^5 + 74550Z^4 - 215250Z^3 + 374850Z^2 - 356580Z \\
&\quad + 136080, \\
P_{12}(Z) &= 455Z^7 - 5460Z^6 + 30030Z^5 - 99575Z^4 + 217140Z^3 - 308700Z^2 \\
&\quad + 255045Z - 88935, \\
P_{13}(Z) &= 105Z^8 - 1365Z^7 + 8190Z^6 - 30030Z^5 + 74760Z^4 - 131040Z^3 \\
&\quad + 157500Z^2 - 115080Z + 36960, \\
P_{14}(Z) &= 15Z^9 - 210Z^8 + 1365Z^7 - 5460Z^6 + 15015Z^5 - 29925Z^4 + 43860Z^3 \\
&\quad - 45675Z^2 + 29925Z - 8910, \\
P_{15}(Z) &= Z^{10} - 15Z^9 + 105Z^8 - 455Z^7 + 1365Z^6 - 3003Z^5 + 4990Z^4 \\
&\quad - 6285Z^3 + 5775Z^2 - 3430Z + 952.
\end{aligned}$$

*In particular the generalized Hamming weights of  $C(2, 3)_2$  are given by  $d_1 = 4, d_2 = 6, d_3 = 7, \dots, d_{10} = 15$ .*

PROOF. This is a direct consequence of Theorem 16 and the computations done in Sections 3-6.

Finally, from Theorem 6, we get the higher weight spectra.

**Theorem 22.** *The non-zero higher weight spectra of the code  $C(2, 3)_2$  are*

$$\begin{array}{llll}
A_4^{(1)} = 105 & A_6^{(1)} = 280 & A_8^{(1)} = 435 & A_{10}^{(1)} = 168 \\
A_{12}^{(1)} = 35 & A_6^{(2)} = 525 & A_7^{(2)} = 2520 & A_8^{(2)} = 6405 \\
A_9^{(2)} = 156805 & A_{10}^{(2)} = 29610 & A_{11}^{(2)} = 38640 & A_{12}^{(2)} = 39830 \\
A_{13}^{(2)} = 26880 & A_{14}^{(2)} = 11865 & A_{15}^{(2)} = 2296 & A_7^{(3)} = 135 \\
A_8^{(3)} = 5565 & A_9^{(3)} = 323405 & A_{10}^{(3)} = 147000 & A_{11}^{(3)} = 479850 \\
A_{12}^{(3)} = 1135470 & A_{13}^{(3)} = 1840020 & A_{14}^{(3)} = 1845600 & A_{15}^{(3)} = 861735 \\
A_8^{(4)} = 15 & A_9^{(4)} = 4900 & A_{10}^{(4)} = 63378 & A_{11}^{(4)} = 497700 \\
A_{12}^{(4)} = 2650900 & A_{13}^{(4)} = 9436140 & A_{14}^{(4)} = 20484030 & A_{15}^{(4)} = 20606924 \\
A_{10}^{(5)} = 3003 & A_{11}^{(5)} = 70980 & A_{12}^{(5)} = 899535 & A_{13}^{(5)} = 7046760 \\
A_{14}^{(5)} = 32555145 & A_{15}^{(5)} = 68646228 & A_{11}^{(6)} = 1365 & A_{12}^{(6)} = 52325 \\
A_{13}^{(6)} = 968310 & A_{14}^{(6)} = 9721470 & A_{15}^{(6)} = 43000517 & A_{12}^{(7)} = 455 \\
A_{13}^{(7)} = 25410 & A_{14}^{(7)} = 599340 & A_{15}^{(7)} = 5722510 & A_{13}^{(8)} = 105 \\
A_{14}^{(8)} = 7455 & A_{15}^{(8)} = 166691 & A_{14}^{(9)} = 15 & A_{15}^{(9)} = 1008 \\
A_{15}^{(10)} = 1 & & & 
\end{array}$$

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