



RESEARCH PAPER

INTEGRABILITY PROPERTIES OF INTEGRAL
TRANSFORMS VIA MORREY SPACES

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Abstract

We show that integrability properties of integral transforms with kernel depending on the product of arguments (which include in particular, popular Laplace, Hankel, Mittag-Leffler transforms and various others) are better described in terms of Morrey spaces than in terms of Lebesgue spaces. Mapping properties of integral transforms of such a type in Lebesgue spaces, including weight setting, are known. We discover that local weighted Morrey and complementary Morrey spaces are very appropriate spaces for describing integrability properties of such transforms. More precisely, we show that under certain natural assumptions on the kernel, transforms under consideration act from local weighted Morrey space to a weighted complementary Morrey space and vice versa, where an interplay between behavior of functions and their transforms at the origin and infinity is transparent. In case of multidimensional integral transforms, for this goal we introduce and use anisotropic mixed norm Morrey and complementary Morrey spaces.

MSC 2010: Primary 46E30, 42C20; Secondary 44A05, 44A10, 44A30

Key Words and Phrases: integral transforms; Laplace transform; Hankel transform; Mittag-Leffler transform; fractional calculus; harmonic analysis; Morrey spaces; vanishing Morrey spaces; anisotropic Morrey spaces; operators with homogeneous kernels; dilation invariant operators

1. Introduction

The main goal of this paper is to show that weighted local Morrey spaces and the so called complementary Morrey spaces provide a very natural language for describing integrability properties of integral transforms

$$Af(x) = \int_0^\infty k(xt)f(t)dt, \quad x > 0, \tag{1.1}$$

in particular, Laplace transform

$$\mathcal{L}f(x) = \int_0^\infty e^{-xt}f(t)dt, \quad x > 0. \tag{1.2}$$

Integral transforms of type (1.1) are well known to be widely used in various fields of mathematics, including fractional calculus, and various applications, with a variety of books on this topic. We refer, for instance, to the books [9], [10], [12], [13], [36], [40] and [48], articles [22], [27], [26], [28], [37], [44], [46], [47] and references therein. Integral transforms of such a type are well known to be used in the study of differential equations of fractional order, see for instance [5]. Operators of the form (1.1), besides the Laplace transform, include in particular various forms of Bessel transform, Mittag-Leffler transform and others. Many concrete transforms of the form (1.1) are particular cases of so called *H*-transforms (see [25]).

It is well known that the scale of Lebesgue spaces is not well adjusted for mapping properties of the Laplace transform. This concerns in general operators of the form (1.1). By dilation arguments it is easy to show that if

$$A : L^p(\mathbb{R}_+) \hookrightarrow L^q(\mathbb{R}_+),$$

then necessarily $q = p' := \frac{p}{p-1}$, $p, q \in [1, \infty]$, so that $L^p \rightarrow L^p$ mapping is possible only for $p = 2$. For the Laplace transform the condition $q = p'$ is necessary and sufficient, when $1 \leq p \leq 2$, as shown by G.H. Hardy [23, Theorem 9]. In [23], G.H. Hardy also proved the one-weight $L^p \rightarrow L^p$ boundedness of operators *A* in the form

$$\int_0^\infty |Af(x)|^p dx \leq c \int_0^\infty |f(x)|^p x^{p-2} dx$$

and

$$\int_0^\infty |Af(x)|^p x^{p-2} dx \leq c \int_0^\infty |f(x)|^p dx$$

under some condition on the kernel $k(x)$ (necessary and sufficient when $k(x) \geq 0$). Various studies were aimed at improving candidates for the target space admitting more general weighted $L^p(u) \rightarrow L^q(v)$ -setting, see for instance [4] and [7], and also in the frameworks of rearrangement invariant spaces in [3].

Since the Laplace transform is well defined on appropriate functions f when x is replaced by $z = x + iy$ with $x > 0$, integrability of Laplace transforms of L^p functions was also studied on the half plane $\{(x, y) : x > 0\}$. We refer to [39] and references therein.

In the real value setting, recently, within the frameworks of rearrangement invariant function spaces there was achieved a progress in characterizing the best possible domain-target candidates, see [11] and [15] and references therein.

To our surprise, Laplace transform or any integral transform of the form (1.1) was never studied in Morrey spaces, up to our knowledge. Meanwhile, local Morrey spaces and their complementary counterparts provide a very natural language for domain and target spaces for integral transforms of the form (1.1), because functions in weighted local Morrey space have better behavior at the origin and worse at infinity in comparison with functions in weighted Lebesgue spaces, and vice versa for complementary Morrey space. We refer to Section 2 for definition of local Morrey and complementary Morrey spaces. In particular, we show that for any operator A of the form (1.1), under certain conditions on its kernel $k(t)$, there hold the inequalities

$$\sup_{r>0} r^\lambda \int_r^\infty x^b |Af(x)|^p dx \leq C \sup_{r>0} \frac{1}{r^\lambda} \int_0^r x^a |f(x)|^p dx, \quad \lambda \geq 0, \quad (1.3)$$

$$\sup_{r>0} \frac{1}{r^\lambda} \int_0^r x^b |Af(x)|^p dx \leq C \sup_{r>0} r^\lambda \int_r^\infty x^a |f(x)|^p dx, \quad \lambda \geq 0, \quad (1.4)$$

where $a, b \in \mathbb{R}$ are related to each other by the (necessary and sufficient) condition

$$a + b = p - 2;$$

for positive kernels $k(t)$ we also find sharp constants in the above inequalities. Moreover, we also show that

$$\lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_0^r x^a |f(x)|^p dx = 0 \implies \lim_{r \rightarrow \infty} r^\lambda \int_r^\infty x^b |Af(x)|^p dx = 0, \quad \lambda > 0 \quad (1.5)$$

and

$$\lim_{r \rightarrow \infty} r^\lambda \int_r^\infty x^a |f(x)|^p dx = 0 \implies \lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_0^r x^b |Af(x)|^p dx = 0, \quad \lambda > 0, \quad (1.6)$$

under the same conditions on the kernel.

We also prove some multi-dimensional versions of such statements. The main attention is paid to integral transforms on $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}$ with kernel depending on $x \circ y = (x_1 y_1, \dots, x_n y_n)$, which are mostly used in applications. The well known Morrey spaces on \mathbb{R}^n or domain in \mathbb{R}^n , where there are measured averages of functions over balls or cubes, are not suitable for this goal. To this end, we introduce anisotropic Morrey spaces with measuring averages over n -dimensional rectangles $\mathcal{R}_h = \{x \in \mathbb{R}_+^n : 0 < x_i < h_i, i = 1, \dots, n\}$, $h = (h_1, \dots, h_n) \in \mathbb{R}_+^n$.

The proofs are based on our results on integral operators, commuting with dilations, in Morrey spaces, obtained in [41]. It is used in the one-dimensional case and in the isotropic case in Subsection 4.1. For the study of multidimensional integral transforms with kernel of the form $k(x \circ y)$, we adapt results from [41] for the case of anisotropic Morrey spaces.

The paper is structured as follows. In Section 2 we provide necessary preliminaries on local Morrey and complementary Morrey spaces and operators with a kernel homogeneous of degree $-n$ in \mathbb{R}^n , in Morrey spaces. The main results for the one-dimensional case are given in Section 3, where in particular, we prove statements (1.3) - (1.6) in Theorems 3.1 and 3.2. In Subsection 3.1 we give general theorems and in Subsection 3.2 we consider application to some concrete integral transforms, Mittag-Leffler transform in particular. In Section 4 we consider multidimensional versions of integral transforms. In Subsection 4.1 we briefly consider the radial Laplace transform. Subsection 4.2 contains definition of anisotropic mixed norm Morrey and complementary Morrey spaces. In Subsection 4.3 we extend some results from [41] to the anisotropic setting. Subsection 4.4 contains the main statements for integral transforms with the kernel $k(x \circ y)$. Finally, a brief Subsection 4.5 contains some additional remarks.

2. Preliminaries

2.1. Weighted local Morrey and complementary Morrey spaces.

For a function f on \mathbb{R}^n we introduce the notation for the following modular

$$\mathfrak{M}_0^{p,\lambda,\gamma}(f,r) := \frac{1}{r^\lambda} \int_{|x|<r} (|f(x)||x|^\gamma)^p dx, \tag{2.1}$$

where $1 \leq p < \infty, \lambda \geq 0$ and $\gamma \in \mathbb{R}$.

Weighted local Morrey spaces $L^{p,\lambda,\gamma}(\mathbb{R}^n)$ are defined by the norm

$$\|f\|_{L^{p,\lambda,\gamma}} = \sup_{r>0} \left(\mathfrak{M}_0^{p,\lambda,\gamma}(f,r) \right)^{\frac{1}{p}}.$$

Recall that

$$L^{p,\lambda,\gamma}(\mathbb{R}^n)|_{\lambda=0} = L^{p,\gamma}(\mathbb{R}^n) =: \left\{ \int_{\mathbb{R}^n} |f(x)|x|^\gamma|^p dx < \infty \right\}.$$

Vanishing weighted local Morrey space $V_0L^{p,\lambda,\gamma}(\mathbb{R}^n)$, $\lambda > 0$, is defined as the set of functions in $L^{p,\lambda,\gamma}(\mathbb{R}^n)$, which satisfy the condition

$$\lim_{r \rightarrow 0} \mathfrak{M}_0^{p,\lambda,\gamma}(f, r) = 0. \quad (2.2)$$

The set $V_0L^{p,\lambda,\gamma}(\mathbb{R}^n)$ is a closed subspace of $L^{p,\lambda,\gamma}(\mathbb{R}^n)$.

We also need weighted complementary Morrey spaces ${}^cL^{p,\lambda,\gamma}(\mathbb{R}^n)$ similarly defined via the following modular

$$\mathfrak{M}_\infty^{p,\lambda,\gamma}(f, r) := r^\lambda \int_{|x|>r} (|f(x)||x|^\gamma)^p dx \quad (2.3)$$

by the norm

$$\|f\|_{{}^cL^{p,\lambda,\gamma}(\mathbb{R}^n)} = \sup_{r>0} \left(\mathfrak{M}_\infty^{p,\lambda,\gamma}(f, r) \right)^{\frac{1}{p}},$$

and the vanishing complementary weighted Morrey space

$$V_\infty {}^cL^{p,\lambda,\gamma}(\mathbb{R}^n)$$

is defined by the condition $\lim_{r \rightarrow \infty} \mathfrak{M}_\infty^{p,\lambda,\gamma}(f, r) = 0$.

Complementary Morrey spaces were introduced in [19] and [20]. Like Morrey spaces they are also known to be used in analysis, see for instance [21] and references therein.

In the one-dimensional case $n = 1$ we consider Morrey and complementary Morrey spaces on the semi-axis \mathbb{R}_+ instead of \mathbb{R} , so that the corresponding modulars in this case are

$$\mathfrak{M}_0^{p,\lambda,\gamma}(f, r) = \frac{1}{r^\lambda} \int_0^r |x^\gamma f(x)|^p dx \quad \text{and} \quad \mathfrak{M}_\infty^{p,\lambda,\gamma}(f, r) = r^\lambda \int_r^\infty |x^\gamma f(x)|^p dx.$$

We refer to the books [1], [35], [45] and surveying papers [38], [42] for Morrey spaces. Note that Morrey and Morrey-Campanato spaces attract increasing attention of researchers during last decades, due to both the interesting structure of spaces and various their applications, and a big variety of interesting papers annually appears; we refer for instance to [2], [32], [33].

We shall also use local anisotropic Morrey spaces on $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}$, but we find it more appropriate to introduce them later, in Section 4.2.

2.2. On operators commuting with dilation and rotations in Morrey spaces. In the paper [41], within the frameworks of Morrey spaces, there were studied integral operators

$$Kf(x) = \int_{\mathbb{R}^n} \mathcal{K}(x, y)f(y)dy, \quad x \in \mathbb{R}^n$$

with the kernel homogeneous of degree $-n$, i.e. $\mathcal{K}(tx, ty) = t^{-n}\mathcal{K}(x, y), t > 0$, and invariant with respect to rotations in $\mathbb{R}^n : \mathcal{K}[\omega(x), \omega(y)] = \mathcal{K}(x, y)$, where $\omega : x \rightarrow \omega(x), |\omega(x)| = |x|$, is an arbitrary rotation.

For the study of such operators in Lebesgue spaces, corresponding to the case $\lambda = 0$, we refer to [24].

Everywhere in the sequel we use the notation $e_1 = (1, 0, \dots, 0)$.

Denote

$$\varkappa(p, \lambda, \gamma) := \int_{\mathbb{R}^n} |\mathcal{K}(e_1, y)| \frac{dy}{|y|^{\frac{n-\lambda}{p} + \gamma}}, \quad n \geq 2 \tag{2.4}$$

with one-dimensional modification

$$\varkappa(p, \lambda, \gamma) := \int_0^\infty |\mathcal{K}(1, y)| \frac{dy}{y^{\frac{1-\lambda}{p} + \gamma}} < \infty, \quad \text{for } n = 1. \tag{2.5}$$

The statement of the following Proposition A, is derived from Theorem 4.2 and Corollary 4.3 in [41]: results in [41] concern the case $\gamma = 0$, which easily leads to the statement of the proposition since $\frac{|x|^\gamma}{|y|^\gamma} \mathcal{K}(x, y)$ satisfies the above dilation and rotation conditions, if $\mathcal{K}(x, y)$ does.

PROPOSITION A.

Let $1 \leq p < \infty, \lambda \geq 0$ and $\gamma \in \mathbb{R}$. Under the condition $\varkappa(p, \lambda, \gamma) < \infty$, the operator K is bounded in the spaces $L^{p, \lambda, \gamma}$ and $V_0L^{p, \lambda, \gamma}$ and

$$\|Kf\|_{L^{p, \lambda, \gamma}} \leq \varkappa(p, \lambda, \gamma) \|f\|_{L^{p, \lambda, \gamma}}. \tag{2.6}$$

When $\mathcal{K}(x, y)$ is non-negative, the condition $\varkappa(p, \lambda, \gamma) < \infty$ is also necessary for the boundedness in both the spaces $L^{p, \lambda, \gamma}$ and $V_0L^{p, \lambda, \gamma}$. Moreover, $\varkappa(p, \lambda, \gamma)$ is the sharp constant and when $\lambda > 0, f(x) = |x|^{\frac{\lambda-n}{p} - \gamma}$ is the minimizing function in the case of the space $L^{p, \lambda, \gamma}$.

2.3. Auxiliaries: On isometry between weighted local Morrey and complementary Morrey spaces. Let

$$Q_\ell f(x) = \frac{1}{|x|^\ell} f\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^n \setminus \{0\}, \ell \in \mathbb{R}, \tag{2.7}$$

so that $Q_\ell^2 = I$.

LEMMA 2.1. *Let $1 \leq p < \infty$, $\lambda \geq 0$ and $\gamma \in \mathbb{R}$. Then the following relations hold:*

$$\mathfrak{M}_0^{p,\lambda,\gamma}(Q_\ell f, r) = \mathfrak{M}_\infty^{p,\lambda,\delta}\left(f, \frac{1}{r}\right) \tag{2.8}$$

and

$$\|Q_\ell f\|_{L^{p,\lambda,\gamma}} = \|f\|_{cL^{p,\lambda,\delta}}, \tag{2.9}$$

where $\gamma + \delta = \ell - \frac{2n}{p}$.

P r o o f. The equality (2.9) for norms follows from (2.8). For (2.8) we have

$$\mathfrak{M}_0^{p,\lambda,\gamma}(Q_\ell f, r) = \frac{1}{r^\lambda} \int_{|x|<r} \left| |x|^{\gamma-\ell} f\left(\frac{x}{|x|^2}\right) \right|^p dx = \frac{1}{r^\lambda} \int_{|y|>\frac{1}{r}} \| |y|^{\ell-\gamma} f(y) \|^p \frac{dy}{|y|^{2n}}$$

via the change of variables $x = \frac{y}{|y|^2}$ with the Jacobian $|y|^{-2n}$, and we arrive at (2.8). □

For the dilation operator

$$\Pi_t f(x) = f(tx), \quad t > 0,$$

we have

$$\|\Pi_t f\|_{L^{p,\lambda,\gamma}} = \frac{1}{t^{\frac{1-\lambda}{p}+\gamma}} \|f\|_{L^{p,\lambda,\gamma}} \tag{2.10}$$

and

$$\|\Pi_t f\|_{cL^{p,\lambda,\gamma}} = \frac{1}{t^{\frac{1+\lambda}{p}+\gamma}} \|f\|_{cL^{p,\lambda,\gamma}} \tag{2.11}$$

for $1 \leq p < \infty$, $\lambda \geq 0$ and $\gamma \in \mathbb{R}$, which is a matter of direct verification.

3. Main results: one-dimensional case

3.1. **Main theorems.** In Theorem 3.1 we demonstrate a certain advantage of Morrey spaces in describing mapping properties of integral transforms of the form (1.1). Namely, we consider mapping properties

$$A : L^{p,\lambda,\gamma} \hookrightarrow cL^{p,\lambda,\delta} \tag{3.1}$$

and

$$A : cL^{p,\lambda,\gamma} \hookrightarrow L^{p,\lambda,\delta}, \tag{3.2}$$

where $1 \leq p < \infty$, $\lambda \geq 0$ and $\delta, \gamma \in \mathbb{R}$. Mappings (3.1), (3.2) and properties (3.10) and (3.11), proved in the sequel, clearly show how behavior of functions at the origin (at infinity) influences on behavior of the transform at infinity (at the origin, respectively).

We start with the following lemma on relation between the weight exponents δ and γ .

LEMMA 3.1. *Each of the mapping properties (3.1) and (3.2) may hold only in the case when*

$$\delta + \gamma = \frac{p - 2}{p}. \tag{3.3}$$

P r o o f. Suppose that (3.1) takes place:

$$\|Af\|_{cL^{p,\lambda,\delta}} \leq C\|f\|_{L^{p,\lambda,\gamma}}.$$

To show that then (3.3) necessarily holds, we use the dilation trick, effectively working for integral operators with kernels having any kind of homogeneity, and well known for instance for Riesz potential operators in Lebesgue spaces, see [43]. By our assumption we also have that

$$\|A\Pi_t f\|_{cL^{p,\lambda,\delta}} \leq C\|\Pi_t f\|_{L^{p,\lambda,\gamma}}$$

for all $t > 0$. It is easy to see that $A\Pi_t = \frac{1}{t}\Pi_{\frac{1}{t}}A$. Consequently,

$$\|\Pi_{\frac{1}{t}}Af\|_{cL^{p,\lambda,\delta}} \leq Ct\|\Pi_t f\|_{L^{p,\lambda,\gamma}}$$

Applying the formulas (2.10) and (2.11) we arrive at

$$\|Af\|_{cL^{p,\lambda,\delta}} \leq Ct^{\frac{p-2}{p}-\delta-\gamma}\|f\|_{L^{p,\lambda,\gamma}}.$$

Hence (3.3) should hold.

The case of the mapping property (3.2) is similarly treated. □

In view of Lemma 3.1, all the exponents δ and γ appearing in the sequel, will be related to each other by the condition (3.3).

Denote

$$\varkappa_0(p, \lambda, \gamma) := \int_0^\infty |k(t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\gamma}}. \tag{3.4}$$

and

$$\varkappa_\infty(p, \lambda, \gamma) := \int_0^\infty |k(t)| \frac{dt}{t^{\frac{1+\lambda}{p}+\gamma}}. \tag{3.5}$$

THEOREM 3.1. *Let $1 \leq p < \infty$, $\lambda \geq 0$ and $\gamma \in \mathbb{R}$. If $\varkappa_0(p, \lambda, \gamma) < \infty$, then $A : L^{p,\lambda,\gamma} \rightarrow cL^{p,\lambda,\frac{p-2}{p}-\gamma}$ and*

$$\|Af\|_{cL^{p,\lambda,\frac{p-2}{p}-\gamma}} \leq \varkappa_0(p, \lambda, \gamma)\|f\|_{L^{p,\lambda,\gamma}}. \tag{3.6}$$

If $\varkappa_\infty(p, \lambda, \gamma) < \infty$, then $A : cL^{p,\lambda,\gamma} \rightarrow L^{p,\lambda,\frac{p-2}{p}-\gamma}$ and

$$\|Af\|_{L^{p,\lambda,\frac{p-2}{p}-\gamma}} \leq \varkappa_\infty(p, \lambda, \gamma)\|f\|_{cL^{p,\lambda,\gamma}}. \tag{3.7}$$

If $k(x) \geq 0$, $x \in \mathbb{R}_+$, then the conditions $\varkappa_0(p, \lambda, \gamma) < \infty$ and $\varkappa_\infty(p, \lambda, \gamma) < \infty$ are also necessary for the boundedness (3.6) and (3.7), respectively, and

the constants in (3.6) and (3.7) are sharp, and when $\lambda \neq 0$, the minimizing functions are $f(x) = x^{\frac{\lambda-1}{p}-\gamma}$ and $f(x) = x^{-\frac{\lambda+1}{p}-\gamma}$, respectively.

P r o o f. With the notation $Q_1 = Q_\ell|_{\ell=1}$ we have

$$Q_1 A = K,$$

where

$$Kf(x) = \int_0^\infty \mathcal{K}(x, t)f(t)dt, \quad \mathcal{K}(x, t) = \frac{1}{x} k\left(\frac{t}{x}\right).$$

Hence

$$\|Q_1 Af\|_{L^{p,\lambda,\gamma}} = \|Kf\|_{L^{p,\lambda,\gamma}}.$$

By Lemma 2.1 we then have

$$\|Af\|_{cL^{p,\lambda,\frac{p-2}{p}-\gamma}} = \|Kf\|_{L^{p,\lambda,\gamma}}. \tag{3.8}$$

By Proposition A, the operator K is bounded in $L^{p,\lambda,\gamma}$, if

$$\int_0^\infty |\mathcal{K}(1, t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\gamma}} = \int_0^\infty |k(t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\gamma}},$$

i.e. $\varkappa_0(p, \lambda, \gamma) < \infty$. Therefore, under this condition from (3.8) by Proposition A, we have

$$\|Af\|_{cL^{p,\lambda,\frac{p-2}{p}-\gamma}} \leq \varkappa_0(p, \lambda, \gamma) \|f\|_{L^{p,\lambda,\gamma}}$$

with the sharp constant when $k(x) \geq 0, x \in \mathbb{R}_+$, which proves (3.6).

To prove (3.7) we proceed as follows

$$\begin{aligned} Af(x) &= \int_0^\infty k(xt)f(t)dt = \int_0^\infty k\left(\frac{x}{t}\right) f\left(\frac{1}{t}\right) \frac{dt}{t^2} = \int_0^\infty \mathcal{K}^*(x, t)(Q_1 f)(t)dt \\ &=: K^*Q_1 f(x), \text{ where } \mathcal{K}^*(x, t) = \frac{1}{t} k\left(\frac{x}{t}\right). \end{aligned}$$

Hence

$$\|Af\|_{L^{p,\lambda,\delta}} = \|K^*Q_1 f\|_{L^{p,\lambda,\delta}}, \tag{3.9}$$

where we chose $\delta = \frac{p-2}{p} - \gamma$. By Proposition A, the operator K^* is bounded in $L^{p,\lambda,\delta}$ if

$$\int_0^\infty |\mathcal{K}^*(1, t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\delta}} = \int_0^\infty \left|k\left(\frac{1}{t}\right)\right| \frac{dt}{t^{1+\frac{1-\lambda}{p}+\delta}} < \infty,$$

which is nothing else but the condition $\varkappa_\infty(p, \lambda, \gamma) < \infty$. Therefore by the boundedness of the operator K^* in $L^{p,\lambda,\delta}$ and isometry provided by Lemma 2.1, we arrive at (3.7).

Necessity of the conditions $\varkappa_0(p, \lambda, \gamma) < \infty$ and $\varkappa_\infty(p, \lambda, \gamma) < \infty$ for the corresponding mapping properties immediately follows from the representations $Q_1 = K, A = K^*Q_1$, the same isometry and Proposition A.

The choice of the minimizing function $f(x) = x^{-\frac{\lambda+1}{p}-\gamma}$ for (3.6) is dictated by Proposition A, via the identity $Q_1A = K$. As regards the minimizing function f for (3.7), from the identity $A = K^*Q_1$, it is clear that it should be chosen so that

$$Q_1f = x^{\frac{\lambda-1}{p}-\delta},$$

from which there follows that $f(x) = x^{-\frac{\lambda+1}{p}-\gamma}$. □

THEOREM 3.2. *Let $1 \leq p < \infty$, $\lambda > 0$ and $\gamma \in \mathbb{R}$. Then the operator A is bounded from $V_0L^{p,\lambda,\gamma}$ to $V_\infty^cL^{p,\lambda,\delta}$ and from $V_\infty^cL^{p,\lambda,\gamma}$ to $V_0L^{p,\lambda,\delta}$, where $\gamma + \delta = \frac{p-2}{p}$, under the conditions $\varkappa_0(p, \lambda, \gamma) < \infty$ and $\varkappa_\infty(p, \lambda, \gamma) < \infty$, respectively, so that there hold the following “regularity properties”*

$$\lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_0^r |y^\gamma f(y)|^p dy = 0 \Rightarrow \lim_{r \rightarrow \infty} r^\lambda \int_r^\infty |y^{-\gamma} Af(y)|^p y^{p-2} dy = 0, \tag{3.10}$$

when $\varkappa_0(p, \lambda, \gamma) < \infty$, and

$$\lim_{r \rightarrow \infty} r^\lambda \int_r^\infty |y^\gamma f(y)|^p dy = 0 \Rightarrow \lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_0^r |y^{-\gamma} Af(y)|^p y^{p-2} dy = 0, \tag{3.11}$$

when $\varkappa_\infty(p, \lambda, \gamma) < \infty$.

P r o o f. The statements of the theorem follow from Proposition A, and the relation (2.8). □

3.2. Application to concrete integral transforms. The reader can easily derive corresponding results for various concrete examples of integral transforms from Theorems 3.1 and 3.2. We formulate such a corollary in detail for the Laplace transform because of its wide popularity, and briefly sketch arising conditions for mapping properties for such famous integral transforms as Bessel-type and Mittag-Leffler ones.

Laplace transform

For the Laplace transform

$$\mathcal{L}f(x) = \int_0^\infty e^{-xt} f(t) dt$$

we arrive at the following corollary.

COROLLARY 3.1. *Let $1 \leq p < \infty$ and $\gamma, \delta \in \mathbb{R}$. The Laplace transform satisfies mapping properties (3.1) and (3.2) if and only if $\gamma + \delta = \frac{p-2}{p}$ and $1 - \lambda < (1 - \gamma)p$ and $1 + \lambda < (1 - \gamma)p$, respectively, and*

$$\sup_{r>0} r^\lambda \int_r^\infty |x^\delta \mathcal{L}f(x)|^p dx \leq \Gamma \left(1 - \gamma + \frac{\lambda - 1}{p} \right)^p \sup_{r>0} \frac{1}{r^\lambda} \int_0^r |x^\gamma f(x)|^p dx, \quad \lambda \geq 0, \quad (3.12)$$

$$\sup_{r>0} \frac{1}{r^\lambda} \int_0^r |x^\delta \mathcal{L}f(x)|^p dx \leq \Gamma \left(1 - \gamma - \frac{\lambda + 1}{p} \right)^p \sup_{r>0} r^\lambda \int_r^\infty |x^\gamma f(x)|^p dx, \quad \lambda \geq 0, \quad (3.13)$$

with the best constants in (3.12) and (3.13), and also

$$\lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_0^r |x^\gamma f(x)|^p dx = 0 \implies \lim_{r \rightarrow \infty} r^\lambda \int_r^\infty |x^\delta \mathcal{L}f(x)|^p dx = 0, \quad \lambda > 0 \quad (3.14)$$

$$\lim_{r \rightarrow \infty} r^\lambda \int_r^\infty |x^\gamma f(x)|^p dx = 0 \implies \lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_0^r |x^\delta \mathcal{L}f(x)|^p dx = 0, \quad \lambda > 0, \quad (3.15)$$

where $1 - \lambda < (1 - \gamma)p$ in (3.12) and (3.14), and $1 + \lambda < (1 - \gamma)p$ in (3.13) and (3.15).

Bessel transform of Hankel-type

For the Bessel-type transform

$$B_{\mu,\nu} f(x) = \int_0^\infty (xy)^\mu J_\nu(xy) f(y) dy, \quad \mu, \nu \in \mathbb{R}, \quad \nu > -\frac{1}{2},$$

where

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

is the Bessel function of the first kind, the sufficient condition for the mappings (3.1) and (3.2) with $\gamma + \delta = \frac{p-2}{p}$ are

$$\gamma - \mu - \nu < \frac{1}{p'} + \frac{\lambda}{p} < \gamma - \mu + \frac{1}{2}$$

and

$$\gamma - \mu - \nu < \frac{1}{p'} - \frac{\lambda}{p} < \gamma - \mu + \frac{1}{2},$$

respectively.

Bessel transform with McDonald function in the kernel (Meijer transform)

Another well known (see for instance [13]) Bessel-type transform is

$$\mathcal{K}_{\mu,\nu}f(x) = \int_0^\infty (xt)^\mu K_\nu(xt)f(t)dt, \tag{3.16}$$

where

$$K_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^{-\nu} \int_0^\infty t^{\nu-1} e^{-t-\frac{x^2}{4t}} dt, \nu > 0$$

is the Bessel-type function known as the McDonald function.

It is known that

$$\int_0^\infty \frac{K_\nu(x)}{x^\beta} dx = 2^{-\beta-1} \Gamma\left(\frac{1-\beta+\nu}{2}\right) \Gamma\left(\frac{1-\beta-\nu}{2}\right), \beta + \nu < 1, \tag{3.17}$$

see [18, 8.432(6)].

From Theorems 3.1 and 3.2, for the transform $\mathcal{K}_{\mu,\nu}$ we obtain the following.

The operator $\mathcal{K}_{\mu,\nu}$ possesses the mapping properties (3.6) and (3.10) if and only if

$$\gamma + \nu - \mu < \frac{\lambda}{p} + \frac{1}{p'},$$

with the sharp constant

$$\varkappa_0 = 2^{\frac{\lambda-1}{p} + \gamma - \mu - 1} \Gamma\left(\frac{\frac{\lambda}{p} + \frac{1}{p'} - \gamma + \mu + \nu}{2}\right) \Gamma\left(\frac{\frac{\lambda}{p} + \frac{1}{p'} - \gamma + \mu - \nu}{2}\right)$$

in (3.6). In the formula for the sharp constant we used the relation (3.17). For the mapping properties (3.7) and (3.11) the statement is formulated in the same way, with only change that λ should be replaced by $-\lambda$ in all the conditions and formulas.

Mittag-Leffler transform

Another popular integral transform with various forms is the Mittag-Leffler transform with the Mittag-Leffler function in the kernel, see for instance [14] and [48, Ch. 16]. Here we consider such a transform in the form

$$\mathcal{E}_{\alpha,\beta}f(x) = \int_0^\infty E_{\alpha,\beta}(-xt)f(t)dt, \tag{3.18}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in \mathbb{C}$$

is the (two-index) Mittag-Leffler function. A comprehensive source on Mittag-Leffler functions is e.g. the book [16]. We refer to [8], where for the goals of inversion, it was shown that the transform (3.18) may be regarded as a particular case of integral *H-transforms*.

We consider the Mittag-Leffler transform (3.18) under the condition

$$0 \leq \alpha \leq \min\{1, \beta\}. \tag{3.19}$$

It is known that under this assumption the following facts hold.

For the Mittag-Leffler function $E_{\alpha,\beta}(-x)$, $x > 0$, the following facts are known:

1) $E_{\alpha,\beta}(-x) = \frac{1}{\Gamma(\beta-\alpha)}\frac{1}{x} + O\left(\frac{1}{x^2}\right)$ as $x \rightarrow \infty$, see the asymptotic for the Mittag-Leffler function in [14] and [16, Theorem 4.3] (Note: $\alpha < 1$ as formulated in Theorem 4.3 of [16], but this asymptotic is true for $\alpha = 1$ as well which can be easily derived from [16, Lemma 4.26]);

2) $E_{\alpha,\beta}(-x) > 0$ ($E_{\alpha,\beta}(-x)$ is even completely monotonic in this case, see [16, p.90]).

By means of these two facts, from the general Theorems 3.1 and 3.2 we derive the following statement for the Mittag-Leffler transform (3.18).

Let $1 \leq p < \infty$, $\gamma + \delta = \frac{p-2}{p}$ and $0 \leq \alpha \leq \min\{1, \beta\} < \infty$. Then the transform $\mathcal{E}_{\alpha,\beta}$ possesses the mapping properties (3.6) and (3.10) if and only if

$$\frac{\lambda - 1}{p} < \gamma < \frac{\lambda}{p} + \frac{1}{p'}$$

and the mapping properties (3.7) and (3.11) if and only if

$$-\frac{\lambda + 1}{p} < \gamma < \frac{1}{p'} - \frac{\lambda}{p}.$$

The sharp constant in (3.6) is equal to

$$\varkappa_0(p, \lambda, \gamma) = I_{\alpha,\beta}(\sigma) := \int_0^\infty E_{\alpha,\beta}(-x) \frac{dx}{x^\sigma},$$

where $\sigma = \frac{1-\lambda}{p} + \gamma$, $0 < \sigma < 1$.

Calculation of the integral $I_{\alpha,\beta}(\sigma)$ for all admissible α and β seems to be a difficult task. Any way it is possible to reduce this integral of two parametric Mittag-Leffler function to a similar integral of one parametric Mittag-Leffler function:

$$\begin{aligned} I_{\alpha,\beta}(\sigma) &= \frac{\alpha\Gamma(\alpha\sigma)}{\Gamma(\beta - \alpha + \alpha\sigma)} \int_0^\infty E_{\alpha,\alpha}(-x) \frac{dx}{x^\sigma} \\ &= \frac{\alpha^2\sigma\Gamma(\alpha\sigma)}{\Gamma(\beta - \alpha + \alpha\sigma)} \int_0^\infty \frac{1 - E_\alpha(-x)}{x^{\sigma+1}} dx, \quad E_\alpha(x) = E_{\alpha,1}(x), \end{aligned} \tag{3.20}$$

and we calculate this sharp constant for the case $\alpha = 1$:

$$I_{1,\beta}(\sigma) = \frac{\pi}{\sin \sigma \pi} \frac{1}{\Gamma(\sigma + \beta - 1)} \tag{3.21}$$

(The calculation being easy for $\alpha = 0$).

The formula (3.21) follows from the first equality in (3.20), since $E_{1,1}(-x) = e^{-x}$, and consequently

$$\int_0^\infty E_{\alpha,\alpha}(-x) \frac{dx}{x^\sigma} \Big|_{\alpha=1} = \Gamma(1 - \sigma).$$

We prove formulas (3.20) and (3.21) in Appendix, in order not to overload the main body of the paper.

As for the sharp constant \varkappa_∞ for the mapping property (3.7), it has the same form $I_{\alpha,\beta}(\sigma) := \int_0^\infty E_{\alpha,\beta}(-x) \frac{dx}{x^\sigma}$, but with $\sigma = \frac{1+\lambda}{p} + \gamma \in (0, 1)$.

In a similar way, the reader can derive the corresponding corollaries from Theorems 3.1 and 3.2 for various other integral transforms with a kernel depending on the product of arguments.

4. Main results: Multidimensional operators

Notation :

$$\begin{aligned} \mathbb{R}_+^n &:= \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}; \\ \mathcal{R}_h &= \{x \in \mathbb{R}_+^n : 0 < x_i < h_i, i = 1, \dots, n\}, h = (h_1, \dots, h_n) \in \mathbb{R}_+^n. \\ x \circ y &= (x_1 y_1, \dots, x_n y_n) \text{ for } x, y \in \mathbb{R}^n; \\ x^{\vec{\lambda}} &= x_1^{\lambda_1} \dots x_n^{\lambda_n}, x \in \mathbb{R}_+^n, \vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n; \\ S^{n-1} &= \{x \in \mathbb{R}^n : |x| = 1\}, |S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{r(\frac{n}{2})}; \\ \vec{p} &= (p_1, \dots, p_n), \vec{\frac{1}{p}} = \left(\frac{1}{p_1}, \dots, \frac{1}{p_n}\right). \end{aligned}$$

4.1. Radial integral transforms. One may consider integral transforms of type (1.1) of functions $f = f(\varrho\sigma)$, $\varrho > 0$, $\sigma \in S^{n-1}$, with respect to the radial variable ϱ , i.e.

$$Af(x) = \int_{\mathbb{R}^n} a(|x| \cdot |y|) f(y) dy, \quad x \in \mathbb{R}^n, \tag{4.1}$$

and deal with mapping properties between Morrey and complementary Morrey spaces in the following form

$$\sup_{r>0} r^\lambda \int_{|x|>r} ||x|^\delta Af(x)|^p dx \leq C_1 \sup_{r>0} \frac{1}{r^\lambda} \int_{|x|<r} ||x|^\gamma f(x)|^p dx, \tag{4.2}$$

$$\sup_{r>0} \frac{1}{r^\lambda} \int_{|x|<r} ||x|^\delta Af(x)|^p dx \leq C_2 \sup_{r>0} r^\lambda \int_{|x|>r} ||x|^\gamma f(x)|^p dx, \tag{4.3}$$

when $\lambda \geq 0$, and

$$\lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_{|y|<r} ||y|^\gamma f(y)|^p dy = 0 \Rightarrow \lim_{r \rightarrow \infty} r^\lambda \int_{|y|>r} ||y|^{-\gamma} Af(y)|^p y^{p-2} dy = 0, \tag{4.4}$$

$$\lim_{r \rightarrow \infty} r^\lambda \int_{|y|>r} ||y|^\gamma f(y)|^p dy = 0 \Rightarrow \lim_{r \rightarrow 0} \frac{1}{r^\lambda} \int_{|y|<r} ||y|^{-\gamma} Af(y)|^p y^{p-2} dy = 0, \tag{4.5}$$

when $\lambda > 0$.

Such mapping properties may take place only under the condition

$$\gamma + \delta = n \frac{p-2}{p}, \tag{4.6}$$

which is proved similarly to 3.1 if we take into account that

$$A\Pi_t = \frac{1}{t^n} \Pi_t A, \quad \|\Pi_t f\|_{L^{p,\lambda;\gamma}} = \frac{1}{t^{\frac{n-\lambda}{p}+\gamma}} \|f\|_{L^{p,\lambda;\gamma}} \quad \text{and}$$

$$\|\Pi_t f\|_{cL^{p,\lambda;\gamma}} = \frac{1}{t^{\frac{n+\lambda}{p}+\gamma}} \|f\|_{cL^{p,\lambda;\gamma}}.$$

The condition (4.6) and the condition $\gamma + \delta = \ell - \frac{2n}{p}$ coincide under the choice $\ell = n$. The application of the operator Q_n to the operator (4.1) yields

$$Q_n Af = \int_{\mathbb{R}^n} \mathcal{K}(x, y) f(y) dy, \quad \mathcal{K}(x, y) = \frac{1}{|x|^n} a\left(\frac{|y|}{|x|}\right). \tag{4.7}$$

The operator on the right hand side is of the type covered by Proposition A. Applying Proposition A, for brevity we restrict ourselves to the case of the radial Laplace transform, i.e. $a(|x| \cdot |y|) = e^{-|x| \cdot |y|}$. By the relation (4.7), Lemma 2.1 and Proposition A, after direct calculations we arrive at the following statement.

THEOREM 4.1. *Let $1 \leq p < \infty$, $\lambda \geq 0$ and the weight exponents γ and δ satisfy the condition (4.6). The operator (4.1) with the kernel $a(|x| \cdot |y|) = e^{-|x| \cdot |y|}$ satisfies the properties (4.2) - (4.5), if and only if*

$$\gamma < \frac{n}{p'} + \frac{\lambda}{p},$$

in the case of (4.2) and (4.4), and

$$\gamma < \frac{n}{p'} - \frac{\lambda}{p},$$

in the case of (4.3) and (4.5), with the sharp constants

$$C_1 = |S^{n-1}| \Gamma\left(\frac{n}{p'} + \frac{\lambda}{p} - \gamma\right) \text{ and } C_2 = |S^{n-1}| \Gamma\left(\frac{n}{p'} - \frac{\lambda}{p} - \gamma\right) \text{ in (4.2) and (4.4), respectively, } |S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

In the same way the reader can arrive at a similar statement for general kernel $a(|x| \cdot |y|)$.

We find more interesting to concentrate on other multidimensional integral transforms which are much more known in analysis and used in applications. To this end, in Subsection 4.4 we prove a theorem of type of Proposition A, for integral operators commuting with anisotropic dilation

$$\Pi_t f(x) = f(t \circ x), \quad x \in \mathbb{R}^n, \quad t = (t_1, \dots, t_n) \in \mathbb{R}_+^n, \tag{4.8}$$

where $\mathbb{R}_+^n = \{x \in \mathbb{R}^n, x_1 > 0, \dots, x_n > 0\}$ and $t \circ x = (t_1 x_1, \dots, t_n x_n)$.

First, we introduce the corresponding spaces which well suit for this goal, namely, local anisotropic Morrey spaces on \mathbb{R}_+^n and their corresponding complementary versions.

4.2. Local anisotropic Morrey spaces. By \mathcal{R}_h , $h \in \mathbb{R}_+^n$, we denote the n -dimensional rectangle

$$\mathcal{R}_h = \{x \in \mathbb{R}_+^n : 0 < x_i < h_i, \quad i = 1, \dots, n\} \quad h = (h_1, \dots, h_n).$$

We use the standard notation for monomials:

$$h^{\vec{\lambda}} := h_1^{\lambda_1} \cdots h_n^{\lambda_n}, \quad \text{where } h \in \mathbb{R}_+^n, \quad \vec{\lambda} = (\lambda_1, \dots, \lambda_n), \quad \lambda_1 \geq 0, \dots, \lambda_n \geq 0.$$

The weighted local anisotropic Morrey space $L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$ is defined as the set of measurable function with the finite norm

$$\|f\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)} = \sup_{h \in \mathbb{R}_+^n} \frac{1}{h^{\vec{\lambda}}} \|(\cdot)^{\vec{\gamma}} f(\cdot)\|_{L^{\vec{p}}(\mathcal{R}_h)},$$

where the mixed $L^{\vec{p}}$ -norm is defined in the usual way (see [6], [34]):

$$\|f\|_{L^{\vec{p}}(\mathcal{R}_h)} = \left(\int_0^{h_n} \cdots \left(\int_0^{h_2} \left(\int_0^{h_1} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \cdots dx_n \right)^{\frac{1}{p_n}}.$$

Such anisotropic Morrey spaces on \mathbb{R}^2 were used in [31] for the study of mixed Hardy operators.

The complementary anisotropic mixed norm Morrey spaces are defined by the norm

$$\|f\|_{cL^{\vec{p},\vec{\lambda};\vec{\gamma}}(\mathbb{R}_+^n)} = \sup_{h \in \mathbb{R}_+^n} h^{\vec{\lambda}} \|(\cdot)^{\vec{\gamma}} f(\cdot)\|_{L^{\vec{p}}(\mathcal{R}_h^{\perp})},$$

where

$$\mathcal{R}_h^{\perp} := \{x \in \mathbb{R}_+^n : x_1 > h_1, \dots, x_n > h_n\}, \quad h = (h_1, \dots, h_n) \in \mathbb{R}_+^n.$$

The above defined Morrey and complementary Morrey spaces coincide with the Lebesgue mixed norm space $L^{\vec{p}}(\mathbb{R}_+^n)$ when $\vec{\lambda} = (0, \dots, 0)$.

When $h \in \mathbb{R}_+^n$ is fixed, we also use the notation

$$N_{\vec{p},\vec{\lambda};\vec{\gamma}}(f, h) = \frac{1}{h^{\vec{\lambda}}} \|(\cdot)^{\vec{\gamma}} f\|_{L^{\vec{p}}(\mathcal{R}_h)} \quad \text{and} \quad {}^cN_{\vec{p},\vec{\lambda};\vec{\gamma}}(f, h) = h^{\vec{\lambda}} \|(\cdot)^{\vec{\gamma}} f\|_{L^{\vec{p}}(\mathcal{R}_h^{\perp})}.$$

The corresponding vanishing Morrey space $V_0L^{\vec{p},\vec{\lambda};\vec{\gamma}}(\mathbb{R}_+^n)$ and vanishing complementary Morrey space $V_{\infty}{}^cL^{\vec{p},\vec{\lambda};\vec{\gamma}}(\mathbb{R}_+^n)$ are defined as the sets of functions in $L^{\vec{p},\vec{\lambda};\vec{\gamma}}(\mathbb{R}_+^n)$ and ${}^cL^{\vec{p},\vec{\lambda};\vec{\gamma}}(\mathbb{R}_+^n)$, which satisfy the conditions

$$\lim_{h^{\vec{\lambda}} \rightarrow 0} N_{\vec{p},\vec{\lambda};\vec{\gamma}}(f, h) = 0 \tag{4.9}$$

and

$$\lim_{h^{\vec{\lambda}} \rightarrow +\infty} {}^cN_{\vec{p},\vec{\lambda};\vec{\gamma}}(f, h) = 0, \tag{4.10}$$

respectively. In the standard way it is proved that these sets are closed subspaces in the spaces $L^{\vec{p},\vec{\lambda};\vec{\gamma}}(\mathbb{R}_+^n)$ and ${}^cL^{\vec{p},\vec{\lambda};\vec{\gamma}}(\mathbb{R}_+^n)$, respectively.

Note, that the tendency to zero of the monomial $h^{\vec{\lambda}} = h_1^{\lambda_1} \dots h_n^{\lambda_n}$ that the rectangle \mathcal{R}_h degenerates, "clinging" to coordinate hyper-planes $x_i = 0$, $i = 1, \dots, n$, in an arbitrary way. Note also that

$$h^{\vec{\lambda}} \rightarrow 0 \Leftrightarrow |\mathcal{R}_h| \rightarrow 0$$

however, geometrically this is the same in the sense that the rectangle \mathcal{R}_h also "clings" to the coordinate hyper-planes when $h^{\vec{\lambda}} \rightarrow 0$ but not necessarily $|\mathcal{R}_h| \rightarrow 0$, and vice versa.

4.3. Integral operators commuting with anisotropic dilation (4.8).

If an integral operator

$$Kf(x) := \int_{\mathbb{R}_+^n} \mathcal{K}(x, y) f(y) dy, \quad x \in \mathbb{R}_+^n$$

commutes with dilation (4.8): $\Pi_t K = K \Pi_t$, $t \in \mathbb{R}_+^n$, then its kernel $\mathcal{K}(x, y)$ satisfies the condition

$$\mathcal{K}(\Pi_t x, \Pi_t y) = t^{-1} \mathcal{K}(x, y), \quad \text{where} \quad t^{-1} = t_1^{-1} \dots t_n^{-1}. \tag{4.11}$$

THEOREM 4.2. *Let $1 \leq p_i < \infty$, $\lambda_i \geq 0$, $i = 1, \dots, n$, and let the condition (4.11) be satisfied. The operator K is bounded in the space $L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$, if*

$$\varkappa := \int_{\mathbb{R}_+^n} \frac{|\mathcal{K}(e, y)| dy}{y^{\frac{1}{p} - \vec{\lambda} + \vec{\gamma}}} < \infty, \tag{4.12}$$

where $e = (1, \dots, 1)$ and $\frac{1}{p} = \left(\frac{1}{p_1}, \dots, \frac{1}{p_n}\right)$. If $\mathcal{K}(x, y) \geq 0$, then the condition (4.12) is necessary for the boundedness, and \varkappa is the sharp constant for the boundedness: $\|K\| = \varkappa$.

If $\lambda_i > 0$, $i = 1, \dots, n$, then under the same condition (4.12) the operator K preserves the vanishing subspace $V_0 L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$.

P r o o f. I. Boundedness in the space $L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$.

Sufficiency part. By the change of variables $y = \Pi_x z = (x \circ z)$ we obtain

$$Kf(x) = x_1 \dots x_n \int_{\mathbb{R}_+^n} \mathcal{K}(x, \Pi_x z) f(x \circ z) dz.$$

Since $x = \Pi_x e$, by the condition (4.11) we get

$$Kf(x) = \int_{\mathbb{R}_+^n} \mathcal{K}(e, z) f(x \circ z) dz.$$

Hence,

$$\|Kf\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}} \leq \int_{\mathbb{R}_+^n} |\mathcal{K}(e, z)| \|\Pi_z f\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}} dz. \tag{4.13}$$

(Note that the Minkowsky inequality applied above, is valid for Morrey norms since it is valid for mixed norm Libegue spaces and $\sup \int \leq \int \sup$.)

It is not hard to calculate that $\|\Pi_z f\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}} = \frac{1}{t^{\frac{1}{p} - \vec{\lambda} + \vec{\gamma}}} \|f\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}}$. We then obtain the sufficiency of the condition (4.12).

Necessity part. Let all $\lambda_i > 0$, $i = 1, \dots, n$. In this case we have the direct minimizing function

$$f_0(x) = \frac{1}{x^{\frac{1}{p} - \vec{\lambda} + \vec{\gamma}}}.$$

Direct calculation shows that $f_0 \in L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$. Moreover, via the dilation change of variables and relation (4.11) we obtain

$$Kf_0(x) = xf_0(x) \tag{4.14}$$

when $\mathcal{K}(x, y) \geq 0$. Hence, $\|Kf_0\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}} = \varkappa_0 \|f_0\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}}$, which proves the necessity and sharpness of the constant.

If $\lambda_i = 0$ for some i , then (4.14) also holds but $f_0 \notin L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$. In this case one should use minimizing sequence instead of the minimizing function f_0 . Thus, if, for instance, $\lambda_1 = 0$, but $\lambda_i > 0, i = 2, \dots, n$, then we choose the minimizing sequence in the form

$$f_{0,\varepsilon}(x) = \frac{1}{x_1^{\eta_1 - \varepsilon}} \prod_{i=2}^n \frac{1}{x_i^{\eta_i}}, \quad \varepsilon > 0, \quad \eta_i = \frac{1}{p_i} - \lambda_i + \gamma_i.$$

Then the arguments in the necessity part with the use of this minimizing sequence are standard, on the base of Fatou lemma, like for instance in [41] in the non-anisotropic case. We omit details.

II. To prove the invariance of the vanishing subspace $V_0 L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$ with respect to the operator K , we apply the Minkowsky inequality with norm $N_{\vec{p}, \vec{\lambda}; \vec{\gamma}}(f, h)$ similarly to actions in (4.13). After direct calculations we obtain

$$N_{\vec{p}, \vec{\lambda}; \vec{\gamma}}(Kf, h) \leq \int_{\mathbb{R}_+^n} \frac{|\mathcal{K}(e, y)|}{y^{\vec{1} - \vec{\lambda} + \vec{\gamma}}} N_{\vec{p}, \vec{\lambda}; \vec{\gamma}}(f, h \circ y) dy.$$

It remains to observe that we can pass to the limit under the integral sign on the right hand side by the Lebesgue dominated convergence theorem. As regards the necessity of the condition (4.12) for the preservation of the vanishing subspace $V_0 L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$, it is proved via the technique of minimizing sequences, usual for such goals. The minimizing sequence in this case is

$$f_{0,\varepsilon}(x) = \frac{1}{x^{\vec{1} - \vec{\lambda} + \vec{\gamma} - \vec{\varepsilon}}}, \quad \vec{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in V_0 L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n).$$

Similar details for isotropic case on \mathbb{R}^n may be found in [41]. □

4.4. Multidimensional integral transforms with the kernel $K(x \circ y)$.

We consider integral transforms

$$Af(x) = \int_{\mathbb{R}_+^n} k(x \circ y) f(y) dy = \int_{\mathbb{R}_+^n} k(x_1 y_1, \dots, x_n y_n) f(y) dy, \quad x \in \mathbb{R}_+^n. \quad (4.15)$$

Multidimensional integral transforms such as n -dimensional Laplace transform, n -dimensional Hankel transform (known also as Koh transform) and others, occur in various applications, see for instance [30], [29] and [17]. We aim at the study of mapping properties of operators of form (4.15) within the frameworks of anisotropic Morrey and complementary Morrey spaces, introduced in Section 4.2. More precisely, we consider mapping properties

$$A : L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n) \hookrightarrow {}^c L^{\vec{p}, \vec{\lambda}; \vec{\delta}}(\mathbb{R}_+^n) \quad (4.16)$$

and

$$A : {}^c L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n) \hookrightarrow L^{\vec{p}, \vec{\lambda}; \vec{\delta}}(\mathbb{R}_+^n). \quad (4.17)$$

The arising necessary relation between the weight exponents $\vec{\gamma}$ and $\vec{\delta}$ now takes the form

$$\gamma_i + \delta_i = \frac{p_i - 2}{p_i}, \quad i = 1, \dots, n, \tag{4.18}$$

which is checked as usual via dilation arguments with the use of dilation operator $\Pi_t f(x) = f(t \circ x)$, $t \in \mathbb{R}_+^n$.

THEOREM 4.3. *Let $1 \leq p_i < \infty$, $\lambda_i \geq 0$, $\gamma_i \in \mathbb{R}$, $i = 1, \dots, n$, and the condition (4.18) be satisfied. The operator (4.15) satisfies the mapping properties (4.16) and (4.17) if*

$$\varkappa_0 := \int_{\mathbb{R}_+^n} |k(y)| \frac{dy}{y^{\frac{1}{p} - \vec{\lambda} + \vec{\gamma}}} < \infty \quad \text{and} \quad \varkappa_\infty := \int_{\mathbb{R}_+^n} |k(y)| \frac{dy}{y^{\frac{1}{p} + \vec{\lambda} + \vec{\gamma}}} < \infty, \tag{4.19}$$

respectively. When $k(y) \geq 0$, $y \in \mathbb{R}_+^n$, these conditions are also necessary and \varkappa_0 and \varkappa_∞ are sharp constants for the boundedness (4.16) and (4.17), respectively.

P r o o f. We introduce the transformation

$$Qf(x) = \frac{1}{x_1 \dots x_n} f\left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right), \quad x = (x_1, \dots, x_n) \in \mathbb{R}_+^n.$$

The operator Q maps the anisotropic space $L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$ onto ${}^cL^{\vec{p}, \vec{\lambda}; \vec{\delta}}(\mathbb{R}_+^n)$, where $\vec{\gamma}$ and $\vec{\delta}$ are related by the condition (4.18). Indeed, it is easy to check that

$${}^cN_{\vec{p}, \vec{\lambda}; \vec{\gamma}}(Qf, h) = N_{\vec{p}, \vec{\lambda}; \vec{\gamma}}(f, h^*), \quad h^* = \left(\frac{1}{h_1}, \dots, \frac{1}{h_n}\right). \tag{4.20}$$

Therefore, under the condition (4.18) we have the isometry

$$\|Qf\|_{{}^cL^{\vec{p}, \vec{\lambda}; \vec{\delta}}} = \|f\|_{L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}} \tag{4.21}$$

and consequently

$$\|Qf\|_{L^{\vec{p}, \vec{\lambda}; \vec{\delta}}} = \|f\|_{{}^cL^{\vec{p}, \vec{\lambda}; \vec{\gamma}}}, \tag{4.22}$$

since $Q^{-1} = Q$.

On the other hand, applying the operator Q to the operator A , we obtain

$$QA = K, \tag{4.23}$$

where

$$Kf(x) = \int_{\mathbb{R}_+^n} \mathcal{K}(x, y) f(y) dy, \quad \mathcal{K}(x, y) = \frac{1}{x_1 \dots x_n} k\left(\frac{y_1}{x_1}, \dots, \frac{y_n}{x_n}\right).$$

Applying Theorem 4.2 to the operator K , making use of the isometry (4.21) via the identity (4.23), we arrive at the first condition in (4.19) for the

mapping (4.16), together with the necessity of this condition and sharpness of the constant \varkappa_0 , when the kernel $k(x)$ is non-negative.

Similarly, the remaining part of the theorem for the mapping (4.17) via consideration of the composition AQ instead of QA . We omit details. \square

In the next theorem we consider “vanishing properties” of type (3.10) and (4.4) for anisotropic setting of this subsection. We restrict ourselves to the case $n = 2$ for better transparency of result, extension of writing for $n > 2$ being evident.

THEOREM 4.4. *Let $1 \leq p_i < \infty$, $\lambda_i \geq 0$, $\gamma_i \in \mathbb{R}$, $i = 1, \dots, n$. Under the condition (4.18) and the condition $\varkappa_0 < \infty$ (with $n = 2$), besides the mapping (4.16), there holds the property: if*

$$\lim_{h_1^{\lambda_1} h_2^{\lambda_2} \rightarrow 0} \frac{1}{h_1^{\lambda_1} h_2^{\lambda_2}} \left(\int_0^{h_2} \left(\int_0^{h_1} |x_1^{\gamma_1} x_2^{\gamma_2} f(x_1, x_2)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{1}{p_2}} = 0 \quad (4.24)$$

then

$$\lim_{h_1^{\lambda_1} h_2^{\lambda_2} \rightarrow 0} \frac{1}{h_1^{\lambda_1} h_2^{\lambda_2}} \left(\int_{h_2}^{\infty} \left(\int_{h_1}^{\infty} |x_1^{\delta_1} x_2^{\delta_2} Af(x_1, x_2)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{1}{p_2}} = 0. \quad (4.25)$$

When the kernel $k(x)$, $x \in \mathbb{R}_+^2$, is non-negative, the condition $\varkappa_0 < \infty$ is also necessary for the property (4.24) to imply the property (4.25).

P r o o f. We only have to show that (4.24) implies (4.25). To this end we use the identity (4.23). We apply Theorem 4.2 to the operator K with respect to the vanishing subspace $V_0 L^{\vec{p}, \vec{\lambda}; \vec{\gamma}}(\mathbb{R}_+^n)$. Then via the relation (4.20) we arrive at statement of this theorem. \square

4.5. Additional remarks. *1° Integral transforms in mixtures of anisotropic Morrey and complementary Morrey spaces*

One can consider Morrey-type spaces which are mixture of usual Morrey spaces and complementary Morrey spaces, where integration in the norm is over $(0, h_i)$ for a part of variables or over (h_j, ∞) for the remaining part of variables. As can be seen from the proof of Theorem 4.2 it may be extended to such a setting. Then application of Theorem 4.2 via the identity (4.23) allows to obtain mapping properties for integral transforms of the form (4.15). In particular, it is not hard to obtain conditions on the kernel $k(y)$ for the property that the condition

$$\lim_{h_1^{\lambda_1} h_2^{\lambda_2} \rightarrow 0} \frac{h_2^{\lambda_2}}{h_1^{\lambda_1}} \left(\int_{h_2}^{\infty} \left(\int_0^{h_1} |x_1^{\gamma_1} x_2^{\gamma_2} f(x_1, x_2)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{1}{p_2}} = 0$$

implies

$$\lim_{h_1^{\lambda_1} h_2^{\lambda_2} \rightarrow 0} \frac{h_1^{\lambda_1}}{h_2^{\lambda_2}} \left(\int_0^{h_2} \left(\int_{h_1}^{\infty} |x_1^{\delta_1} x_2^{\delta_2} Af(x_1, x_2)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{1}{p_2}} = 0,$$

with $\gamma_i + \delta_i = \frac{p_i - 2}{p_i}$, $i = 1, 2$.

2° Admission of $\lambda_i = 0$ for vanishing subspaces

As noted in Subsection 4.2, anisotropic Morrey and complementary Morrey spaces coincide with the mixed norm Lebesgue space space, if $\lambda_i = 0$ for all $i = 1, \dots, n$. An intermediate situation when λ_i may be equal to zero for some i , i.e. when we have just Lebesgue integration over $(0, +\infty)$ in part of variables, is also of interest. This possibility is allowed for norm inequalities in Theorems 4.2 and 4.3. In Theorem 4.4, given for brevity for $n = 2$, we supposed for simplicity that both $\lambda_1 > 0$ and $\lambda_2 > 0$. Analyzing the proof, it is not hard to see that the statement of Theorem 4.4 remains valid when either $\lambda_1 = 0$ $\lambda_2 > 0$ or $\lambda_1 > 0$, $\lambda_2 = 0$. The reader can also easily obtain statements of type of Theorem 4.4 for an arbitrary n with admission of $\lambda_i = 0$ for some i , but not all.

5. Appendix

Proof of the first equality in (3.20).

Let first $\beta > \alpha$. It is known that the Mittag-Leffler function $E_{\alpha,\beta}(-x)$ with $\beta > \alpha$ can be expressed in terms of fractional integration of order $\beta - \alpha$ of the function $E_{\alpha,\alpha}(-x)$:

$$E_{\alpha,\beta}(-x) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^1 \left(1 - t^{\frac{1}{\alpha}}\right)^{\beta - \alpha - 1} E_{\alpha,\alpha}(-tx) dt, \tag{5.26}$$

see for instance [16, Lemma 4.26]. Then

$$\begin{aligned} I_{\alpha,\beta}(\sigma) &= \frac{1}{\Gamma(\beta - \alpha)} \int_0^{\infty} \frac{dx}{x^\sigma} \int_0^1 \left(1 - t^{\frac{1}{\alpha}}\right)^{\beta - \alpha - 1} E_{\alpha,\alpha}(-tx) dt \\ &= \frac{1}{\Gamma(\beta - \alpha)} \int_0^1 \left(1 - t^{\frac{1}{\alpha}}\right)^{\beta - \alpha - 1} \int_0^{\infty} \frac{E_{\alpha,\alpha}(-tx)}{x^\sigma} dx. \end{aligned}$$

Then after the change of variable $xt = \xi$ we have

$$I_{\alpha,\beta}(\sigma) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^1 (1 - t^{\frac{1}{\alpha}})^{\beta - \alpha - 1} t^{\sigma - 1} dt I_{\alpha,\alpha}(\sigma).$$

Hence the first equality in (3.20) follows by easy calculations. It remains to add the case $\beta = \alpha$. To this end, it suffices to note that both the left and right hand sides of the first equality in (3.20) are analytic with respect to $\beta \in \mathbb{C}$.

Proof of the second equality in (3.20).

It is known that $E_{\alpha,\alpha}(-x) = -\alpha \frac{d}{dx} E_{\alpha,\alpha}(-x)$, see [16, Lemma 4.25]. Therefore,

$$\begin{aligned} \int_0^{\infty} \frac{E_{\alpha,\alpha}(-x)}{x^{\sigma}} dx &= -\alpha \int_0^{\infty} \frac{1}{x^{\sigma}} \frac{d}{dx} E_{\alpha}(-x) dx = -\alpha \int_0^{\infty} \frac{1}{x^{\sigma}} d[E_{\alpha}(-x) - 1] \\ &= \alpha \sigma \int_0^{\infty} \frac{1 - E_{\alpha}(-x)}{x^{\sigma+1}} dx, \end{aligned}$$

which proves the second equality in (3.20).

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Received: November 11, 2019, Revised: February 28, 2020

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Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **23**, No 5 (2020), pp. 1274–1299,
 DOI: 10.1515/fca-2020-0064