

# Strictly non-proportional geodesically equivalent metrics have $h_{\text{top}}(g) = 0$

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## 1. Definition and main results

**Definition 1.** *Two ( $C^\infty$ -smooth) Riemannian metrics  $g$  and  $\bar{g}$  on a manifold  $M^n$  are said to be **geodesically equivalent** if their geodesics coincide as unparameterized curves. They are **strictly non-proportional at  $x \in M^n$** , if the polynomial  $\det(g|_x - t\bar{g}|_x)$  has only simple roots.*

The question of whether two different metrics can have the same geodesics is natural and, therefore, classical. The first examples are due to E. Beltrami [B], a local descriptions of geodesically equivalent metrics was understood by U. Dini [Di] and T. Levi-Civita [LC]. We will recall Levi-Civita's Theorem in Section 2.1. For more historical details, see the surveys [Mi, Am], or/and the introductions to the papers [M1, M4].

The main result of our paper is (for definition and properties of  $h_{\text{top}}$  we refer to [Bo, KH, Ma]):

**Theorem 1.** *Suppose the Riemannian metrics  $g$  and  $\bar{g}$  on a closed connected manifold  $M^n$  are geodesically equivalent and strictly non-proportional at least at one point. Then the topological entropy  $h_{\text{top}}(g)$  of the geodesic flow of  $g$  vanishes.*

The condition that the metrics are strictly non-proportional is important: for example, the product metric on a closed product manifold  $M = M_1 \times M_2$  admits a family  $g_1 + tg_2$  of non-proportional metrics (but not strictly non-proportional if  $\dim M > 2$ ) with the same geodesics. But if at least one factor has fundamental group with positive exponential growth (for instance if  $M_1$  is hyperbolic), then by the Dinaburg Theorem any geodesic flow on  $M$  has  $h_{\text{top}}(g) > 0$ .

Vanishing of the topological entropy of a  $C^\infty$ -smooth flow implies a lot of dynamical restrictions. For example, the ball volume grows sub-exponentially with its radius (Manning's inequality [Mn]), the number of geodesic arcs joining two generic points grows sub-exponentially with its maximal length (Mañé's formula [Ma]) and the volume of a compact submanifold propagated by the geodesic flow also changes sub-exponentially (Yomdin's Theorem [Y]), see also [P2].

Probably even more interesting are topological restrictions implied by  $h_{\text{top}}(g) = 0$ . The subexponential growth of  $\pi_1(M^n)$  (Dinaburg's Theorem [D]) is not very intriguing under the assumptions of Theorem 1, since it is known [M3] that in this case the fundamental group is virtually abelian. But the restriction coming from the Gromov-Paternain Theorem [G, P1] and from [PP1] are new, nontrivial and interesting: Namely in the simply connected case the manifold  $M^n$  is **rationally elliptic**, i.e.  $\pi_*(M^n) \otimes \mathbb{Q}$  is finite-dimensional. This is a very restrictive property since by the results of [FHT, Pa] a rationally elliptic manifold  $M^n$  enjoys the following properties:

1.  $\dim \pi_*(M^n) \otimes \mathbb{Q} \leq n$ ,  $\dim H_*(M^n, \mathbb{Q}) \leq 2^{n-1}$ ,  $\dim H_i(M^n, \mathbb{Q}) \leq \frac{1}{2} \binom{n}{i}$  ( $i = 1, \dots, n-1$ ),
2. The Euler characteristic  $\chi(M^n)$  satisfies  $2^n - n + 1 \geq \chi(M^n) \geq 0$ . Moreover,  $\chi(M^n) > 0$  iff  $H_{\text{odd}}(M^n, \mathbb{Q}) = 0$ .

A manifold  $M$  with finite  $\pi_1(M)$  is called **rationally hyperbolic**, if its universal cover is not rationally elliptic. Thus, as a consequence of Theorem 1, we get

**Corollary 1.** *A rationally hyperbolic closed manifold  $M^n$  does not admit two geodesically equivalent Riemannian metrics  $g$  and  $\bar{g}$  which are strictly non-proportional at least at one point.*

Rational hyperbolicity means nothing in dimensions less than 4, since all closed 4-manifolds with finite fundamental group are rational-elliptic. Note that the topology of closed 2- and 3-manifolds admitting non-proportional geodesically equivalent metrics is completely understood: In dimension 2, such manifolds are homeomorphic to the sphere, the projective plane, the torus or the Klein bottle [MT2]. In dimension 3, such manifolds are homeomorphic to lens spaces or to Seifert manifolds with zero Euler number [M2].

Starting from dimension 4, almost all simply-connected manifolds are rationally hyperbolic. For example, in dimension 4, up to homeomorphism, there exist infinitely many simply-connected closed manifolds, and only five of them are rationally elliptic:  $S^4$ ,  $S^2 \times S^2$ ,  $\mathbb{C}P^2$ ,  $\mathbb{C}P^2 \# \mathbb{C}P^2$  and  $\overline{\mathbb{C}P^2 \# \mathbb{C}P^2}$ . It is possible to construct geodesically equivalent metrics on  $S^4$  and  $S^2 \times S^2$  that are strictly non-proportional at least at one point. We conjecture here that these two are the only closed simply-connected 4-manifolds admitting strictly non-proportional geodesically equivalent metrics. In dimension 5, a closed rational-elliptic manifold has rational homotopy type of  $S^2 \times S^3$  or  $S^5$  (there are infinitely many homotopy types for simply-connected 5-manifolds). By recent results of [PP1] (see Theorem E there), a closed manifold admitting a metric with zero topological entropy is  $S^5$ ,  $S^3 \times S^2$ ,  $SU(3)/SO(3)$  or the nontrivial  $S^3$ -bundle over  $S^2$ . We conjecture that  $S^3 \times S^2$  and  $S^5$  are the only closed simply-connected connected 5-manifolds admitting geodesically equivalent metrics which are strictly non-proportional at least at one point.

In Section 5 we announce the restrictions on the topology of non-simply-connected manifolds (admitting geodesically equivalent metrics which are strictly non-proportional at least at one point) that follows from Corollary 1.

Now let us comment the proof of Theorem 1. The main ingredients are Theorems 2, 3 and Corollary 2, which imply that the geodesic flow of  $g$  is Liouville-integrable.

Precisely the same integrable systems were recently actively studied in mathematical physics, in the framework of the theory of separation of variables. Depending on the school, they are called L-systems [Be], Benenti-systems [IMM] and quasi-bi-hamiltonian systems [CST].

But Liouville integrability does not immediately imply vanishing of the topological entropy; counterexamples can be found in [BT1, BT2, Bu1, Bu2, K, KT]. If the singularities of the integrable system behave sufficiently good (non-degenerate in the sense of Williamson-Vey-Eliasson-Ito [E, I], see [P1], or the Taimanov conditions [T]), or if the system has a lot of symmetries (for example, as in collective integrability [BP, P1]), then  $h_{\text{top}}(g) = 0$ . But for other situations nothing is known (at least if  $n > 2$ , see [P0]), even if the integrals are real-analytic or polynomial in momenta.

It is worth mentioning that geodesically equivalent metrics are usually not real-analytic: Levi-Civita's Theorem from Section 2.1 shows the existence of an infinite-dimensional space of non-analytic  $C^\infty$ -perturbations in the class of geodesically-equivalent metrics. Also the set of singular points of the constructed integrals for the corresponding Hamiltonian system can be quite complicated. For instance, the projection of the singularities in  $TM^n$  to the base  $M^n$  is surjective for  $n > 2$  and its restriction to a singular Liouville fiber can have image which is locally the product of the Cantor set and the  $(n - 1)$ -dimensional disk.

The logic of our proof for Theorem 1 is as follows:

1. We show that the topological entropy is supported on the singularities, which we describe.
2. We show that dynamics on them can be considered as a subsystem of the geodesic flow
  - on a lower-dimensional closed submanifold
  - admitting geodesically equivalent metrics which are strictly non-proportional at least at one point.

Therefore we can apply induction by the dimension.

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## 2 Geometry behind the geodesic equivalence

In what follows we always assume that the manifold  $M^n$  is connected and that the Riemannian metrics  $g$  and  $\bar{g}$  on  $M^n$  are geodesically equivalent and strictly non-proportional at least at one point.

### 2.1 Integrability and Levi-Civita's Theorem

A Riemannian metric  $g$  determines the map  $b_g : TM \rightarrow T^*M$  with the inverse  $\sharp^g : T^*M \rightarrow TM$ . Consider the (1,1)-tensor (automorphism field)  $L : TM \rightarrow TM$  given by the formula

$$L = (\det(\sharp^{\bar{g}} \circ b_g))^{-\frac{1}{n+1}} \cdot (\sharp^{\bar{g}} \circ b_g). \quad (1)$$

In local coordinates,  $L_i^j = \sqrt[n+1]{(\det(\bar{g})/\det(g))} g_{i\alpha} \bar{g}^{\alpha j}$ . This tensor  $L$  determines the family  $S_t \in C^\infty(T^*M \otimes TM)$ ,  $t \in \mathbb{R}$ , of (1,1)-tensors

$$S_t := \det(L - t \text{Id}) \cdot (L - t \text{Id})^{-1}. \quad (2)$$

**Remark 1.** *Although  $(L - t \text{Id})^{-1}$  is not defined for  $t \in \text{Sp}(L)$ , the tensor  $S_t$  is well-defined for every  $t \in \mathbb{R}$ . In fact, it is the adjunct matrix of  $(L - t \text{Id})$ . Thus by the Laplace main minors formula,  $S_t$  is a polynomial in  $t$  of degree  $n - 1$  with coefficients being (1,1)-tensors.*

The isomorphism  $b^g$  allows us to identify the tangent and cotangent bundles of  $M^n$ . This identification allows us to transfer the natural Poisson structure and the Hamiltonian system  $H(x, p) = \frac{1}{2} p \cdot \sharp^g(p)$  from  $T^*M^n$  to  $TM^n$ .

**Theorem 2 ([MT1]).** *If  $g, \bar{g}$  are geodesically equivalent, then, for every  $t_1, t_2 \in \mathbb{R}$ , the functions*

$$I_{t_i} : TM^n \rightarrow \mathbb{R}, \quad I_{t_i}(v) := g(S_{t_i}(v), v) \quad (3)$$

*are commuting integrals for the geodesic flow of  $g$ .*

Since  $L$  is self-adjoint with respect to both  $g$  and  $\bar{g}$ , the spectrum  $\text{Sp}(L)$  is real at every point  $x \in M^n$ . Denote it by  $\lambda_1(x) \leq \dots \leq \lambda_n(x)$ . Every eigenvalue  $\lambda_i(x)$  is at least continuous functions on  $M^n$ , and is smooth near the points where it is a simple eigenvalue.

**Theorem 3 ([M1]).** *Let  $(M^n, g)$  be a geodesically complete connected Riemannian manifold. Let a Riemannian metric  $\bar{g}$  on  $M^n$  be geodesically equivalent to  $g$ . Then, for every  $i \in \{1, \dots, n - 1\}$  and for all  $x, y \in M^n$ , the following holds:*

1.  $\lambda_i(x) \leq \lambda_{i+1}(y)$ .
2. If  $\lambda_i(x) < \lambda_{i+1}(x)$ , then  $\lambda_i(z) < \lambda_{i+1}(z)$  for almost every point  $z \in M^n$ .
3. If  $\lambda_i(x) = \lambda_j(y)$  for a certain  $j \neq i$ , then there exists  $z \in M^n$  such that  $\lambda_i(z) = \lambda_j(z)$ .

**Corollary 2 ([MT3]).** *Let  $(M^n, g)$  be a connected Riemannian manifold. Suppose a Riemannian metric  $\bar{g}$  on  $M^n$  is geodesically equivalent to  $g$  and is strictly non-proportional to  $g$  at least at one point. Then, for every mutually-different  $t_1, t_2, \dots, t_n \in \mathbb{R}$ , the integrals  $I_{t_i}$  are functionally independent almost everywhere, i.e. the differentials  $dI_{t_i}$  are linearly independent a.e. in  $TM$ .*

Let us describe the local form of the integrals  $I_t$ . For every  $x \in M^n$  consider coordinates in  $T_x M^n$  such that the metric  $g$  is given by the diagonal matrix  $\text{diag}(1, 1, \dots, 1)$  and the tensor  $L$  is given by the diagonal matrix  $\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ . Then the tensor (2) reads:

$$\begin{aligned} S_t &= \det(L - t \text{Id})(L - t \text{Id})^{(-1)} \\ &= \text{diag}(\Pi_1(t), \Pi_2(t), \dots, \Pi_n(t)), \end{aligned}$$

where the polynomials  $\Pi_i(t)$  are given by the formula

$$\Pi_i(t) \stackrel{\text{def}}{=} \prod_{j \neq i} (\lambda_j - t).$$

Hence, for every  $\xi = (\xi_1, \dots, \xi_n) \in T_x M^n$ , the polynomial  $I_t(x, \xi)$  is given by

$$I_t = \xi_1^2 \Pi_1(t) + \xi_2^2 \Pi_2(t) \cdots + \xi_n^2 \Pi_n(t). \quad (4)$$

For further use, let us consider the one parameter family of functions

$$I_t' \stackrel{\text{def}}{=} \frac{d}{dt}(I_t).$$

For every fixed  $t \in \mathbb{R}$  this function is an integral of the geodesic flow for  $g$ .

Let us now formulate (a weaker version of) the classical Levi-Civita's Theorem.

**Theorem 4 (Levi-Civita [LC]).** *Consider two Riemannian metrics on an open subset  $U^n \subset M^n$  and the tensor  $L$  given by (1). Suppose the spectrum  $\text{Sp}(L)$  is simple at every point  $x \in U^n$ .*

*Then the metrics are geodesically equivalent on  $U^n$  if and only if around each point  $x \in U^n$  there exist coordinates  $x_1, x_2, \dots, x_n$  in which the metrics have the following model form:*

$$ds_g^2 = |\Pi_1(\lambda_1)| dx_1^2 + |\Pi_2(\lambda_2)| dx_2^2 + \cdots + |\Pi_n(\lambda_n)| dx_n^2, \quad (5)$$

$$ds_{\bar{g}}^2 = \rho_1 |\Pi_1(\lambda_1)| dx_1^2 + \rho_2 |\Pi_2(\lambda_2)| dx_2^2 + \cdots + \rho_n |\Pi_n(\lambda_n)| dx_n^2, \quad (6)$$

where the functions  $\rho_i$  are given by

$$\rho_i \stackrel{\text{def}}{=} \frac{1}{\lambda_1 \lambda_2 \cdots \lambda_n} \frac{1}{\lambda_i}.$$

and  $\lambda_i = \lambda_i(x_i)$  are smooth functions of one variable.

**Definition 2.** *The above coordinates will be called **Levi-Civita coordinates** and the neighborhoods where the coordinates are defined will be called **Levi-Civita charts**.*

In Levi-Civita coordinates the tensor  $L$  is diagonal  $\text{diag}(\lambda_1, \dots, \lambda_n)$ , so the notations in the Levi-Civita Theorem are compatible with those in the beginning of the section.

**Corollary 3 ([M1, BM]).** *Suppose the Riemannian metrics  $g, \bar{g}$  are geodesically equivalent on  $M$ . Then, the Nijenhuis torsion of the tensor  $L$  given by (1) vanishes:  $N_L = 0$ .*

If the metrics are strictly non-proportional at least at one point, Corollary 3 follows from the above version of Levi-Civita's theorem. In the general case, Corollary 3 follows from the original version of Levi-Civita's Theorem [LC] and was proven in [M1] and [BM].

Combining formulae (5) and (4), we see that in the Levi-Civita coordinates the function  $I_t$  is given by

$$I_t = \sum_i |\Pi_i(\lambda_i(x))| \Pi_i(t) \xi_i^2 \quad (7)$$

In particular, the function  $I_{\lambda_i(x)}$  as the function on the cotangent bundle is equal to  $(-1)^{i-1} p_i^2$ .

## 2.2 Distributions of eigenvectors: submanifolds $M_A$

We begin with investigation of the set of points from the Levi-Civita charts, the union of which is the open dense set

$$\text{Reg}(M) = \{x \in M : \lambda_i(x) \neq \lambda_j(x) \text{ for } i \neq j\}.$$

This set can be represented as the intersection  $\text{Reg}(M) = \bigcap_A \text{Reg}_A(M)$  by all (proper) subsets  $A \subset \{1, 2, \dots, n\}$ , where we denote

$$\text{Reg}_A(M) = \{x \in M : \forall i \in A \forall j \notin A \lambda_i(x) \neq \lambda_j(x)\}.$$

At every point  $x \in \text{Reg}_A(M)$  denote by  $D_A(x)$  the subspace of  $T_x M^n$  spanned by the eigenspaces with the eigenvalues  $\lambda_i$ , where  $i \in A$ . Since the eigenvalues  $\lambda_i$  for  $i \in A$  do not bifurcate with the eigenvalues  $\lambda_j$  for  $j \notin A$ ,  $D_A$  is a smooth distribution on  $\text{Reg}_A(M)$ . By Corollary 3 it is integrable. We will denote by  $M_A(x)$  its integral submanifold containing  $x \in \text{Reg}_A(x) \subset M^n$ .

**Lemma 1.** *For  $x \in \text{Reg}_A(M)$  the following statements hold:*

1. *The restrictions of  $g$  and  $\bar{g}$  to  $M_A(x)$  are geodesically equivalent.*
2.  *$g|_{M_A(x)}$  and  $\bar{g}|_{M_A(x)}$  are strictly non-proportional at least at one point.*
3. *For  $i \in A$  the  $i^{\text{th}}$  eigenvector of  $L$  (corresponding to  $\lambda_i$ ) coincides with the respective eigenvector of the operator  $L_A$ , constructed via (1) for the metrics  $g|_{M_A(x)}$  and  $\bar{g}|_{M_A(x)}$ .*
4. *There exists a universal along  $M_A(x)$  constant  $c$  (calculated explicitly in the proof) such that the part of  $c \cdot \text{Sp}(L)$ , corresponding to  $A$ , coincides with the spectrum of the operator  $L_A$ , constructed by the restricted to  $M_A(x)$  metrics.*
5. *In particular, if an eigenvalue  $\lambda_i$ ,  $i \in A$  is constant, then the corresponding eigenvalue of the operator  $L_A$ , constructed for the restrictions of  $g$  and  $\bar{g}$  to  $M_A(x)$ , is constant on  $M_A(x)$ .*

**Proof:** The distribution  $D_A$  defines a foliation on  $\text{Reg}_A(M)$  and on its open dense subset  $\text{Reg}(M)$ . Then it is sufficient to prove the first, third and the fourth statements of the lemma at the points of this subset. By Theorems 3, 4 in a neighborhood of every point  $x \in \text{Reg}(M)$ , there exist Levi-Civita coordinates such that the metrics  $g$ ,  $\bar{g}$  are given by formulas (5)-(6). In these coordinates,  $M_A(x)$  is the coordinate plaque of the coordinate collection  $x_\alpha$  with  $\alpha \in A = \{\alpha_1, \dots, \alpha_m\}$ . Then the restrictions of the metrics to  $M_A(x)$  are given by:

$$\begin{aligned} g|_{M_A} &= |\Pi_{\alpha_1}(\lambda_{\alpha_1})|dx_{\alpha_1}^2 + |\Pi_{\alpha_2}(\lambda_{\alpha_2})|dx_{\alpha_2}^2 + \dots + |\Pi_{\alpha_m}(\lambda_{\alpha_m})|dx_{\alpha_m}^2, \\ \bar{g}|_{M_A} &= \rho_{\alpha_1}|\Pi_{\alpha_1}(\lambda_{\alpha_1})|dx_{\alpha_1}^2 + \rho_{\alpha_2}|\Pi_{\alpha_2}(\lambda_{\alpha_2})|dx_{\alpha_2}^2 + \dots + \rho_{\alpha_m}|\Pi_{\alpha_m}(\lambda_{\alpha_m})|dx_{\alpha_m}^2. \end{aligned}$$

Since  $\lambda_j$  is constant on  $M_A(x)$  for every  $j \notin A$ , every factor of  $\Pi_{\alpha_i}$  of the form  $\lambda_j - \lambda_{\alpha_i}$  can be “hidden” in  $dx_{\alpha_i}^2$ . We see that then the first metric is already in the Levi-Civita form, and the second metric becomes in the Levi-Civita’s form after multiplication by

$$C \stackrel{\text{def}}{=} \prod_{j \notin A} \lambda_j, \tag{8}$$

which is constant on  $M_A(x)$ . Hence, by Levi-Civita’s Theorem, the restrictions of the metrics to  $M_A$  are geodesically equivalent.

Direct calculations show that in local coordinates the tensor  $L_A$  is given by:

$$C^{1/(m+1)} \text{diag}(\lambda_{\alpha_1}, \dots, \lambda_{\alpha_m}). \tag{9}$$

The third and the fourth statements of the lemma follow.

Now let us prove the second statement. Suppose the restriction of the metrics are not strictly non-proportional at every point of a certain  $M_A(x)$ . Then, by Theorem 3, there exist  $\alpha_1, \alpha_2 \in A$

such that  $\lambda_{\alpha_1} \equiv \lambda_{\alpha_2}$  on  $M_A(x)$ . Consider the set  $B := \{1, \dots, n\} \setminus A$ . Take the union of all leaves  $M_B$  containing at least one point of  $M_A(x)$ . Clearly, this union contains an open subset of  $M^n$ . Since the eigenvalues  $\lambda_{\alpha_1}, \lambda_{\alpha_2}$  are constant along  $M_B$ , in view of (9) and Theorem 3, at every point of this open subset we have  $\lambda_{\alpha_1} = \lambda_{\alpha_2}$ , which contradicts Theorem 3. Lemma 1 is proven.

**Lemma 2.** *Suppose the eigenvalue  $\lambda_i$  is not a constant. Take a point  $y \in M^n$  such that*

$$\max_{x \in M} \lambda_{i-1}(x) < \lambda_i(y) < \min_{x \in M} \lambda_{i+1}(x).$$

(We assume by definition that  $\min_{x \in M} \lambda_{n+1}(x) = \infty$  and  $\max_{x \in M} \lambda_0(x) = -\infty$ .)

Let  $C(i) := \{1, 2, \dots, n\} \setminus \{i\}$ . Then,  $M_{C(i)}(y)$  is a closed submanifold.

The conditions that the eigenvalue is not constant and that  $\lambda_i$  is neither maximum nor minimum are important: one can construct counterexamples, if one of these conditions is omitted.

**Proof of Lemma 2:** Since  $\max_{x \in M} \lambda_{i-1}(x) < \lambda_i(y) < \min_{x \in M} \lambda_{i+1}(x)$ , there exist  $c_{\text{small}}, c_{\text{big}} \in \mathbb{R}$  such that

- $c_{\text{small}} < \lambda_i(y) < c_{\text{big}}$ ,
- at least one of the numbers  $c_{\text{small}}, c_{\text{big}}$  is a regular value of the function  $\lambda_i$ ,
- the other number is not a critical value of  $\lambda_i$  (i.e. is either a regular value or is equal to  $\lambda_i$  at no point.)

Denote by  $N$  the connected component of the set

$$\{x \in M^n : c_{\text{small}} \leq \lambda_i(x) \leq c_{\text{big}}\},$$

containing the point  $y$ . Then  $N \subset \text{Reg}_{C(i)}(M)$  is a connected manifold with boundary. Therefore,  $D_{C(i)}$  is a smooth distribution on  $N$ . Since it is integrable by Corollary 3, it defines a foliation. By Corollary 3, the function  $\lambda_i$  is constant on the leaves of the foliation. Then, every connected component of the boundary of  $N$  is a leaf of the foliation.

At every  $x \in M^n$ , consider the vector  $v_i$  satisfying

$$\begin{cases} L(v_i) &= \lambda_i(x)v_i \\ g(v_i, v_i) &= |\Pi_i(\lambda_i)|. \end{cases} \quad (10)$$

By definition of  $N$ , the function  $|\Pi_i(\lambda_i)|$  is nonzero and smooth at every point of  $N$ . Thus  $v_i$  vanishes nowhere in  $N$ . Hence, at least on the double-cover of  $N$ , it is defined globally up to a sign and is smooth. The double-cover projection maps closed submanifolds into closed ones. Therefore, without loss of generality we can assume that the vector field  $v_i$  is globally defined already on  $N$ .

Consider the flow of the vector field  $v_i$ . It takes leaves to leaves. Indeed, it is sufficient to prove this almost everywhere, for instance in Levi-Civita charts. In Levi-Civita coordinates the leaves of the foliation are the plaques of the coordinates  $x_\alpha$ , where  $\alpha \in C(i)$ , and the vector field  $v_i$  is  $\pm \frac{\partial}{\partial x_i}$ , so the claim is trivial.

Since the leaves are  $(n-1)$ -dimensional and the flow of  $v_i$  shuffles them, the flow acts transitively and all leaves are homeomorphic. Every connected component of the boundary of  $B$  is compact and is a leaf, whence all leaves are compact. In particular,  $M_{C(i)}(y)$  is compact. Lemma 2 is proven.

### 2.3 Bifurcation of eigenvalues: submanifolds $\text{Sing}_i^j$

The spectrum  $\text{Sp}(L)$  is simple in  $\text{Reg}(M)$ , i.e. almost everywhere in  $M^n$ . But at certain points the multiplicity of some  $\lambda_i$  can become greater than one. Such points will be called **the bifurcation points** of  $\lambda_i$ . By Theorem 3 the following types of bifurcations of the eigenvalue  $\lambda_i$  are possible.

**Case 1:** The eigenvalues  $\lambda_i$  and  $\lambda_{i+1}$  are not constant and there exists  $x \in M$  such that  $\lambda_i(x) = \lambda_{i+1}(x)$ . Denote  $\bar{\lambda}_i = \max \lambda_i(x) = \min \lambda_{i+1}(x)$ . Let us consider the set

$$\text{Sing}_i^1 \stackrel{\text{def}}{=} \{x \in M^n : (\lambda_i(x) - \bar{\lambda}_i)(\lambda_{i+1}(x) - \bar{\lambda}_i) = 0\}.$$

This set was studied in [M1] (see Theorem 6 there). It was shown that  $\text{Sing}_i^1$  is a connected closed totally geodesic submanifold of codimension one. The restrictions of the metrics to it are strictly non-proportional at least at one point. Note that not all points of  $\text{Sing}_i^1$  are points of bifurcation of the eigenvalues  $\lambda_i, \lambda_{i+1}$ .

**Case 2:** There exists  $x \in M$  and  $i \in \{2, \dots, n-1\}$  such that  $\lambda_{i-1}(x) = \lambda_{i+1}(x)$ . In this case, the eigenvalue  $\lambda_i$  is constant. Let us consider the set

$$\text{Sing}_i^2 \stackrel{\text{def}}{=} \{x \in M^n : (\lambda_{i-1}(x) - \lambda_i)(\lambda_{i+1}(x) - \lambda_i) = 0\}.$$

This set was also studied in [M1] (see Theorem 6 there). It was shown that  $\text{Sing}_i^2$  is a connected closed totally geodesic submanifold of codimension two. The restrictions of the metrics to it are strictly non-proportional at least at one point. Moreover, the set of the points  $x \in \text{Sing}_i^2$  such that  $\lambda_{i-1}(x) = \lambda_{i+1}(x)$  is nowhere dense in  $\text{Sing}_i^2$ .

**Case 3a:** The eigenvalue  $\lambda_i$  is constant, there exists  $x \in M$  such that  $\lambda_i = \lambda_{i+1}(x)$  and there exists no  $y$  such that  $\lambda_{i-1}(y) = \lambda_i$ .

**Case 3b:** The eigenvalue  $\lambda_i$  is constant, there exists  $x \in M$  such that  $\lambda_i = \lambda_{i-1}(x)$  and there exists no  $y$  such that  $\lambda_{i+1}(y) = \lambda_i$ .

In Cases 3a, 3b, let us consider respectively the sets

$$\text{Sing}_i^3 = \{x \in M^n : \lambda_i = \lambda_{i+1}(x)\} \quad \text{or} \quad \text{Sing}_i^3 = \{x \in M^n : \lambda_i = \lambda_{i-1}(x)\}.$$

The next lemma shows that, similar to Cases 1 and 2,  $\text{Sing}_i^3$  is a submanifold of codimension 2 and the restrictions of the metrics to  $\text{Sing}_i^3$  are geodesically equivalent and strictly non-proportional at least at one point. Note that, contrast to the previous cases, the set  $\text{Sing}_i^3$  is not necessary connected.

**Lemma 3.** *Under assumptions of Cases 3a or 3b, the set  $\text{Sing}_i^3$  is a*

- (1) *totally geodesic*
- (2) *closed submanifold of codimension 2.*
- (3) *Moreover, the restrictions of the metrics to  $\text{Sing}_i^3$  are strictly non-proportional at least at one point.*

Here we will proof that  $\text{Sing}_i^3$  is a closed submanifold of codimension 2 such that the restrictions of the metrics to it are strictly non-proportional at least at one point. The first statement of the lemma, namely that  $\text{Sing}_i^3$  is totally geodesic, will follow immediately from Theorem 6, see Remark 2. Before Theorem 6, Lemma 3 will be used only once, namely in the proof of Theorem 5. Since the proof of Theorem 6 does not require Theorem 5, no logical loop appears.

**Proof of statements 2,3 of Lemma 3:** We consider Case 3a, the other case is completely analogous. By definition, the set  $\text{Sing}_i^3$  is closed and, therefore, compact.

Let us show that locally  $\text{Sing}_i^3$  is a submanifold of codimension 2. Let  $A = \{i, i+1\}$ . Take a point  $x_0$  such that  $\lambda_i = \lambda_{i+1}(x_0)$ . Then  $x_0 \in \text{Reg}_A(M)$  and we can consider the set  $M_A(x_0)$ . By Lemma 1, the restrictions of the metrics to  $M_A(x_0)$  are geodesically equivalent and strictly non-proportional at least at one point. Since  $M_A(x_0)$  is two-dimensional, the set of points, where these restrictions are proportional, is discrete [MT2]. In view of Lemma 1, the restrictions of the metrics are proportional at  $x_0$ . Then in a small neighborhood of  $x_0$ , there exists no other point  $x \in M_A(x_0)$  such that  $\lambda_i = \lambda_{i+1}(x)$ . Denote by  $B$  the set  $\{1, 2, \dots, n\} \setminus A$ . For every point  $x$  of a small neighborhood of  $x_0$  in  $M_A(x_0)$ , consider the set  $M_B(x)$ . It is a submanifold of codimension two. Since the eigenvalues  $\lambda_i, \lambda_{i+1}$  are constant along  $M_B$ , in a small neighborhood of  $x_0$  the set  $\text{Sing}_i^3$  coincides with  $M_B(x_0)$ . Thus it is a submanifold of codimension 2.

By the second statement of Lemma 1, the restrictions of the metrics to  $\text{Sing}_i^3$  are strictly non-proportional at least at one point. The 2<sup>nd</sup> and 3<sup>d</sup> statements of Lemma 3 are proven.

Let us note that for a fixed  $i$  only one of the submanifolds  $\text{Sing}_i^j$ ,  $j = 1, 2, 3$ , can be non-empty.

### 3 Description of singular points

Consider some mutually-different numbers  $t_1, \dots, t_n \in \mathbb{R}$  and the respective integrals  $I_{t_1}, \dots, I_{t_n}$ . Consider the Poisson action of the the group  $(\mathbb{R}^n, +)$  on  $TM^n$ : an element  $(a_1, \dots, a_n) \in \mathbb{R}^n$  acts by time-one shift along the Hamiltonian vector field of the function  $a_1 I_{t_1} + \dots + a_n I_{t_n}$ . Since the functions are commuting integrals, the action is well-defined, smooth, symplectic, preserves the integrals  $I_t$  and the Hamiltonian of the geodesic flow, see §49 of [A] for details.

A point  $(x, \xi) \in TM$  is called **singular** if the differentials  $dI_{t_1}, \dots, dI_{t_n}$  are linearly dependent at  $(x, \xi)$ . An orbit of the action is called **singular** if it has a singular point. All points of a singular orbit are singular and have the same coefficients of the linear dependence.

Although the Poisson action depends on the choice of constants  $t_1, \dots, t_n$ , the property of  $(x, \xi)$  being singular does not depend on the choice of  $t_i$  as far as these numbers are all different.

#### 3.1 Singular points in Levi-Civita coordinates

The next theorem describes singular points that lie over a Levi-Civita chart  $U^n \subset \text{Reg}(M^n)$ . Fix a point  $x \in \text{Reg}(M^n)$  and denote by  $\bar{\lambda}_1, \dots, \bar{\lambda}_n$  the constants  $\lambda_1(x), \dots, \lambda_n(x)$  respectively.

**Theorem 5.** *Let the metrics  $g$  and  $\bar{g}$  be given by formulas (5)-(6) in a neighborhood  $U^n \subset M^n$ . If the point  $(y, \xi) = (x_1, \dots, x_m, \xi_1, \dots, \xi_m) \in T \text{Reg}(M^n)$  is singular, then there exists  $i \in \{1, \dots, n\}$  such that  $dI_{\bar{\lambda}_i} = 0$ . Then  $I_{\bar{\lambda}_i}(x, \xi) = 0$  and at least one of the following statements holds:*

1. The derivative  $\frac{\partial \lambda_i(x)}{\partial x_i}$  vanishes at  $x$ .
2. The function  $I'_{\bar{\lambda}_i}$  vanishes at  $(x, \xi)$ .

Moreover, if  $M_{C(i)}(y)$  is compact, the whole geodesic passing through  $y$  with the velocity vector  $\xi$  is contained in  $M_{C(i)}(y)$ , where  $C(i)$  is the same as in Lemma 2.

Actually, the assumption that  $M_{C(i)}(y)$  is compact is not necessary: Theorem 5 remains true, if we replace this condition by the condition that  $y \notin \text{Sing}_i^1$ . Our stronger assumption makes the proof shorter.

**Proof of Theorem 5:** Suppose the point  $(y, \xi)$  is singular. Then, there exist constants  $(\mu_1, \dots, \mu_n) \neq (0, \dots, 0)$  such that at  $(y, \xi)$  it holds:

$$\mu_1 dI_{\bar{\lambda}_1} + \dots + \mu_n dI_{\bar{\lambda}_n} = 0.$$

We will show that for every  $i$  such that  $\mu_i \neq 0$  the differential  $dI_{\bar{\lambda}_i}$  vanishes at  $(y, \xi)$ . For every  $j \in \{1, \dots, n\}$  consider the function  $I_{\lambda_j(x)}(x, \eta) := (I_t(x, \eta))|_{t=\lambda_j(x)}$ . In a small neighborhood of  $y$ , the function  $\lambda_j$  is smooth. Hence the function  $I_{\lambda_j(x)}$  is smooth as well. At the point  $(y, \xi)$  we have:

$$dI_{\lambda_j(y)} = dI_{\bar{\lambda}_j} + I'_{\bar{\lambda}_j} \cdot d\lambda_j.$$

We will work on the cotangent bundle to  $M^n$ . As we explained in Section 2.1, the function  $I_{\lambda_j(x)}$  is equal to  $(-1)^{j-1} p_j^2$  and its differential has coordinates

$$\underbrace{(0, \dots, 0)}_{n+j-1}, 2 \cdot (-1)^{j-1} \cdot p_j, 0, \dots, 0).$$

Since the function  $\lambda_j$  depends on  $x_j$  only, its differential is

$$\underbrace{(0, \dots, 0)}_{j-1}, \frac{\partial \lambda_j}{\partial x_j}, 0, \dots, 0).$$



Thus  $dI_{\bar{\lambda}_j}$  at  $(y, \xi)$  is given by

$$\left(\underbrace{0, \dots, 0}_{j-1}, I'_{\bar{\lambda}_j} \cdot \frac{\partial \lambda_j}{\partial x_j}, \underbrace{0, \dots, 0}_{n-1}, 2 \cdot (-1)^{j-1} \cdot p_j, 0, \dots, 0\right).$$

We see that the differentials  $dI_{\bar{\lambda}_j}$  do not combine: If  $\mu_i \neq 0$ , then  $dI_{\bar{\lambda}_i} = 0$ . Therefore,  $p_i = 0$  (i.e.  $\xi_i = 0$ ), which is equivalent to  $I_{\bar{\lambda}_i}(x, \xi) = 0$ , and at least one of the following holds:  $\frac{\partial \lambda_i}{\partial x_i}(x) = 0$  or  $I'_{\bar{\lambda}_i}(x, \xi) = 0$ . The first part of the theorem is proven.

Now let us show that the geodesic  $\gamma$  such that  $(\gamma(0), \dot{\gamma}(0)) = (y, \xi)$  is contained in  $M_{C(i)}(y)$ . Since  $M_{C(i)}(y)$  is compact, it is sufficient to prove that at almost every point of the geodesic the velocity vector of the geodesic is contained in  $D_{C(i)}$ . Since  $\text{Sing}_k^j$  are totally geodesic submanifolds, the geodesic  $\gamma$  intersect them transversally, and it is sufficient to prove that the velocity vector of the geodesic lies in  $D_{C(i)}$  in Levi-Civita's charts.

Since  $I_{\bar{\lambda}_i}$  is an integral and  $dI_{\bar{\lambda}_i} = 0$  at  $(y, \xi)$ , we obtain that  $dI_{\bar{\lambda}_i}$  vanishes at every point  $(\gamma(t), \dot{\gamma}(t))$ . Then, as we explained above, in the Levi-Civita chart, the component  $\xi_i$  equals zero, so that the velocity vector of the geodesic lies in  $D_{C(i)}$ . Finally, the geodesic stays in  $M_{C(i)}$  forever. Theorem 5 is proven.

### 3.2 Removable singularities

Our next goal is to show that certain singular points are artificially singular: if we use a finite cover and choose the integrals appropriate, they become regular.

Suppose the eigenvalue  $\lambda_i$  is constant. From the proof of Theorem 5 it follows that for every  $x \in \text{Reg}_{\{\lambda_i\}}(M)$  and  $\xi \in D_{C(i)}(x) \subset T_x M^n$  the differential  $dI_{\lambda_i}$  vanishes at  $(x, \xi)$ . We will show that this singularity is **removable**, in the sense that on an appropriate finite cover we can find a linear in velocities function  $J_i$  such that  $J_i^2 = (-1)^{i-1} I_{\lambda_i}$ . This relation immediately implies that  $J_i$  commutes with the functions  $I_t$ . Since  $I_{\lambda_i}$  is an integral,  $J_i$  is an integral as well. Since it is linear in velocities, it corresponds to a Killing vector field. We will show that this Killing vector field is nonzero at  $x$ , which automatically implies that the differential of this integral does not vanish at  $(x, \xi)$ .

In the Levi-Civita coordinates  $I_{\lambda_i} = (-1)^{i-1} p_i^2$  and we can put  $J_i = \pm p_i$ . Clearly, in the Levi-Civita coordinate system,  $J_i(\eta) := g(v_i, \eta)$ , where  $v_i = \pm \frac{\partial}{\partial x_i}$ .

Note that the vector field  $\frac{\partial}{\partial x_i}$  satisfies conditions (10), and that near every regular point every vector field satisfying (10) is the vector field  $\frac{\partial}{\partial x_i}$  of a certain Levi-Civita coordinate system.

Thus, in order to show that (at least on a finite cover) there exists a smooth function  $J_i$  such that it is linear in velocities and such that  $J_i^2 = (-1)^{i-1} I_{\lambda_i}$ , it is sufficient to prove

**Theorem 6.** *Suppose  $\lambda_i$  is constant. Then at least on a double cover of  $M^n$  there exists a smooth vector field  $v_i$  satisfying (10) at every point  $x \in M^n$ .*

**Remark 2.** *Conditions (10) imply that the zeros of  $v_i$  coincide with  $\cup_{j=2,3} \text{Sing}_i^j$ . Since  $v_i$  is a Killing vector field,  $\text{Sing}_i^3$  is a totally-geodesic submanifold.*

**Proof of Theorem 6:** First we show that at least on the double-cover there exists a continuous vector field  $v_i$  with the required properties. In order to do this, it is sufficient to prove the following semi-local statement:

(S) Locally near every point  $x$  there exist precisely two continuous vector fields  $v_i$  satisfying (10).

If  $\lambda_{i-1}(x) \neq \lambda_i \neq \lambda_{i+1}(x)$ , then  $y \in \text{Reg}_{\{\lambda_i\}}(M)$ . Then,  $\Pi_i(\lambda_i) \neq 0$ . Hence,  $v_i \neq 0$  in a small neighborhood of  $x$  and the statement (S) is trivial.

Let us consider  $x \in \text{Sing}_i^j$ , where  $j = 2$  or  $3$ , and prove the statement in a small disk neighborhood  $U^n \ni x$ .

First of all, if a vector field  $v_i$  satisfies (10), then the vector field  $-v_i$  satisfies (10) as well. Since  $\text{Sing}_i$  is nowhere dense, the fields do not coincide. Therefore we obtain at least two different required vector fields.

Next, there exist no more than two such vector fields. Indeed, such a vector field  $v_i$  must vanish along  $\text{Sing}_i^j$ , since  $\Pi_i(\lambda_i)$  equals zero there, and it is non-zero in the complement. This complement is connected, because  $\text{Sing}_i^j$  has codimension 2 (by proven part of Lemma 3 and as we explained in Section 2.3), and the claim follows.

At last, let us prove that such continuous field  $v_i$  exists in the small disk neighborhood  $U^n \ni x$ . Since  $U^n \setminus \text{Sing}_i^j$  is connected, we can define  $v_i$  in one of two possible ways at some point  $x_0$  and extend by continuity along paths in  $U^n \setminus \text{Sing}_i^j$ . We need to show that the result is well-defined.

In order to do this we connect two paths  $\phi_0, \phi_1$  from  $x_0$  to  $x_1$  in  $U^n \setminus \text{Sing}_i^j$  by a homotopy  $\phi_\tau$  in  $U^n$ . The paths and the homotopy can be assumed smooth. Since  $\text{Sing}_i^j$  has codimension 2, we can perturb homotopy and make it to be transversal to  $\text{Sing}_i^j$ . Thus, the intersection of  $\text{Image}_{\phi_\tau}$  with  $\text{Sing}_i^j$  is a finite set  $\{(t_k, \tau_k)\} \in [0, 1] \times [0, 1]$  and it suffices to consider only one point of intersection  $y_0 = \phi_{\tau_0}(t_0) = \phi(t_0, \tau_0) \in \text{Sing}_i^j$ . If we can find the required field  $v_i$  on a transversal 2-dimensional disk at  $y_0$ , we are done.

As we explained in Section 2.3, at almost every point  $y \in \text{Sing}_i^j$  we have  $\lambda_{i-1}(y) \neq \lambda_{i+1}(y)$ . (Actually, for  $j = 3$  this is true at every point.) Thus, without loss of generality, we can assume that  $\lambda_{i-1}(y_0) \neq \lambda_{i+1}(y_0)$ .

Assume  $\lambda_{i-1}(y_0) \neq \lambda_i = \lambda_{i+1}(y_0)$ . The case  $\lambda_{i-1}(y_0) = \lambda_i \neq \lambda_{i+1}(y_0)$  is completely analogous.

Let  $A = \{i, i+1\}$ . Then  $y_0 \in \text{Reg}_A(M)$ . Consider the leaf  $M_A(y_0)$ . This is a 2-dimensional manifold transverse to  $\text{Sing}_i^j$  at  $y_0$ . The homotopy can be perturbed to have the image locally coinciding with  $M_A(y_0)$ . Since  $v_i \in D_A$ , the problem, thanks to Lemma 1, is reduced to a local 2-dimensional question on  $M_A(y_0)$ .

Consider the restriction of the metrics to  $M_A(y_0)$ . Denote by  $L_A$  the tensor (1) constructed for the restrictions of the metrics. We denote by  $\lambda_A \leq \lambda'_A$  its eigenvalues. By Lemma 1,  $\lambda_A$  is constant,  $\lambda'_A$  is not. If there exists a (continuous) vector field  $v_A$  on  $M_A$  such that it vanishes precisely at  $y_0$ , such that it is eigenvector of  $L_A$  with eigenvalue  $\lambda_A$ , and such that its length is  $\sqrt{\lambda'_A - \lambda_A}$ , we are done. Indeed, by Lemma 1 the vector field  $v_i$  given by

$$\sqrt{C^{-1/3} \left| \prod_{\alpha \neq i, i+1} (\lambda_i - \lambda_\alpha) \right|} v_A,$$

where  $C$  is given by (8), satisfies the conditions (10). Since

$$\sqrt{C^{-1/3} \left| \prod_{\alpha \neq i, i+1} (\lambda_i - \lambda_\alpha) \right|}$$

is a smooth positive function, the existence of  $v_A$  implies the existence of  $v_i$ .

Let us prove the existence of such vector field  $v_A$ . At every  $y \in M_A(y_0)$ ,  $y \neq y_0$ , denote by  $l_A$  the eigenspace of  $L_A$  corresponding to  $\lambda_A$ . Let us show that for every geodesic  $\gamma$  on  $M_A(y_0)$  passing through  $y_0$  the velocity vector  $\dot{\gamma}(t)$  is orthogonal (in the restriction of  $g$ ) to  $l_A$  at every  $\gamma(t) \neq y_0$ . Indeed, let  $I_t^A$  be the one-parametric family of the integrals from Theorem 2 constructed for the restrictions of  $g$  and  $\bar{g}$  to  $M_A(y_0)$ . Consider the integral  $I_{\lambda_A}^A$ . At the tangent plain to every point  $z$  consider the coordinates such that the restriction of  $g$  to  $M_A(y_0)$  is given by  $\text{diag}(1, 1)$  and  $L_A$  is  $\text{diag}(\lambda_A, \lambda'_A)$ . In this coordinates, the integral  $I_t^A$  equals  $(\lambda'_A - t)\xi_1^2 + (\lambda_A - t)\xi_2^2$ , so that  $I_{\lambda_A}^A$  is equal to  $(\lambda'_A - \lambda_A)\xi_1^2$ . We see that the integral vanishes on every geodesic  $\gamma$  passing through  $y_0$ . Because  $\lambda'_A(z) \neq \lambda_A(z)$  for  $z \neq y_0$ , we obtain that the component  $\xi_1$  of the velocity vector of  $\gamma$  at  $z$  vanishes, which means that the eigenvalue of  $L_A$  corresponding to  $\lambda_A$  is orthogonal to  $\gamma$ .

Clearly, in  $M_A(y_0) \setminus y_0$  there exists a vector field of length 1 such that it is orthogonal to the geodesics passing through  $y_0$ , see Figure 1.

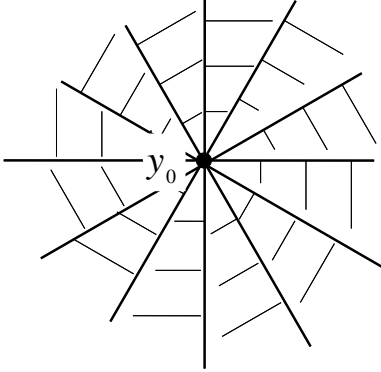


Figure 1: In dimension 2, there exists a vector field orthogonal to all geodesics containing  $y_0$ .

Multiplying this vector field by  $\sqrt{\lambda'_A - \lambda_A}$ , we obtain a required vector field  $v_A$  on  $M_A(y_0) \setminus y_0$ . We put  $v_A = 0$  at point  $y_0$ . Since  $\sqrt{\lambda'_A - \lambda_A}$  converges to 0 when  $x$  tends to  $y_0$ , the result is a required continuous vector field  $v_A$  on  $M_A(y_0)$ . Therefore, there exists a vector field  $v_i$  along  $M_A(y_0)$  (satisfying (10)). Thus, the vector  $v_i$  at  $x_1$  does not depend on the choice of path connecting  $x_0$  and  $x_1$ . Finally,  $v_i$  is well-defined at the whole  $U^n \setminus \text{Sing}_i^j$ , and is at least continuous on it.

At the points of  $U^n \cap \text{Sing}_i^j$  let us put  $v_i$  equal to zero. Since  $\Pi_i(\lambda_i)$  tends to 0 when  $x$  approaches  $\text{Sing}_i^j$ , the vector field is continuous on  $U^n$ . Statement **(S)** is proven.

Then, at least on the double cover of  $M^n$ , there exists a continuous vector field  $v_i$  satisfying (10). Without loss of generality, we can assume that the vector field  $v_i$  is defined already on  $M^n$ .

Now let us prove that the vector field  $v_i$  is actually smooth. Clearly, it is smooth on the compliment to  $\text{Sing}_i^j$ , because it coincides with the appropriate field  $\frac{\partial}{\partial x_i}$  there. Denote by  $F_t$  the flow of the vector field  $v_i$  on  $M^n \setminus (\text{Sing}_i^2 \cup \text{Sing}_i^3)$ . This flow is globally (=for every value of  $t$ ) defined. Indeed, if  $x \notin \text{Sing}_i^2 \cup \text{Sing}_i^3$ , then  $\lambda_{i-1}(x) < \lambda_i < \lambda_{i+1}(x)$ . Since  $v_i$  is an eigenvector of  $L$  with eigenvalue  $\lambda_i$  and the Nijenhuis tensor  $N_L$  vanishes (Corollary 3), for every  $t$  we have:  $\lambda_{i-1}(F_t(x)) = \lambda_{i-1}(x)$ ,  $\lambda_{i+1}(F_t(x)) = \lambda_{i+1}(x)$ . Therefore, the trajectory of the flow passing through  $x$  never approaches the set  $\text{Sing}_i^2 \cup \text{Sing}_i^3$ .

The function  $J(\eta) := g(v_i, \eta)$  is a linear in velocities integral of the geodesic flow, which implies that  $F_t$  acts by isometries on  $M^n \setminus (\text{Sing}_i^2 \cup \text{Sing}_i^3)$ . Since  $M^n \setminus (\text{Sing}_i^2 \cup \text{Sing}_i^3)$  is everywhere dense in  $M^n$ , the map  $F_t$  can be extended by completeness to act by isometries on the whole  $M^n$ . Thus, there exists a Killing vector field on  $M^n$  coinciding with  $v_i$  almost everywhere. Since every Killing vector field is smooth, the vector field  $v_i$  is smooth. Theorem 6 is proven.

## 4 Proof of Theorem 1

We use induction by the dimension. If dimension of the manifold is  $n < 2$ , Theorem 1 is trivial. Assume that for every dimension less than  $n$  Theorem 1 is true and consider  $\dim M = n$ .

Vanishing of the topological entropy for the lift of a dynamical system to a finite cover (of a closed manifold) implies vanishing of the topological entropy of the original system. Thus, we assume that already on  $M^n$  for every constant eigenvalue  $\lambda_i$  we can associate a global vector field  $v_i$  from Theorem 6. Therefore for every constant  $\lambda_i$  we globally define the integral  $J_i$  such that its differential does not vanish over the points of  $\text{Reg}(M^n)$ , it commutes with all integrals  $I_t$ , it is functionally dependent with the integral  $I_{\lambda_i}$ .

By geodesic flow we will understand the restriction of the Hamiltonian system on  $TM^n$  with the Hamiltonian  $H(\xi) := g(\xi, \xi)$  to  $T_1M^n = \{\xi \in TM^n : H(\xi) = 1\}$ . The symplectic form on

$TM^n$  came from  $T^*M^n$  via standard identification by  $g$ .

Since  $T_1M^n$  is compact, the variational principle (see, for example, Theorem 4.5.3 of [KH]) holds, and we obtain

$$h_{\text{top}}(g) = \sup_{\mu \in \mathfrak{B}} h_{\mu}(g).$$

Here  $\mathfrak{B}$  is the set of all invariant ergodic probability measures on  $T_1M^n$  and  $h_{\mu}$  is the entropy of an invariant measure  $\mu$ . Recall that a measure is called **ergodic**, if  $\mu(B)(1 - \mu(B)) = 0$  for all  $\mu$ -measurable invariant Borel sets  $B$ .

Therefore, in order to prove Theorem 1, it is sufficient to prove that  $h_{\mu}(g) = 0$  for all  $\mu \in \mathfrak{B}$ . Fix one such measure and let  $\text{Supp}(\mu)$  be its support (the set of  $x \in M^n$  such that every neighborhood  $U_{\epsilon}(x)$  has positive measure).

Since the measure is ergodic, its support lies on a level surface of every invariant continuous function. Then,  $\text{Supp}(\mu)$  is included into a Liouville leaf  $\Upsilon$  (Recall that a *Liouville leaf* is a connected component of the set  $\{I_{t_1} = c_1, \dots, I_{t_n} = c_n\}$ , where  $c_1, \dots, c_n$  are constants.)

Suppose a point  $\xi \in \text{Supp}(\mu)$  is nonsingular, or is a removable singular point (in the sense that every  $I_{\lambda_i}$  such that  $dI_{\lambda_i} = 0$  can be replaced by a linear integral  $J_i$  such that  $dJ_i \neq 0$ ). Then, a small neighborhood  $U(\xi)$  of  $\xi$  in  $\text{Supp}(\mu)$

- has positive measure in  $\mu$ ,
- contains only points that are nonsingular or removable-singular.

We will show that these two conditions imply that the entropy of  $\mu$  is zero.

By implicit function Theorem,  $\Upsilon$  is  $n$ -dimensional near  $\xi$ . Denote by  $O(\xi)$  the orbit of the Poisson action of  $(\mathbb{R}^n, +)$  containing  $\xi$ . Since it is also  $n$ -dimensional, in a small neighborhood of  $\xi$  it coincides with  $\Upsilon$ . Thus,  $U(\xi) \subset O(\xi)$ .

The orbits of the Poisson action and the dynamic on them are well-studied (see, for example, §49 of [A]). There exists a diffeomorphism to

$$T^k \times \mathbb{R}^{n-k} = \underbrace{S^1 \times \dots \times S^1}_k \times \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n-k}$$

with the standard coordinates  $\phi_1, \dots, \phi_k \in (\mathbb{R} \bmod 2\pi)$ ,  $t_{k+1}, \dots, t_n \in \mathbb{R}$  such that in these coordinates (the push-forward of) every trajectory of the geodesic flow is given by the formula

$$(\phi_1(\tau), \dots, \phi_k(\tau), t_{k+1}(\tau), \dots, t_n(\tau)) = (\phi_1(0) + \omega_1\tau, \dots, \phi_k(0) + \omega_k\tau, t_{k+1}(0) + \omega_{k+1}\tau, \dots, t_n(0) + \omega_n\tau),$$

where the constants  $\omega_1, \dots, \omega_n$  are universal on  $T^k \times \mathbb{R}^{n-k}$ .

We see that if at least one of the constants  $\omega_{k+1}, \dots, \omega_n$  is not zero, every point of  $U(\xi)$  is wandering in  $\text{Supp}(\mu)$  (see §3 in Chapter 3 of [KH] for definition), which contradicts the invariance of the measure. Then, the entropy of  $\mu$  is zero.

If all constants  $\omega_{k+1}, \dots, \omega_n$  are zero, the coordinates  $t_{k+1}, \dots, t_n$  are constants on the trajectories of the geodesic flow. Since  $\mu$  is ergodic, they are constant on the points of  $\text{Supp}(\mu)$ . Then,  $\text{Supp}(\mu)$  is (diffeomorphic to) the torus  $T^{\bar{k}}$  of dimension  $\bar{k} \leq k$ , and the dynamics on  $\text{Supp}(\mu)$  is (conjugate to) the linear flow on  $T^{\bar{k}}$ . Then, the entropy of  $\mu$  is zero, see for example Proposition 3.2.1 of [KH].

Now suppose that  $\text{Supp}(\mu)$  contains only singular points which are not removable. If all of them belong to  $\cup_{i,j} T\text{Sing}_i^j$ , then (because the measure is ergodic)  $\text{Supp}(\mu)$  is a subset of a certain  $T\text{Sing}_i^j$ . Since  $\text{Sing}_i^j$  is totally geodesic, and since by induction hypothesis the topological entropy on  $\text{Sing}_i^j$  is zero, the entropy of  $\mu$  is also zero.

The last case is when  $\text{Supp}(\mu)$  contains a singular point which is not removable and which does not belong to  $\cup_{i,j} T\text{Sing}_i^j$ . Then, since all  $\text{Sing}_i^j$  are totally geodesic, and since there are finitely many of them,  $\text{Supp}(\mu)$  contains a singular point  $\xi$  which is not removable and such that its projection does not belong to  $\cup_{i,j} \text{Sing}_i^j$ . Then, the projection of a small neighborhood  $U(\xi) \subset \text{Supp}(\mu)$  of  $\xi$  does not contain points of  $\cup_{i,j} \text{Sing}_i^j$ .

From Theorems 5,6 it follows, that for certain  $\bar{\lambda}_i$  such that  $\lambda_i$  is not constant the differentials of  $I_{\bar{\lambda}_i}$  vanish at  $\xi$ . Since the number of such  $\bar{\lambda}_i$  is finite, and since the measure is ergodic, we obtain that there exists  $i$  such that

- $dI_{\bar{\lambda}_i} = 0$  at every point of  $\text{Supp}(\mu)$ ,
- the eigenvalue  $\lambda_i$  satisfies the assumptions of Lemma 2. (Otherwise the singularity is removable or  $\xi$  lies in  $\cup_{i,j} T\text{Sing}_i^j$ .)

Hence, by Lemma 2, for every point  $y$  from the projection of  $U(\xi)$  we have that  $M_{C(i)}(y)$  is compact. Then, by Theorem 5, for every  $\eta \in U(\xi)$ , the projection of the trajectory of the geodesic flow passing through  $\eta$  stays on the corresponding  $M_{C(i)}$ . Since all  $M_{C(i)}$  passing through the projection of  $U(\xi)$  are compact and do not intersect one another, a trajectory staying in one  $T_1M_{C(i)}$  never approaches another  $T_1M_{C(i)}$ . Thus, since  $\mu$  is ergodic, all points of  $\text{Supp}(\mu)$  belong to a certain  $T_1M_{C(i)}(y)$ . Then, the dynamics on  $\text{Supp}(\mu)$  is a subsystem of the geodesic flow for the restriction of  $g$  to  $M_{C(i)}(y)$ . (Indeed, if a geodesic of a metric lies on a submanifold, then it is a geodesic in the restriction of the metric to the submanifold.) Finally, by induction assumptions, the entropy of  $\mu$  is zero.

Thus, for every ergodic probabilistic invariant measure  $\mu$  its entropy is zero. Finally, the topological entropy is zero. Theorem 1 is proven.

## 5 Topological restrictions for manifolds with infinite fundamental group: announcement

**Theorem 7.** *Suppose the Riemannian metrics  $g$  and  $\bar{g}$  on a closed connected manifold  $M^n$  are geodesically equivalent and strictly non-proportional at least at one point. Then some finite cover of  $M^n$  is diffeomorphic to the product  $Q^k \times T^{n-k}$  of a rational-elliptic manifold and the torus.*

The proof of this theorem is lengthy and will appear elsewhere ( for small dimensions, in view of Theorem 1, Theorem 7 follows from [PP2]). Here we sketch the proof only. It uses Corollary 1, methods developed in [M1, M4] and classical results of [CG].

In [M1], it was shown that if a manifold with non-proportional geodesically equivalent metrics has an infinite fundamental group, it admits a local product structure (= a new Riemannian metric and two orthogonal foliations of complementary dimensions  $B_k$  and  $B_{n-k}$  such that in a small neighborhood of almost every point all three object look as they come from the Riemannian product of two Riemannian manifolds). In [M4] (see Lemma 2 there), it was shown that (assuming that the initial metrics  $g$  and  $\bar{g}$  are strictly non-proportional at least at one point), the restriction of the local-product metric to the leaves of the foliations admits a metric which is geodesically equivalent to it and strictly non-proportional to it at almost every point. By applying the same construction to the leaves, we obtain that  $M^n$  admits a Riemannian metric  $h$  and  $m$  orthogonal foliations  $B_{k_1}, B_{k_2}, \dots, B_{k_m}$  of complementary dimension  $k_1 + k_2 + \dots + k_m = n$  such that

- the restriction of the metric  $h$  to  $B_{k_1}$  is flat,
- the leaves of  $B_{k_2}, B_{k_3}, \dots, B_{k_m}$  are compact and have finite fundamental group (this is actually the lengthy part of the proof; its proof is similar to the proof of Theorem 2 from [M1], but one can not apply Theorem 2 from [M1] directly and should essentially repeat all steps of its proof in a slightly different setting.)
- the restriction of  $h$  to each of  $B_{k_2}, B_{k_3}, \dots, B_{k_m}$  admits a metric which is geodesically equivalent to it and is strictly non-proportional to it at least at one point.
- locally, in a neighborhood of every point, the metric  $h$  and the foliations  $B_{k_i}$  look as they (simultaneously) came from the direct product of  $m$  Riemannian manifolds.

Then, by Corollary 1, the universal cover of  $B_{k_2} \times B_{k_3} \times \dots \times B_{k_m}$  is rational elliptic, and Theorem 7 follows from Theorem 9.2 of [CG].

## 6 Vanishing of the entropy pseudonorm: announcement

An action  $\Phi : (\mathbb{R}^n, +) \rightarrow \text{Diff}(W)$  determines the following **entropy pseudonorm** [K] :

$$\rho_\Phi(v) := h_{\text{top}}(\Phi(v)).$$

The triangle inequality is based on the Hu's formula [H].

In particular, for the Poisson action  $\Phi : (\mathbb{R}^n, +) \rightarrow \text{Symp}(W^{2n}, \omega)$  associated with a Liouville-integrable Hamiltonian system one gets a certain pseudonorm  $\rho_\Phi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ .

This pseudonorm is degenerate for most examples of integrable geodesic flows with positive entropy ( $W^{2n} = TM^n$ ), but it is possible to construct a Liouville-integrable Hamiltonian system such that  $\rho_\Phi$  is a norm [K].

**Theorem 8.** *Suppose the Riemannian metrics  $g$  and  $\bar{g}$  on a closed connected manifold  $M^n$  are geodesically equivalent and strictly non-proportional at least at one point. Let  $\Phi$  be the Poisson action constructed by the integrals  $I_{t_1}, \dots, I_{t_n}$ , where the numbers  $t_i$  are mutually different. Then,  $\rho_\Phi(v) = 0$  for every  $v \in R^n$ .*

The proof of this theorem will be published elsewhere.

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