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Symmetric Ideals

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Abstract

Polynomials appear in many different fields such as statistics, physics and optimization. However, when the degrees or the number of variables are high, it generally becomes quite difficult to solve polynomials or to optimize polynomial functions. An approach that can often be helpful to reduce the complexity of such problems is to study symmetries in the problems. A relatively new field, that has gained a lot of traction in the last fifteen years, is the study of symmetry in polynomial rings in increasingly many variables. By considering the action of the symmetric groups on these polynomial rings, one can for instance show that certain sequences of symmetric ideals in increasingly larger polynomial rings are finitely generated up to orbits.

In this thesis we will investigate some properties of such sequences. In particular the Hilbert Series and Gröbner bases of Specht ideals, a class of ideals arising from the representation theory of the symmetric group. We prove a conjectured Gröbner basis for Specht ideals of shape $(n-k, 1^k)$ and give two different criteria for verifying the conjecture for other Specht ideals. We also build on a result from the representation theory of the symmetric group by showing that the leading monomials of the standard Specht polynomials span the vector space of leading monomials of Specht polynomials.

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List of Notations

- $\mathbb{N} = \{1, 2, 3, ...\}$
- $\mathbb{N}_0 = \{0, 1, 2, ...\}$
- $[n] = \{1, 2, ..., n\}$
- $X_n = \{x_1, x_2, ..., x_n\}$
- $X_n^\diamond$ - the free commutative monoid generated by X_n
- $R_n = K[X_n]$
- $\mathbf{x} = x_1 x_2 \cdots x_n$
- $\subset$ - strict subset
- $<$ - strictly less than
- $\triangleleft$ - "ideal of" or "dominated by"
- $\mathrm{supp}(f)$ - the set of all the variables appearing in $f \in R_n$

Introduction

Polynomials are some of the most fundamental objects in mathematics and have a long history of being studied by mathematicians from all over the world. From Diophantus of Alexandria around 200 AD to the modern day mathematician with the aid of a computer, polynomials have been a great source of inspiration and frustration for a lot of people over the years. And yet with all of these people looking to find a way to solve or optimize polynomial functions, not one has found an approach that can be claimed to be completely satisfactory.

Although, maybe this is a good thing since so many different mathematical theories owe their existence to the complexity of such questions. For instance, would Galois theory exist and what would take its place as an inspiration to develop group theory? Most likely, group theory would have a much smaller significance, and Galois theory, if it existed, would not be all that interesting.

On the other hand, due to the importance of polynomials in most scientific branches, a lot of problems would be easier to deal with. They appear in statistics as polynomials in stochastic variables, in finance they can be used to model how interest accumulates, physicist may use them to describe the trajectory of objects and engineers can use them to model a robots movements. Thus understanding polynomials better could significantly impact the world.

A principle that is often helpful to understand a problem or to find optimal solutions, is to notice and exploit an inherent symmetry that many problems possess. For instance, if we are looking for zero sets of symmetric polynomials, then as soon as we have found one zero point we know that the orbits of the point must be a zero point as well. Thus it suffices to find a single zero point per orbit. That is, if we have a polynomial or polynomial function with some symmetry, we can try to use the symmetry to reduce the complexity of solving the polynomial or to optimize the polynomial function.

It is at this intersection of polynomials and symmetries that this thesis takes place. We will be looking at ideals consisting of polynomials that are not necessarily symmetric, but stable under the permutation action of the symmetric group. That is, polynomial ideals that contain all the orbits of the polynomials.

Furthermore, we will be considering these ideals in polynomial rings in an increasing number of variables, thereby obtaining sequences of symmetric ideals. We will see that several algebraic properties stabilize in such sequences. That is, at some point increasing the number of variables does not seem to bring anything new to the situation. Additionally, restricting to symmetric ideals appear to be a good way to deal with the polynomial ring in infinitely many variables. It ensures, for instance, that the ideals are finitely generated up to the action of the symmetric group, as was shown in [5], [1] and [13]. Thus the limiting object of such sequences behave rather nicely.

In particular we will be looking at a class of ideals arising from the representation theory of the symmetric group, called Specht ideals. These ideals are generated by the irreducible representations of the symmetric group and are therefore a natural class of ideals to consider seeing as we are working with the action of the symmetric group.

We will be investigating the Hilbert series of some of the sequences of Specht ideals and the equivariant Hilbert series of the projective limit of these sequences. We also investigate a conjectured Gröbner basis of some of the sequences and the equivariant Gröbner basis of their limiting ideal. To facilitate this investigation we make use of some results regarding Specht ideals from [16] and [21], where they also studied these ideals. We also make use of some results regarding general sequences of symmetric ideals from [17] and [10].

Towards the end of the thesis we present some further research areas for these sequences, for instance free resolutions and the Castlenuovo-Mumford regularity. This can be thought of as an invariant that describes the complexity of a minimal free resolution of an ideal. Thus, with regards to our sequences, it describes how this complexity increases with the increase in the number of variables.

The first three chapters review some known results regarding polynomial rings and representation theory, after which we move on to some original results in Chapter 4. We begin, in Chapter 1, with an introduction to some essential concepts from commutative algebra along with some tools that we will need later. We go through the definitions of Hilbert series and Gröbner bases and introduce some classical results regarding these.

Afterwards, in Chapter 2, we introduce representation theory of the symmetric group. We present some fundamental results from representation theory of finite groups, before looking at the symmetric group in particular. We will show how the regular representation of the symmetric group can be decomposed into irreducible representations. Chapter 3 will be used to describe the general framework for working with sequences of symmetric ideals and symmetric ideals in the infinite polynomial ring. In this chapter we extend some of the definitions and ideas from the finite polynomial rings, that we introduced in Chapter 1, to the infinite polynomial ring.

Then we move on to Chapter 4 where we look at sequences of Specht ideals. This is the main part of this thesis and contains several new results. Firstly, in section 4.1, we compute the Hilbert series for some Specht ideals corresponding to partitions of the form $(n-k, k)$. This was also studied in the article [25], but the work we present is independent of this. Then, in Theorem 4.2.1, we show that the Specht ideals of shape $(n-k, 1^k)$ has a very natural Gröbner basis which supports a more general conjecture regarding Gröbner bases of Specht ideals. This extends previous studies of this class of ideals done in [9], and could be of interest outside the study of Specht ideals. We also use these Gröbner bases, and a result from [10], to give the equivariant Hilbert series of the corresponding sequences in Theorem 4.2.2.

Furthermore, we provide a reduction to the Gröbner basis conjecture in Lemma 4.3.1 and give two different criterion for verifying the conjectured Gröbner basis for other Specht ideals in Theorem 4.3.2 and 4.4.1. One that can be thought of as a variation of Buchbergers' criterion, except modified for symmetric sequences, and one that specifically applies to the Specht ideals of shape $(n-k, k)$. The former is not restricted to Specht ideals and hence may be of interest in the study of symmetric sequences in general.

In Theorem 4.3.1, we extend a result from representation theory by showing that the leading monomials of the standard Specht polynomials represent all the leading monomials of the Specht polynomials. Lastly, in Proposition 4.5.1, we give some conditions for the existence of an "equivariant Hilbert polynomial" before presenting some possibilities for further research in Chapter 5.

Chapter I / Polynomial rings

We begin this chapter by recalling some basic definitions of rings and ideals and proceed to give a general framework for working in polynomial rings. Concepts like Gröbner bases, Krull dimension, Hilbert series and decompositions of ideals are introduced and provided examples for.

Krull dimension and Hilbert series are both ways of describing the size of an ideal, although the Krull dimension relates the size to more fundamental ideals (prime ideals) and the Hilbert series describes the size of an ideal by breaking it down to finite dimensional vector spaces. Primary decomposition is a way of describing it as an intersection of primary ideals, a generalization of prime ideals. Gröbner bases are generating sets of ideals that help us determine when an element is in the ideal or not, which is generally not a trivial question. They are also often used to compute intersections and Hilbert series.

Thus, all of these concepts are introduced to deal with various notions of the size of an ideal, what the fundamental components of an ideal is, when is an element a member of an ideal and so forth. In essence they are ways of describing some of the fundamental properties of an ideal.

In this chapter we will assume the reader has some previous knowledge about sets and set operations, equivalence classes, monoids and groups. We will give the definition of a ring and ideal for instance, but it will be helpful to have seen such objects before. All of these concepts are generally included in any introductory book on algebra, for instance in [8].

1.1. Ideals, varieties and dimension

Recall that a ring, {R,+,·}, is a set R together with an additive and a multiplicative operation such that {R,+} is an abelian group, {R,·} is a monoid, and multiplication distributes with respect to addition:

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

$$(b + c) \cdot a = b \cdot a + c \cdot a.$$

For brevity we denote the ring by R.

We will mostly work with the polynomial ring $R_n = K[X_n]$, where X_n is the set of variables $\{x_1, x_2, ..., x_n\}$, with $n \in \mathbb{N}$. If $n = 0$ we will just let R_n denote K, where K is a field (a commutative ring with multiplicative inverses). We will restrict to the case where K is a field of characteristic zero. That is, if 1 and 0 are the multiplicative and additive identity elements of K respectively, then $\min\{k \in \mathbb{N} | k \cdot 1 = 0\}$ does not exist.

We define R_n to be the set of all finite sums of the form $\sum_{\alpha \in M} a_\alpha \mathbf{x}^\alpha$, where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $a_\alpha \in K$ and M is a finite subset of $\mathbb{N}_0^n$. When we have a particular set of polynomials, we often tend to work with the ideals they generate instead of the polynomials themselves. Recall that an *ideal*, $I \triangleleft R$, of a ring, R, is an additive subgroup of R such that $ra \in I \ \forall \ r \in R$ and $a \in I$. So if we have the polynomials $f_1, \ f_2, ..., \ f_r \in R_n$, we say that the ideal they generate is the set

$$\langle f_1, \ f_2, ..., \ f_r \rangle = \left\{ \sum_{i=1}^{r} h_i f_i \mid h_i \in R_n \ \forall \ i \right\}.$$

When we work with rings and ideals we often need to make use of a *ring homomorphism*. That is, a map, $\phi : R \to R'$, such that $\phi(ab + c) = \phi(a)\phi(b) + \phi(c)$ for all $a, b, c \in R$. Note that on the left hand side of the equation we are using the ring operations of R and on the left we use the ring operations of R'.

A closely related object to an ideal that we often consider is the quotient ring, R/I. This ring is defined as the ring of equivalence classes, $[a] := a + I$, under the equivalence relation $a \sim b$ if $a - b \in I$. The ring operations are defined as follows:

$$(a + I)(b + I) = ab + I,$$
$$(a + I) + (b + I) = (a + b) + I.$$

One can think about the quotient ring as the the image of the ring homomorphism $q(a) = a + I \ \forall \ a \in R$.

Associated to sets of polynomials and ideals we have what is called a *variety*. This is the set of points in K^n where all the polynomials in the set/ideal, vanish. It is easy to see that for a set of polynomials $F = \{f_1, \ f_2, ..., \ f_r\}$, its variety, $V(F)$, is equal to the variety of the ideal that it generates, that is $V(F) = V(\langle F \rangle)$. Thus, if we are considering a set of polynomials, we can replace the set with the ideal that it generates, and this won't affect the associated variety. Since it can often be advantageous to work with ideals instead of sets, we will often be doing this.

In fact we can take this one step further and consider the ideal of all polynomials vanishing on the variety of our polynomials: $I(V(F)) = \{f \in R_n | f(\mathbf{a}) = 0 \ \forall \ \mathbf{a} \in K^n\}$. Although, in general, this ideal will not be the same as $\langle F \rangle$ and also the above description does not give us an algebraic description of the ideal $I(V(F))$ in terms of a generating set of the ideal. The following theorems will help us with that.

First we present a fundamental property that some rings possess called *Noetherianity*. This is a crucial property for the Hilbert basis theorem and the Lasker-Noether theorem. It is defined as follows:

Definition 1.1.1. *A ring S is Noetherian if any sequence of ideals of S, $I_1 \subseteq I_2 \subseteq ...$, stabilizes, that is, if there exist an integer n such that $I_n = I_{n+1} = ...$ This is often called the* ACC, *or the* ascending chain condition.

Theorem 1.1.1. *The ring R_n is Noetherian.*

Thus if we have an ideal, I, generated by elements $\{f_1, f_2, ...\} \subseteq R_n$, then we can construct the sequence of ideals $\langle f_1 \rangle \subseteq \langle f_1, f_2 \rangle \subseteq ...$, and due to the Noetherianity of R_n, we know that it must stabilize. Thus $I = \langle f_1, f_2, ..., f_k \rangle$, for some $k \in \mathbb{N}$. In fact one can argue that the theorem above is equivalent to the Hilbert basis theorem:

Theorem 1.1.2 (Hilbert Basis Theorem). *Every ideal in R_n is generated by finitely many elements.*

Thus the ideal $I(V(F))$ is finitely generated. The next theorem gives a different way of characterizing the ideal. Before that however, we need a definition: The *radical* of an ideal, I of R_n, denoted $\sqrt{I}$, is the ideal consisting of all polynomials $f \in R_n$ such that $f^m \in I$, for some positive integer m. An ideal, I, is called radical if $I = \sqrt{I}$.

Theorem 1.1.3 (Hilbert's (strong) Nullstellensatz). *If K is an algebraically closed field and $I \lhd K[X_n]$, then $\sqrt{I} = I(V(I))$.*

The proofs of theorems 1.1.1, 1.1.2 and 1.1.3 along with a more in-depth deliberation on the topic of varieties and radical ideals can be found in [7], chapters 2 and 4.

Next we introduce the notion of the Krull dimension of an ideal.

The Krull dimension of a ring is defined as the supremum of the lengths of chains of prime ideals: $P_0 \subset P_1 \subset ... \subset P_r$, where the inclusions are strict and the length is the integer r. Recall that a prime ideal is the anologue of ideals generated by prime numbers in $\mathbb{Z}$, that is, they are the ideals, I, such that if $pq \in I$, then $p \in I$ or $q \in I$. We define the Krull dimension of ideal, $I \lhd R_n$, to be the Krull dimension of the quotient ring R_n/I. An ideal, $J \lhd R_n/I$ is of the form $q(L)$, where $L \lhd R_n$ contains I and q is the quotient map. It can also be shown that J is prime if L is prime. Thus the Krull dimension of I is the supremum of the lengths of chains of prime ideals that contain I.

Note that even if a ring is Noetherian it need not be of finite Krull dimension since the Krull dimension is ascertained by taking the supremum over a possibly infinite set of sequences of ideals (see for instance [15], Example 5.96).

Although the definition of the Krull dimension might seem somewhat non intuitive, we can take advantage of a correspondence between prime ideals and what we call irreducible varieties to rephrase it in more geometric terms.

Definition 1.1.2. *An irreducible variety $V \subset K^n$, is a variety with the property that if $V = W_1 \cup W_2$ for some varieties W_1 and W_2, then $V = W_1$ or $V = W_2$.*

Theorem 1.1.4. *Let K be an algebraically closed field, $V \subseteq K^n$ be a variety and $I \lhd R_n$. Then $I(V)$ is prime if and only if V is irreducible. Similarly $V(I)$ is irreducible if and only if I is prime.*

Proof. See Corollary 4 in [7] Chapter 4.5. □

If we have two ideals $I, J \lhd R_n$, with $I \subseteq J$, then clearly $V(J) \subseteq V(I)$. Similarly, if we have two varieties $V, U \subseteq K^n$, with $V \subseteq U$, then $I(U) \subseteq I(V)$. Thus the Krull dimension of an ideal $I \lhd R_n$ can be thought of as the supremum of lengths of chains of irreducible varieties where the inclusions are the reverse of the inclusions of the corresponding prime ideals.

Lastly, in this section, we introduce the concept of primary decomposition. Geometrically this can be thought of as a way of understanding an object by looking at its fundamental components. The fact that we can do such a thing, is again a consequence of the Noetherian property of R_n.

We start with a definition of a generalization of prime ideals:

Definition 1.1.3. *An ideal $I \lhd R_n$ is* primary *if whenever $fh \in I$ either $f \in I$ or $h^k \in I$ for some $k \in \mathbb{N}$.*

Just like prime ideals are a generalization of prime numbers, primary ideals is a generalization of powers of prime numbers, that is, $\langle p^k \rangle$ is a primary ideal in $\mathbb{Z}$ if p is a prime number.

Definition 1.1.4. *A* primary decomposition *of an ideal I is an intersection of primary ideals, $\cap_{i=0}^k P_i$, equal to the ideal I. The decomposition is minimal if $\cap_{i \neq j} P_i \not\subset P_j \ \forall j$ and if the $\sqrt{P_i}$'s are distinct.*

Thus, if we continue with our analogous situation in $\mathbb{Z}$ with ideals of powers of prime numbers, we may think of a primary decomposition as being similar to factoring numbers into powers of prime numbers. Also, note that the intersection of two ideals is an ideal itself. To see this let $r \in R$ and $a \in I \cap J$ for some $I, J \lhd R$, then, since $ra \in I$ and $ra \in J$, we have $ra \in I \cap J$. Similarly if $a, b \in I \cap J$, then $a + b \in I$ and $a + b \in J$, thus $a + b \in I \cap J$ so $I \cap J$ is an ideal.

From Definition 1.1.3, we can see that the radical of a primary ideal is a prime ideal. Also, note that taking the radical commutes with taking intersections, that is, if $I = \cap_{i=1}^{k} P_i$, then $\sqrt{I} = \cap_{i=1}^{k} \sqrt{P_i}$ (see Proposition 16 in Chapter 4.3 in [7] for a proof). Thus if we have a minimal primary decomposition of a radical ideal, then all the primary ideals in the decomposition must be prime ideals.

We finish this section with the main result of this topic:

Theorem 1.1.5 (Lasker-Noether). *Every ideal, I, in R_n has a minimal primary decomposition, $\cap_{i=0}^{k} P_i$, and the $\sqrt{P_i}$'s are uniquely determined by I.*

Proof. See Theorem 7 and 9 in Chapter 4.7 of [7]. □

Thus in the case of radical ideals we can speak of the minimal primary decomposition.

1.2. Orderings and Gröbner bases

When dealing with univariate polynomials we are in the nice position that it is obvious when one monomial is larger than the other. Thus division of polynomials makes perfect sense and the Euclidean algorithm provides us with a method of computing a quotient and a remainder. However, in the multivariate case, the question of which of two monomials is larger needs to be settled before general division makes sense. But even after settling on a particular order, uniqueness of remainder can still fail and thus we need something extra to deal with that. This is where Gröbner bases comes into the picture.

First we present the definition of a monomial order: A monomial order, "$\leq$" on R_n is a relation on the set of monomials of R_n, denoted $X_n^\diamond$, such that:

i "$\leq$" is a total order, that is,

a) $\forall\ \mathbf{x}^\alpha, \mathbf{x}^\beta \in X_n^\diamond$, either $\mathbf{x}^\alpha \leq \mathbf{x}^\beta$ or $\mathbf{x}^\beta \leq \mathbf{x}^\alpha$,

b) if $\mathbf{x}^\alpha \leq \mathbf{x}^\beta$ and $\mathbf{x}^\beta \leq \mathbf{x}^\alpha$, then $\alpha = \beta$,

c) if $\mathbf{x}^\alpha \leq \mathbf{x}^\beta$ and $\mathbf{x}^\beta \leq \mathbf{x}^\gamma$, then $\mathbf{x}^\alpha \leq \mathbf{x}^\gamma$,

ii $\forall\ \mathbf{x}^\alpha, \mathbf{x}^\beta, \mathbf{x}^\gamma \in X_n^\diamond$ such that $\mathbf{x}^\alpha \leq \mathbf{x}^\beta$, we have that $\mathbf{x}^{\alpha+\gamma} \leq \mathbf{x}^{\beta+\gamma}$,

iii $\forall\ \mathbf{x}^\alpha \in X_n^\diamond$, $1 \leq \mathbf{x}^\alpha$

When these conditions are fulfilled it may be proven that "$\leq$" is a well-ordering, that is, that there are no infinte strictly decreasing sequences of monomials. For the proof of this claim see [7], Corollary 6.

A very common monomial order to use is the lexicographical order, also called the "lex" order. This order is defined as the relation $\mathbf{x}^\alpha < \mathbf{x}^\beta$ if the leftmost nonzero entry in the vector $\beta - \alpha$ is positive. The lex order gives us the following relation on the variables: $x_1 > x_2 > ... > x_n$, thereby the name lexicographical. We will also have use for an order called the invlex order. It is defined as follows: $\mathbf{x}^\alpha < \mathbf{x}^\beta$ if the rightmost nonzero entry in the vector $\beta - \alpha$ is positive. Thus, the invlex order gives the following relation on the variables: $x_1 > x_2 > ... > x_n$.

Now that we have a good definition of what an ordering is, we can start dividing polynomials. But first some notation, let $f \in R_n$, then $\mathrm{lm}_{\leq}(f)$ denotes the leading monomial of f with respect to the chosen order. We will usually forgo the subscript if there is no ambiguity regarding the order. Similarly, $\mathrm{lt}(f)$ and $\mathrm{lc}(f)$, denotes the leading term and the coefficient of the leading term respectively, thus $\mathrm{lt}(f) = \mathrm{lc}(f)\,\mathrm{lm}(f)$.

We will describe the division algorithm by using an example. A more thorough exposition can be found in [7], Chapter 2.3, but we just need to know that such an algorithm exist and what can be problematic with the algorithm.

Theorem 1.2.1. *Let $f, h_1, h_2, ..., h_k \in R_n$ and let "$\leq$" be an ordering on $X_n^\diamond$. Then there exists polynomials $r, q_1, q_2, ...q_k \in R_n$ such that $f = \sum_{i=1}^k h_i q_i + r$ and no monomial of r is divisible by any of the monomials* $\mathrm{lm}(h_1), \mathrm{lm}(h_2), ..., \mathrm{lm}(h_k)$.

We will call r the remainder of f on division by $H = \{h_1, h_2, ..., h_k\}$.

Proof. See Theorem 3 in [7] Chapter 2.3. $\square$

Let $f = x_1 x_2^2 + x_1 x_2 \in R_2$ and let $X_2^\diamond$ be ordered lexicographically. We will divide f by the polynomials $h_1 = x_1 + x_2$ and $h_2 = x_1 x_2 + x_1$. We start with the following setup:

$$
\begin{array}{r|l}
q_1 : & \\
q_2 : & \\
\hline
\begin{array}{c} x_1 + x_2 \\ x_1 x_2 + x_1 \end{array} & x_1 x_2^2 + x_1 x_2
\end{array}
,
$$

where q_1 and q_2 are the quotients, that is, $f = h_1q_1 + h_2q_2 + r$, where r is the remainder. Since $\mathrm{lm}(f) > \mathrm{lm}(h_1)$ and $\mathrm{lm}(h_1) | \mathrm{lm}(f)$, we write

$$
\begin{array}{r|l}
q_1: & x_2^2 \\
q_2: & \\
\hline
\begin{array}{r} x_1 + x_2 \\ x_1x_2 + x_1 \end{array} & x_1x_2^2 + x_1x_2 \\
 & \underline{x_1x_2^2 + x_2^3} \\
 & x_1x_2 - x_2^3
\end{array}
.
$$

What we have left, $x_1x_2 - x_2^3$, still has a leading monomial that is both larger than and divisible by $\mathrm{lm}(h_1)$. Thus we repeat the same process.

$$
\begin{array}{r|l}
q_1: & x_2^2 + x_2 \\
q_2: & \\
\hline
\begin{array}{r} x_1 + x_2 \\ x_1x_2 + x_1 \end{array} & x_1x_2^2 + x_1x_2 \\
 & \underline{x_1x_2^2 + x_2^3} \\
 & x_1x_2 - x_2^3 \\
 & \underline{x_1x_2 + x_2^2} \\
 & -x_2^3 - x_2^2
\end{array}
$$

The leading monomial of $-x_2^3 - x_2^2$ is smaller than the leading monomial of both h_1 and h_2, thus we stop here and let $r = -x_2^3 - x_2^2$. That is, we have obtained the following expression

$$f = x_1x_2^2 + x_1x_2 = (x_1 + x_2)(x_2^2 + x_2) - x_2^3 - x_2^2 = h_1q_1 + r.$$

However, note that $f = x_2h_2$, thus, if we were to the division with h_1 and h_2 reordered, we would get that $x_1x_2^2 + x_1x_2 = x_2(x_1x_2 + x_1)$. That is, both the remainder and the quotients depend on the order in which we list the polynomials we wish to divide over. However, we will see that uniqueness of remainder can be achieved by the use of Gröbner bases.

Given an ideal $I \lhd R_n$, we will let $\mathrm{lm}(I)$ denote the *leading monomial ideal* of I, that is, the ideal generated by the leading monomials of I, $\mathrm{lm}(I) := \langle \mathrm{lm}(f) | f \in I \rangle$. We have the following definition of a Gröbner basis of I:

Definition 1.2.1. *A finite subset $G \subseteq I \lhd R_n$ is a Gröbner basis of I if $\langle \mathrm{lm}(G) \rangle := \langle \mathrm{lm}(g) | g \in G \rangle = \mathrm{lm}(I)$.*

Note that the definition of a Gröbner basis is dependant on a particular monomial order on $X_n^{\diamond}$. Thus if G is a Gröbner basis of $I \lhd R_n$ with respect to a monomial order "$\leq$", it may not be a Gröbner basis with respect to a different order.

Ideals of R_n have the following useful property:

Proposition 1.2.1. *Given an ideal $I \lhd R_n$ and a monomial order, there exists a Gröbner basis, G, for I and $I = \langle G \rangle$.*

Proof. See Corollary 6 in Chapter 2.5 of [7]. □

Let us look again at our example of dividing the polynomial $f = x_1x_2^2 + x_1x_2$ by $h_1 = x_1 + x_2$ and $h_2 = x_1x_2 + x_1$. Notice that the reason we were unable to end up with a zero remainder was that the leading monomial of $r = -x_2^3 - x_2^2$, was smaller than the leading monomial of both h_1 and h_2, thus $\mathrm{lm}(r) \notin \langle \mathrm{lm}(h_1), \mathrm{lm}(h_2) \rangle$. So even though the definition of a Gröbner basis may not seem an obvious one, it gets us right to the central issue with the division algorithm. Thus we have the following property for Gröbner bases:

Proposition 1.2.2. *If G is a Gröbner basis of $I \lhd R_n$, then the remainder, $\overline{f}^G$, of $f \in R_n$ on division by G, is unique. In particular, if $f \in I$, then $\overline{f}^G = 0$.*

Proof. For the second statement, note that if $\overline{f}^G \neq 0$, then $\mathrm{lm}(\overline{f}^G) \in \langle \mathrm{lm}(G) \rangle$ since G is a Gröbner basis. Also, $\mathrm{lm}(\overline{f}^G) = h\,\mathrm{lm}(g)$, where $g \in G$ and $h \in R_n$ is a nonzero monomial. However, due to Theorem 1.2.1, this is a contradiction, thus $\mathrm{lm}(\overline{f}^G) = 0$.

For the first statement let $f \in R_n$ such that $f \notin I$ and let r_1 and r_2 be two distinct nonzero remainders of f on division by G. Note that $f - r_1$ and $f - r_2$ are both in I and have a zero remainder on division by G. Thus, $h = (f - r_1) - (f - r_2)$ is a nonzero polynomial with $\overline{h}^G = 0$. But $h = r_2 - r_1$ and due to Theorem 1.2.1 none of the terms of r_1 and r_2 are divisible by any of the leading monomials in $\mathrm{lm}(G)$. Since $\overline{h}^G = 0$ and none of the terms of h are divisible by any elements in $\mathrm{lm}(G)$, then $h = 0 \Rightarrow r_1 = r_2$, which is a contradiction. □

Let us now turn to the question of determining whether a given set of polynomials is a Gröbner basis for an ideal. Recall that in the previous example, we had that $\{h_1, h_2\}$ was not a Gröbner basis of $\langle h_1, h_2\rangle$ since we found an element, $r = -x_2^3 - x_2^2 \in \langle h_1, h_2\rangle$, with the property that $\mathrm{lm}(r) \notin \langle \mathrm{lm}(h_1), \mathrm{lm}(h_2)\rangle$. That is, we were able to construct an element in the ideal with a leading monomial smaller than the leading monomial of the generators. This is exactly the property that Buchberger's criterion focuses on so we present it here:

Theorem 1.2.2 (Buchberger's criterion)**.** *Let G be a finite subset of $I \lhd R_n$. Then G is a Gröbner basis of I if and only if for all pairs $\{g_1, g_2\} \subseteq G$, we have that $\overline{S(g_1, g_2)}^G = 0$, where*

$$S(g_1, g_2) = \frac{\mathrm{lcm}(\mathrm{lm}(g_1), \mathrm{lm}(g_2))}{\mathrm{lt}(g_1)} g_1 - \frac{\mathrm{lcm}(\mathrm{lm}(g_1), \mathrm{lm}(g_2))}{\mathrm{lt}(g_2)} g_2,$$

is the S-polynomial *of g_1 and g_2.*

Proof. See Theorem 6 in Chapter 2.6 of [7]. □

This criterion can be relaxed a bit by focusing only on the leading monomials and not on the entire factorization. Before we state it we introduce some new notation. If a polynomial $f \in R_n$ can be written of the form $f = \sum h_i g_i$ with $\mathrm{lm}(f) \geq \mathrm{lm}(h_i g_i)\ \forall\ i$, where $h_i \in R_n$ and $g_i \in G$, then we write $f \to_G 0$. Then clearly we have that $\overline{f}^G = 0 \Rightarrow f \to_G 0$.

Theorem 1.2.3. *Let G be a finite subset of $I \lhd R_n$. Then G is a Gröbner basis of I if and only if for all pairs $\{g_1, g_2\} \subseteq G$, we have that $S(g_1, g_2) \to_G 0$..*

Proof. See Theorem 3 in Chapter 2.9 of [7]. □

There are also algorithms for computing a Gröbner basis of an ideal, but we will not be needing them here. What we will be needing is to determine generating sets of intersections of ideals with certain subrings. This is also something that Gröbner bases can help us with.

The setup is as follows: let $I \lhd R_n$ and G be a Gröbner basis of I for some order. Suppose we would like to describe the ideal $I \cap K[x_2, x_3, ..., x_n]$, what would a generating set be for this ideal? In general it is not sufficient to simply look at the intersection of a generating set of I, and $K[x_2, x_3, ..., x_n]$, however, if we consider a particular kind of Gröbner bases, this exactly what we can do. For this we need a definition:

Definition 1.2.2. *An order "$\leq$", on $X_n^\diamond$, is called an* elimination order of type l*, where $l \in [n]$, if any monomial divisible by x_i, for $1 \leq i \leq l$ is larger than any monomial in $K[x_{l+1}, x_{l+2}, ..., x_n]$.*

We clearly have that the lex order is an elimination order of maximal type for any ring R_n. By limiting to these orderings we get the Gröbner bases we are looking for. In fact it turns out that we get a stronger property than we were looking for.

Theorem 1.2.4. *Fix an ordering of type $l \in [n]$ on $X_n^\diamond$ and let G be a Gröbner basis of $I \triangleleft R_n$, then $G \cap K[x_{l+1}, x_{l+2}, ..., x_n]$ is a Gröbner basis of $I \cap K[x_{l+1}, x_{l+2}, ..., x_n]$.*

Proof. See Theorem 2 in Chapter 3.1 of [7] and the corresponding exercises. □

1.3. Hilbert Series and exact sequences

To understand the size of an ideal in a more intuitive way than looking at its Krull dimension, we consider the Hilbert series of an ideal. For instance, if we consider the polynomial ring R_n, then we may view the d^{th} graded component of R_n as the vector space spanned by the polynomials of degree d in R_n. Then the Hilbert series would give us the dimension of each degree d component. Thus Hilbert series is a way of considering an ideal to be "the sum of its parts".

Before defining Hilbert series we start with the definition of a graded ring and homogeneous ideals.

Definition 1.3.1. *A polynomial $f \in R_n$ is* homogeneous *if all the monomials appearing in f have the same degree. An ideal generated by homogeneous polynomials is a homogeneous ideal.*

We have the following useful property for homogeneous ideals, which is sometimes used as the definition of a homogeneous ideal:

Proposition 1.3.1. *An ideal $I \triangleleft R_n$ is homogeneous if and only if for any $f \in I$, all the homogeneous components of f are also in I.*

Proof. Let $I \lhd R_n$ be a homogeneous ideal generated by $f_1, f_2, ..., f_k$. Then any $f \in I$ can be written of the form $f = \sum_{i=1}^{k} h_i f_i$, with $h_i \in R_n$ for all i. Write each h_i, as the sum of its homogeneous components, $h_i = \sum_j g_{i,j}$. Then we can write f as the sum of its homogeneous components in the following way: For each $d \leq \deg f$ let A_d be the set of pairs (i,j) such that $\deg g_{i,j} + \deg f_i = d$, then $f = \sum_{d \leq \deg f} \sum_{(i,j) \in A_d} g_{i,j} f_i$. Then clearly any homogeneous component of f can be written on the form $\sum_{(i,j)\in A_d} g_{i,j} f_i$, which is an element of I.

Conversely, if all homogeneous components of any element $f \in I$ is in I, then any generator of I can be replaced it with its homogeneous components. □

Thus we can say that a homogeneous ideal is equal to the span over K of all the homogeneous polynomials in the ideal. This observation leads to the concept of a graded ring.

A ring S, is called *graded* (or $\mathbb{N}_0$-graded) if it can be written as a direct sum of its graded components: $S = \bigoplus_{d \geq 0} S_d$, where the graded components are abelian groups and $S_i S_j \subseteq S_{i+j}$. For instance, for the polynomial ring R_n we consider its d^{th}-graded component to be the vector space of the homogeneous polynomials of degree d. Note, however, that we also need to include the zero element in the d^{th} graded component.

Similarly, we say that an ideal, $I \lhd S$, is graded if S is a graded ring and $I = \bigoplus_{d \geq 0}(I \cap S_d)$. Thus we say that I inherits its grading from S. Due to Propostition 1.3.1 we can see that the graded ideals in a graded ring are the homogeneous ideals. Also, note that the quotient ring S/I inherits a grading from S by summing over the quotient groups $S_d/(I \cap S_d)$.

Before introducing the Hilbert series we consider a more general construction called a *generating funtion*. If we have a sequence of numbers given by the function $f(n)$, that is, a sequence of the form $(f(n))_{n \in \mathbb{N}_0}$, then its generating function is the power series $F(t) = \sum_{n \geq 0} f(n) t^n$. Thus a generating function can be considered as a more compact way of writing a sequence of numbers.

The Hilbert series of an ideal $I \lhd R_n$ depends on a given grading and is defined as the generating function $H_I(t) = \sum_{d \geq 0} HF_I(d) t^d$, where $HF_I(d) = \mathrm{Dim}(I_d)$ is the *Hilbert function* of I. $HF_I(d)$ denotes the dimension of the dth-graded component where the grading of I is inherited from the grading of R_n. Equivalently, we can consider the Hilbert series of the quotient ring R_n/I: $H_{R_n/I}(t) = \sum_{d \geq 0} HF_{R_n/I}(d) t^d$. We can say the following about the Hilbert series of a homogeneous ideal:

Theorem 1.3.1 (Hilbert-Serre)**.** *The Hilbert series of a homogeneous ideal* $I \lhd R_n$ *is a rational function of the form* $\frac{f(t)}{(1-t)^d}$*, where* $f(t)$ *is a polynomial with integer coefficients.*

Proof. See [2] Theorem 11.1. □

It turns out that if we employ a more general result about generating functions, the Hilbert-Serre theorem is equivalent to the Hilbert function being eventually polynomial. That is, that there exist some $m \in \mathbb{N}_0$ such that $HF_I(n) = HP_I(n)$ for $n \geq m$ and where $HP_I(n)$ is a polynomial. We define the polynomial, $HP_I(n)$, to be the *Hilbert polynomial* and in fact it can be shown that $HP_I(n)$ has integer coefficients (see [7], Proposition 3, Chapter 9).

Lemma 1.3.1. *A sequence of numbers in* $\mathbb{C}$, $(f(n))_{n\in\mathbb{N}_0}$, *is eventually polynomial if and only if* $\sum_{n\geq 0} f(n)t^n = \frac{h(t)}{(1-t)^d}$ *for some polynomial* h *and some* $d \in \mathbb{N}_0$.

Proof. See Corollary 4.1.7 in Chapter 4 of [3]. □

Thus we have an equivalence between the existence of the Hilbert polynomial and the rationality of the Hilbert series. We proceed by looking at how one might calculate the Hilbert series of an ideal. Firstly we have the following useful lemma:

Lemma 1.3.2. *The Hilbert series of a homogeneous ideal* $I \lhd R_n$ *is equal to the Hilbert series of* $\operatorname{lm}(I)$.

Proof. Follows from Proposition 9 in [7], Chapter 9.3. □

Thus, if we already have a Gröbner basis of an ideal, we may instead consider its initial ideal when we compute the Hilbert series. This is often a good idea seeing as monomial ideals tend to be easier to handle. For instance, similar to Proposition 1.3.1, we have the following property regarding monomial ideals:

Lemma 1.3.3. *An ideal* $I \trianglelefteq R_n$ *is monomial if and only if for any* $f \in I$ *all the terms of* f *are also in* I.

Proof. See Lemma 3 of [7] Chapter 2.4. □

Due to this property we can show that computing intersections of monomial ideals is quite straightforward.

Proposition 1.3.2. *Let I be a monomial ideal generated by the monomials $m_1, ..., m_k$ and let J be a monomial ideal generated by the monomials $n_1, ..., n_l$. Then $I \cap J$ is a monomial ideal generated by the monomials* $\mathrm{lcm}(m_i, n_j)$ $\forall\ i \in [k],\ j \in [l]$.

Proof. If $f \in P = I \cap J$, then $f \in I$ and $f \in J$. By Lemma 1.3.3, all the terms of f lie in I and in J, thus all the terms of f lie in P. Thus, by Lemma 1.3.3, we have that P is a monomial ideal.

Clearly $\mathrm{lcm}(m_i, n_j) \in P\ \forall\ i \in [k]$ and $j \in [l]$, so we will show that any monomial of P is divisible by some monomial of the form $\mathrm{lcm}(m_i, n_j)$. So let m be a monomial of P, then $m_i|m$ and n_j for some $i \in [k]$ and $j \in [l]$, thus $\mathrm{lcm}(m_i, n_j)|m$. Therefore we have that $P = \langle\{\mathrm{lcm}(m_i, n_j)|i \in [k],\ j \in [l]\}\rangle$. □

Thus if we write a monomial ideal, I, as the sum of two monomial ideals, J_1 and J_2, then we get that $H_I(t) = H_{J_1}(t) + H_{J_2}(t) - H_{J_1 \cap J_2}(t)$. This follows from considering the d^{th} graded components of the ideals I, J_1, J_2 and $J_1 \cap J_2$ as finite dimensional vector spaces. Thus, if we know the Hilbert series of J_1, J_2 and if it is easier to compute the Hilbert series of $J_1 \cap J_2$ than of I, then we can compute the Hilbert series of $J_1 \cap J_2$ to determine the Hilbert series of I.

Another object that can be of great use when computing Hilbert series is an exact sequence. Essentially the idea with exact sequences is to relate an ideal to different (easier) ideals via a sequence of maps and deduce from those ideals what the Hilbert series must be.

Definition 1.3.2. *An exact sequence is a sequence of homomorphisms*

$$... \longrightarrow S_{n+1} \xrightarrow{\partial_{n+1}} S_n \xrightarrow{\partial_n} S_{n-1} \longrightarrow ...,$$

where $\mathrm{Im}(\partial_{n+1}) = \mathrm{Ker}(\partial_n)$.

An easy example of an exact sequence of rings is the following. Let I be an ideal of R_n, then the following is an exact sequence:

$$0 \xrightarrow{id} I \xrightarrow{\iota} R_n \xrightarrow{q} R_n/I \xrightarrow{0} 0,$$

where ι is the inclusion map and q is the quotient map. Then $id(0) = \{0\} = \mathrm{Ker}(\iota)$, $\iota(I) = I = \mathrm{Ker}(q)$ and $q(R_n) = R_n/I = \mathrm{Ker}(0)$. We usually skip writing down the homomorphisms id and 0.

Note that if we are using the standard grading for all the rings in the above sequence, the maps are all degree-preserving. That is, the degree of the image of an element is the same as the degree of the element itself. Thus if we restrict the map to the d^{th}-graded components, we get a map between finite dimensional vector spaces:

$$0 \xrightarrow{id} I_d \xrightarrow{\iota|_{I_d}} (R_n)_d \xrightarrow{q|_{(R_n)_d}} (R_n/I)_d \xrightarrow{0|_{(R_n/I)_d}} 0,$$

where $\iota|_{I_d}$ denotes the restriction of ι to the d^{th}-graded component of I and similar for the other maps.

Clearly $\mathrm{Dim}((R_n)_d) = \mathrm{Dim}(\mathrm{Im}(q|_{(R_n)_d})) + \mathrm{Dim}(\mathrm{Ker}(q|_{(R_n)_d}))$, and so we have that $\mathrm{Dim}((R_n)_d) = \mathrm{Dim}(\mathrm{Ker}(0|_{(R_n/I)_d})) + \mathrm{Dim}(\mathrm{Im}(\iota|_{I_d})) = \mathrm{Dim}((R_n/I)_d) + \mathrm{Dim}(I_d)$. Thus we get the following identity:

$$H_{R_n}(t) = H_{R_n/I}(t) + H_I(t).$$

So if we know the Hilbert series of two of the rings above, then we can find the Hilbert series for the third.

In general, if we are looking for the Hilbert series of an ideal, we may start by looking for related objects that we understand better and try to construct a sequence just like we did for this example. Then we can apply the same argument as above in a more general setting to get the following result:

Proposition 1.3.3. *Let*

$$0 \longrightarrow S_n \xrightarrow{\partial_n} \ \dots \ \longrightarrow S_2 \xrightarrow{\partial_2} S_1 \longrightarrow 0,$$

be an exact sequence of graded rings, then $H_{S_1}(t) = \sum_{i=2}^{n}(-1)^i H_{S_i}(t)$.

Proof. We restrict to the dth-graded components:

$$0 \longrightarrow (S_n)_d \xrightarrow{\partial_n|_{(S_n)_d}} \ \dots \ \longrightarrow (S_2)_d \xrightarrow{\partial_2|_{(S_2)_d}} (S_1)_d \longrightarrow 0.$$

Then, due to the exactness of the sequence, we get that

$$\mathrm{Dim}((S_1)_d) = \mathrm{Dim}(Im(\partial_2|_{(S_2)_d})) = \mathrm{Dim}((S_2)_d) - \mathrm{Dim}(Ker(\partial_2|_{(S_2)_d})) =$$

$$\mathrm{Dim}((S_2)_d) - \mathrm{Dim}(Im(\partial_3|_{(S_3)_d})) = \mathrm{Dim}((S_2)_d) - \mathrm{Dim}((S_3)_d) + \mathrm{Dim}(Ker(\partial_3|_{(S_3)_d}) =$$

$$\dots \ = \sum_{i=2}^{n}(-1)^i \,\mathrm{Dim}((S_i)_d) + (-1)^{k+1}\,\mathrm{Dim}(Ker(\partial_n|_{(S_n)_d})) =$$

$$\sum_{i=2}^{n}(-1)^i \,\mathrm{Dim}((S_i)_d) + 0 = \sum_{i=2}^{n}(-1)^i \,\mathrm{Dim}((S_i)_d).$$

□

We can apply this principle to show the following:

Lemma 1.3.4. *Let $n > 0$, then the Hilbert Series of R_n is $H_{R_n}(t) = \frac{1}{(1-t)^n}$.*

Proof. We give an inductive proof of this statement. For $n = 1$, we clearly we have that $H_{R_1}(t) = 1 + t + t^2 + ... = \sum_{i=0}^{\infty} t^i = \frac{1}{1-t}$.

Now assuming the statement is true for $n = k - 1 \geq 1$, we show that it holds true for $n = k$:

We clearly have that $R_{k-1} \simeq R_k/\langle x_k \rangle$. We also have the following exact sequence:

$$0 \longrightarrow R_k(1) \xrightarrow{\times x_k} R_k \xrightarrow{q} R_k/\langle x_k \rangle \longrightarrow 0,$$

where $R_k(1)$ denotes the ring R_k except that we have shifted the grading to start at 1. This is often called the 1^{th} *twist* of R_k. We do this so that when an element $f \in R_n(1)$ of degree d is sent to R_k via multiplication by x_k, then $f \times x_k$ also has degree d.

By Proposition 1.3.3 we have $H_{R_k}(t) = H_{R_k/\langle x_k \rangle} + H_{R_k(1)}(t)$. Since $R_k(1)$ is the same as R_k, except for a twist in the grading by 1, we have that $H_{R_k(1)}(t) = tH_{R_k}(t)$. And since $R_k/\langle x_k \rangle \simeq R_{k-1}$, we have that $H_{R_k/\langle x_k \rangle}(t) = H_{R_{k-1}}(t)$, which by the induction hypothesis is equal to $\frac{1}{(1-t)^{k-1}}$. Thus $H_{R_k}(t) = H_{R_{k-1}} + tH_{R_k}(t)$, which implies that $H_{R_k}(t) = \frac{1}{(1-t)}H_{R_{k-1}}(t) = \frac{1}{(1-t)^k}$. $\square$

This result, together with the previous example following Definition 1.3.2, shows that we can easily pass from the series $H_I(t)$ to $H_{R_n/I}(t)$ by simply subtracting it from $H_{R_n}(t)$. Thus we may use whichever one we prefer.

We will now turn to a particular type of rings that exploit the principle of exact sequences very nicely, namely *complete intersection rings*. We start with a definition, but first recall that a is a zero divisor of a ring S, if there exists a nonzero $b \in S$ such that $ab = 0$.

Definition 1.3.3. *A regular sequence in R_n, is a sequence of homogeneous polynomials $(f_1, f_2, ..., f_k)$ such that f_i is not a zero divisor of $R_n/\langle f_1, f_2, ..., f_{i-1} \rangle$ for all i.*

A ring, R_n/I, is a complete intersection ring if the ideal, I, is generated by a regular sequence. Note that the ring, $R_k/\langle x_k \rangle$, from the proof of Lemma 1.3.4, is a complete intersection ring.

Complete intersection rings have the following useful property:

Proposition 1.3.4. *Let R_n/I be a complete intersection ring and let I be generated by the regular sequence $(f_1, f_2, ..., f_k)$ of degrees $d_1, d_2, ..., d_k$, then $H_{R_n/I}(t) = \frac{\prod_{i=1}^{k}(1-t^{d_i})}{(1-t)^n}$.*

Proof. Let $i \in [k]$, then, since f_i is a nonzero divisor of $R_n/\langle f_1, ..., f_{i-1}\rangle$, the following sequence is exact:

$$0 \longrightarrow (R_n/\langle f_1, ..., f_{i-1}\rangle)(d_i) \xrightarrow{\times f_i} R_n/\langle f_1, ..., f_{i-1}\rangle \xrightarrow{q} R_n/\langle f_1, .., f_i\rangle \longrightarrow 0.$$

By Proposition 1.3.3, we have that

$$H_{R_n/\langle f_1,...,f_i\rangle}(t) = H_{R_n/\langle f_1,...,f_{i-1}}(t) - H_{(R_n/\langle f_1,...,f_{i-1}\rangle)(d_i)}(t) =$$

$$(1 - t^{d_i})H_{R_n/\langle f_1,...,f_{i-1}\rangle}(t).$$

Thus, inductively, we have

$$H_{R_n/\langle f_1,...,f_k\rangle}(t) = \prod_{i=1}^{k}(1 - t^{d_i})H_{R_n}(t).$$

So by Lemma 1.3.4 we get that

$$H_{R_n/\langle f_1,...,f_k\rangle}(t) = \prod_{i=1}^{k}(1 - t^{d_i})H_{R_n}(t) = \frac{\prod_{i=1}^{k}(1 - t^{d_i})}{(1 - t)^n}.$$

□

Chapter II / Representations

In this chapter we turn to representation theory, a topic that can be thought of as a generalization of vector spaces. For instance, if we have a vector space V and a scalar c, then multiplying all the elements of V, by c, can be thought of as a way of stretching or shrinking the space (depending on c). Now, if we would like to allow for more possibilities in how we affect the vector space V, be it by turning the space clockwise or reflecting it along a line etc., then we need representation theory.

The first section of this chapter serves as a quick introduction to representation theory for finite dimensional vector spaces and will also describe the relation between a few similar concepts such as group actions, representations and modules. We will introduce a couple of central results for representation theory, namely Masche's Theorem and Schur's Lemma and describe how to classify components of the vector space that behaves similarly when we start affecting it by rotations, reflections or similar actions.

Section two and three will go into the representation theory for the symmetric group. The theory for the symmetric group is quite well studied and therefore offers a lot of useful tools that we can take advantage of in the later chapters. It will also provide some context for why we consider Specht ideals in Chapter 4. Section two will focus on the finite dimensional case, which will be of use when we apply it to the polynomial ring in section three.

2.1. Group actions, representations and modules

A group action of a group, G, on a set X can be though of as a way to identify the group elements as automorphisms of X. Thus the group elements are "acting" on X according to specific automorphisms.

For instance, we can define the symmetric group, S_2, to be acting on $\mathbb{R}^2$, by swapping basis vectors. That is, if $\sigma \in S_2$ and $v = (v_1, v_2) \in \mathbb{R}^2$, then we define the action of S_2 by $\sigma(v) = (v_{\sigma(1)}, v_{\sigma(2)})$. Clearly, with this definition, the group elements correspond to automorphisms of $\mathbb{R}^2$.

Formally, we define a left group action G, on X, to be a map $\alpha : G \times X \to X$, that satisfies the following:

$$\alpha(id_G, x) = id_X(x),$$

$$\alpha(g, \alpha(h, x)) = \alpha(g \cdot h, x),$$

$$\forall g, h \in G \text{ and } x \in X.$$

Usually we denote $\alpha(g)(x)$ by gx. Equivalently, we can define a group action to be a group homomorphism $\alpha : G \to \text{Aut}(X)$.

A representation of a group G is similar to a group action, except we consider the action on a vector space and not a general set. Thus a representation of G on the vector space V is a triple (ρ, G, V), where $\rho : G \to \text{Aut}(V)$ is a group homomorphism.

A common representation that we will be using a lot is the following: Given an integer $n \in \mathbb{N}$, we let S_n denote the symmetric group on the set $[n]$. Let $\rho : S_n \to \text{Aut}(R_n)$ be the map defined by $\rho(\sigma)f(x_1, x_2, ...x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) \ \forall \ \sigma \in S_n$ & $f \in R_n$, then (ρ, S_n, R_n) is a representation of S_n on the infinite dimensional vector space R_n. Notice that this action of the symmetric group is degree-preserving, so we can equivalently define the representations $(\rho_d, S_n, R_{n,d})$ in a similar way for each degree d component. Then the representation, ρ, is just the direct sum of the degree d representations, ρ_d.

This suggests the following definition of a *subrepresentation*: Given a representation (ρ, G, V) and a subspace W of V, $(\rho|^W, G, W)$, is a subrepresentation of (ρ, G, V) if W is stable under the action of G, that is, if $gw \in W \ \forall \ w \in W$ & $g \in G$. The reason we need it to be stable is simply to ensure that the restriction of ρ to the codomain W is an automorphisms of W.

A particular kind of subrepresentation will be of interest to us, these are called *irreducible* representations. These are the nonzero representations that doesn't contain any smaller subrepresentations except the trivial subspace. The irreducible representations allow for the following nice description of a representation:

Theorem 2.1.1 (Maschke's Theorem)**.** *Let (ρ, G, V) be a representation of a finite group G on a finite dimensional vector space V. If V is a vector space over a field of a characteristic that does not divide the order of G, then ρ can be written as a direct sum of irreducible representations.*

Proof. See Theorem 1.5.3 in [20]. $\square$

We will usually write such a decomposition of a representation (ρ, G, V) as $V = \bigoplus_{i=1}^{k} W^{(i)}$, where each $W^{(i)}$ is irreducible, and only define the corresponding maps, $\rho_i : G \to W^{(i)}$, if necessary.

A decomposition of a representation into irreducible representations will not in general be unique. Just consider the representation of S_2 on $\mathbb{R}^2$ defined by $\sigma(v) = \text{Sgn}(\sigma)v$. Then, if e_1 and e_2 are the standard basis vectors, we can decompose the vector space into the direct sum of the span of e_1 and the span of e_2, alternatively we can decompose it as the span of $e_1 + e_2$ and the span of $e_1 - e_2$. Clearly these decompositions are irreducible, but they do not give the same subspaces.

On the bright side, a decomposition into irreducibles is unique up to isomorphism. That is, a decomposition of a representation, (ρ, G, V), of a finite group G on a finite vector space can be written as $V = \bigoplus_i m_i W^{(i)}$, where each $W^{(i)}$ is irreducible and m_i denotes the number of times $W^{(i)}$ occurs in V. This is called an *isotypic decomposition* of V, and each collection of isomorphic irreducibles, $m_i W^{(i)}$, is called an isotypic component. The fact that this decomposition is unique follows from another central result in representation theory called Schur's Lemma:

Theorem 2.1.2 (Schur's Lemma)**.** *Let (ρ, G, V) and (ϕ, G, W) be two irreducible representations of a finite group G on finite dimensional vector spaces V and W. If $\theta : V \to W$ is a* G-homomorphism*, that is, a linear mapping such that $\theta \circ \rho(g) = \phi(g) \circ \theta \ \forall \ g \in G$, then either θ is a G-isomorphism, or it is the zero map.*

Proof. See Theorem 1.6.5 in [20]. □

Corollary 2.1.1. *A representation, (ρ, G, V), of a finite group G on a finite vector space V, has a unique isotypic decomposition.*

Proof. Let V have two decompositions into irreducibles and let $V = \bigoplus_i m_i W^{(i)}$ and $V = \bigoplus_i n_i U^{(i)}$ be the corresponding isotypic decompositions. Let $U^{(k)}$ be isomorphic to $W^{(j)}$ for some k and j. We have that $id : V \to V$ be a G-automorphism and let $p_i : V \to m_i W^{(i)}$ be a projection onto $m_i W^{(i)}$. Clearly the restriction of $p_i \circ id$ from $U^{(k)}$ to any of the copies of $W^{(i)}$ is a G-homomorphism. Thus when $i \neq j$, Schur's Lemma says that $(p_i \circ id)|_{U^{(k)}}^{W^{(i)}}$ is the zero map. Therefore we have that $\theta(U^{(k)}) \subseteq m_j W^{(j)}$.

Since the argument was done for an arbitray irreducible component, we get that $id(n_i U^{(i)}) \subseteq m_l W^{(l)}$, when $U^{(i)}$ is isomorphic to $W^{(l)}$ and since $id(V) = \bigoplus id(n_i U^{(i)}) = V = \bigoplus_i m_i W^{(i)}$ we get that $n_i = m_l$. Thus the two isotypic decompositions are the same. □

Lastly in this section we will look at a different way of talking about a representation, namely as a module. A module is similar to a vector space except that instead of having a scalar field, we have a ring taking its place. We introduce modules because it can sometimes be more convenient to consider a module rather than a representation.

Definition 2.1.1. *Let R be a ring and G an abelian group. Then G is a left R-module if we can define a "ring action" on M, that is, a multiplicative map $\cdot : R \times G \to G$, such that:*

$$\cdot(id_R, g) = g,$$
$$\cdot(r, g) + \cdot(s, g) = \cdot(r + s, g),$$
$$\cdot(r, \cdot(s, g)) = \cdot(rs, g),$$
$$\cdot(r, g) + \cdot(r, h) = \cdot(r, g + h),$$
$$\forall\ r, s \in R \text{ and } g, h \in G.$$

Usually we replace the notation $\cdot(r, g)$ by rg, and since we will mainly be working with the scenario that R is a commutative ring we will just call a left R-module for an R-module since then the left and right modules coincide.

To see the connection between modules and representations, consider a representation, (ρ, G, V), where V is a vector space over the field, K, and construct the *group ring* of K and G, $K[G]$. $K[G]$ is defined to be the set of finite formal sums $\{\sum_{i=1}^{k} c_i g_i | c_i \in K, g_i \in G\}$, where the additive and multiplicative in $K[G]$ are natural extensions of the ring and group operations on K and G. That is, for $c, d \in K$ and $g, h \in G$, let $(cg) \cdot (dh) = (cd)(gh)$, where the product (cd) is taken to be the product in K and (gh) is the product in G.

With this definition $K[G]$ is a (commutative) ring. Since V is a vector space over K and we have defined a representation of G on V that gives us K-linear automorphisms of V we can easily see that V is a $K[G]$-module by extending the map, ρ, by letting $\rho(cg) = c\rho(g)$, for all $c \in K$ and $g \in G$. Thus we may refer to the $K[G]$-module V, instead of the representation (ρ, G, V), but still be referring to the same thing.

2.2. Decomposition of the symmetric group representation

In this section we will consider the representation of the symmetric group, S_n, and we will show how one can construct the complete list of irreducible representations. We start by introducing a representation that contains all possible irreducible representations (up to isomorphism) and then we describe each such irreducible component. We will mostly present the relevant results and not go too much into any argumentation in this section.

The representation we will focus on is generally called the *regular representation* and is constructed by having a group G act on itself in a natural way. Firstly, we construct the group ring $K[G]$ and note that we may consider it a vector space over K where the group elements are the basis vectors. Thus if G is finite, then $K[G]$ becomes a finite dimensional vector space. Then we let G act on $K[G]$ the following way: for $g \in G$ and $f = \sum_{i=1}^{k} c_i g_i \in K[G]$, we let $gf = g(\sum_{i=1}^{k} c_i g_i) = \sum_{i=1}^{k} c_i(gg_i)$, where gg_i is taken as the multiplication in G.

Before we state the first result, recall that a conjugacy class of G, $[a]$, is a subset of G consisting of all elements of G that are conjugate to a. That is, $[a] = \{b \in G | b = gag^{-1} \text{ for some } g \in G\}$. Note that this defines an equivalence relation on G, so the conjugacy classes are disjoint.

Proposition 2.2.1. *Let $K[G] = \bigoplus_i m_i W^{(i)}$, where the $W^{(i)}$'s form a complete list of irreducible representations and $W^{(i)} \not\cong W^{(j)}$ for $i \neq j$. Then $m_i = \dim(W^{(i)})$ and $K[G] = \bigoplus_{i=1}^{k} m_i W^{(i)}$, where k is the number of conjugacy classes of G.*

Proof. See [20] Proposition 1.10.1. □

Thus to find the number of irreducible representations of S_n we simply have to find the number of conjugacy classes of S_n. It is well known that the cycle types of the elements of S_n determines the conjugacy classes (see for instance Chapter 1 of [20]). Given an element $\sigma \in S_n$ such that $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1, \sigma(4) = 6, \sigma(6) = 4$ and $\sigma(i) = i$ for the rest, then we express it in cycle notation in the following way $\sigma = (1,2,3)(4,6)(5)(7)...(n) = (1,2,3)(4,6)$. Then we define its cycle type to be the ordered tuple of the lengths of each subcycle in descending order. Thus the cycle type of σ is $(3,2,1,1,...,1)$.

Note that if we take a product of two cycles $\tau_1 = (1,2)$ and $\tau_2 = (2,3)$, then we have to write the product, $\tau_1\tau_2 = (1,2)(2,3)$, as disjoint cycles before we can find its cycle type. Thus the cycle type of $\tau_1\tau_2 = (1,2,3)$ is $(3,1,1,...,1)$ and not $(2,2,1,1,...,1)$.

Clearly, if we sum up the components of the cycle type of an element $\sigma \in S_n$, it is equal to n. Thus the cycle type is a partition of n, that is, a decreasingly ordered tuple of positive integers that sum up to n. We usually write $\lambda \vdash n$, when $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ is a partition of n. Similarly, if we have a partition of n, then it defines the cycle type of an element in S_n For instance, the partition $(1,1,...,1) \vdash n$ describes the cycle type of the identity of S_n and the partition $(n) \vdash n$ describes the cycle type of $(1,2,...,n) \in S_n$.

Thus the number of irreducible components of the representation of S_n is equal to the number of partitions of n. Unfortunately, no one has found a closed formula that gives the number of partitions of n. However, it can be computed using a generating function. More information on this can be found in [11].

It will not be necessary for us to know the number of irreducible components, so we will not go into that. What is of interest to us, is that the partitions give us a particularly nice way of describing each irreducible component of our representation. Thus for the remainder of this section we will take advantage of this bijection between partitions of n and irreducible representations of S_n to describe the irreducible representations.

To start with note that the set of partitions of n can be equipped with a partial order thus making it a *poset*, that is, a partially-ordered set. A partial order on a set, S, is defined as a relation, "$\leq$", that satisfies the following for all $a, b, c \in S$:

$$a \leq a,$$

$$a \leq b \;\&\; b \leq a \;\Rightarrow a = b,$$
$$a \leq b \;\&\; b \leq c \;\Rightarrow a \leq c.$$

Note that from the definition above, there is a possibility that neither $a \leq b$ nor $b \leq a$ for two elements of a, b of S.

We will make use of a partial order on the set of partitions of n, called the *dominance order*, when we get to the Specht ideals. We say that $\lambda = (\lambda_1, ..., \lambda_l)$ dominates $\mu = (\mu_1, ..., \mu_k)$, written $\mu \trianglelefteq \lambda$, if $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$ for all $j \leq \min\{l, k\}$. This is clearly a partial order on the partitions of n.

To construct the irreducible representation corresponding to a particular partition we associate, to each partition $\lambda \vdash n$, a set of *tableaux*:

Definition 2.2.1. *Let $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$, then a tableau, T, of shape λ is an array of n boxes indexed by $[n]$ arranged in the following way:*

$$T = \begin{array}{|c|c|c|c|c|c|}
\hline
\alpha_{1,1} & \alpha_{1,2} & \cdots & \cdots & \cdots & \alpha_{1,\lambda_1} \\
\hline
\alpha_{2,1} & \alpha_{2,2} & \cdots & \cdots & \alpha_{2,\lambda_2} \\
\cline{1-5}
\cdots & \cdots & \cdots & \cdots \\
\cline{1-4}
\cdots & \cdots & \cdots & \cdots \\
\cline{1-4}
\alpha_{l,1} & \cdots & \alpha_{l,\lambda_l} \\
\cline{1-3}
\end{array} \quad ,$$

where $\alpha_{i,j} \in [n]$ and $\alpha_{i,j} \neq \alpha_{r,s}$ when $(i, j) \neq (r, s)$. We will let, $\mathrm{Tab}(\lambda)$ *denote the set of tableaux of shape λ.*

Given a partition $\lambda \vdash n$ we say that two tableaux, $T, T' \in \mathrm{Tab}(\lambda)$, are *row equivalent* if the i^{th} row of each tableaux contain the same indices for all i. This is an equivalence relation on the set of tableaux of shape λ and we call the equivalence classes, $[T]$, *tabloids*. We can also define these equivalence classes by letting S_n act on the tableaux by permuting the indices of a tableau. Then, if we let S_{R_i} be the subgroup of S_n that only permutes the indices of the i^{th} row of T, and if we let $R_T := S_{R_1} \times S_{R_2} \times ... \times S_{R_l}$, then we can define $[T]$ to be the set $R_T T$. We usually call R_T the row stabilizer of T.

This action on the tableaux can be used to define an action on the tabloids as well. Just note that if $\tau(T)$ is row equivalent to the tableau T for $\tau = (\tau_1 ... \tau_k)$, then for any $\sigma \in S_n$, $\sigma(\tau(T)) = \sigma\tau(T) = (\sigma(\tau_1)...\sigma(\tau_k))\sigma(T)$, where $(\sigma(\tau_1)...\sigma(\tau_k))$ is a row stabilizer of $\sigma(T)$. Thus, if we define the S_n-action on the tabloid $[T]$, to be given by $\sigma([T]) := [\sigma(T)]$, then this action is well-defined. This tells us that we can consider the vector space $M^\lambda := \{\sum_{i=1}^{l} c_i[T_i] \mid c_i \in K,\ T \in \mathrm{Tab}(\lambda)\}$, spanned by the equivalence classes $[T]$, as an S_n-module.

Similarly to the row stabilizer, we define C_T to be the column stabilizer of T and we let $e_T := \sum_{\sigma \in C_T} \mathrm{Sgn}(\sigma)\sigma([T])$ be the *polytabloid* corresponding to T. Since we have that the action of S_n on the tabloids is well-defined, then we have that $e_T = e_{T'}$ when T and T' are row equivalent. Another useful observation on the polytabloids is the following:

Lemma 2.2.1. *Let T be a tableau and $\sigma \in S_n$, then $e_{\sigma T} = \sigma e_T$.*

Proof. See lemma 2.3.3 in Chapter 2 of [20]. □

The polytabloids of shape λ generates a module, S^λ, that we call the *Specht module* corresponding to λ. We have the following result regarding the Specht modules:

Theorem 2.2.1. *When $K = \mathbb{C}$, the Specht modules form a complete list of irreducible representations of S_n.*

Proof. See Theorem 2.4.6 in [20]. □

Although we need the field to be algebraically closed to guarantee that the Specht modules are the irreducible representations, the next result holds even if it is not. Before that we will define a tableau to be *standard* if the indices of the rows are increasing (from left to right) and the indices of the columns are increasing(from top to bottom).

Theorem 2.2.2. *The polytabloids corresponding to the standard tableaux of shape λ is a basis of S^λ.*

Proof. See Theorem 2.5.2 in [20]. □

2.3. Representation in the polynomial ring and Specht ideals

When representing the symmetric group S_n in R_n we are in a slightly different situation seeing as the vector space is now infinite dimensional. However, the theory for the finite case we'll still be of great help. We will still be using the idea of associating a partition to each irreducible representation and their generators will correspond to the tableaux of that partition. However, we begin with a bit of invariant theory.

The idea is to first identify all the invariant polynomials in R_n, that is, the ones that are left fixed by the action of S_n. Since each invariant polynomial can be considered as a generator for the trivial subrepresentation, we will group them together and just consider the remaining part of the ring. Then the remainder of the ring will contain the more "interesting" irreducible representations, so we can focus on this part.

We will consider the natural representation of S_n on R_n given by $\sigma f(x_1, x_2, ..., x_n)$ $= f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$ $\forall\ \sigma \in S_n$ and $f \in R_n$. Then the invariant, or *symmetric*, polynomials are the ones with the property that $\sigma f = f\ \forall\ \sigma \in S_n$, and we denote the set of invariant polynomials as $R_n^{S_n}$. Note that since the action of σ is a homomorphism, we have that the sum and product of two invariant polynomials are invariant. Thus $R_n^{S_n}$ has a ring structure and is therefore called the *invariant ring* of S_n.

Of particular interest are the *elementary symmetric polynomials*. They are defined as $e_1 = \sum_{i\in[n]} x_i$, $e_2 = \sum_{i,j\in[n],i<j} x_i x_j$, ..., $e_n = x_1 x_2 \cdots x_n$. It can be shown that any symmetric polynomial, $f \in R_n$, can be written uniquely as a polynomial in the e_i's, that is, we have the following theorem:

Theorem 2.3.1. *Let $f \in R_n^{S_n}$, then $f(\boldsymbol{x}) = h(e_1(\boldsymbol{x}), ..., e_n(\boldsymbol{x}))$, where $h \in K[e_1, ..., e_n]$ is unique.*

Proof. See Theorem 3 in Chapter 7.1 of [7]. □

In addition to focusing on the non-symmetric polynomials, we can reduce the set of polynomials we consider a little more by making the following observation: If $W \subseteq R_n$ is irreducible, then for any symmetric polynomial f, we have that fW is an isomorphic copy of W since $\sigma(fw) = \sigma(f)\sigma(w) = f\sigma(w)\ \forall\ \sigma \in S_n$ and $w \in W$. Thus we can construct many copies of the same irreducible representation. Therefore we can instead consider the covariant ring, $R_{n_{S_n}} = R_n/\langle e_1(\mathbf{x}), ..., e_n(\mathbf{x})\rangle$. We know the following about the covariant ring:

Theorem 2.3.2. *The covariant ring $R_{n_{S_n}}$ is the regular representation of S_n.*

Proof. See [22], Proposition 4.9. □

Note that in [22], the result above is proven for the complex numbers, however, as mentioned in Chapter 8.1 of [4], the result also holds for the real numbers and thus it holds for our case. In the same chapter they also give the following decomposition of the polynomial ring: $R_n \simeq R_{n_{S_n}} \otimes_{K[S_n]} R_n^{S_n}$, where $\otimes_{K[S_n]}$ denotes the tensor product of the $K[S_n]$-modules $R_{n_{S_n}}$ and $R_n^{S_n}$. We will give a quick definition of tensor products for clarity, but for our purposes it will simply serve as a nice way of separating the "uninteresting" symmetric polynomials and the more "interesting" covariant ring.

Definition 2.3.1. *Let M and N be R-modules, where R is some commutative ring. Then the tensor product of M and N over R, $M \otimes_R N$, is the R-module with generators $\{m \otimes n | m \in M, n \in N\}$ and that satisfies $(am + bm') \otimes (cn + dn') = ac(m \otimes n) + ad(m \otimes n') + bc(m' \otimes n) + bd(m' \otimes n')$ for all $m, m' \in M, n, n' \in N$ and $a, b, c, d \in R$.*

It may be easiest to think of the tensor product as the product $M \times N$, except that we impose some extra relations on the elements. Note, however, that the elements of $M \otimes_R N$ are of the form $\sum_{i=1}^{k} r_i(m_i \otimes n_i)$, with $m_i \in M, n_i \in N$ and $r_i \in R$, and not generally on the form $m \otimes n$. Thus the analogy with the product space has its limitations.

Returning to the topic at hand, based on the theory we went through in the last section, Theorem 2.3.2 tells us that we will find all irreducible representations of S_n in the covariant ring. We also know that we can use the partitions of n to identify all the distinct components.

To construct each irreducible component we define the *Specht polynomials*:

Definition 2.3.2. *Let $\lambda \vdash n$ and $T \in \mathrm{Tab}(\lambda)$, then, if $\lambda_2 > 0$, the Specht polynomial corresponding to T is the polynomial $f_T := \prod_{c=1}^{\lambda_2} \prod_{i<j} (x_{\alpha_{i,c}} - x_{\alpha_{j,c}})$, where $\alpha_{i,c}$ denotes the index of the box in the i^{th} row and c^{th} column of T. When $\lambda_2 = 0$, we define the Specht polynomial to be $f_T := 1$.*

Example 2.3.1. *Let $(3, 2, 1) \vdash 6$ and $T \in \mathrm{Tab}((3, 2, 1))$ be the tableau*

$$T = \begin{array}{|c|c|c|} \hline 3 & 6 & 4 \\ \hline 1 & 5 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array} ,$$

then $f_T = (x_3 - x_1)(x_3 - x_2)(x_1 - x_2)(x_6 - x_5)$.

A different way of defining the Specht polynomials is to take the product of the Vandermonde determinants corresponding to the indices of each column. Also, note that in the preceding example $f_{(12)T} = (12)f_T = -f_T$. In fact we can see that when $\sigma \in C_T$, we have that $\sigma f_T = \pm f_T$, and in general we have $f_{\sigma(T)} = \sigma f_T$ for any tableau of any shape. Note that this is the just like the equation we had in Lemma 2.2.1 in the previous section, that is, $e_{\sigma T} = \sigma e_T$. This fact has an immediate consequence:

Corollary 2.3.1. *Let* $\lambda \vdash n$ *and* $K = \mathbb{C}$, *then* $\mathrm{span}_K\{f_T | T \in \mathrm{Tab}(\lambda)\}$ *form an irreducible representation of* S_n.

Proof. Define the map $\theta : \mathrm{span}_K\{e_T | T \in \mathrm{Tab}(\lambda)\} \to S^\lambda$ to be the map given by $\theta(f_T) = e_T$. By the comment preceding the Lemma we have that $e_{\sigma T} = \sigma e_T$ and $f_{\sigma T} = \sigma f_T$, thus $\theta(\sigma f_T) = \theta(f_{\sigma T}) = e_{\sigma T} = \sigma e_T = \sigma\theta(f_T)$. Therefore we have that θ is an S_n-isomorphism. From Theorem 2.2.1 we know that S^λ is an irreducible representation. Thus $\mathrm{span}_K\{f_T | T \in \mathrm{Tab}(\lambda)\}$ is also an irreducible representation. □

Due to Corollary 2.3.1 we will also call $\mathrm{span}_K\{e_T | T \in \mathrm{Tab}(\lambda)\}$ the Specht module. Another consequence of this identification is the following:

Corollary 2.3.2. *If* $K = \mathbb{C}$ *then the Specht modules* $\mathrm{span}_K\{f_T | T \in \mathrm{Tab}(\lambda)\}$ *form a complete list of irreduble representations of the covariant ring* $R_{n_{S_n}}$.

Proof. By Theorem 2.3.2 the covariant ring is the regular representation and thus contains all irreducible representations (up to isomorphism). From Theorem 2.2.1 we know that the Specht modules form a complete list of irreducible representations of S_n and by Corollary 2.3.1 we know that the Specht module $\mathrm{span}_K\{f_T | T \in \mathrm{Tab}(\lambda)\}$ is isomorphic to the Specht module S^λ. □

Thus, to sum up, we can write the S_n-module R_n as the tensor product $R_{n_{S_n}} \otimes_{K[S_n]} R_n^{S_n}$, where the covariant ring $R_{n_{S_n}}$ decomposes into the Specht modules spanned by the Specht polynomials and each element of the invariant ring $R_n^{S_n}$ spans its own trivial representation. Essentially the Specht modules play an important role in the representation theory of the symmetric group and this is what motivates the following definition:

Definition 2.3.3. *The Specht ideal of shape* λ *is the ideal generated by the Specht polynomials of shape* λ.

We call a Specht polynomial for a standard Specht polynomial if the corresponding tableaux is standard. In the last section we ended with a result regarding a basis of the Specht modules which was independent of whether K was algebraically closed or not. We have a similar result for the Specht ideals:

Proposition 2.3.1. *Let $\lambda \vdash n$. The standard Specht polynomials corresponding λ generate the Specht ideal of shape λ.*

Proof. See Theorem 1.1 in [18]. $\square$

Chapter III / Symmetric filtrations

In this chapter we consider how to deal with the polynomial ring in infinitely many variables, $R = K[X]$, where $X = \{x_i | i \in \mathbb{N}\}$. This ring can be viewed as the the union of all the finite polynomial rings, $R = \cup_{n \in \mathbb{N}_0} R_n$, so its natural to interpret it as a limiting object of the sequence of polynomial rings in finitely many variables. Similarly, we can consider ideals in this ring as limits of sequences of ideals in the finite variable polynomial rings. Thus we can define a variation of Hilbert series for this ring by considering the corresponding sequences of Hilbert series for the finite polynomial rings.

We will see that dealing with these sequences become easier when the sequences of ideals are symmetric since this allows us to consider R as a module over the group ring $R[S_\infty]$, where S_∞ is the infinite symmetric group. With this group action defined we can define a variation of Gröbner bases for ideals in this ring "up to orbits".

However, before we go into Hilbert Series and Gröbner bases, we should establish some fundamental properties of the ring R. We introduce a variation of Noetherianity for R and look at how we can order the monomials in a sensible way.

3.1. Noetherianity up to symmetry

Although we cannot say that the infinte polynomial ring is Noetherian as a ring, we can consider it as an $R[S_\infty]$-module which luckily makes it Noetherian as a module. Thus we circumvent that problem nicely. However, we must be careful regarding how we choose to order the elements of R since the Noetherianity of the module is dependant on the ordering.

We will go through the main steps that establishes R as Noetherian since it will highlight some principles that will be useful to keep in mind when working in R. The proof follows (in principle) the same approach as they do in [7] for the ring R_n, but some more care needs to be taken, particularly with regards to the ordering. Thus we will start with some definitions and properties regarding orderings, continue with Gröbner bases and use the relation between Gröbner bases and orderings to establish finite generation of ideals. That is, we establish a variation of Hilbert basis theorem which can be seen as equivalent to the module being Noetherian. It is also worth pointing out that most of what follows is taken from [1] except they do this in a more general setting where the set of variables is not necessarily countable.

The elements of R is defined in the similarly to the finite case, that is, $R = \{\sum_{i=1}^{m} k_i w_i \mid k_i \in K,\ w_i \in X^\diamond,\ m \in \mathbb{N}\}$, where $X^\diamond$ is the free commutative monoid generated by X. The infinite symmetric group we define as $S_\infty = \{\sigma : \mathbb{N} \to \mathbb{N} \mid \sigma$ is a bijection that fixes all but a finite set of elements of $\mathbb{N}\}$. Note that in a similar way as the ring R can be thought of as the union of the finite rings R_n, we can think of S_∞ as the union of all the finite symmetric groups, S_n, where S_n is embedded into S_{n+1} as the stabilizer of $n+1$.

To consider R as a module we define the group ring of S_∞ and R as the ring $R[S_\infty] = \{\sum_{i_1}^{m} r_i\sigma_i \mid r_i \in R,\ \sigma_i \in S_\infty\}$. So R is an $R[S_\infty]$-module, where elements of S_∞ acts on elements of R by permuting the indices of the variables in X. Thus when we consider ideals in R, we need them to be closed under the action of S_∞. We usually denote this condition by $S_\infty I \subseteq I$, or say that I is a stable ideal. Thus stable ideals are the submodules of R. We call the ideals that are stable with respect to the symmetric group, *symmetric ideals*.

Just like the ring R can be considered a limit of the sequence $(R_n)_{\mathbb{N}_0}$, we can consider a stable ideal $I \triangleleft R$ as the limit of the sequence $(I_n)_{\mathbb{N}_0}$, where $I_n = I \cap R_n$ is an ideal in R_n. The ideals I_n are clearly stable under the action of S_n, thus we get a sequence of symmetric ideals.

To establish Noetherianity we also need an ordering on $X^\diamond$ that behaves nicely with the group action we just introduced. We start with a definition:

Definition 3.1.1. *A relation "$\leq$" on a set, Y, is a* quasi-order *if it is reflexive and transitive. That is,*

$$y \leq y \ \forall\ y \in Y,$$

and

$$y_1 \leq y_2 \ \&\ y_2 \leq y_3 \implies y_1 \leq y_3,\ \forall\ y_1, y_2, y_3 \in Y.$$

In addition it is called a well-quasi-order *if any subset of non-related elements, is finite.*

For a quasi ordered set Y, we will let $F(Z)$ denote the final segment generated by $Z \subseteq Y$. $F(Z)$ is defined to be a final segment if for any $y \in Y$ such that there exists a $z \in Z$ with $z \leq y$, then $y \in Z$. Then it can be shown that a quasi-order set has the following properties(see [1], proposition 2.1):

Lemma 3.1.1. *Let "$\leq$" is a quasi-order on a set Y, then the following are equivalent:*

i "$\leq$" is a well-quasi-order

ii Each final segment is finitely generated.

Given an order "$\leq$" on $X^{\diamond}$ we can build a new order called the *symmetric cancellation order* that will be useful when we consider R as a module. It is defined as follows: $w \preceq u$ if $w \leq u$ and there exists a group element, $\sigma \in S_{\infty}$, such that $\sigma w | u$ with $\sigma w' \leq \sigma w$ for any $w' \leq w$.

Thus, if we use the symmetric cancellation order, we can think of the set of leading monomials of a stable ideal as our version of a final segment. Then, due to Lemma 3.1.1, if we have an ordering on $X^{\diamond}$ such that the corresponding symmetric cancellation order is a well-quasi-order, we get that the sets of leading monomials are finitely generated.

To find such an ordering we start with a definition:

Definition 3.1.2. *An ordering, "$\leq$", on X is a cardinal-well ordering if the complement of any nontrivial final segment has strictly smaller cardinality than the cardinality of X.*

Note that if we take the lex order on $X^{\diamond}$ and restrict it to X we do not get a cardinal-well order. To see this consider for example the final segment $Y = \{x_1, x_2, x_3\} \subseteq X$, then the complement Y^c has the same cardinality as X. However, if we consider the invlex order on $X^{\diamond}$ it restricts to a cardinal-well ordering of X. This is important due to the following theorem:

Theorem 3.1.1. *If "$\leq$" is a lexicographical ordering of $X^{\diamond}$ that restricts to a cardinal-well ordering of X, then the corresponding symmetric cancellation order is a well-quasi-order.*

Proof. See theorem 2.20 in [1]. Note that by "a lexicographical ordering" it is meant that if X is ordered by $\leq^*$, then the corresponding lexicographical ordering on $X^{\diamond}$ is defined as follows: Let $w, u \in X^{\diamond}$, then $w, u \in X_n^{\diamond}$ for some $n \in \mathbb{N}$, thus $w = \mathbf{x}^{\alpha}$ and $u = \mathbf{x}^{\beta}$. Let $x_i = \max_{\leq^*}\{x_j | x_j \in \operatorname{supp}(w) \cup \operatorname{supp}(u) \ \&\ \alpha_j \neq \beta_j\}$, then $w > u$ if $\alpha_i > \beta_i$. $\square$

The next step is to define a notion of a Gröbner basis for ideals of R. But first we introduce the following notation: If Y is a subset of a ring A, then $\langle Y \rangle_A := \{\sum_{i=1}^{k} a_i y_i \mid a_i \in A,\ y_i \in Y\}$. We usually skip the subscript when the ring A is clear from context.

Definition 3.1.3. *A subset G of a stable ideal $I \triangleleft R$ is a Gröbner basis if $\langle F(G) \rangle_K$ and $\langle \operatorname{lm}(f) \mid f \in I \rangle_K$ are equal.*

Note that the sets in the definition are generated over K, not $R[S_\infty]$. The distinction is an important one since if we have an polynomial, f, in a stable ideal I, and a group element $\sigma \in S_\infty$, we do not necessarily have an $h \in I$ with $\text{lm}(h) = \text{lm}(\sigma f)$. Similarly to the finite case we have the following properties regarding Gröbner bases.

Lemma 3.1.2. *Let $G \subset I \triangleleft R$ be a Gröbner basis of a stable ideal I, then $\langle G \rangle_{R[S_\infty]} = I$.*

Proof. Let $J_n = \langle G \rangle_{R[S_\infty]} \cap R_n$, then we have that $\text{lm}(J_n) = \text{lm}(I_n)$ since $\langle F(G) \rangle_K = \langle \text{lm}(f) \mid f \in I \rangle_K$. Since $\text{lm}(J_n) = \text{lm}(I_n)$ and $J_n \subset I_n$, then by Proposition 1.2.1 we have that $I_n = J_n$. Thus $I = \cup_{n \in \mathbb{N}_0} I_n = \cup_{n \in \mathbb{N}_0} J_n = \langle G \rangle_{R[S_\infty]}$. □

Proposition 3.1.1. *Let "$\leq$" be an order on $X^\diamond$ such that the corresponding symmetric cancellation order is a well-quasi-order. Then every stable ideal of R has a finite Gröbner basis.*

Proof. By statement *ii* of Lemma 3.1.1, we have that $F(G)$ is finitely generated since $F(G)$ is defined to be a final segment and the symmetric cancellation order is a well-quasi-order. □

By the comments preceding Theorem 3.1.1 and the theorem itself we have that the symmetric cancellation ordering corresponding to the invlex order is a well-quasi-order. And due to Proposition 3.1.1 we get that every stable ideal has a finite Gröbner basis. Thus if we have an increasing sequence of stable ideals, $I_1 \subseteq I_2 \subseteq ...$, then $I = \cup_{n \in \mathbb{N}_0} I_n$ has a finite Gröbner basis $G \subseteq I$. Since $G \subseteq I_n$ for some $n \in \mathbb{N}_0$, then $I_n = I_{n+1} = ...$ Thus we have the following theorem:

Theorem 3.1.2. *The $R[S_\infty]$-module, R, is Noetherian.*

As was mentioned in the beginning of this section, if we take the intersection of a stable ideal I and R_n for all $n \in \mathbb{N}_0$, we get a sequence of symmetric ideals. Similarly, if $(I_n)_{n \in \mathbb{N}_0}$ is a sequence of symmetric ideals $I_n \triangleleft R_n$ with $R_m S_m(I_n) \subseteq I_m \ \forall \ m \geq n$, then we get the stable ideal $I = \cup_{n \in \mathbb{N}_0} I_n \triangleleft R$. Such sequences are called (S_n-)*invariant filtrations.* We say that a filtration stabilizes if there exist a $k \in \mathbb{N}$ such that $I_n = R_n S_n(I_k)$, for all $n > k$.

Proposition 3.1.2.

i There is a bijection between stable ideals of R and invariant filtrations $(I_n)_{n\in\mathbb{N}_0}$ with the property that $I_n \cap R_m \subseteq I_m \ \forall\ m < n$.

ii Every filtration stabilizes.

Proof. See [1] Lemma 4.6 and Theorem 4.7. □

We will mostly be working with filtrations generated by some symmetric ideal $I_k \triangleleft R_k$, that is, a sequence, $(I_n)_{n\in\mathbb{N}_0}$, where

$$I_n = \begin{cases} \langle S_n(I_k)\rangle_{R_n}, & \text{for } n \geq k, \\ I_k \cap R_n, & \text{else,} \end{cases}$$

thus stabilization is not an interesting question. However, it is worth asking whether, for such a filtration, we have the property described in 3.1.2 part *i*. Let us call this the *intersection property.*

The following is an example that not all such filtrations has the intersection property:

Example 3.1.1. *Let I_2 be the symmetric ideal $\langle x_1 + x_2\rangle \triangleleft R_2$ and let (I_n) be the filtration it generates. Then $I_3 = \langle S_3 I_2\rangle = \langle x_1 + x_2, x_1 + x_3, x_2 + x_3\rangle$.*

Since $(x_1+x_2)-(x_2+x_3) = x_1 - x_3 \in I_3$, then $\frac{1}{2}((x_1+x_3)+(x_1-x_3)) = x_1 \in I_3$ and since I_3 is symmetric, then $I_3 = \langle x_1, x_2, x_3\rangle$. Clearly $\{x_1, x_2, x_3\}$ is a Gröbner basis with respect to the invlex order, thus I_2 is a strict subset of $I_3 \cap R_2 = \langle x_1, x_2\rangle$.

This raises the question of how we can determine if a given filtration has the intersection property.

3.2. Equivariant Gröbner bases and Hilbert series

Even though the R is Noetherian as an $R[S_\infty]$-module, it turns out that viewing it as an $R[S_\infty]$-module is not a very practical perspective to work with. The main reason is that there is no order on $X^\diamond$ such that the act of taking the leading monomial commutes with the group action of S_∞. That is, if $f = x_1 + x_2 \in R$ and $\sigma = (12) \in S_\infty$, then it is not true that $\operatorname{lm}(\sigma f) = \sigma \operatorname{lm}(f)$ for any order. Thus, as was mentioned in the previous section, the leading monomial ideal is not generally a symmetric ideal.

Therefore we will present an alternative way of working with R, that takes care of this problem without giving us restrictions with regards to the ideals we can consider. This will involve a different notion of a Gröbner basis than what we previously considered, called an *equivariant Gröbner basis*. We will also introduce a variant of Hilbert series for the infinite case, called *equivariant Hilbert series*.

We start by defining a monoid action in a very similar to the way we defined a group action in Chapter 2.

Definition 3.2.1. *A left monoid action of a monoid,* Π*, on a set* S*, is a map,* $\alpha : \Pi \times S \to S$*, such that*

$$\alpha(id_\Pi, s) = id_S s,$$

and

$$\alpha(g, \alpha(h, s)) = \alpha(g \cdot h, s),$$

for all $g, h \in \Pi$ *and* $s \in S$*.*

Now let us consider an action of a monoid Π on the ring R with the property that $\pi(cw) = c\pi(w)$, when $\pi \in \Pi$, $c \in K$ and $w \in X^\diamond$. Then, as a generalization of the group ring in the last section, we introduce the *skew-monoid ring*:

$$R[\Pi] = \left\{ \sum_{i=1}^{k} r_i \pi_i \mid r_i \in R,\ \pi_i \in \Pi \right\}.$$

We define addition in $R[\Pi]$ the same way as with the group ring, but multiplication is defined differently. For $r_1\pi_1$ and $r_2\pi_2$ we define the product to be $r_1\pi_1 * r_2\pi_2 = r_1\pi_1(r_2)\pi_1\pi_2$, then we extend it my letting it distribute with respect to addition.

To address the problem we mentioned in the beginning of this section we introduce a new type of ordering:

Definition 3.2.2. *We say that a monomial order, "$\leq$", on* $X^\diamond$ *is a* Π-order *if for all* $w, u \in X^\diamond$ *with* $w \leq u$ *and all* $\pi \in \Pi$*, we have that* $\pi(w) \leq \pi(u)$*.*

Since we do not have S_∞-orders, we need to find an appropriate monoid to replace it with. The following monoid will do that for us:

Definition 3.2.3. *Let* $\mathrm{Inc}(\mathbb{N})$ *be the monoid* $\{\phi : \mathbb{N} \to \mathbb{N} \mid \phi(a) < \phi(b)\ \forall\ a < b\}$*, with composition as the binary operation.*

We will let the action of Inc($\mathbb{N}$) on R be defined by $\phi(x_i) = x_{\phi(i)}$ for any $x_i \in X$ and $\phi \in \text{Inc}(\mathbb{N})$. Since any monomial $w \in X^\diamond$ is contained in $X_n^\diamond$ for some $n \in \mathbb{N}$, then, for any $\phi \in \text{Inc}(\mathbb{N})$, there exist a $\sigma \in S_\infty$ such that $\phi(w) = \sigma(w)$. Therefore the Inc($\mathbb{N}$)-action takes on the appearance of that of a submonoid of S_∞. Thus, if $I \triangleleft R$ is stable with respect to S_∞, then it is stable with respect to Inc($\mathbb{N}$).

In addition, we clearly have that the invlex order is an Inc($\mathbb{N}$)-order, and so the leading monomial ideals of stable ideals are stable under the Inc($\mathbb{N}$)-action. So we have resolved the issue that the leading monomial ideals are not generally symmetric, but now we have to show that the results from the last section also holds true for the monoid Inc($\mathbb{N}$). We start by introducing a variation of Gröbner basis for Π-stable ideals of R.

Definition 3.2.4. *If "$\leq$" is a Π-order and $I \triangleleft R$ is a stable ideal with respect to* Π*, then $G \subseteq I$ is a* Π-equivariant Gröbner basis *of I if $\langle \text{lm}(\Pi G) \rangle = \text{lm}(I)$.*

Also, for $w, u \in X^\diamond$, we will let $w|_\Pi u$ denote that $\pi(w)$ divides u for some $\pi \in \Pi$. We can consider the partial-order $|_{\text{Inc}(\mathbb{N})}$ as the analogue of the symmetric cancellation order from the last section. Then clearly an Inc($\mathbb{N}$)-equivariant Gröbner basis of an $I \triangleleft R$ with $S_\infty I \subseteq I$ is also a Gröbner basis with respect to the symmetric group in the sense of Definition 3.1.3.

We also have the following properties from [12], Proposition 2.10 and Theorem 2.14:

Theorem 3.2.1. *Let $I \triangleleft R$ with $S_\infty I \subseteq I$, then:*

i Inc($\mathbb{N}$)$I \subseteq I$,

ii *I is finitely generated over $R[S_\infty]$ if and only if I is finitely generated over* $R[\text{Inc}(\mathbb{N})]$,

iii *R is Noetherian as an $R[S_\infty]$-module if it is Noetherian as an* $R[\text{Inc}(\mathbb{N})]$*-module,*

iv *R is Noetherian as an* $R[\text{Inc}(\mathbb{N})]$*-module.*

The same article gives an algorithm for computing Inc($\mathbb{N}$)-equivariant Gröbner bases. Also, since the leading monomial ideals are stable with respect to the Inc($\mathbb{N}$)-action, it follows from property *iv* of 3.2.1 that the leading monoial ideals are finitely generated. Hence there is a finite equivariant Gröbner basis for any Inc($\mathbb{N}$) stable ideal. Thus the properties above tells us that we can work with R as an $R[\text{Inc}(\mathbb{N})]$-module without restricting ourselves with regards to the ideals that we consider. Most importantly we have a more practical setting to work with S_∞-stable ideals.

One thing that works a little differently from considering R as an $R[S_\infty]$-module occurs when we consider Inc(ℕ)-filtrations. Recall that filtrations of symmetric ideals generated by a symmetric ideal $I_k \lhd R_k$ were generated by taking intersections and symmetrizing, that is, constructing the ideals $I_k \cap R_n$ for $n < k$ and $\langle S_n(I_k)\rangle_{R_n}$ for $n > k$. In the Inc(ℕ)-setting we can still do that and by property *ii* of 3.2.1 we know have that the limiting ideals are finitely generated. Alternatively we can construct Inc(ℕ)-*filtrations* for any ideal $I_k \lhd R_k$ in the following way:

$$I_n = \begin{cases} I_k \cap R_n \text{ for } n < k, \\ \langle \mathrm{Inc}(\mathbb{N})_{k,n}(I_k)\rangle_{R_n} \text{ for } n > k, \end{cases}$$

where $\mathrm{Inc}(\mathbb{N})_{k,n} = \{\phi \in \mathrm{Inc}(\mathbb{N}) \mid \phi(k) \leq n\}$. Then the ideal $I = \cup_{n\in\mathbb{N}_0} I_n$ is Inc(ℕ)-stable. However, such ideals need not be symmetric so we get a larger class of ideals and sequences that we can consider.

Although, if we start with a symmetric ideal $I_k \lhd R_k$ then the symmetric filtration and the Inc(ℕ)-filtration it generates are the same filtration. To see this let $f \in I_k$ and $\sigma \in S_n$. Then, as stated in [12] preceding Proposition 2.10, $\sigma f = (\phi \circ \tau) f$ for some $\tau \in S_k$ and $\phi \in \mathrm{Inc}(\mathbb{N})_{k,n}$. Thus $\langle S_n(I_k)\rangle_{R_n} = \langle \mathrm{Inc}(\mathbb{N})_{k,n}(I_k)\rangle_{R_n}$, so the sequences are the same. We state this as a lemma:

Lemma 3.2.1. *The symmetric filtration generated by a symmetric ideal $I_k \lhd R_k$ is equal to the* Inc(ℕ)*-filtration generated by I_k.*

One caveat regarding Lemma 3.2.1 worth mentioning is that even if $B \subset I_k$ is a generating set for the symmetric filtration it need not be a generating set for the Inc(ℕ)-filtration. Just consider the filtration $(I_n)_{n\in\mathbb{N}_0}$, where $I_0 = \{0\}$ and $I_n = \langle S_n x_1\rangle_{R_n}\ \forall\ n \geq 1$. It can be generated by x_2, as a symmetric filtration, but not as an Inc(ℕ)-filtration.

We can use this way of defining sequences to give the following partial answer to the question raised at the end of the last section:

Proposition 3.2.1. *Let $I_k \lhd R_k$ be a symmetric ideal and $(I_n)_{n\in\mathbb{N}_0}$ a symmetric filtration generated by I_k. Let $X^\diamond$ be equipped with the invlex order and $G \subseteq I = \cup_{n\in\mathbb{N}_0} I_n$ be a finite* Inc(ℕ)*-equivariant Gröbner basis of I contained in I_d. Then $I_n \cap R_m \subseteq I_m\ \forall\ m < n \leq k$ and $d \leq m < n$.*

Proof. Since $I_m = I_k \cap R_m$ for $m \leq k$, it is clearly true for $m < n \leq k$. For $d \leq m < n$, note that since G is an Inc(ℕ)-equivariant Gröbner basis of I, then $\mathrm{Inc}(\mathbb{N})_{d,m}(G)$ is a Gröbner basis of $I \cap R_m$. Since $S_m(G) \subseteq I_m$, then $\mathrm{Inc}(\mathbb{N})_{d,m}(G) \subseteq I_m$, thus $I_m = I \cap R_m \implies I_m = I_n \cap R_m$. □

Thus the condition that $d = k$ in the above proposition, is sufficient (although maybe not necessary?) for the sequence to have the intersection property. As we saw in Example 3.1.1, the generating ideal $I_2 = \langle x_1 + x_2 \rangle$ does not contain an equivariant Gröbner basis, but the next ideal $I_3 = \langle x_1, x_2, x_3 \rangle$ does contain one, namely $\{x_1\}$. That is, d was not equal to k in this example and the sequence did not have the intersection property.

We will leave the topic of Gröbner bases and turn to the topic of Hilbert series for the infinite situation. This will be a natural continuation of the finite case and is defined for the $\operatorname{Inc}(\mathbb{N})$-action, that is, for the more general class of ideals.

Definition 3.2.5. *The* equivariant Hilbert series *of an* $\operatorname{Inc}(\mathbb{N})$*-filtration* $(I_n)_{n\in\mathbb{N}_0}$*, that is, a sequence of ideals with* $\operatorname{Inc}(\mathbb{N})_{m,n}(I_m) \subseteq I_n$ *for all* $m < n$*, is defined as the generating function*

$$H_{(I_n)_{n\in\mathbb{N}_0}}(s,t) = \sum_{n\geq 0} H_{I_n}(t) s^n.$$

Similarly, we define the equivariant Hilbert series of an $\operatorname{Inc}(\mathbb{N})$*-stable ideal* $I \lhd R$ *as the equivariant Hilbert series of the* $\operatorname{Inc}(\mathbb{N})$*-filtration* $(I \cap R)_{n\in\mathbb{N}_0}$*.*

Recall that in the finite case we could define the Hilbert series of I_n equivalently as the Hilbert series of the quotient ring R_n/I_n, since $H_{I_n}(t) = H_{R_n}(t) - H_{R_n/I_n}(t)$ and we know $H_{R_n}(t)$ to be equal to $\frac{1}{(1-t)^n}$. We can do the same for the infinite case by calculating the equivariant Hilbert series of the filtration $(R_n)_{n\in\mathbb{N}_0}$.

Example 3.2.1. *We calculate the equivariant Hilbert series of the filtration* $(R_n)_{n\in\mathbb{N}_0}$*,*

$$H_R(s,t) = \sum_{n\geq 0} \frac{s^n}{(1-t)^n} = \frac{1}{1-\frac{s}{1-t}} = \frac{1-t}{1-t-s}.$$

Also similar to the finite case, we have a theorem that ensures that the equivariant Hilbert series are rational.

Theorem 3.2.2. *Let* $(I_n)_{n\in\mathbb{N}_0}$ *be an* $\operatorname{Inc}(\mathbb{N})$*-filtration of graded ideals, then the equivariant Hilbert series is a rational function of the form*

$$H_{(R_n/I_n)_{n\in\mathbb{N}_0}}(s,t) = \frac{g(s,t)}{(1-t)^d \cdot \prod_{j=0}^{k}((1-t)^{c_j} - f_j(t)s)},$$

with $f_j(t) \in \mathbb{Z}[t]$ *with* $c_j, k \in \mathbb{N}_0$*,* $f_j(1) > 1$ *and* $c_j \leq 1\ \forall\ j$ *and* $g(s,t) \in \mathbb{Z}[s,t]$*.*

Proof. See [17] Proposition 7.2. □

Chapter IV / Filtrations of Specht ideals

The topic of the last chapter is relatively new, and so it makes sense to investigate a class of ideals that have a strong connection with the representation theory of the symmetric group. Thus we will be considering the Specht ideals that was introduced in Chapter 2. We will investigate a conjecture Gröbner basis and calculate the Hilbert series for some sequences of Specht ideals. We mostly focus on Specht ideals of shape $(n-k,k)$ and $(n-k,1^k)$, but will also present some results regarding Specht ideals of a general shape.

In the first two sections we compute the Hilbert series of Specht ideals of shape $(n-k,k)$. Then we prove a general form of a Gröbner basis for the Specht ideals of shape $(n-k,1^k)$ before using this to get the equivariant Hilbert series of the corresponding sequences.

Afterwards we investigate which parts of the proof may generalize to other partitions and use the results to present a criterion for verifying a conjectured Gröbner basis for the Specht ideals of any shape. We also get a reduced conjecture from this investigation and show that the leading monomials of the standard Specht polynomials are sufficient to describe the leading monomials of any Specht polynomial.

At the end we present a specific criterion for the Specht ideals of shape $(n-k,k)$ that is based on the Hilbert series of these ideals. We finish by analyzing some of the equivariant Hilbert series we obtained.

4.1. Hilbert series: Specht ideals of shape $(n-k,k)$

In this section we look at the Hilbert series of some of the Specht ideals of shape $(n-k,k)$ and eventually the equivariant Hilbert series of the sequences they generate. However, before we start with the computations we can make an observation to reduce the computation time a little.

The nice thing about considering Specht ideals in the setting of sequences of ideals is that the image of a Specht ideal in n variables after applying the group action of S_m, for $m \geq n$, is that it corresponds to another Specht ideal in m variables. So they naturally induce a sequence of ideals themselves. That is,

Lemma 4.1.1. *let $\lambda \vdash n$ be the partition $(\lambda_1, \lambda_2, ..., \lambda_l)$, with $\lambda_1 = \lambda_2$, and let $\lambda^m = (\lambda_1 + m, \lambda_2, ..., \lambda_l)$, with $n, l, m \in \mathbb{N}$. Then $\langle S_{n+m} I_\lambda \rangle_{R_{n+m}} = I_{\lambda^m}$.*

Proof. Let h be a generator of $\langle S_{n+m} I_\lambda \rangle_{R_{n+m}}$, then $h = \sigma(f_T)$, where $T \in \mathrm{Tab}(\lambda)$ and $\sigma \in S_{n+m}$. Let

$$T = \begin{array}{|c|c|c|c|}
\hline
\alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,k} \\
\hline
\alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{1,k} \\
\hline
\cdot\cdot & \cdot\cdot & \cdot\cdot & \multicolumn{1}{c}{} \\
\cline{1-3}
\cdot\cdot & \cdot\cdot & \multicolumn{2}{c}{} \\
\cline{1-2}
\alpha_{l,1} & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}$$

Consider the tableau $T' \in Tab(\lambda^m)$, where

$$T' = \begin{array}{|c|c|c|c|c|c|c|}
\hline
\sigma(\alpha_{1,1}) & \sigma(\alpha_{1,2}) & \cdots & \sigma(\alpha_{1,k}) & \alpha_{1,k+1} & \cdots & \alpha_{1,k+m} \\
\hline
\sigma(\alpha_{2,1}) & \sigma(\alpha_{2,2}) & \cdots & \sigma(\alpha_{1,k}) & \multicolumn{3}{c}{} \\
\cline{1-4}
\cdot\cdot & \cdot\cdot & \cdot\cdot & \multicolumn{4}{c}{} \\
\cline{1-3}
\cdot\cdot & \cdot\cdot & \multicolumn{5}{c}{} \\
\cline{1-2}
\sigma(\alpha_{l,1}) & \multicolumn{6}{c}{} \\
\cline{1-1}
\end{array} ,$$

and $\alpha_{1,k+s}$ are the remaining indices of $[n+m]$, that is $\alpha_{1,k+s} \neq \sigma(\alpha_{i,j})$ for $j \leq k$. Clearly $\sigma(f_T) = f_{T'} \in I_{\lambda^m}$ since the columns of length one does not contribute to $f_{T'}$, thus $\langle S_{n+m} I_\lambda \rangle_{R_{n+m}} \subseteq I_{\lambda^m}$.

For the reverse inclusion, we start with a tableau $T' \in \mathrm{Tab}(\lambda^m)$ indexed by $\alpha_{i,j}$ and apply a group element $\tau \in S_{n+m}$ such that $\tau(\alpha_{i,j}) \in [n]$ for all the indices (i,j) that make up a tableau of shape λ. Then $\tau(f_{T'}) = f_T$ for some $T \in \mathrm{Tab}(\lambda)$, thus $f_{T'} = \tau^{-1}(f_T) \in S_{n+m} I_\lambda$. Hence $I_{\lambda^m} \subseteq \langle S_{n+m} I_\lambda \rangle_{R_{n+m}}$ and therefore $I_{\lambda^m} = \langle S_{n+m} I_\lambda \rangle_{R_{n+m}}$. $\square$

Thus when we consider the sequence of ideals generated by a Specht ideal, we are in fact considering sequences of Specht ideals. Combining this observation and the fact that the standard Specht polynomials generate the Specht Ideals (see Lemma 2.3.1), we get an easier job when running computations. That is, we can simply define the ideals generated by the standard Specht polynomials of a given shape, instead of having to compute the orbit of a Specht polynomial.

With this observation we are ready to look at some calculations. First we present the approach and then we look at some calculations.

Method

Let $\lambda = (\lambda_1, \lambda_2, ...\lambda_k) \vdash n$, where $n, k \in \mathbb{N}$ and $\lambda_1 = \lambda_2$. Define

$$I_m := \begin{cases} I_\lambda \cap R_m, \text{ if } m \leq n, \\ \langle S_m I_\lambda \rangle_{R_m}, \text{ else.} \end{cases}$$

Step 1: We define the ideal I_λ in some computer algebra system as the ideal generated by the set $SF_\lambda = \{f_T | T \in \text{Tab}(\lambda)\}$. Then we calculate the Hilbert series of I_λ.

Step 2: For the next ideal in the sequence generated by I_λ, $\langle S_{n+1} I_\lambda \rangle_{R_{n+1}}$, we and define the partitions $\lambda^m := (\lambda_1 + m, \lambda_2, ..., \lambda_k)$. By Lemma 4.1.1 we know that $I_{n+m} = I_{\lambda^m} = \langle SF_{\lambda^m} \rangle$, thus we calculate the Hilbert series of I_{n+1}.

Step 3: Repeat step 2 for $m = 2, m = 3, ...$ until a pattern emerges (if a pattern emerges). If a pattern does not appear, then we have to abort.

If the ideal $I_m \neq \langle 0 \rangle$ for $m < n$, and we are interested in the equivariant Hilbert series of the filtration of $I = \cup_{m \in \mathbb{N}_0} I_m$, we also need to look at the intersections.

Step 4: Let $b = \min\{m | I_m \neq \langle 0 \rangle\}$. Then we calculate a Gröbner basis, G, of I_λ with respect to an elimination order of type $n - b$ (for instance the lex order). Then we calculate the Hilbert series of $I_m = \langle G \cap R_m \rangle$ for all $b \leq m < n$.

Step 5: Assuming a pattern emerged in step 3, calculate the equivariant Hilbert Series of the filtration.

Remark 4.1.1. *Note that step 4 may be omitted if we are not interested in the whole filtration. The resulting expression will still be a rational function, since it is equal to the equivariant Hilbert series of the filtration minus a finite number of Hilbert Series'.*

We will begin by looking at partitions of the form (k,k). Although for $k=1$, we will not need to use the method above. But for step 5 we will need the following lemma. Firstly we define $\binom{m}{k}$ to be zero when $m<k$.

Lemma 4.1.2. *Let $k \geq 0$ then $\sum_{m\geq 0}\binom{m}{k}s^m = \frac{s^k}{(1-s)^{k+1}}$.*

Proof. See (4.1.6) in Chapter 4 of [3]. □

Partition (1,1)

In this case $I_m = \langle x_1 - x_i | i \in [m]\backslash\{1\}\rangle$ for $m \geq 2$ and $I_1 = I_0 = \langle 0 \rangle$. If we consider the quotient ring R_m/I_m, we can identify it with R_1 by identifying x_i with x_1 for all $i>1$. From Lemma 1.3.4 we know that $H_{R_1}(t) = \frac{1}{1-t}$, that is

$$H_{R_m/I_m}(t) = \begin{cases} 1, \text{ if } m = 0, \\ \frac{1}{1-t}, \text{ else.} \end{cases}$$

Thus

$$H_{R/I}(s,t) = 1 + \sum_{i=1}^{\infty}\frac{s^i}{1-t} = 1 + \frac{1}{1-t}\bigg(-1+\sum_{i=0}^{\infty}s^i\bigg) =$$

$$1+\frac{1}{1-t}\bigg(-1+\frac{1}{1-s}\bigg) = 1 + \frac{s}{(1-t)(1-s)} = \frac{st-t+1}{(1-t)(1-s)}$$

Since $H_R(s,t) = \frac{1-t}{1-s-t}$, we get that the equivariant Hilbert series of the ideal is

$$H_I(s,t) = H_R(s,t) - H_{R/I}(s,t) = \frac{1-t}{1-s-t} - \frac{st-t+1}{(1-t)(1-s)} =$$

$$\frac{(1-t)^2(1-s)-(st-t+1)(1-s-t)}{(1-t)(1-s)(1-s-t)} = \frac{s^2t}{(1-t)(1-s)(1-s-t)}.$$

For the next partition we start by including the Macaulay2-code that was used for the calculation. This will not be included for the other partitions seeing as they follow the same procedure.

Partition (2,2)

Let $I_m = \langle f_T | T \in \mathrm{StdTab}((m-2,2)) \rangle$ for $m \geq 4$ and $I_m = I_4 \cap R_m$ for $m < 4$.

The following code uses the package "SpechtModule" in Macaulay2.

```
i1 : R_4 = QQ[x_0..x_3]
i2 : p = new Partition from{2,2};
i3 : T = standardTableaux(p);
i4 : I_4 = ideal(spechtPolynomial(T_0,R_4),
     spechtPolynomial(T_1,R_4));
i5 : hilbertSeries(I_4, Reduce => true)

                  2
       1 + 2T + T
o5   = -----------
                2
         (1 - T)
```

Note that the function "hilbertSeries", gives the Hilbert series of the quotient ring. Thus $H_{R_4/I_4}(t) = \frac{1+2t+t^2}{(1-t)^2}$.

For step 2 and 3 we use a similar code and get these results:

$$H_{R_5/I_5}(t) = \frac{1+3t+t^2}{(1-t)^2},$$

$$H_{R_6/I_6}(t) = \frac{1+4t+t^2}{(1-t)^2},$$

$$H_{R_7/I_7}(t) = \frac{1+5t+t^2}{(1-t)^2}.$$

It appears as thought the Hilbert series of R_m/I_m is $H_{R_m/I_m}(t) = \frac{1+(m-2)t+t^2}{(1-t)^2}$ for $m \geq 4$.

The ideal I_4 intersects non trivially with R_3, so we do step 4 as well.

```
i1 : R_4 = QQ[x_0..x_3, MonomialOrder => Lex];
    i2 : p = new Partition from{2,2};
    i3 : T = standardTableaux(p);
    i4 : I_4 = ideal(spechtPolynomial(T_0,R_4),
         spechtPolynomial(T_1,R_4));
    i5 : G = gens gb I_4
```

```
o5 = | x_1^2x_2-x_1^2x_3-x_1x_2^2+x_1x_3^2+x_2^2x_3-x_2x_3^2
       x_0x_2-x_0x_3-x_1x_2+x_1x_3 x_0x_1-x_0x_3-x_1x_2+x_2x_3 |
i6 : I_3 = ideal(sub(G_0_0,R_3));
i7 : hilbertSeries(I_3, Reduce => true)



                  2
         1 + T + T
o7 =    ----------
                 2
          (1 - T)
```

Thus the Hilbert series of R_3/I_3 is $H_{R_3/I_3}(t) = \frac{1+t+t^2}{(1-t)^2}$, and there are no more non trivial intersections. Therefore the Hilbert series of R_m/I_m appears to be

$$H_{R_m/I_m}(t) = \begin{cases} \frac{1+(m-2)t+t^2}{(1-t)^2}, & \text{for } m \geq 3, \\ \frac{1}{(1-t)^m}, & \text{for } 0 \leq m < 3. \end{cases}$$

Then the last step is calculating the equivariant Hilbert series.

$$H_{R/I}(t) = \sum_{m\geq 0} H_{R_m/I_m}(t)s^m = \sum_{m\geq 0}^{2} \frac{s^m}{(1-t)^m} + \sum_{m\geq 3} \frac{(1+(m-2)t+t^2)s^m}{(1-t)^2} =$$
$$\frac{1-\frac{s^3}{(1-t)^3}}{1-\frac{s}{(1-t)}} + \frac{(1+t^2)s^3}{(1-t)^2}\sum_{m\geq 0} s^m + \frac{ts^2}{(1-t)^2}\sum_{m\geq 1} ms^m =$$
$$\frac{(1-t)^3-s^3}{(1-t)(1-t-s)} + \frac{(1+t^2)s^3}{(1-t)^2(1-s)} + \frac{ts^3}{(1-t)^2(1-s)^2} =$$
$$\frac{(1-t)^2+s(1-t+s)}{(1-t)^2} + \frac{(1+t^2)s^3}{(1-t)^2(1-s)} + \frac{ts^3}{(1-t)^2(1-s)^2} =$$
$$\frac{-s^4t^2+s^3t^2+s^2t^2-s(2t^2-3t+1)+(t-1)^2}{(1-t)^2(1-s)^2}$$

To solve the sums above, we used Lemma 4.1.2 with $k=0$ and $k=1$. We can get an easier expression as if we compute $H_I(s,t)$,

$$H_I(s,t) = H_R(s,t) - H_{R/I}(s,t) =$$
$$\frac{1-t}{1-s-t} - \frac{-s^4t^2+s^3t^2+s^2t^2-s(2t^2-3t+1)+(t-1)^2}{(1-t)^2(1-s)^2} =$$
$$\frac{-s^3t^2(s^2+s(t-2)-t)}{(1-t)^2(1-s)^2(1-s-t)}.$$

Partition (3,3)

Let $I_m = \langle f_T | T \in \mathrm{StdTab}((m-3,3)) \rangle$ for $m \geq 6$ and $I_m = I_6 \cap R_m$ for $m < 6$.

We follow the same method and use a similar code as for partition (2,2), thus we just list the results here.

The ideal, I_m, is nonzero for $m \geq 4$, thus step doing 1 to 4 gives us these results:

$$H_{R_4/I_4}(s,t) = \frac{1+t+t^2+t^3+t^4+t^5}{(1-t)^3},$$

$$H_{R_5/I_5}(s,t) = \frac{1+2t+3t^2+4t^3}{(1-t)^3},$$

$$H_{R_6/I_6}(s,t) = \frac{1+3t+6t^2+5t^3}{(1-t)^3},$$

$$H_{R_7/I_7}(s,t) = \frac{1+4t+10t^2+6t^3}{(1-t)^3},$$

$$H_{R_8/I_8}(s,t) = \frac{1+5t+15t^2+7t^3}{(1-t)^3}.$$

It appears as if $H_{R_m/I_m}(s,t) = \frac{1+(m-3)t+\binom{m-2}{2}t^2+(m-1)t^3}{(1-t)^3}$, for $m \geq 5$, that is

$$H_{R_m/I_m}(s,t) = \begin{cases} \frac{1+(m-3)t+\binom{m-2}{2}t^2+(m-1)t^3}{(1-t)^3}, & \text{for } m \geq 5, \\ \frac{1+t+t^2+t^3+t^4+t^5}{(1-t)^3}, & \text{for } m = 4, \\ \frac{1}{(1-t)^m}, & \text{for } 0 \leq m < 4. \end{cases}$$

To compute the equivariant Hilbert series, we will need Lemma 4.1.2 again.

$$H_{R/I}(s,t) = \sum_{m\geq 0} H_{R_m/I_m}(t)s^m = \sum_{m=0}^{3} \frac{s^m}{(1-t)^m} + \frac{(1+t+t^2+t^3+t^4+t^5)s^4}{(1-t)^3} +$$

$$\sum_{m\geq 5} \frac{1+(m-3)t+\binom{m-2}{2}t^2+(m-1)t^3}{(1-t)^3} s^m =$$

$$\frac{1-\frac{s^4}{(1-t)^4}}{1-\frac{s}{(1-t)}} + \frac{(1+t+t^2+t^3+t^4+t^5)s^4}{(1-t)^3} + \frac{s^5}{(1-t)^3}\sum_{m\geq 0} s^m +$$

$$\frac{ts^3}{(1-t)^3}\sum_{m\geq 2} ms^m + \frac{t^2s^2}{(1-t)^3}\sum_{m\geq 3}\binom{m}{2}s^m + \frac{t^3s}{(1-t)^3}\sum_{m\geq 4} ms^m =$$

$$\frac{(1-t)^4-s^4}{(1-t)^3(1-t-s)} + \frac{(1+t+t^2+t^3+t^4+t^5)s^4}{(1-t)^3} + \frac{s^5}{(1-t)^3}\sum_{m\geq 0} s^m +$$

$$\frac{ts^3}{(1-t)^3}\left(\left(\sum_{m\geq 1} ms^m\right) - s\right) + \frac{t^2s^2}{(1-t)^3}\left(\left(\sum_{m\geq 2}\binom{m}{2}s^m\right) - s^2\right) +$$

$$\frac{t^3s}{(1-t)^3}\left(\left(\sum_{m\geq 1} ms^m\right) - s - 2s^2 - 3s^3\right) =$$

$$\frac{s(s(s-t+1)+(1-t)^2)+(1-t)^3+(1+t+t^2+t^3+t^4+t^5)s^4}{(1-t)^3} +$$

$$\frac{s^5}{(1-t)^3(1-s)} + \frac{ts^4 - ts^4(1-s)^2}{(1-t)^3(1-s)^2} + \frac{t^2s^4 - t^2s^4(1-s)^3}{(1-t)^3(1-s)^3} +$$

$$\frac{t^3s^2 - t^3s(s+2s^2+3s^3)(1-s)^2}{(1-t)^3(1-s)^2} =$$

$$\frac{\begin{array}{r}-s^7t^3(t^2+t-2)+s^6t^3(3t^2+3t-4)-s^5(3t^5+3t^4-t^3)+s^4t^3(t^2+t+1)\\ +s^3t^3+s^2(-3t^3+6t^2-4t+1)+s(-1+t)^2(-2+3t)+(1-t)^3\end{array}}{(1-s)^3(1-t)^3}$$

Thus,

$$H_I(s,t) = H_R(s,t) - H_{R/I}(s,t) =$$

$$\frac{\begin{array}{l}-s^4t^3(s^4(t^2+t-2)+s^3(t^3-3t^2-6t+6)\\ \quad -s^2(3t^3-3t^2-10t+5)+st(3t^2-t-5)-t^3)\end{array}}{(1-s)^3(1-t)^3(1-s-t)}.$$

Partition (4,4)

Let $I_m = \langle f_T | T \in \mathrm{StdTab}((m-4,4)) \rangle$ for $m \geq 8$ and $I_m = I_8 \cap R_m$ for $m < 8$.

In this case the ideal, I_m, is nonzero for $m \geq 5$, thus doing step 1 to 4 gives these results:

$$H_{R_5/I_5}(s,t) = \frac{1+t+t^2+t^3+t^4+t^5+t^6+t^7+t^8+t^9}{(1-t)^4},$$

$$H_{R_6/I_6}(s,t) = \frac{1+2t+3t^2+4t^3+5t^4+6t^5+2t^6-2t^7-t^8}{(1-t)^4},$$

$$H_{R_7/I_7}(s,t) = \frac{1+3t+6t^2+10t^3+15t^4}{(1-t)^4},$$

$$H_{R_8/I_8}(s,t) = \frac{1+4t+10t^2+20t^3+21t^4}{(1-t)^4},$$
$$H_{R_9/I_9}(s,t) = \frac{1+5t+15t^2+35t^3+28t^4}{(1-t)^4}.$$

It seems as if $H_{R_m/I_m}(t) = \frac{1+(m-4)t+\binom{m-3}{2}t^2+\binom{m-2}{3}t^3+\binom{m-1}{2}t^4}{(1-t)^4}$ for $m \geq 7$, thus

$$H_{R_m/I_m}(t) = \begin{cases} \frac{1+(m-4)t+\binom{m-3}{2}t^2+\binom{m-2}{3}t^3+\binom{m-1}{2}t^4}{(1-t)^4}, & \text{for } m \geq 7, \\ \frac{1+2t+3t^2+4t^3+5t^4+6t^5+2t^6-2t^7-t^8}{(1-t)^5}, & \text{for } m = 6, \\ \frac{1+t+t^2+t^3+t^4+t^5+t^6+t^7+t^8+t^9}{(1-t)^5}, & \text{for } m = 5, \\ \frac{1}{(1-t)^m}, & \text{for } 0 \leq m < 5. \end{cases}$$

Then we compute the equivariant Hilbert series:

$$H_{R/I}(s,t) = \sum_{m\geq 0} H_{R_m/I_m}(t)s^m = \sum_{m=0}^{4} \frac{s^m}{(1-t)^m} +$$
$$\frac{1+t+t^2+t^3+t^4+t^5+t^6+t^7+t^8+t^9}{(1-t)^5}s^5 +$$
$$\frac{1+2t+3t^2+4t^3+5t^4+6t^5+2t^6-2t^7-t^8}{(1-t)^5}s^6 +$$
$$\sum_{m\geq 7} \frac{1+(m-4)t+\binom{m-3}{2}t^2+\binom{m-2}{3}t^3+\binom{m-1}{2}t^4}{(1-t)^4}s^m.$$

The expressions become a bit long for this one so we will just jump right to the results:

$$H_{R/I}(s,t) = \frac{g_{R/I}(s,t)}{(1-s)^4(1-t)^4},$$

where

$$g_{R/I}(s,t) = -s^{10}t^4(t^2+4t+5)(1-t)^2 + s^9t^4(t^5+5t^4+9t^3-7t^2-23t+15)-$$
$$s^8t^4(4t^5+10t^4+16t^3-8t^2-32t+13) + s^7t^4(6t^5+10t^4+14t^3-2t^2-18t+1)$$
$$-s^6t^4(4t^5+5t^4+6t^3+2t^2-2t-1) + s^5t^4(t^5+t^4+t^3+t^2+t+1)+$$
$$s^4t^4 - s^3(4t^4-10t^3+10t^2-5t+1) + s^2(6t^2-8t+3)(1-t)^2$$
$$s(4t-3)(1-t)^3 + (1-t)^4,$$

and

$$H_I(s,t) = H_R(s,t) - H_{R/I}(s,t) = \frac{g_I(s,t)}{(1-s)^4(1-t)^4(1-s-t)},$$

where

$$g_I(s,t) = s^5t^4(s^6(t^2+4t+5)(1-t)^2 + s^5(4t^4+13t^3-3t^2-34t+20)+$$
$$s^4(t^6-6t^4-32t^3-8t^2+70t-28) - s^3(4t^6-4t^4-38t^3-22t^2+63t-14)+$$
$$s^2t(6t^5-t^3-22t^2-18t+21) - st^2(4t^4-5t-5)+t^6).$$

Partition (5,5)

Let $I_m = \langle f_T | T \in \mathrm{StdTab}((m-5,5)) \rangle$ for $m \geq 10$ and $I_m = I_{10} \cap R_m$ for $m < 10$.

The ideal, I_m, is nonzero if $m \geq 6$, so step 1 to 4 gives us these results:

$$H_{R_6/I_6}(s,t) = \frac{1+t+t^2+...+t^{14}}{(1-t)^5},$$

$$H_{R_7/I_7}(s,t) =$$
$$\frac{1+2t+3t^2+4t^3+5t^4+6t^5+7t^6+8t^9+9t^8-4t^9-3t^{10}-2t^{11}-t^{12}}{(1-t)^5},$$

$$H_{R_8/I_8}(s,t) =$$
$$\frac{1+3t+6t^2+10t^3+15t^4+21t^5+28t^6-6t^7-11t^8-t^9+3t^{10}+t^{11}}{(1-t)^5},$$

$$H_{R_9/I_9}(s,t) = \frac{1+4t+10t^2+20t^3+35t^4+56t^5}{(1-t)^5},$$

$$H_{R_{10}/I_{10}}(s,t) = \frac{1+5t+15t^2+35t^3+70t^4+84t^5}{(1-t)^5}.$$

It seems as if $H_{R_m/I_m}(t) = \frac{1+(m-5)t+\binom{m-4}{2}t^2+\binom{m-3}{3}t^3+\binom{m-2}{4}t^4+\binom{m-1}{3}t^5}{(1-t)^5}$, for $m \geq 9$, thus,

$$H_{R_m/I_m}(t) = \begin{cases} \frac{1+(m-5)t+\binom{m-4}{2}t^2+\binom{m-3}{3}t^3+\binom{m-2}{4}t^4+\binom{m-1}{3}t^5}{(1-t)^5}, & \text{for } m \geq 9, \\ \frac{1+3t+6t^2+10t^3+15t^4+21t^5+28t^6-6t^7-11t^8-t^9+3t^{10}+t^{11}}{(1-t)^5}, & \text{for } m = 8, \\ \frac{1+2t+3t^2+4t^3+5t^4+6t^5+7t^6+8t^9+9t^8-4t^9-3t^{10}-2t^{11}-t^{12}}{(1-t)^5}, & \text{for } m = 7, \\ \frac{1+t+t^2+...+t^{14}}{(1-t)^5}, & \text{for } m = 6. \end{cases}$$

Then we compute the equivariant Hilbert series:

$$H_{R/I}(s,t) = \sum_{m\geq 0} H_{R_m/I_m}(t)s^m = \sum_{m=0}^{5} \frac{s^m}{(1-t)^m} + \frac{1+t+t^2+...t^{14}}{(1-t)^5}s^6 +$$

$$\frac{1+2t+3t^2+4t^3+5t^4+6t^5+7t^6+8t^9+9t^8-4t^9-3t^{10}-2t^{11}-t^{12}}{(1-t)^5}s^7 +$$

$$\frac{1+3t+6t^2+10t^3+15t^4+21t^5+28t^6-6t^7-11t^8-t^9+3t^{10}+t^{11}}{(1-t)^5}s^8 +$$

$$\sum_{m\geq 9} \frac{h_m(t)}{(1-t)^5}s^m,$$

where $h_m(t) = \frac{1+(m-5)t+\binom{m-4}{2}t^2+\binom{m-3}{3}t^3+\binom{m-2}{4}t^4+\binom{m-1}{3}t^5}{(1-t)^5}$.

Again, the expressions become quite long, so we will go right to the results.

$$H_{R/I}(s,t) = \frac{g_{R/I}(s,t)}{(1-s)^5(1-t)^5},$$

where

$$\begin{gathered} g_{R/I}(s,t) = s^{13}t^5(t^3+6t^2+14t+14)(1-t)^3 + \\ s^{12}t^5(t^5+9t^4+35t^3+60t^2+21t-56)(-1+t)^2 - \\ s^{11}t^5(t^9+t^8+6t^7+21t^6+46t^5+11t^4-154t^3-99t^2+246t-79) + \\ s^{10}t^5(5t^9+5t^8+15t^7+35t^6+65t^5+35t^4-195t^3-135t^2+215t-41) - \\ s^9t^5(10t^9+10t^8+20t^7+35t^6+55t^5+45t^4-135t^3-100t^2+80t-1) + \\ s^8t^5(10t^9+10t^8+15t^7+21t^6+28t^5+29t^4-46t^3-36t^2+3t+1) - \\ s^7t^5(5t^9+5t^8+6t^7+7t^6+8t^5+9t^4-4t^3-3t^2-2t-1) + \\ s^6t^5(t^9+t^8+t^7+t^5+t^4+t^6+t^3+t^2+t+1) + s^5t^5 - \\ s^4(5t^5-15t^4+20t^3-15t^2+6t-1) + \\ s^3(10t^3-20t^2+15t-4)(-1+t)^2 + \\ s^2(10t^2-15t+6)(1-t)^3 + \\ s(5t-4)(1-t)^4 + (1-t)^5, \end{gathered}$$

and

$$H_I(s,t) = H_R(s,t) - H_{R/I}(s,t) = \frac{g_I(s,t)}{(1-s)^5(1-t)^5(1-s-t)},$$

where

$$g_I(s,t) = s^6t^5(-s^8(t^3+6t^2+14t+14)(1-t)^3 - s^7(5t^4+32t^3+68t^2+35t-70)(1-t)^2 +$$

$$s^6(t^9 + 10t^6 + 65t^5 - 180t^3 - 270t^2 + 74t^4 + 435t - 135)+$$
$$s^5(t^10 - 5t^9 - 10t^6 - 100t^5 - 200t^4 + 250t^3 + 480t^2 - 540t + 120)-$$
$$s^4(5t^{10} - 10t^9 - 5t^6 - 85t^5 - 275t^4 + 195t^3 + 450t^2 - 336t + 42)+$$
$$s^3t(10t^9 - 10t^8 - t^5 - 38t^4 - 209t^3 + 81t^2 + 216t - 84)-$$
$$s^2t^2(10t^8 - 5t^7 - 7t^3 - 84t^2 + 14t + 42) + st^4(5t^6 - t^5 - 14) - t^{10}).$$

The above calculations are reliant on the patterns of the Hilbert series holding up, which we have not confirmed. But we can make this observation:

Remark 4.1.2. *Looking a little closer at the patterns that appeared in each calculation above, we can see that they all fall under the following general pattern: Given a partition $(n-k,k) \vdash n$, with $k < n$, then it seems as if*

$$H_{R_n/I_n}(t) = \frac{\sum_{i=0}^{k-1} \binom{n-k-1+i}{i} t^i + \binom{n-1}{r-1} t^k}{(1-t)^k}.$$

We can also make the following observation on the equivariant Hilbert series above:

Remark 4.1.3. *If we look closer at the equivariant Hilbert series, $H_I(s,t)$, for the filtration of $I_{(k,k)}$, then, based on the above calculations, they appear to be of the form $H_I(s,t) = \frac{g_I(s,t)}{(1-s)^k(1-t)^k(1-s-t)}$, where $s^{k+1}t^k | g_I$. It also appears as if we can write g_I as $g_I(s,t) = s^{k+1}t^k \sum_{i=0}^{2(k-1)} f_i(t)s^i$, where $f_i(t) \neq 0$, $\deg(f_i(t)) \leq \sum_{j=0}^{k-1}$, $\forall\ i$, and $\sum_{i=0}^{2(k-1)} f_i(t) = 1$.*

Some effort was put into proving the pattern in Remark 4.1.2, but it was unsuccessful, however, it turns out that it was proven in [21], Theorem 2.1. Thus, the equivariant Hilbert series above are correct.

The proof of that theorem was reliant on the following recurrance relation between the Specht ideals:

$$H_{R_n/I_{(n-k,k)}}(t) = H_{R_{n-1}/I_{(n-k-1,k)}}(t) + \frac{t}{1-t} H_{R_{n-1}/I_{(n-k,k-1)}}(t).$$

Thus it was natural to look for similar patterns for partitions of length three. And although it seemed like they were often related in some way, no consistent pattern was found. However, the search led to the following observation on the Hilbert series of Specht ideals corresponding to partitions of shape $(n-k, 1^k) := (n-k, 1, 1, ..., 1)$:

Remark 4.1.4. *Let $(n-k, 1^k) \vdash n$, with $k < n$, then it appears as if*

$$H_{R_n/I_{(n-k,1^k)}}(t) = (1+t+t^2+...+t^{k-1})H_{R_{n-1}/I_{(n-k-1,1^k)}}(t)+\frac{t^k}{1-t}H_{R_{n-1}/I_{(n-k,1^{k-1})}}(t).$$

To prove this relation we will first prove a conjectured Gröbner basis of these ideals, which in turn will give us a sequence of leading monomial ideals. Then the recurrance relation will follow from a result in [10] on monomial ideals.

4.2. Gröbner bases: Specht ideals of shape $(n-k, 1^k)$

In this section we investigate a conjectured Gröbner basis for the Specht ideals and give a proof of this conjecture for the ideals $I_{(n-k,1^k)}$. But first we present a result concerning the relationship between Specht ideals corresponding to different partitions.

Lemma 4.2.1. *Let $\lambda, \mu \vdash n$ with $\lambda \trianglerighteq \mu$, then $I_\mu \subseteq I_\lambda$.*

Proof. See [16], Theorem 1. □

Thus, if one experiments with some Gröbner basis calculations of the Specht ideals, the following conjecture is quite natural:

Conjecture 4.2.1. *Let $\lambda \vdash n$, then the Specht polynomials of shape $\mu \vdash n$, where $\lambda \trianglerighteq \mu$, form a Gröbner basis of I_λ.*

In the first part of this section we prove this conjecture for the partitions $(n-k, 1^k)$ and then we look at some implications regarding the Hilbert series of such Specht ideals. In the next section we investigate the proof in detail to see which parts of it may generalize and in particular we argue that Conjecture 4.2.1 is equivalent to a more reduced conjecture.

We will prove the following theorem for the partitions $(n-k, 1^k)$:

Theorem 4.2.1. *Let $k \in \mathbb{N}$ and $\lambda_{n,k} = (n-k, 1^k)$ be a partition of n, with $0 < k < n$. Then the set $F_{\lambda_{n,k}} = \{f_T | T \in \mathrm{Tab}(\lambda_{n,k})\}$ form a Gröbner basis of the Specht ideal $I_{\lambda_{n,k}} = \langle F_{\lambda_{n,k}} \rangle$ with respect to any permutation of the lex order.*

We first have to establish some preliminary results that will be useful for the main proof. But first let us define what is meant by a "permutation of the lex order":

Definition 4.2.1. *Let $\sigma \in S_n$ and "$<_{\sigma,lex}$" be the monomial order on the set of monomials of R_n defined by in the following way: Let $\boldsymbol{x}^\alpha,\ \boldsymbol{x}^\beta \in X_n^\diamond$, then*

$$\boldsymbol{x}^\alpha <_{\sigma,lex} \boldsymbol{x}^\beta$$

if the first nonzero entry of $\sigma(\beta) - \sigma(\alpha)$ is positive. Call this the σ-lex order.

The σ-lex order that orders the variables such that the indices are decreasing is the invlex order. Next we do the following exercise from [7] Chapter 2.

Lemma 4.2.2. *Let $f, g \in R_n$ and $\boldsymbol{x}^\alpha, \boldsymbol{x}^\beta \in X_n^\diamond$, then $S(\boldsymbol{x}^\alpha f, \boldsymbol{x}^\beta g) = \boldsymbol{x}^\gamma S(f,g)$, where $\boldsymbol{x}^\gamma = \frac{\mathrm{lcm}(\boldsymbol{x}^\alpha \mathrm{lm}(f), \boldsymbol{x}^\beta \mathrm{lm}(g))}{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}$.*

Proof. Since $\mathrm{lm}(pq) = \mathrm{lm}(p)\,\mathrm{lm}(q)\ \forall\ p, g \in R_n$ and $\mathrm{lm}(\mathbf{x}^\kappa) = \mathbf{x}^\kappa\ \forall\ \mathbf{x}^\kappa \in X_n^\diamond$, we have

$$S(\mathbf{x}^\alpha f, \mathbf{x}^\beta g) = \frac{\mathrm{lcm}(\mathrm{lm}(\mathbf{x}^\alpha f), \mathrm{lm}(\mathbf{x}^\beta g))}{\mathrm{lm}(\mathbf{x}^\alpha f)}\mathbf{x}^\alpha f - \frac{\mathrm{lcm}(\mathrm{lm}(\mathbf{x}^\alpha f), \mathrm{lm}(\mathbf{x}^\beta g))}{\mathrm{lm}(\mathbf{x}^\beta g)}\mathbf{x}^\beta g =$$

$$\frac{\mathrm{lcm}(\mathbf{x}^\alpha \mathrm{lm}(f), \mathbf{x}^\beta \mathrm{lm}(g))}{\mathbf{x}^\alpha \mathrm{lm}(f)}\mathbf{x}^\alpha f - \frac{\mathrm{lcm}(\mathbf{x}^\alpha \mathrm{lm}(f), \mathbf{x}^\beta \mathrm{lm}(g))}{\mathbf{x}^\beta \mathrm{lm}(g)}\mathbf{x}^\beta g =$$

$$\frac{\mathrm{lcm}(\mathbf{x}^\alpha \mathrm{lm}(f), \mathbf{x}^\beta \mathrm{lm}(g))}{\mathrm{lm}(f)} f - \frac{\mathrm{lcm}(\mathbf{x}^\alpha \mathrm{lm}(f), \mathbf{x}^\beta \mathrm{lm}(g))}{\mathrm{lm}(g)} g =$$

$$\frac{\mathrm{lcm}(\mathbf{x}^\alpha \mathrm{lm}(f), \mathbf{x}^\beta \mathrm{lm}(g))}{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}\left(\frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lm}(f)} f - \frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lm}(g)} g\right) =$$

$$\frac{\mathrm{lcm}(\mathbf{x}^\alpha \mathrm{lm}(f), \mathbf{x}^\beta \mathrm{lm}(g))}{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))} S(f,g) = \mathbf{x}^\gamma S(f,g).$$

□

Given a tableau T of shape $\lambda_{n,k}$, let $\mathbf{x}_T$ denote the squarefree monomial consisting of all the variables corresponding to the first column of T. That is, if

$$T = \begin{array}{|c|c|c|}\hline i_1 & i_6 & i_7 \\ \hline i_2 & \multicolumn{2}{c}{} \\ \cline{1-1} i_3 & \multicolumn{2}{c}{} \\ \cline{1-1} i_4 & \multicolumn{2}{c}{} \\ \cline{1-1} i_5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

is a tableau of shape $\lambda_{7,4}$, then $\mathbf{x}_T = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}$. Given two tableaux T and T', we let $\mathbf{x}_{T,T'}$ denote the least common multiple of $\mathbf{x}_T$ and $\mathbf{x}_{T'}$, or equivalently, $\mathbf{x}_{T,T'} = \prod_{x_i \in U} x_i$, where $U = \operatorname{supp}(\mathbf{x}_T) \cup \operatorname{supp}(\mathbf{x}_{T'})$.

As a corollary to Lemma 4.2.2 we show that:

Corollary 4.2.1. *If $T, T' \in \operatorname{Tab}(\lambda_{n,k})$, then $S(\boldsymbol{x}_T f_T, \boldsymbol{x}_{T'} f_{T'}) = \boldsymbol{x}_{T,T'} S(f_T, f_{T'})$.*

Proof. Due to Lemma 4.2.2, we just need to show that $\frac{\operatorname{lcm}(\mathbf{x}_T \operatorname{lm}(f_T), \mathbf{x}_{T'} \operatorname{lm}(f_{T'}))}{\operatorname{lcm}(\operatorname{lm}(f_T), \operatorname{lm}(f_{T'}))} = \mathbf{x}_{T,T'}$. If we let $\operatorname{lm}(f_T) = \mathbf{x}^\alpha$ and $\operatorname{lm}(f_{T'}) = \mathbf{x}^\beta$, then we can write $\mathbf{x}_T \operatorname{lm}(f_T) = \mathbf{x}^{\alpha'}$ and $\mathbf{x}_{T'} \operatorname{lm}(f_{T'}) = \mathbf{x}^{\beta'}$, where

$$\alpha'_i = \begin{cases} 1 + \alpha_i, & \text{if } x_i | \mathbf{x}_T \\ \alpha_i, & \text{else,} \end{cases}$$

and

$$\beta'_i = \begin{cases} 1 + \beta_i, & \text{if } x_i | \mathbf{x}_{T'} \\ \beta_i, & \text{else.} \end{cases}$$

Similarly, if we let $\operatorname{lcm}(\mathbf{x}^\alpha, \mathbf{x}^\beta) = \mathbf{x}^\gamma$, we set $\operatorname{lcm}(\mathbf{x}_T\mathbf{x}^\alpha, \mathbf{x}_{T'}\mathbf{x}^\beta) = \mathbf{x}^{\gamma'}$, where

$$\gamma'_i = \begin{cases} 1 + \gamma_i, & \text{if } x_i | \mathbf{x}_{T,T'} \\ \gamma_i, & \text{else.} \end{cases}$$

Since $\operatorname{supp}(\mathbf{x}^{\alpha'}) = \operatorname{supp}(\mathbf{x}_T)$ and $\operatorname{supp}(\mathbf{x}^{\beta'}) = \operatorname{supp}(\mathbf{x}_{T'})$, we have $\operatorname{supp}(\mathbf{x}^{\gamma'}) = \operatorname{supp}(\mathbf{x}_{T,T'})$. Thus,

$$\frac{\operatorname{lcm}(\mathbf{x}_T \operatorname{lm}(f_T), \mathbf{x}_{T'} \operatorname{lm}(f_{T'}))}{\operatorname{lcm}(\operatorname{lm}(f_T), \operatorname{lm}(f_{T'}))} = \frac{\operatorname{lcm}(\mathbf{x}^{\alpha'}, \mathbf{x}^{\beta'})}{\operatorname{lcm}(\mathbf{x}^\alpha, \mathbf{x}^\beta)} = \frac{\mathbf{x}^{\gamma'}}{\mathbf{x}^\gamma} =$$

$$\prod_{x_i \in \operatorname{supp}(\mathbf{x}^{\gamma'})} x_i = \prod_{x_i \in \operatorname{supp}(\mathbf{x}_{T,T'})} x_i = \mathbf{x}_{T,T'}.$$

□

Lemma 4.2.3. *Let $\phi : R_n \to R_n$ be the ring homomorphism defined by*

$$\phi(x_j) = \begin{cases} x_1, & \textit{for } j = 1, \\ x_j + x_1, & \textit{else.} \end{cases}$$

Then ϕ is a ring automorphism.

Proof. If we let $\theta : R_n \to R_n$, be the ring homomorphism

$$\theta(x_j) = \begin{cases} x_1, & \text{for } j = 1, \\ x_j - x_1, & \text{else,} \end{cases}$$

then it is clear that $\phi \circ \theta = \theta \circ \phi = id_{R_n}$, so ϕ is a bijection. $\square$

Lemma 4.2.4. *The ideal of leading monomials of $F_{\lambda_{n,k}}$ with respect to the invlex order is*

$$\langle \mathrm{lm}(F_{\lambda_{n,k}}) \rangle = \langle x_{i_1} x_{i_2}^2 \cdots x_{i_k}^k | 1 < i_1 < i_2 < \ldots < i_k \leq n \rangle.$$

This is also the ideal $\langle \mathrm{lm}(SF_{\lambda_{n,k}}) \rangle$, where $SF_{\lambda_{n,k}} = \{f_T | T \in \mathrm{StdTab}(\lambda_{n,k})\}$.

A column-standard tableau, is a tableau where the indices of the columns are strictly increasing. Given a tableau, T, of shape $\lambda \vdash n$, there exists a column-standard tableau T' of shape λ such that $f_T = \pm f_{T'}$. To see this, just note that permuting the indices of any column of T has the effect of multiplying f_T with ± 1.

Proof. Due to the preceding comment we may restrict our attention to column-standard tableaux. So let $T \in \mathrm{Tab}(\lambda_{n,k})$ be column-standard, then

$$T = \begin{array}{|c|c|c|c|c|} \hline i_{k+1} & i_{k+2} & .. & .. & i_n \\ \hline i_1 \\ \cline{1-1} i_2 \\ \cline{1-1} .. \\ \cline{1-1} .. \\ \cline{1-1} i_k \\ \cline{1-1} \end{array},$$

with $1 \leq i_{k+1} < i_1 < i_2 < \ldots < i_k \leq n$, and the leading monomial of f_T is $\mathrm{lm}(f_T) = x_{i_1} x_{i_2}^2 \cdots x_{i_k}^k$.

Conversely, given a tuple $(i_1, i_2, \ldots, i_k) \in \mathbb{N}^k$, with
$1 < i_1 < i_2 < \ldots < i_k \leq n$, we can construct a column-standard tableau $T \in$

$\mathrm{Tab}(\lambda_{n,k})$

$$T = \begin{array}{|c|c|c|c|c|}
\hline
i_{k+1} & i_{k+2} & .. & .. & i_n \\
\hline
i_1 \\
\cline{1-1}
i_2 \\
\cline{1-1}
.. \\
\cline{1-1}
.. \\
\cline{1-1}
i_k \\
\cline{1-1}
\end{array},$$

with $i_{k+1} < i_1$ and the other indices are the remaining indices in $[n]$.

For the second statement, we do the same thing, except that we let $i_{k+1} = 1$ and order the indices $i_{k+2}, ..., i_n$ increasingly. $\square$

Proof of theorem 1. Since the collection of Specht polynomials is symmetric, we will just prove the theorem for the invlex order, and the rest will follow. In fact, we will prove the stronger statement that $SF_{\lambda_{n,k}}$ is a Gröbner basis of $I_{\lambda_{n,k}}$ using induction on k.

For $k = 1$, $SF_{\lambda_{n,1}} = \{x_1 - x_i | i \in [n]\backslash\{1\}\}$. Let $f_T = (x_1 - x_j)$,
$f_{T'} = (x_1 - x_i) \in SF_{\lambda_{n,1}}$, with $i \neq j$, then $S(f_T, f_{T'}) = x_i x_1 - x_j x_1$. Thus, if $i > j$, $S(f_T, f_{T'}) = -x_1 f_{T'} + x_1 f_T$, and $\mathrm{lm}(S(f_T, f_{T'})) = x_i x_1 = \mathrm{lm}(-x_1 f_{T'}) > \mathrm{lm}(x_1 f_T)$. So $S(f_T, f_{T'}) \rightarrow_{SF_{\lambda_{n,1}}} 0$ and $SF_{\lambda_{n,1}}$ is a Gröbner basis of $I_{\lambda_{n,1}}$.

Now, assuming the statement is correct for $k \geq 1$ we show that it is true for $k+1$.

Let $\mathrm{StdTab}_{[n+1]\backslash\{1\}}(\lambda_{n,k})$ be the standard tableaux of shape $\lambda_{n,k}$ with indices in $[n+1]\backslash\{1\}$. By lemma 2.3.1 and 4.2.3, we know that $\phi(SF_{\lambda_{n+1,k+1}})$ is a generating set for $\phi(I_{\lambda_{n+1,k+1}})$. Note that if $i < j$ and $i, j \in \mathbb{N}$, then

$$\phi(x_i - x_j) = \begin{cases} -x_j, & \text{if } i = 1, \\ x_i - x_j, & \text{else.} \end{cases}$$

Thus, $\phi(SF_{\lambda_{n+1,k+1}}) \subseteq \{(-1)^{k-1}\mathbf{x}_T f_T | T \in \mathrm{Tab}_{[n+1]\backslash\{1\}}(\lambda_{n,k})\}$. To show that $\phi(SF_{\lambda_{n+1,k+1}}) = \{\pm\mathbf{x}_T f_T | T \in \mathrm{Tab}_{[n+1]\backslash\{1\}}(\lambda_{n,k})\}$, let
$T \in \mathrm{Tab}_{[n+1]\backslash\{1\}}(\lambda_{n,k})$, then $f_T = \pm f_{T'}$, where T' is a column-standard tableau and the $n - k - 1$ rightmost indices of T' are ordered increasingly.

Consider the tableau $T'' \in \mathrm{StdTab}(\lambda_{n+1,k+1})$:

$$T'' = \begin{array}{|c|c|c|c|c|}\hline 1 & i_{k+2} & .. & .. & i_n \\ \hline i_1 \\ \cline{1-1} i_2 \\ \cline{1-1} .. \\ \cline{1-1} .. \\ \cline{1-1} i_{k+1} \\ \cline{1-1} \end{array} ,$$

where

$$T' = \begin{array}{|c|c|c|c|c|}\hline i_1 & i_{k+2} & .. & .. & i_n \\ \hline i_2 \\ \cline{1-1} .. \\ \cline{1-1} .. \\ \cline{1-1} i_{k+1} \\ \cline{1-1} \end{array} .$$

Clearly $\phi(f_{T''}) = \mathbf{x}_{T'} f_{T'}$, thus

$$\phi(SF_{\lambda_{n+1,k+1}}) = \{\pm \mathbf{x}_T f_T | T \in \mathrm{Tab}_{[n+1]\setminus\{1\}}(\lambda_{n,k})\}.$$

Since it is irrelevant for our argument whether the polynomials are multiplied by plus or minus 1, we will assume they are always multiplied by 1.

We show that $\phi(SF_{\lambda_{n+1,k+1}})$ is a Gröbner basis of $\phi(I_{\lambda_{n+1,k+1}})$. Let $\mathbf{x}_T f_T, \mathbf{x}_{T'} f_{T'} \in \phi(SF_{\lambda_{n+1,k+1}})$. By Corollary 4.2.1,

$$S(\mathbf{x}_T f_T, \mathbf{x}_{T'} f_{T'}) = \mathbf{x}_{T,T'} S(f_T, f_{T'}),$$

By the induction hypothesis $F_{\lambda_{n,k}} = \{f_T | T \in \mathrm{Tab}_{[n+1]\setminus\{1\}}(\lambda_{n,k})\}$ is a Gröbner basis with respect to any permutation of lex order. Let $U = \mathrm{supp}(\mathbf{x}_{T,T'})$ and $R_U = K[U]$. Since all permutations of the lex order are elimination orders of maximal type, $F_{\lambda_{n,k}} \cap R_U$ is a Gröbner basis with respect to a permutation of the lex order with $x_i < x_j \ \forall i \in U, \ j \notin U$.

However, $F_{\lambda_{n,k}} \cap R_U$ is symmetric with respect to the symmetry group on the set U, S_U, and therefore it is a Gröbner basis with respect to the τ-lex order for any

$\tau \in S_U$. Since there exists a $\tau \in S_U$ such that the τ-lex order on $U^\diamond$ is equivalent to the restriction of the invlex order to $U^\diamond$, we can just continue to work with the invlex order.

Thus we can write $S(f_T, f_{T'}) = \sum h_j f_{T_j}$ with $f_{T_j} \in F_{\lambda_{n,k}} \cap R_U$ and $h_j \in R_U$ and $\mathrm{lm}(S(f_T, f_{T'})) \geq_{invlex} \mathrm{lm}(h_j f_{T_j})$. Since $\mathbf{x}_{T,T'} f_{T_j} \in H$ for any $f_{T_j} \in F_{\lambda_{n,k}} \cap R_U$, we have that

$$S(\mathbf{x}_T f_T, \mathbf{x}_{T'} f_{T'}) = \mathbf{x}_{T,T'} S(f_T, f_{T'}) = \sum h_j \mathbf{x}_{T,T'} f_{T_j}$$

factors over $\phi(SF_{\lambda_{n+1,k+1}})$ and

$$\mathrm{lm}(S(\mathbf{x}_T f_T, \mathbf{x}_{T'} f_{T'})) = \mathbf{x}_{T,T'} \mathrm{lm}(S(f_T, f_{T'})) \geq_{invlex}$$

$$\mathbf{x}_{T,T'} \mathrm{lm}(h_j f_{T_j}) = \mathrm{lm}(h_j \mathbf{x}_{T,T'} f_{T_j}).$$

Thus $\phi(SF_{\lambda_{n+1,k+1}})$ is a Gröbner basis of $\phi(I_{\lambda_{n+1,k+1}})$.

If we use the index set $[n+1]\backslash\{1\}$, instead of $[n]$, then we know from Corollary 4.2.1 and the induction hypothesis that

$$\mathrm{lm}(I_{\lambda_{n,k}}) = \langle x_{i_2} x_{i_3}^2 \cdots x_{i_{k+1}}^k | 2 < i_2 < i_3 < ... < i_{k+1} \leq n+1 \rangle.$$

Since the elements of $\phi(SF_{\lambda_{n+1,k+1}})$ is of the form $\mathbf{x}_T f_T$, with $T \in \mathrm{Tab}_{[n+1]\backslash\{1\}}(\lambda_{n,k})$, we have that

$$\langle \mathrm{lm}(\phi(SF_{\lambda_{n+1,k+1}})) \rangle = \langle x_{i_1} x_{i_2}^2 \cdots x_{i_{k+1}}^{k+1} | 1 < i_1 < i_2 < ... < i_{k+1} \leq n+1 \rangle.$$

As we can see this is equal to $\langle \mathrm{lm}(SF_{\lambda_{n+1,k+1}}) \rangle$. Since $\phi(SF_{\lambda_{n+1,k+1}})$ is a Gröbner basis of $\phi(I_{\lambda_{n+1,k+1}})$, we have that

$$\mathrm{lm}(\phi(I_{\lambda_{n+1,k+1}})) = \langle \mathrm{lm}(\phi(SF_{\lambda_{n+1,k+1}})) \rangle = \langle \mathrm{lm}(SF_{\lambda_{n+1,k+1}}) \rangle.$$

We have that $I_{\lambda_{n+1,k+1}}$ and $\phi(I_{\lambda_{n+1,k+1}})$ are homogeneous ideals, thus, according to Lemma 1.3.2, their Hilbert Series are equal to the Hilbert series of their leading monomial ideals. Also, since ϕ is a degree-preserving isomorphism, we get

$$H_{\mathrm{lm}(I_{\lambda_{n+1,k+1}})}(t) = H_{I_{\lambda_{n+1,k+1}}}(t) = H_{\phi(I_{\lambda_{n+1,k+1}})}(t) =$$

$$H_{\mathrm{lm}(\phi(I_{\lambda_{n+1,k+1}}))}(t) = H_{\langle \mathrm{lm}(SF_{\lambda_{n+1,k+1}}) \rangle}(t).$$

Since $\langle \mathrm{lm}(SF_{\lambda_{n+1,k+1}}) \rangle \subseteq \mathrm{lm}(I_{\lambda_{n+1,k+1}})$ and their Hilbert series agree, we get an equality, that is $\langle \mathrm{lm}(SF_{\lambda_{n+1,k+1}}) \rangle = \mathrm{lm}(I_{\lambda_{n+1,k+1}})$. Thus $SF_{\lambda_{n+1,k+1}}$(and therefore $F_{\lambda_{n+1,k+1}}$) is a Gröbner basis of $I_{\lambda_{n+1,k+1}}$. $\square$

4.2.1. Hilbert series revisited

As mentioned after Remark 4.1.4, the Gröbner basis would give us the recurrence relation of Hilbert series. Thus, having established Theorem 4.2.1, we will look into how this works so that we may get the Hilbert series of the ideals $I_{\lambda_{n,k}}$. But first we need a few preliminary results.

Corollary 4.2.2. *Let* $(I_m)_{m\in\mathbb{N}_0}$ *be the filtration of* $I_{\lambda_{k+1,k}}$*, for* $k>0$*. Then the sequence of leading monomial ideals* $(\mathrm{lm}(I_m))_{m\in\mathbb{N}_0}$ *is the* $\mathrm{Inc}(\mathbb{N})$*-invariant chain given by*

$$\mathrm{lm}(I_m) = \begin{cases} \langle \mathrm{Inc}(\mathbb{N}) x_2 x_3^2 \cdots x_{k+1}^k \rangle \cap R_n, & \textit{for } n > k, \\ \langle 0 \rangle, & \textit{else.} \end{cases}$$

Proof. For $n > k$ it follows from Lemma 4.2.4 and Theorem 4.2.1. For $n \leq k$, note that $I_{k+1} = I_{\lambda_{k+1,k}}$ is a principal ideal with support $\{x_1, x_2, ..., x_{k+1}\}$, thus $I_n = I_{k+1} \cap R_n = \langle 0 \rangle$. □

Interestingly this immediately implies the following:

Corollary 4.2.3. *The following holds for the filtration of* $I_{\lambda_{k+1,k}}$*:*

i The polynomial f_T*, where* $T \in \mathrm{StdTab}(\lambda_{k+1,k})$ *is an equivariant Gröbner basis of* $\cup_{n>k} I_{\lambda_{n,k}}$*,*

ii the filtration of $I_{\lambda_{k+1,k}}$ *has the intersection property.*

Proof. The first statement follows from Corollary 4.2.2 and the second follows from the first statement if we employ Proposition 3.2.1. □

From Corollary 2.2 in [10], we get the recursive relation of the Hilbert series of $\mathrm{lm}(I_{\lambda_{n,k}})$ and since the ideals, $I_{\lambda_{n,k}}$, are homogeneous, Lemma 1.3.2 tells us that this relation holds for the ideals, $I_{\lambda_{n,k}}$, as well. That is, we have the following lemma:

Lemma 4.2.5. *For* $n > k+1$ *and* $k > 1$*, we have*

$$H_{R_n/I_{\lambda_{n,k}}}(t) = \left(\sum_{j=0}^{k-1} t^j \right) H_{R_{n-1}/I_{\lambda_{n-1,k}}}(t) + \frac{t^k}{(1-t)} H_{R_{n-1}/I_{\lambda_{n-1,k-1}}}(t).$$

Proof. Comes from using corollary 2.2 in [10] with $r = k-1, \delta_r = 1$, $a_r = k$, $I_n = I_{\lambda_{n,k}}$ and $J_{n-1} = I_{\lambda_{n-1,k-1}}$. □

To make good use of this recurrence relation, we first need some initial states.

Lemma 4.2.6. *For $n > 1$ we have that $H_{R_n/I_{\lambda_{n,1}}}(t) = \frac{1}{1-t}$, and for $n > 2$, we have that $H_{R_n/I_{\lambda_{n,n-1}}}(t) = \frac{\sum_{i=0}^{n(n-1)/2-1} t^i}{(1-t)^{(n-1)}}$.*

Proof. We have shown the first statement already in Section 4.1.

For the second statement note that $I_{\lambda_{n,n-1}}$ is a principal ideal generated by a polynomial of degree $\sum_{j=1}^{n-1} j = \frac{n(n-1)}{2}$. This can easily be seen by using the definition of f_T, where $T \in \mathrm{Tab}(\lambda_{n,n-1})$, and by counting the factors. Since $I_{\lambda_{n,n-1}}$ is a principal ideal, then $R_n/I_{\lambda_{n,n-1}}$ is a complete intersection ring. Thus by Proposition 1.3.4 we have

$$H_{R_n/I_{\lambda_{n,n-1}}}(t) = \frac{1-t^{n(n-1)/2}}{(1-t)^n} = \frac{\sum_{i=0}^{n(n-1)/2-1} t^i}{(1-t)^{(n-1)}}.$$

□

With these initial states and the recurrence relation we can calculate the Hilbert series of $I_{\lambda_{n,k}}$. We will not be using this to provide a general form of the Hilbert series for all $\lambda_{n,k}$ since they quickly become quite messy, but we will use this for the first few sequences:

Example 4.2.1. *We know that for the partitions $(n-1,1)$ the Hilbert series are $H_{R_n/I_{(n-1,1)}} = \frac{1}{1-t}$, so we will look at the partitions $(n-2,1,1)$.*

First we rewrite the recurrence relation in a more useful form:

$$H_{R_n/I_{\lambda_{n,k}}}(t) = \left(\sum_{j=0}^{k-1} t^j\right) H_{R_{n-1}/I_{\lambda_{n-1,k}}}(t) + \frac{t^k}{(1-t)} H_{R_{n-1}/I_{\lambda_{n-1,k-1}}}(t) =$$

$$\left(\sum_{j=0}^{k-1} t^j\right)\left(\left(\sum_{j=0}^{k-1} t^j\right) H_{R_{n-2}/I_{\lambda_{n-2,k}}}(t) + \frac{t^k}{(1-t)} H_{R_{n-2}/I_{\lambda_{n-2,k-1}}}(t)\right) +$$

$$\frac{t^k}{(1-t)} H_{R_{n-1}/I_{\lambda_{n-1,k-1}}}(t) = \left(\sum_{j=0}^{k-1} t^j\right)^2 H_{R_{n-2}/I_{\lambda_{n-2,k}}}(t) +$$

$$\left(\sum_{j=0}^{k-1} t^j\right) \frac{t^k}{(1-t)} H_{R_{n-2}/I_{\lambda_{n-2,k-1}}}(t) + \frac{t^k}{(1-t)} H_{R_{n-1}/I_{\lambda_{n-1,k-1}}}(t) =$$

$$\cdots$$

$$\cdots$$

$$\left(\sum_{j=0}^{k-1} t^j\right)^{n-k-1} H_{R_{k+1}/I_{\lambda_{k+1,k}}}(t) + \sum_{i=1}^{n-k-1} \left(\sum_{j=0}^{k-1} t^j\right)^{i-1} \frac{t^k}{(1-t)} H_{R_{n-i}/I_{\lambda_{n-i,k-1}}}(t). \quad (1)$$

Using Lemma 4.2.6 we get that for the partitions $(n-2,1,1)$ the Hilbert series are:

$$H_{R_n/I_{\lambda_{n,2}}}(t) = (1+t)^{n-3}\frac{1+t}{(1-t)^2} + \frac{t^2}{(1-t)^2}\sum_{i=1}^{n-3}(1+t)^{i-1} =$$

$$\frac{(1+t)^{n-2} + t^2\sum_{i=1}^{n-3}(1+t)^{i-1}}{(1-t)^2}$$

Example 4.2.2. *We calculate the Hilbert series for the partitions $\lambda_{n,3}$. Using the relation 1 from the previous example we get*

$$H_{R_n/I_{\lambda_{n,3}}}(t) =$$

$$(1+t+t^2)^{n-k-1} H_{R_4/I_{\lambda_{4,3}}}(t) + \sum_{i=1}^{n-4}(1+t+t^2)^{i-1}\frac{t^3}{(1-t)} H_{R_{n-i}/I_{\lambda_{n-i,2}}}(t).$$

Using Lemma 4.2.6 and the Hilbert series of the partitions $\lambda_{n,2}$ we get

$$H_{R_n/I_{\lambda_{n,3}}}(t) = (1+t+t^2)^{n-k-1}\frac{1+t+t^2+t^3+t^4+t^5}{(1-t)^3} +$$

$$\sum_{i=1}^{n-4}(1+t+t^2)^{i-1}\frac{t^3}{(1-t)}\frac{(1+t)^{n-i-2} + t^2\sum_{j=1}^{n-i-3}(1+t)^{i-1}}{(1-t)^2} =$$

$$\frac{(1+t+t^2)^{n-k-1}(1+t+t^2+t^3+t^4+t^5)}{(1-t)^3} +$$

$$\frac{t^3\sum_{i=1}^{n-4}(1+t+t^2)^{i-1}(1+t)^{n-i-2} + t^2\sum_{j=1}^{n-i-3}(1+t)^{i-1}}{(1-t)^3}.$$

We can also use Theorem 2.4 in ([10]) to get the equivariant Hilbert series of the filtration of $I_{\lambda_{n,k}}$:

Theorem 4.2.2. *Let $k > 0$ and $I_{\lambda_k} = \cup_{n>k} I_{\lambda_{n,k}} \lhd R$. Then*

$$H_{R/I_{\lambda_k}}(s,t) = \frac{g_{\lambda_k}(s,t)}{(1-t)^k \prod_{i=1}^{k}(1-s(1+t+...+t^{i-1}))},$$

where

$$g_{\lambda_k}(s,t)(1-t-s) = (1-t)\prod_{i=1}^{k}(1-s-t+st^i) - s^{k+1}t^{k(k+1)/2}$$

and the above fraction is the reduced form of $H_{R/I_{\lambda_k}}(s,t)$.

Proof. Let $\mathrm{lm}(I_{\lambda_k}) = \cup_{n>k} \mathrm{lm}(I_{\lambda_{n,k}})$. Since $H_{I_{\lambda_{n,k}}}(t) = H_{\mathrm{lm}(I_{\lambda_{n,k}})}(t)$, we have that $H_{I_{\lambda_k}}(s,t) = H_{\mathrm{lm}(I_{\lambda_k})}(s,t)$. Now we use Theorem 2.4 in [10] with $g_{r,\underline{a},\underline{u}}(s,t) = g_{\lambda_k}(s,t), r = k, u_r = k+1$ and $a_i = i \ \forall \ i$. $\square$

Since the equivariant Hilbert series is a sum of Hilbert series multiplied by powers of s, we may use it to find a particular Hilbert series by taking derivatives with respect to s, divide by the factorial of number of derivatives, and then evaluate the resulting sum for $s = 0$.

4.3. Reduced conjecture and standard Specht polynomials

We will continue to consider Conjecture 4.2.1 for other partitions than $(n-k, 1^k)$. To begin with we investigate whether the proof for the partitions $(m-k, 1^k)$ can be generalized to other partitions. The answer is unfortunately no, and we begin by explaining some of the difficulties. However, parts of the proof work for other partitions as well and will be useful in reducing Conjecture 4.2.1.

Additionally, it will provide us with a strengthening of Proposition 2.3.1. That is, we will see that the leading monomials of the standard Specht polynomials are all the leading monomials of Specht polynomials. Lastly we use these two results to give us a condition for verifying the conjectured Gröbner basis for any filtration of a Specht ideal.

We start with a simple example that illustrates some of the problems with generalizing the proof.

Example 4.3.1. *Let $\lambda = (3,2)$, then the Specht polynomial of any partition dominated by λ is a multiple of the Specht polynomials corresponding to λ or $\mu = (3,1,1)$. We will use the invlex order and begin by restricting to the standard tableaux of shape λ and μ. Then we pass them through the map $\phi : R_5 \to R_5$ given by $x_1 \mapsto x_1$ and $x_i \mapsto x_i + x_1$ for $i > 1$.*

Let $B = \phi(f_T | T \in \mathrm{StdTab}(\lambda) \cup \mathrm{StdTab}(\mu))$ and let

$$S = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \quad T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}.$$

Also, let

$$S' = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 5 \\ \cline{1-1}\end{array}, \quad T' = \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array},$$

then

$$S(\phi(f_S), \phi(f_T)) = \phi(x_4 f_{S'}, x_2 f_{T'}) = x_2 x_4 f_{S'} - x_5 x_2 f_{T'} =$$
$$x_2(x_4 f_{S'} - x_5 f_{T'}) = x_2 S(f_{S'}, f_{T'}).$$

Already we can see that Corollary 4.2.1 is of no use to us, otherwise we would be able to write $S(\phi(f_S), \phi(f_T)) = x_2 x_4 S(f_{S'}, f_{T'})$, but let us keep going to see what else might go wrong. By using an induction similar to the proof we would restrict to the variables $\mathrm{supp}(f_{S'}) \cup \mathrm{supp}(f_{T'}) = \{x_2, x_3, x_4, x_5\}$ and write $S(f_{S'}, f_{T'}) = \sum h_i f_{T_i}$ with $T_i \in \mathrm{Tab}_{\{2,3,4,5\}}((3,1))$, where we assume that the set of f_{T_i}'s is a Gröbner basis. However, that would allow for the possibility that $T_i = \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$, for some i, and $x_2 f_{T_i}$ is not in B. Thus we would potentially end up with polynomials that may not be in the image of ϕ.

Thus we have given an example that shows two main steps where the proof is difficult to generalize. The first one forces us to put restrictions on the h_i's and the second requires that we find a way of giving a harsher restriction than simply restricting to the support of the polynomials.

Let us now look at which steps of the proof that does generalize. When we considered the partitions $(n-k, 1^k)$, one of the first things we did was to restrict to the Specht polynomials corresponding to $(n-k, 1^k)$. Thus we where able to restrict our attention to the Specht polynomials of one particular partition.

In general, when we consider a filtration of the Specht ideal corresponding to some partition $\lambda \vdash n$, the conjectured Gröbner basis of the next ideal in the sequence, $I_{\lambda'}$, where $\lambda' = (\lambda_1 + 1, \lambda_2, ..., \lambda_l)$, will contain Specht polynomials of more partitions than I_λ. However, the next thing we will show is that we may restrict our attention to the partitions $\mu = (\mu_1, \mu_2, .., \mu_k) \trianglelefteq \lambda'$, where $\mu_1 = \lambda_1 + 1$. Thus we only need to consider a finite number of partitions per filtration.

Before we show this, let us adopt the convention that if, for a partition $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$, we refer to λ_i, where $i \notin [l]$, then we let $\lambda_i = 0$.

Lemma 4.3.1. *Let $\mu \trianglelefteq \lambda$ be partitions of n, then there exist a partition, $\tau \vdash n$, with $\tau \trianglelefteq \lambda$ and $\tau_1 = \lambda_1$, such that any Specht polynomial of shape μ is a multiple of a Specht polynomial of shape τ.*

Proof. We construct a partition that will satisfy the lemma by using induction on the difference $d = \lambda_1 - \mu_1$.

For $d = 0$, we can simply let $\tau = \mu$. So suppose the lemma is true for $d \geq 0$, then we show that it is true for $d + 1$.

Let $m = \min\{j | \mu_j > \lambda_j\}$ and let $r = \max\{j | \mu_j = \mu_m\}$. Then we define the partition $\sigma = (\mu_1 + 1, \mu_2, ..., \mu_{r-1}, \mu_r - 1, \mu_{r+1}, ..., \mu_l)$, where $l = \text{length}(\mu)$. We will show that $\sigma \trianglelefteq \lambda$. Suppose this is not the case and let $s = \min\{j \in \mathbb{N} | \sum_{i=1}^{j} \lambda_i < \sum_{i=1}^{j} \sigma_i\}$. Then we consider two possibilities:

If $s \geq r$ we have that $\sum_{i=1}^{s} \sigma_i = \mu_1 + 1 + \mu_2 + ... + \mu_{r-1} + \mu_r - 1 + \mu_{r+1} + ... + \mu_s = \sum_{i=1}^{s} \mu_i$. But since $\mu \trianglelefteq \lambda$ we have $\sum_{i=1}^{s} \mu_i \leq \sum_{i=1}^{s} \lambda_i$. Thus $\sum_{i=1}^{s} \sigma_i \leq \sum_{i=1}^{s} \lambda_i$, which contradicts the choice of s.

If $s < r$, we have $\sum_{i=1}^{s} \lambda_i < \sum_{i=1}^{s} \sigma_i = \sum_{i=1}^{s} \mu_i + 1$. Since $\mu \trianglelefteq \lambda$, we have $\sum_{i=1}^{s} \lambda_i \geq \sum_{i=1}^{s} \mu_i$, thus $\sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \mu_i$. Since $\mu_1 < \lambda_1$ and $\mu_j \leq \lambda_j \; \forall \; j < m$, we know that $m \leq s < r$. But we also have that $\sum_{i=1}^{s+1} \lambda_i \leq \sum_{i=1}^{s+1} \mu_i$, which is a contradiction since $\mu_{s+1} > \lambda_{s+1}$ and $\sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \mu_i$.

Since both cases gives a contradiction, we have that $\sigma \trianglelefteq \lambda$. Note that all the columns of σ either have length 1 or are at most the same length as the corresponding column of μ. Thus if f_T is any Specht polynomial of shape μ, then it is of the form $hf_{S'}$, where $S' \in \text{Tab}(\sigma)$ and h is some polynomial.

We have established that $\sigma \trianglelefteq \lambda$ and that $\lambda_1 - \sigma_1 = d$, so by the induction hypothesis $f_{S'} = gf_S$, where g is some polynomial, $S \in \text{Tab}(\tau)$ and τ is a partition dominated by λ with $\tau_1 = \lambda_1$. Thus $f_T = hgf_S$ and the proof is complete. $\square$

Essentially this tells us that when we investigate Conjecture 4.2.1 for a sequence of Specht ideals generated by a partition $\lambda \vdash n$, we may restrict our attention to the orbits of the partitions $\mu \trianglelefteq \lambda$ with $\mu_1 = \lambda_1$. Conjecture 4.2.1 combined with this observation is enough to reduce the conjecture to the following:

Conjecture 4.3.1. *Let $\lambda \vdash n$, then the Specht polynomials of shape $\mu \vdash n$, where $\mu \trianglelefteq \lambda$ with $\mu_1 = \lambda_1$, form a Gröbner basis of I_λ.*

The next reduction we can make is on the tableaux that we consider. Recall that in the proof we did for the partition $(n-k, 1^k)$, we started by restricting to the standard tableaux when using the invlex ordering. This turns out to be something we can generally do, as the following argument will show.

Before we can prove the statement, we need a few preliminary results and some terminology.

Given a partition $\lambda \vdash n$, we let $F_\lambda = \{f_T | T \in Tab(\lambda)\}$ and $SF_\lambda = \{f_T | T \in \mathrm{StdTab}(\lambda)\}$. Two tableaux, T and T', are said to be row equivalent if $T = \sigma(T')$, for some row stabilizer σ of T', that is, the index set of the ith row of T is the same as for the ith row of T' for any i.

Lemma 4.3.2. *Let $\lambda \vdash n$ and let $T, T' \in \mathrm{Tab}(\lambda)$ be two row equivalent column-standard tableaux. Then $\mathrm{lm}(f_T) = \mathrm{lm}(f_{T'})$ with respect to the invlex order.*

Proof. Let c_j denote the j^{th} column of T and let f_{c_j} denote the Vandermonde polynomial corresponding to the j^{th} column. Then we can write f_T as $f_{c_1} f_{c_2} \cdots f_{c_l}$, where $l = \mathrm{length}(\lambda^\perp)$. Then, since $\mathrm{lm}(hg) = \mathrm{lm}(h)\,\mathrm{lm}(g)\ \forall\ h, g \in R_n$, we have that $\mathrm{lm}(f_T) = \mathrm{lm}(f_{c_1})\,\mathrm{lm}(f_{c_2}) \cdots \mathrm{lm}(f_{c_l})$.

We can write any column c_r as

$$c_r = \begin{array}{|c|} \hline \alpha_{1,r} \\ \hline \alpha_{2,r} \\ \hline .. \\ \hline .. \\ \hline \alpha_{k,r} \\ \hline \end{array},$$

where $\alpha_{1,r} < \alpha_{2,r} < ... < \alpha_{k,r}$, since T is column-standard.

Then $f_{c_r} = \prod_{i<j} (x_{\alpha_{i,r}} - x_{\alpha_{j,r}})$ and so

$$\mathrm{lm}(f_{c_r}) = \prod_{i<j} \mathrm{lm}(x_{\alpha_{i,r}} - x_{\alpha_{j,r}}) = \prod_{i<j} x_{\alpha_{j,r}} = \prod_{j} x_{\alpha_{j,r}}^{j-1}.$$

Thus, given an index, s, in the j^{th} row of c_r, the degree of x_s, in $\mathrm{lm}(f_{c_r})$, is $j-1$. And since the index does not appear in any other column, then this is also the degree of x_s in $\mathrm{lm}(f_T)$.

Thus, the number of the row in which an index s appears, is enough to determine the degree of x_s in $\mathrm{lm}(f_T)$. Since T' is row equivalent to T, then this row number is the same for any index s, thus $\mathrm{lm}(f_T) = \mathrm{lm}(f_{T'})$. □

Lemma 4.3.3. *Let $\lambda \vdash n$ and let $T \in \mathrm{Tab}(\lambda)$ be column-standard, then there exists a standard tableau, $T' \in \mathrm{StdTab}(\lambda)$, that is row equivalent to T.*

Proof. We prove this by induction on n. For $n = 1$ there is only one tableau, it is standard, column-standard and of course, row equivalent to itself.

Assuming the statement is true for $n \geq 1$, let $\lambda \vdash n+1$ and $T \in \mathrm{Tab}(\lambda)$ be column-standard. Then the index $n+1$ is in the bottom of a column of T. Let c_i denote said column and let k be the length of this column. Without loss of generality, we may assume that c_i is the rightmost column of length k. If it is not, then we can switch the column with the rightmost column of length k and be left with a row equivalent column-standard tableau.

Let us remove the box in T that contains the index $n+1$. Then we end up with a column-standard tableau $S \in \mathrm{Tab}(\mu)$, where μ is some partition of n. According to the induction hypothesis, there exist a standard tableau $S' \in \mathrm{StdTab}(\mu)$ that is row equivalent to S. Now we can just reattach the box with the index $n+1$ to S in the same position as we removed it, and we have a standard tableaux $T' \in \mathrm{Tab}(\lambda)$ that is row equivalent to T. □

Here is an alternative proof of Lemma 4.3.3.

Proof. Let $T \in \mathrm{Tab}(\lambda)$ be column-standard. Let T' be the row standard tableau we get if we order the indices of each row of T increasingly. We show that $T' \in \mathrm{StdTab}(\lambda)$.

If T' is not standard, then there has to exist a column c_r of T' with two consecutive indices $\alpha_{i,r}$ and $\alpha_{i+1,r}$ such that $\alpha_{i,r} > \alpha_{i+1,r}$. Thus we have the following picture:

$$T' = \begin{array}{|c|c|c|c|c|c|c|}\hline .. & .. & .. & .. & .. & .. & .. \\ \hline \alpha_{i,1} & \alpha_{i,2} & .. & .. & \alpha_{i,r} & .. & .. \\ \hline \alpha_{i+1,1} & \alpha_{i+1,2} & .. & .. & \alpha_{i+1,r} & .. & .. \\ \hline .. & .. & .. & .. & .. & .. & .. \\ \hline \end{array}.$$

Since T' is row-standard, then $\alpha_{i+1,1} < \alpha_{i+1,2} < \ldots < \ldots < \alpha_{i+1,r} < \alpha_{i,r}$ and $\alpha_{i,1} < \alpha_{i,2} < \ldots < \ldots < \alpha_{i,r}$. Thus there are at most $r-1$ indices in the i^{th} row that are smaller than r indices in the $(i+1)^{th}$ row which implies that T was not column-standard to begin with. This is a contradiction, and thus T' must be standard.

□

Theorem 4.3.1. *Let $\lambda \vdash n$, then $\langle \mathrm{lm}(SF_\lambda)\rangle = \langle \mathrm{lm}(F_\lambda)\rangle$ with the invlex order.*

Proof. Clearly $\langle \mathrm{lm}(SF_\lambda)\rangle \subseteq \langle \mathrm{lm}(F_\lambda)\rangle$, so we just have to show the reverse inclusion.

Let $T \in \mathrm{Tab}(\lambda)$, then there exists a column-standard tableau, $T' \in \mathrm{Tab}(\lambda))$, such that $f_T = \pm f_{T'}$. According to Lemma 4.3.3 there exists a tableau $T'' \in \mathrm{StdTab}(\lambda)$, row equivalent to T'. From Lemma 4.3.2 we know that $\mathrm{lm}(f_{T'}) = \mathrm{lm}(f_{T''})$, thus

$$\mathrm{lm}(f_T) = \mathrm{lm}(f_{T'}) = \mathrm{lm}(f_{T''}).$$

Therefore we have that

$$\langle \mathrm{lm}(F_\lambda)\rangle \subseteq \langle \mathrm{lm}(SF_\lambda)\rangle.$$

Together with the first inclusion we get

$$\langle \mathrm{lm}(SF_\lambda)\rangle = \langle \mathrm{lm}(F_\lambda)\rangle.$$

□

Thus, not only do the standard Specht polynomials span the space of Specht polynomials, but their leading monomials also span the space of leading monomials of Specht polynomials. In fact we can say that they provide a basis seeing as no two different standard Specht polynomial has the same leading monomial. This is similar to Theorem 1.1 in [18], where they show that the standard Specht polynomials is in fact a basis for the Specht polynomials.

4.3.1. Gröbner basis criterion

We will introduce a criterion that can be used to verify that a generating set of a symmetric sequence is an equivariant Gröbner basis with respect to the invlex order. In particular this criterion, in combination with the results of the previous section, will be useful for giving a condition to verify the conjectured Gröbner basis for a filtration of any Specht ideal.

We will present a few results before that we need to show the criterion.

Lemma 4.3.4. *Let $f, g \in G \subset R_n$ with supp$(f) \cap \mathrm{supp}(g) = \emptyset$, then* $S(f, g) \to_G = 0$.

Proof. See Proposition 4 in Chapter 2.9 of [7]. □

Before the next lemma we will extend the notation R_n a bit by defining R_U to be the polynomial ring, $K[x_{u_1}, x_{u_2}, ..., x_{u_n}]$, when $U = \{u_1, u_2, ..., u_n\} \subset \mathbb{N}$. Similarly we let S_U denote the symmetric group on the indices $u_1, u_2, ..., u_n$.

Lemma 4.3.5. *Let G be a finite symmetric subset of R_k and let $U \subset \mathbb{N}$ with $|U| = k$. Then for any $m \geq \max(j \in U)$, $(S_m G) \cap R_U = \tau(G)$, where τ is any element of $\mathrm{Inc}(\mathbb{N})$ such that $\tau([k]) = U$.*

Proof. Clearly we have that $\tau(G) \subset R_U$, since $\tau([k]) = U$. Also, $\tau(G) \subseteq S_m G$, since $m \geq \max(j \in U)$ and $\mathrm{Inc}_{k,m}(\mathbb{N})$ acts as a submonoid of S_m on R_k. Thus $\tau(G) \subseteq (S_m G) \cap R_U$.

The set $(S_m G) \cap R_U$ is clearly in bijection with the set G, thus they have the same size. And since τ is an injective map, $|\tau(G)| = |G|$. Combined with the inclusion above, this means that $\tau(G) = (S_m G) \cap R_U$. □

Theorem 4.3.2. *Let $(I_n)_{\mathbb{N}_0}$ be a symmetric filtration generated by I_k for some $k \in \mathbb{N}$. Suppose G is a generating subset of I_k such that $S_{2k-1}G$ is a Gröbner basis of I_{2k-1} with respect to the invlex order, then,*

$$G_m = \begin{cases} (S_k G) \cap R_m, & \textit{if } m < k, \\ S_m G, & \textit{else}, \end{cases}$$

is a Gröbner basis of I_m with respect to any permutation of the lex order.

Proof. Clearly the sequence $(G_n)_{n\in\mathbb{N}_0}$ has the intersection property, thus for $m < 2k-1$, $G_m = G_{2k-1} \cap R_m$ and G_m is a Gröbner basis for $I_{2k-1} \cap R_m$ with respect to the invlex order since the invlex order is an elimination order of maximal type.

Also, if $k \leq m \leq 2k-1$, $I_m = \langle G_m \rangle = \langle G_{2k-1} \cap R_m \rangle = I_{2k-1} \cap R_m$, thus the filtration generated by I_{2k-1} is equal to $(I_n)_{\mathbb{N}_0}$. So, for $m \leq 2k-1$, G_m is a Gröbner basis for I_m with respect to the invlex order.

Let us consider the case when $m > 2k-1$. Given $f_1, f_2 \in G_m$, let U be the index set of $\text{supp}(f_1) \cup \text{supp}(f_2)$. Since $f_1 = \sigma_1 f_1'$ and $f_2 = \sigma_2 f_2'$ for some $f_1', f_2' \in G \subset I_k$ and $\sigma_1, \sigma_2 \in S_m$, we have that $|U| \leq 2k$. If $|U| = 2k$, Lemma 4.3.4 says that $S(f_1, f_2) \to_{G_m} 0$.

If $|U| < 2k$, let R_U be the polynomial ring with index set U and let $G_U = G_m \cap R_U$. Let $f_1, f_2 \in G_U$, then by Lemma 4.3.5 there are $f_1', f_2' \in G_{|U|}$ such that $f_1 = \tau(f_1')$ and $f_2 = \tau(f_2')$, for some $\tau \in \text{Inc}(\mathbb{N})$ with $\tau(|U|) = U$.

Since $|U| < 2k$, $G_{|U|}$ is a Gröbner basis of $I_{|U|}$ with respect to the invlex order, thus we have that $S(f_1', f_2') = \sum h_i g_i$, where $h_i \in R_{|U|}$ and $g_i \in G_{|U|}$ and $\text{lm}(S(f_1', f_2')) \geq \text{lm}(h_i g_i)\ \forall\ i$. Thus, since the invlex order is an $\text{Inc}(\mathbb{N})$-order, we have that

$$S(f_1, f_2) = \text{lcm}(\text{lm}(f_1), \text{lm}(f_2)) \left(\frac{f_1}{\text{lm}(f_1)} - \frac{f_2}{\text{lm}(f_2)} \right) =$$

$$\text{lcm}(\text{lm}(\tau(f_1')), \text{lm}(\tau(f_2'))) \left(\frac{\tau(f_1')}{\text{lm}(\tau(f_1'))} - \frac{\tau(f_2')}{\text{lm}(\tau(f_2'))} \right) =$$

$$\tau(\text{lcm}(\text{lm}(f_1'), \text{lm}(f_2')))\tau \left(\frac{f_1'}{\text{lm}(f_1')} - \frac{f_2'}{\text{lm}(f_2')} \right) =$$

$$\tau(S(f_1', f_2')) = \tau \left(\sum h_i g_i \right) = \sum \tau(h_i)\tau(g_i),$$

where $\tau(h_i) \in R_U$ and $\tau(g_i) \in G_U$.

And since the invlex order is an $\text{Inc}(\mathbb{N})$-order, we have that $\text{lm}(S(f_1, f_2)) = \text{lm}(\tau(S(f_1', f_2'))) \geq \tau(\text{lm}(h_i g_i)) = \text{lm}(\tau(h_i)\tau(g_i))\ \forall\ i$. Thus $S(f_1, f_2) \to_{G_U} 0$ and therefore $S(f_1, f_2) \to_{G_m} 0$, so G_m is a Gröbner basis of I_m. Lastly, since G_m is symmetric for all m, it is a Gröbner basis with respect to any permutation of the lex order. $\square$

Given $\lambda \vdash n$ let G_λ denote the set $\{f_T | T \in \cup_{\substack{\mu \trianglelefteq \lambda, \\ \mu_1 = \lambda_1}} \text{Tab}(\mu)\}$ and let $\lambda^i = (\lambda_1 + i, \lambda_2, ..., \lambda_l)$, for $i \geq 0$. We are now in a position to state the following:

Corollary 4.3.1. *Let $\lambda \vdash n$ and suppose $G_{\lambda^{n-1}}$ is a Gröbner basis of $I_{\lambda^{n-1}}$ with respect to the invlex order. Then, for any permutation of the lex order, $G_\lambda \cap R_m$ is a Gröbner basis of $I_\lambda \cap R_m$ for $m < n$ and G_{λ^i} a Gröbner basis of I_{λ^i} for all $i \geq 0$.*

Proof. From Lemma 4.1.1 we know that $\langle S_m I_\lambda \rangle_{R_m} = I_{\lambda^{m-n}}$ and since G_{λ^0} is symmetric the statement follows from Theorem 4.3.2. □

Thus if we want to verify Conjecture 4.3.1 for a sequence of partitions, $\lambda^i \vdash n+i$ generate by some $\lambda \vdash n$, we just need to check it for one partition, namely λ^{n-1}. Note that by using Theorem 4.3.1 we can improve Corollary 4.3.1 by only consider the Specht polynomials corresponding to the standard tableaux (if we use the invlex order). Before looking at some examples where we make use of Corollary 4.3.1 and Theorem 4.3.1, we will show that verifying this criterion also means that restricting to the standard Specht polynomials gives us an equivariant Gröbner basis of the filtration.

We start by making the following observation:

Lemma 4.3.6. *Let $\lambda \vdash n$, then $SF_{\lambda^i} \subseteq \mathrm{Inc}_{n,n+i}(\mathbb{N})SF_\lambda$.*

Proof. We prove the inclusion by induction on i. Clearly the statement is true when $i = 0$, so we assume it is true for $i \geq 0$ and show that it is true for $i+1$.

Let $T \in \mathrm{StdTab}(\lambda^{i+1})$ and let m be the index in the rightmost box of the first row. If we cut off that box (and the index m) we are left with a tableau $T' \in \mathrm{StdTab}_{[n+i]\setminus m}(\lambda^i)$ and clearly $f_T = f_{T'}$. Let $T'' = \sigma T'$, where $\sigma = (n+i\ n+i-1\ \dots\ m) \in S_{n+i}$, then $T'' \in \mathrm{StdTab}(\lambda^i)$. Let $\tau \in \mathrm{Inc}_{n+i,n+i+1}(\mathbb{N})$ be the map defined by

$$\tau(j) = \begin{cases} j, & \text{if } j < m, \\ j+1, & \text{else.} \end{cases}$$

Clearly $\tau(T'') = T'$, thus $\tau(f_{T''}) = f_{T'} = f_T$ and $f_T \in \mathrm{Inc}_{n+i,n+i+1}(\mathbb{N})SF_{\lambda^i}$ which means that $SF_{\lambda^{i+1}} \subseteq \mathrm{Inc}_{n+i,n+i+1}(\mathbb{N})SF_{\lambda^i}$. By the induction hypothesis, $SF_{\lambda^i} \subseteq \mathrm{Inc}_{n,n+i}(\mathbb{N})SF_\lambda$, thus $SF_{\lambda^{i+1}} \subseteq \mathrm{Inc}_{n+i,n+i+1}(\mathbb{N})SF_{\lambda^i} \subseteq (\mathrm{Inc}_{n+i,n+i+1}(\mathbb{N}) \circ \mathrm{Inc}_{n,n+i}(\mathbb{N}))SF_\lambda$.

To show the final inclusion, let $\phi_1 \in \mathrm{Inc}_{n,n+i}(\mathbb{N})$, $\phi_2 \in \mathrm{Inc}_{n+i,n+i+1}(\mathbb{N})$ and $i \in [n]$. Then $\phi_1(i) \leq n+i$ and $\phi_2(\phi_1(i)) \leq n+i+1$, thus $\phi_2 \circ \phi_1 \in \mathrm{Inc}_{n,n+i+1}(\mathbb{N})$ and $(\mathrm{Inc}_{n+i,n+i+1}(\mathbb{N}) \circ \mathrm{Inc}_{n,n+i}(\mathbb{N})) \subseteq \mathrm{Inc}_{n,n+i+1}(\mathbb{N})$. Therefore we have that $SF_{\lambda^{i+1}} \subseteq (\mathrm{Inc}_{n+i,n+i+1}(\mathbb{N}) \circ \mathrm{Inc}_{n,n+i}(\mathbb{N}))SF_\lambda \subseteq \mathrm{Inc}_{n,n+i+1}(\mathbb{N})SF_\lambda$. □

Keeping the same notation as above, this tells us the following:

Corollary 4.3.2. *If $G_{\lambda^{n-1}}$ is a Gröbner basis of $I_{\lambda^{n-1}}$ with respect to the invlex order, then*

$$SG_\lambda := \{f_T | T \in \cup_{\substack{\mu \trianglelefteq \lambda, \\ \mu_1 = \lambda_1}} \mathrm{StdTab}(\mu)\}$$

is an equivariant Gröbner basis of the filtration of I_λ and thus the filtration has the intersection property.

Proof. Corollary 4.3.1 tells us that G_{λ^m} is a Gröbner basis of I_{λ^m} for $m \geq 0$ and Theorem 4.3.1 tells us that we can restrict to the standard tableaux. Lastly Lemma 4.3.6 tells us that the Inc($\mathbb{N}$)-orbits of SG_λ includes the Specht polynomials corresponding to the standard tableaux.

The second statement follows from Proposition 3.2.1. □

Let us finally use Corollary 4.3.1 and check Conjecture 4.3.1 for some filtrations. Since the conjecture has already been verified for all sequences generated by partitions of $1, 2$ and 3, we start with partitions of 4.

Example 4.3.2 (Partitions of 4)**.** *The only partition of* 4 *that we have not considered is* $(2, 2)$. *This one dominates the partition* $(2, 1, 1)$ *so we include those Specht polynomials. From Corollary 4.3.1 we know that to verify the conjectured Gröbner basis for any partition of the form* $(n-2, 2)$ *we need only check it for the partition* $(5, 2)$ *and we only need to consider tableaux of shape* $(5, 2)$ *and* $(5, 1, 1)$. *Also, if we use the invlex order, then according to Theorem 4.3.1 we can restrict our attention to the standard tableaux of these shapes.*

For practical purposes we will use the lex order and swap all the standard tableaux, T, with the tableaux $\tau(T)$, where $\tau = (1, n)(2, n-1) \cdots (i+1, n-i) \cdots \in S_n$. Note that this is equivalent to using invlex order and standard tableaux since "$\leq_{\tau,lex}$" is equal to "$\leq_{invlex}$". We run the following code on the computer system Magma:

```
//Compute the tableaux
S1:=StandardTableaux([6,2,1]);
S2:=StandardTableaux([6,1,1,1]);
T:=SetToIndexedSet(S1 join S2);

//Define the polynomial ring(default order is lex)
n:=9;
Q:=Rationals();
X:=PolynomialRing(Q,n);

/*Computes the Specht polynomial of the conjugate of the tableau
tau(T), where tau = (1 n)(2 n-1) ... (i+1 n-i)... in S_n.*/
```

```
specht:=function(T);
 j:=NumberOfRows(T);
 tmp:=X!1;
 for i in {1..j} do
  tmp2:=Row(T,i);
  for k in {1..#tmp2-1} do
   for l in {k+1..#tmp2} do
    K:=Integers()!tmp2[k];L:=Integers()!tmp2[l];
    tmp:=tmp*(X.(n+1-K)-X.(n+1-L));
   end for;
  end for;
 end for;
 return tmp;
end function;

/*Compute the Specht polynomials and check if it is
a Gröbner basis*/
G:=[specht(Conjugate(T[i])) : i in [1..#T]];
IsGroebner(G);
```

and the resulting answer is "true", that is, we have confirmed that $\{f_T | T \in \mathrm{StdTab}((5,2)) \cup \mathrm{StdTab}((5,1,1))\}$ *is a Gröbner basis for* $I_{(5,2)}$*. Thus the conjecture is verified for* $I_{(n-2,2)}$ *for all* $n \geq 4$ *and for all* σ*-lex orderings.*

Example 4.3.3 (Partitions of 5)**.** *We will do the same thing as in the previous example for the sequences generated by Specht ideals of partitions of* 5*. The only one that we have not considered is* $(2,2,1)$*. This partition dominates* $(2,1^3)$*. Thus we run the above code again, we just change* n *to be* 9 *and the partitions to be* $(6,2,1)$ *and* $(6,1^3)$*.*

The result is again "true", thus the conjecture holds for the ideals $I_{(2,2,1)}$*.*

Example 4.3.4 (Partitions of 6)**.** *If we apply the method above for the partition* $(3,3)$*, we find that the Specht polynomials of* $(n-3,3),(n-3,2,1)$ *and* $(n-3,1^3)$ *is a Gröbner basis of the ideal* $I_{(n-3,3)}$ *for all* $n \geq 6$*. However, when we considered the other partitions of* 6 *that we have left (partitions* (2^3) *and* $(2,2,1,1)$*), then the computation time exceeded a week and was aborted.*

Example 4.3.4 shows that the method above has its limitations. However, it is worth pointing out that we can use Theorem 4.3.2 for other ideals than the Specht ideals. Thus if a filtration is generated by I_k, for small k, and we have a suggestion for an equivariant Gröbner basis contained in I_k, then we could use the method above to verify our suggestion.

4.4. Gröbner basis criterion: partition $(n-k,k)$

In this section we give a new criterion to verify Conjecture 4.3.1. The criterion might be more efficient than the one we introduced in Corollary 4.3.1, but only applicable to partitions of the form $(n-k,k)$. We will also be applying the new criterion to verify the conjecture for the partition $(n-k,k)$, for some values of k.

Recall that to establish Hilbert series recurrence relation for the partitions of the form $(n-k,1^k)$, we first verified the Gröbner basis conjecture. The main approach in this section is exactly opposite. That is, since we know the Hilbert series of partitions of the form $(n-k,k)$, we can use this to verify the conjecture.

As was mentioned at the end of section 4.1, we have some recurrence relations for the Specht ideals of shape $(n-k,k)$.

Lemma 4.4.1. *Let $n > 2k \geq 2$, then we have the following relation:*

$$H_{R_n/I_{(n-k,k)}}(t) = H_{R_{n-1}/I_{(n-k-1,k)}}(t) + \frac{t}{1-t} H_{R_{n-1}/I_{(n-k,k-1)}}(t).$$

For the partition (k,k) with $k > 1$, we have the following relation:

$$H_{R_{2k}/I_{(k,k)}}(t) = H_{R_{2k-1}/I_{(k-1,k-1,1)}}(t) + \frac{t}{1-t} H_{R_{2k-1}/I_{(k,k-1)}}(t).$$

Proof. See Corollary 5.4 in [21]. □

We will now proceed to give the same recurrence relations for the leading monomials of a subset of the conjectured Gröbner basis. Then we can use this to give a criterion to verify Conjecture 4.3.1 for the partitions of the form $(n-k,k)$.

We will be using the invlex order and define the subsets as follows: Given the partition $(k,k) \vdash 2k \geq 2$, we define

$$B_{(k,k)} := \{f_T | T \in \cup_{\substack{\lambda \trianglelefteq (k,k) \\ \lambda_1 = k}} \mathrm{StdTab}(\lambda) \text{ and } x_{2k}^2 \nmid \mathrm{lm}(f_T)\},$$

and for $n > 2k$,

$$B_{(n-k,k)} := B_{(n-k-1,k)} \cup$$
$$\{f_T | T \in \cup_{\substack{\lambda \trianglelefteq (n-k,k) \\ \lambda_1 = n-k}} \mathrm{StdTab}(\lambda), \text{ and } x_n | \mathrm{lm}(f_T),\ x_n^2 \nmid \mathrm{lm}(f_T)\}.$$

For the partition $(k,k,1)$, with $k \geq 1$, we define

$$B_{(k,k,1)} := \{f_T | T \in \cup_{\substack{\lambda \trianglelefteq (k,k,1) \\ \lambda_1 = k}} \mathrm{StdTab}(\lambda)\}.$$

Thus we have that $B_{(n-k,k)}$ is a subset of the conjectured Gröbner basis of $I_{(n-k,k)}$ and that $B_{(k,k,1)}$ is a subset of the conjectured Gröbner basis of $I_{(k,k,1)}$. Also, since the standard Specht polynomials of shape $(n-k,k)$ is a subset of $B_{(n-k,k)}$, we have that $\langle B_{(n-k,k)}\rangle = I_{(n-k,k)}$ and similarly we have that $\langle B_{(k,k,1)}\rangle = I_{(k,k,1)}$.

Once the recurrence relations of the Hilbert series have been established for the ideals $\langle \mathrm{lm}(B_{(n-k,k)})\rangle$ and $\langle \mathrm{lm}(B_{k,k,1})\rangle$, then we can use this to show the following:

Theorem 4.4.1. *Let $k > 1$, then if $B_{(d,d,1)}$ is a Gröbner basis for the ideal $I_{(d,d,1)}$ for all $d \in [k]$, then $B_{(n-d,d)}$ is a Gröbner basis for $I_{(n-d,d)}$ for any $n \geq 2d$ and any $d \in [k+1]$.*

We start by noting the following relations:

Lemma 4.4.2. *Let $k > 1$ and $n > 2k$, then*

$$\mathrm{lm}(B_{(n-k,k)}) = \mathrm{lm}(B_{(n-k-1,k)}) \cup x_n \,\mathrm{lm}(B_{(n-k,k-1)}),$$

and

$$\mathrm{lm}(B_{(k,k)}) = \mathrm{lm}(B_{(k-1,k-1,1)}) \cup x_{2k} \,\mathrm{lm}(B_{(k,k-1)}).$$

Proof. The first equality is clear from the construction of $B_{(n-k,k)}$, so we only have to show the second equality.

Firstly we show that $B_{(k,k)} \cap R_{2k-1} = B_{(k-1,k-1,1)}$. Clearly $B_{(k-1,k-1,1)}$ is a subset of both R_{2k-1} and $B_{(k,k)}$, so we show the opposite inclusion. Let $f_T \in B_{(k,k)} \cap R_{2k-1}$, then $T \in \mathrm{StdTab}(\lambda)$ with $\lambda \lhd (k,k)$ and $\lambda_1 = k$ and $\lambda_2 < k$. The index $2k$ must be in the rightmost box of the first row so we can cut away this box and get a standard tableaux, S, of the partition $\lambda' = (\lambda_1 - 1, \lambda_2, ..., \lambda_l)$ such that $f_T = f_S$. Since $\lambda'_1 = k-1$, we just have to check that $\lambda' \trianglelefteq (k-1,k-1,1)$.

We know that $\lambda_1 = k$ and $\lambda_2 < k$. Thus $\lambda'_1 = k-1$ and $\lambda'_2 \leq k-1$ and therefore we have that $\lambda_3 \geq 1$. Thus $\lambda' \trianglelefteq (k-1,k-1,1)$ and $f_T = f_S \in B_{(k-1,k-1,1)}$ That is, we know that $B_{k,k} \cap R_{2k-1} \subseteq B_{(k-1,k-1,1)}$ and hence that $B_{(k,k)} \cap R_{2k-1} = B_{(k-1,k-1,1)}$. The next thing to show is that $D := \mathrm{lm}(B_{(k,k)}) \setminus \mathrm{lm}(B_{(k-1,k-1,1)}) =$ $x_{2k}\,\mathrm{lm}(B_{(k,k-1)})$. We start with the inclusion $D \subseteq x_{2k}\,\mathrm{lm}(B_{(k,k-1)})$.

Let $\mathrm{lm}(f_T) \in D$ for some $T \in \mathrm{StdTab}(\lambda)$, where $\lambda \trianglelefteq (k,k)$ and $\lambda_1 = k$. Since $x_{2k} | \mathrm{lm}(f_T)$ and $x_{2k}^2 \not| \mathrm{lm}(f_T)$, then the index $2k$ lies in the rightmost box of the second row of T and in a column of length 2. Thus if we remove this box we get a standard tableau S of the partition $\lambda' = (\lambda_1, \lambda_2 - 1, \lambda_3, ..., \lambda_l)$ such that $\mathrm{lm}(f_T) = x_{2k}\,\mathrm{lm}(f_S)$. Since the partition λ' is constructed by removing 1 from the second part of the partition λ and $(k, k-1)$ can be constructed from (k,k) the same way, we have that $\lambda' \trianglelefteq (k, k-1)$. Therefore $D \subseteq x_{2k}\,\mathrm{lm}(B_{(k-1,k-1,1)})$.

The opposite inclusion is attained by following the exact same procedure backwards. Thus $D = x_{2k}\,\mathrm{lm}(B_{(k-1,k-1,1)})$ and therefore $\mathrm{lm}(B_{(k,k)}) = \mathrm{lm}(B_{(k-1,k-1,1)}) \cup$ $x_{2k}\,\mathrm{lm}(B_{(k,k-1)})$. ∎

This gives us a way of writing the ideals of leading monomials as a sum of ideals of leading monomials. To take advantage of this, we also need to determine what the intersections of the ideals in the right side of the equations in Lemma 4.4.2 is. For this we have the following lemma:

Lemma 4.4.3. *Let $k > 1$ and $n > 2k$, then*

$$\langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n} \cap x_n \langle \mathrm{lm}(B_{(n-k,k-1)}) \rangle_{R_n} = x_n \langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n},$$

and

$$\langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}} \cap x_{2k} \langle \mathrm{lm}(B_{(k,k-1)}) \rangle_{R_{2k}} = x_{2k} \langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}}.$$

Proof. From Proposition 1.3.2 we know that the intersections are generated by the least common multiples of the generators of the ideals. Thus we will look at these generators.

If $m \in \mathrm{lm}(B_{(n-k-1,k)})$ and $n \in x_n \,\mathrm{lm}(B_{(n-k,k-1)})$, then we know that $x_n \nmid m$ and $n = x_n r$ for some $r \in \mathrm{lm}(B_{(n-k,k-1)})$. Thus $\mathrm{lcm}(m,n) = x_n \,\mathrm{lcm}(m,r)$. This means that

$$\langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n} \cap x_n \langle \mathrm{lm}(B_{(n-k,k-1)}) \rangle_{R_n} =$$

$$x_n(\langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n} \cap \langle \mathrm{lm}(B_{(n-k,k-1)}) \rangle_{R_n}).$$

Similarly we can show that

$$\langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}} \cap x_{2k} \langle \mathrm{lm}(B_{(k,k-1)}) \rangle_{R_{2k}} =$$

$$x_{2k}(\langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}} \cap \langle \mathrm{lm}(B_{(k,k-1)}) \rangle_{R_{2k}}).$$

Thus we can focus on the intersections

$$\langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n} \cap \langle \mathrm{lm}(B_{(n-k,k-1)}) \rangle_{R_n}$$

and

$$\langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}} \cap \langle \mathrm{lm}(B_{(k,k-1)}) \rangle_{R_{2k}}.$$

Let $\mathrm{lm}(f_T) \in \langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n}$ for some $T \in \mathrm{StdTab}(\lambda)$, where $\lambda \trianglelefteq (n-k-1,k)$. Then $\mathrm{lm}(f_T)$ is divisible by $\mathrm{lm}(f_S)$ for some $S \in \mathrm{StdTab}(\lambda')$, where $\lambda' = (\lambda_1 + 1, \lambda_2, ..., \lambda_l - 1)$. We will now show that $\lambda' \trianglelefteq (n-k, k-1)$. If this is not the case, then $\lambda'_2 > k-1$. Since $\lambda'_1 = n-k$, then $\lambda'_2 = k$ which means that λ' is a partition of n. This is a contradiction since λ is a partition of $n-1$ and we didn't add anything to λ when we constructed λ'.

Thus we know that $\mathrm{lm}(f_T) \in \langle B_{(n-k,k-1)} \rangle_{R_n}$ and we therefore have that $\langle B_{(n-k-1,k)} \rangle_{R_n} \subseteq \langle B_{(n-k,k-1)} \rangle_{R_n}$. This means that the first intersection is

$$\langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n} \cap \langle \mathrm{lm}(B_{(n-k,k-1)}) \rangle_{R_n} = \langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n}.$$

We can follow the same line of reasoning to show that

$$\langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}} \cap \langle \mathrm{lm}(B_{(k,k-1)}) \rangle_{R_{2k}} = \langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}}.$$

Thus we get that

$$\langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n} \cap x_n \langle \mathrm{lm}(B_{(n-k,k-1)}) \rangle_{R_n} =$$

$$x_n(\langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n} \cap \langle \mathrm{lm}(B_{(n-k,k-1)}) \rangle_{R_n}) = x_n \langle \mathrm{lm}(B_{(n-k-1,k)}) \rangle_{R_n},$$

and that

$$\langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}} \cap x_{2k} \langle \mathrm{lm}(B_{(k,k-1)}) \rangle_{R_{2k}} =$$

$$x_{2k}(\langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}} \cap \langle \mathrm{lm}(B_{(k,k-1)}) \rangle_{R_{2k}}) = x_{2k} \langle \mathrm{lm}(B_{(k-1,k-1,1)}) \rangle_{R_{2k}}.$$

□

Before we can use this to verify the recurrence relation for the Hilbert series we need one more result.

Lemma 4.4.4. *Let I be a nonzero ideal of R_n for $n > 0$ and let $i \in [n]$ and $j \in [n+1]$. Then the following is true:*

$$H_{R_n/x_i I}(t) = \frac{1}{(1-t)^{n-1}} + tH_{R_n/I}(t),$$

$$H_{R_{n+1}/\langle I \rangle_{R_{n+1}}}(t) = \frac{1}{1-t} H_{R_n/I}(t),$$

$$H_{R_{n+1}/x_j \langle I \rangle_{R_{n+1}}}(t) = \frac{1}{(1-t)^n} + \frac{t}{1-t} H_{R_n/I}(t).$$

Proof. For the first equality, note that $H_{x_i I}(t) = tH_I(t)$ since all the polynomials have increased their degree by one when multiplied with x_i. Thus,

$$H_{R_n/x_i I}(t) = H_{R_n}(t) - H_{x_i I}(t) = H_{R_n}(t) - tH_I(t) =$$

$$H_{R_n}(t) - t\Big(H_{R_n}(t) - H_{R_n/I}(t) \Big) = (1-t)H_{R_n}(t) + tH_{R_n}(t) =$$

$$\frac{1}{(1-t)^{n-1}} + tH_{R_n/I}(t).$$

For second equality, note that since x_{n+1} is a non-zero divisor of $R_{n+1}/\langle I\rangle_{R_{n+1}}$ we can construct the following exact sequence:

$$0 \longrightarrow (R_{n+1}/\langle I\rangle_{R_{n+1}})(1) \xrightarrow{\times x_{n+1}} R_{n+1}/\langle I\rangle_{R_{n+1}} \xrightarrow{q} R_{n+1}/\langle I, x_{n+1}\rangle_{R_{n+1}} \longrightarrow 0,$$

where q is the quotient map. The last ring, $R_{n+1}/\langle I, x_{n+1}\rangle_{R_{n+1}}$, is isomorphic to R_n/I, thus, by Proposition 1.3.3, we get that

$$(1-t)H_{R_{n+1}/\langle I\rangle_{R_{n+1}}} = H_{R_n/I} \Rightarrow H_{R_{n+1}/\langle I\rangle_{R_{n+1}}}(t) = \frac{1}{1-t}H_{R_n/I}(t).$$

The last equality comes from applying the two first:

$$H_{R_{n+1}/x_j\langle I\rangle_{R_{n+1}}}(t) = \frac{1}{(1-t)^n} - tH_{R_{n+1}/\langle I\rangle_{R_{n+1}}}(t) =$$

$$\frac{1}{(1-t)^n} + \frac{t}{1-t}H_{R_n/I}(t).$$

□

Now we have enough tools to show the following:

Lemma 4.4.5. *If $k > 1$ and $n > 2k$, then*

$$H_{R_n/\langle \mathrm{lm}(B_{(n-k,k)})\rangle}(t) = H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle}(t) + \frac{t}{1-t}H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle}(t)$$

and

$$H_{R_{2k}/\langle \mathrm{lm}(B_{(k,k)})\rangle}(t) = H_{R_{2k-1}/\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle}(t) + \frac{t}{1-t}H_{R_{2k-1}/\langle \mathrm{lm}(B_{(k,k-1)})\rangle}(t).$$

Proof. We start with the first equality. From Lemma 4.4.2 we know that $\langle \mathrm{lm}(B_{(n-k,k)})\rangle = \langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_n} + x_n\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle_{R_n}$ and from Lemma 4.4.3 we know that $\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_n} \cap x_n\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle_{R_n} = x_n\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_n}$. Thus,

$$H_{R_n/\langle \mathrm{lm}(B_{(n-k,k)})\rangle}(t) = H_{R_n/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_n}}(t) +$$

$$H_{R_n/x_n\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle_{R_n}}(t) - H_{R_n/x_n\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_n}}(t).$$

From Lemma 4.4.4 we get that

$$H_{R_n/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_n}}(t) = \frac{1}{1-t}H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_{n-1}}}(t),$$

$$H_{R_n/x_n\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle_{R_n}}(t) = \frac{1}{(1-t)^{n-1}} + \frac{t}{1-t}H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle_{R_{n-1}}}(t)$$

and

$$H_{R_n/x_n\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_n}}(t) = \frac{1}{(1-t)^{n-1}} + \frac{t}{1-t} H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_{n-1}}}(t).$$

Thus,

$$H_{R_n/\langle \mathrm{lm}(B_{(n-k,k)})\rangle}(t) = \frac{1}{1-t} H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_{n-1}}}(t) +$$
$$\frac{1}{(1-t)^{n-1}} + \frac{t}{1-t} H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle_{R_{n-1}}}(t) -$$
$$\frac{1}{(1-t)^{n-1}} - \frac{t}{1-t} H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_{n-1}}}(t) =$$
$$H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k-1,k)})\rangle_{R_{n-1}}}(t) + \frac{t}{1-t} H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-k,k-1)})\rangle_{R_{n-1}}}(t).$$

Similarly for the second statement we have that

$$\langle \mathrm{lm}(B_{(k,k)})\rangle_{R_{2k}} = \langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k}} + x_{2k}\langle \mathrm{lm}(B_{(k,k-1)})\rangle_{R_{2k}},$$

due to Lemma 4.4.2. And, due to Lemma 4.4.3, we have that

$$\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k}} \cap x_{2k}\langle \mathrm{lm}(B_{(k,k-1)})\rangle_{R_{2k}} = x_{2k}\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k}}.$$

From Lemma 4.4.4 we know that

$$H_{R_{2k}/\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k}}}(t) = \frac{1}{1-t} H_{R_{2k-1}/\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k-1}}}(t),$$

$$H_{R_{2k}/x_{2k}\langle \mathrm{lm}(B_{(k,k-1)})\rangle_{R_{2k}}}(t) = \frac{1}{(1-t)^{2k-1}} + \frac{t}{1-t} H_{R_{2k-1}/\langle \mathrm{lm}(B_{(k,k-1)})\rangle_{R_{2k-1}}}(t),$$

$$H_{R_{2k}/x_{2k}\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k}}}(t) = \frac{1}{(1-t)^{2k-1}} + \frac{t}{1-t} H_{R_{2k-1}/\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k-1}}}(t).$$

Thus

$$H_{R_{2k}/\langle \mathrm{lm}(B_{(k,k)})\rangle}(t) = H_{R_{2k}/\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k}}}(t) +$$
$$H_{R_{2k}/x_{2k}\langle \mathrm{lm}(B_{(k,k-1)})\rangle_{R_{2k}}}(t) - H_{R_{2k}/x_{2k}\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k}}}(t) =$$
$$H_{R_{2k-1}/\langle \mathrm{lm}(B_{(k-1,k-1,1)})\rangle_{R_{2k-1}}}(t) + \frac{t}{1-t} H_{R_{2k-1}/\langle \mathrm{lm}(B_{(k,k-1)})\rangle_{R_{2k-1}}}(t).$$

□

Having established the recurrence relation, we can proceed to prove theorem 4.4.1.

Proof of Theorem 4.4.1. We will do a nested inductive proof on n and d where $d = 1$ and $n > 1$ is our initial step. For $d = 1$, $B_{(n-1,1)} = \{f_T | T\, \mathrm{StdTab}((n-1,1))\}$, and we already know from Theorem 4.2.1 that this is a Gröbner basis of $I_{(n-1,1)}$.

So suppose the theorem is true for the partition $(m-l, l)$ for some $l \in [k]$ and any $m \geq 2l$, let us show it is true for $(2d, d)$ when $d = l + 1$.

We know from Lemma 4.4.5 that

$$H_{R_{2d}/\langle \mathrm{lm}(B_{(d,d)})\rangle}(t) = H_{R_{2d-1}/\langle \mathrm{lm}(B_{(d-1,d-1,1)})\rangle}(t) + \frac{t}{1-t} H_{R_{2d-1}/\langle \mathrm{lm}(B_{(d,d-1)})\rangle}(t)$$

and from Lemma 4.4.1 that

$$H_{R_{2d}/I_{(d,d)}}(t) = H_{R_{2d-1}/I_{(d-1,d-1,1)}}(t) + \frac{t}{1-t} H_{R_{2d-1}/I_{(d,d-1)}}(t).$$

Since we know that $B_{(d-1,d-1,1)}$ is a Gröbner basis for $I_{(d-1,d-1,1)}$ and, due to the inductive hypothesis, that $B_{(d,d-1)}$ is a Gröbner basis for $I_{(d,d-1)}$, then

$$H_{R_{2d-1}/\langle \mathrm{lm}(B_{(d-1,d-1,1)})\rangle}(t) = H_{R_{2d-1}/I_{(d-1,d-1,1)}}(t)$$

and

$$H_{R_{2d-1}/\langle \mathrm{lm}(B_{(d,d-1)})\rangle}(t) = H_{R_{2d-1}/I_{(d,d-1)}}(t).$$

Thus we get that

$$H_{R_{2d}/\langle \mathrm{lm}(B_{(d,d)})\rangle}(t) = H_{R_{2d}/I_{(d,d)}}(t),$$

and since $B_{(d,d)}$ is a subset of $I_{(d,d)}$, it is a Gröbner basis of $I_{(d,d)}$.

Now, assuming the theorem is true for the partition $(m-l, l)$ for some $l \geq 1$ and any $m \geq 2l$ and also for the partition $(m-d, d)$, for some $m \geq 2d$ where $d = l+1$, then we show that it is true for the partition $(n-d, d)$, where $n = m+1$.

We know, by Lemma 4.4.5, that

$$H_{R_n/\langle \mathrm{lm}(B_{(n-d,d)})\rangle}(t) = H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-d-1,d)})\rangle}(t) + \frac{t}{1-t} H_{R_{n-1}/\langle \mathrm{lm}(B_{(n-d,d-1)})\rangle}(t),$$

and from our inductive hypothesis we know that $B_{(n-d,d-1)}$ and $B_{(n-d-1,d)}$ are Gröbner bases of $I_{(n-d,d-1)}$ and $I_{(n-d-1,d)}$ respectively. Thus

$$H_{R_n/\langle \mathrm{lm}(B_{(n-d,d)})\rangle}(t) = H_{R_{n-1}/I_{(n-d-1,d)}}(t) + \frac{t}{1-t} H_{R_{n-1}/I_{(n-d,d-1)}}(t) =$$

$$H_{R_n/I_{(n-d,d)}}(t),$$

where the last equality follows from Lemma 4.4.1. Since $B_{(n-d,d)}$ is a subset of $I_{(n-d,d)}$ with the same Hilbert series, we know that $B_{(n-d,d)}$ is a Gröbner basis of $I_{(n-d,d)}$. $\square$

Thus we can use Theorem 4.4.1 to verify the conjectured Gröbner basis for the Specht ideals of shape $(n-k,k)$. It is also worth noting that if one is able to verify that $B_{(k,k,1)}$ is a Gröbner basis of $I_{(k,k,1)}$ for any $k \geq 1$, then it would also verify the conjecture for the ideals $(n-k,k)$. But for now let us use this criterion for some values of k.

Example 4.4.1 ($k=2$)**.** *We have already verified that $B_{(d,d,1)}$ is a Gröbner basis for $I_{d,d,1}$ when $d \in [2]$, thus $B_{n-d,d}$ is a Gröbner basis of $I_{(n-d,d)}$ when $d \in [3]$. Note that we verified Conjecture 4.3.1 for these ideals in the previous section for $d \in \{2,3\}$, but $B_{(n-d,d)}$ is a strict subset of the conjectured Gröbner basis, so this is a slightly stronger result.*

Example 4.4.2 ($k=3$)**.** *Let us look at the case when $k=3$. Then we have that $B_{(3,3,1)} = \{f_T | T \in \mathrm{StdTab}(3,3,1) \cup \mathrm{StdTab}(3,2,2) \cup \mathrm{StdTab}(3,2,1,1) \cup \mathrm{StdTab}(3,1^4)\}$. To verify that this is a Gröbner basis we use the same code as in Example 4.3.2 with the partitions $(3,3,1),(3,2,2)$, $(3,2,1,1)$ and $(3,1^4)$ as input.*

```
//Compute the tableaux
S1:=StandardTableaux([3,3,1]);
S2:=StandardTableaux([3,2,2]);
S3:=StandardTableaux([3,2,1,1]);
S4:=StandardTableaux([3,1,1,1,1]);
T:=SetToIndexedSet(S1 join S2 join S3 join S4);

//Define the polynomial ring(default order is lex)
n:=7;
Q:=Rationals();
X:=PolynomialRing(Q,n);

/*Computes the Specht polynomial of the conjugate of the tableau
tau(T), where tau = (1 n)(2 n-1) ... (i+1 n-i)... in S_n.*/
specht:=function(T);
 j:=NumberOfRows(T);
 tmp:=X!1;
 for i in {1..j} do
  tmp2:=Row(T,i);
  for k in {1..#tmp2-1} do
   for l in {k+1..#tmp2} do
    K:=Integers()!tmp2[k];L:=Integers()!tmp2[l];
    tmp:=tmp*(X.(n+1-K)-X.(n+1-L));
   end for;
  end for;
 end for;
```

```
 return tmp;
end function;

//Compute the Specht polynomials and check if it is
a Gröbner basis*/
G:=[specht(Conjugate(T[i])) : i in [1..#T]];
IsGroebner(G);
```

The result is "true", that is $B_{(3,3,1)}$ is a Gröbner basis of $I_{(3,3,1)}$ and thus $B_{(n-d,d)}$ is a Gröbner basis of $I_{(n-d,d)}$ $\forall\ n \geq 2d$ and $d \in [4]$.

When we tried doing the same for the partition $(4, 4, 1)$, the computation time exceeded a week and we therefore aborted it.

4.5. Equivariant Hilbert series

We round off this chapter by exploring the equivariant Hilbert series a bit more and comment on the results of this chapter.

Recall from Chapter 1 that the rationality of the Hilbert series is equivalent to the existence of the Hilbert polynomial. Thus it is natural to ask if we can make a similar statement for the infinite case. For instance, an interesting question would be whether $H_{R_n/I_n}(t)$ is eventually polynomial in the variable n if we allow the coefficients to be rational functions in t. We can give an answer to that question if we adjust the proof of Lemma 1.3.1 slightly. But first we define what we mean by a rational function.

Let S be a nonzero commutative ring where for any $s, r \in S$ such that $s * r = 0$, we have that either s or r is zero, then we call S an *integral domain*. When we have an integral domain, S, we can construct the *field of fractions* of S. This is the smallest field that S can be embedded in and can be constructed as follows: Let $L = \{(s, r) | s, r \in S \ \&\ r \neq 0\}$ and equip L with the equivalence relation $(s, r) \sim (s', r')$ if $sr' = s'r$. Then the field of fractions of S is the field $L/\sim$ with the product defined as $(s, r) * (s', r') = (ss', rr')$ and the addition as $(s, r) + (s', r') = (sr' + s'r, rr')$. Note that we often write the elements (s, r) as $\frac{s}{r}$.

The simplest example of a fraction field is to start with the integral domain $\mathbb{Z}$ and construct the field $\mathbb{Q}$ by the method above. We will be considering the fraction field $\mathbb{C}(t) := \{\frac{p(t)}{q(t)} | p(t), q(t) \in \mathbb{C}[t]\}$, but if a more thorough introduction is needed, please see [8], Chapter 7.5.

We start with a couple of lemmas before we go to the main statement.

Lemma 4.5.1. *Let $(I_n)_{n\in\mathbb{N}_0}$ be a symmetric filtration, then there exist an integer $k \in \mathbb{N}_0$ such that the Krull dimension* $\mathrm{Dim}(R_n/I_n)$ *is a polynomial of the form $An + B$ for $n > k$, where $A = 0$ or $A = 1$, and $B \in \mathbb{N}_0$.*

Proof. See [17] Theorem 7.9. □

Lemma 4.5.2. *A sequence, $(p(n))_{n\in\mathbb{N}_0}$, of complex polynomials of $\mathbb{C}[t]$ is given by a polynomial in n of degree $\leq d$ if and only if*

$$\sum_{n\geq 0} p(n)x^n = \frac{h(x)}{(1-x)^{d+1}},$$

for some polynomial $h(x) \in (\mathbb{C}[t])[x]$ of degree $\leq d$.

Proof. See appendix. □

Before the main result, note that when we say eventually polynomial, it will mean eventually polynomial with coefficients in $\mathbb{C}(t)$. Then we can state the following:

Proposition 4.5.1. *Let $(I_n)_{n\in\mathbb{N}_0}$ be a filtration of ideals and let $I = \cup_{n\in\mathbb{N}_0}$, then the sequence of Hilbert series of R_n/I_n is eventually polynomial and has eventually a constant Krull dimension if and only if the equivariant Hilbert series of R/I can be written on the form $\frac{g(s,t)}{(1-t)^d(1-s)^k}$, where $g(s,t) \in \mathbb{C}[s,t]$ and $d, k \in \mathbb{N}_0$.*

Proof. We start by letting $m \in \mathbb{N}_0$ be the smallest index such that $H_{R_n/I_n}(t)$ is the polynomial $f(n) \in (\mathbb{C}(t))[n]$ and the Krull dimension is the constant B, for all $n > m \in \mathbb{N}_0$.

Thus we have that $f(n)(1-t)^B = p(n)$ is a polynomial in $(\mathbb{C}[t])[n]$ for $n > m$. Therefore we can write the the equivariant Hilbert series as

$$H_{R/I}(s,t) = \sum_{n=0}^{m} H_{R_n/I_n}(t)s^n + \frac{1}{(1-t)^B}\sum_{n>m} p(n)s^n =$$

$$\sum_{n=0}^{m}\left(H_{R_n/I_n}(t) - \frac{p(n)}{(1-t)^B}\right)s^n + \frac{1}{(1-t)^B}\sum_{n\geq 0} p(n)s^n.$$

Now we can apply Lemma 4.5.2 and rewrite the last sum:

$$H_{R/I}(s,t) = \sum_{n=0}^{m}\left(H_{R_n/I_n}(t) - \frac{p(n)}{(1-t)^B}\right)s^n + \frac{1}{(1-t)^B}\frac{q(s,t)}{(1-s)^k},$$

where $q(s,t) \in \mathbb{C}[s,t]$. Thus, by gathering all the (finite) terms in one fraction, we get that

$$H_{R/I}(s,t) = \frac{g(s,t)}{(1-t)^d(1-s)^k}.$$

To go the other way, we simply use the multivariate division algorithm (see Theorem 1.2.1) to write $g(s,t)$ as $q(s,t)(1-s)^k + r(s,t)$, where the degree of $r(s,t)$ in s is less than k or $r(s,t) = 0$. Thus we can write the equivariant Hilbert series as

$$H_{R/I}(s,t) = \frac{q(s,t)}{(1-t)^d} + \frac{r(s,t)}{(1-t)^d(1-s)^k}.$$

If $r(s,t) = 0$, then, since the degree of $q(s,t)$ in s is finite we have that $H_{R_n/I_n}(t) = 0$ for large enough n. Thus the sequence of Hilbert series is eventually the zero polynomial and the Krull dimension is zero.

If $r(s,t) \neq 0$, then, since the degree of $r(s,t)$ in s is less than or equal to k, we can use Lemma 4.5.2 to write the second fraction as $\frac{1}{(1-t)^d}\sum_{n\geq 0} p(n)s^n$, with $p(n) \in (\mathbb{C}[t])[n]$. If we let m be the degree of $q(s,t)$ in s, we can write

$$H_{R/I}(s,t) = \frac{q(s,t)}{(1-t)^d} - \frac{1}{(1-t)^d}\sum_{n\leq m} p(n)s^n + \frac{1}{(1-t)^d}\sum_{n>m} p(n)s^n.$$

It is only the last part of the expression above where s occurs to any power greater than m, and thus $\frac{1}{(1-t)^d}\sum_{n>m} p(n)s^n = \sum_{n>m} H_{R_n/I_n}(t)s^n$, and so the Hilbert series is the polynomial $\frac{p(n)}{(1-t)^d}$ for $n > m$. Since $p(n) \in (\mathbb{C}[t])[n]$, d provides a constant bound on the Krull dimension. Since we know, by Lemma 4.5.1, that the Krull dimension is eventually of the form $An + B$, then we know that $A = 0$. □

Since this is not the general form of the equivariant Hilbert series, Proposition 4.5.1 tells us that we cannot generally expect both the Krull dimension to be constant and the Hilbert series to be eventually polynomial. Thus, we were quite lucky when we calculated the Hilbert series of the Specht ideals corresponding to the partitions $(n-k,k)$, since it turned out the Hilbert series were eventually polynomial and of constant Krull dimension. This made it easier for us to recognize the pattern that presented itself.

Conversely, when we considered the Hilbert series of the Specht ideals of shape $(n-k,1^k)$, we didn't get equivariant Hilbert series of the form $\frac{g(s,t)}{(1-t)^d(1-s)^m}$. Thus, since the Krull dimension was constant, we would not have been able to express the Hilbert series as a polynomial. Instead we got expressions like,

$$H_{R_n/I_{(n-2,1^2)}}(t) = \frac{(1-t)^{n-2} + t^2\sum_{i=1}^{n-3}(1+t)^{i-1}}{(1-t)^2},$$

which has a function to the power of n and an increasing number of terms in the numerator. That is, a pattern that is much harder to recognize. This could also go some way to explain why it was difficult (and unsuccessful) to find a general expression for the Hilbert series of the Specht ideals of shape $(n-k, 1^k)$.

Another interesting thing to note is that the Hilbert series of the polynomial ring is not polynomial (or of constant Krull dimension). Thus, even if the sequence of Hilbert series of R_n/I_n, for some filtration $(I_n)_{n\in\mathbb{N}_0}$, is eventually polynomial, then the sequence of Hilbert series of I_n will not be eventually polynomial.

Chapter V / Further research

In this chapter we look at a type of exact sequence called a free resolution. As we have seen before, exact sequences can be nice tools to use when determining the Hilbert series of an ideal. Similarly, resolutions are much used in algorithms for computing the Hilbert series of an ideal, so they are an important class of exact sequences.

Associated to free resolutions we have an invariant that is called the Castlenuovo-Mumford regularity. This is an invariant that gives a bound for the largest degree of a generator in a free resolution. Thus, in some sense, in provides a description of the width of a free resolution. Similarly the projective dimension of an ideal describes the minimal length of a free resolution. Thus the projective dimension and regularity combined describes the minimal size of a free resolution.

Since we are considering filtrations of ideals, studying how the projective dimension and regularity evolve with the increase in variables would give us information on how the complexity of free resolutions evolve with the increase in variables.

We will start this chapter by introducing resolutions and give some standard results on the regularity and projective dimension of an ideal. Then we finish by saying a few words on how this relates to the Specht ideals and present some open questions about Specht ideals and symmetric filtrations.

5.1. Free resolutions and the Castelnuovo-Mumford regularity

Resolutions are constructed using modules, thus before we can introduce resolutions we need some definitions regarding the different types of modules that we have.

Definition 5.1.1. *Let A be a ring. An A-module, M, is*

- *a free A-module if it can be generated by a set of elements that are linearly independent over A, and*
- *a graded A-module if $A_iM_j \subseteq M_{i+j}$.*

If we have a free graded A-module, M, then, since its free, it can be written as $M = \bigoplus_{i=1}^{m} Af_i$, where $\{f_1, ..., f_m\}$ is a linearly independent generating set of M. Also, since it is graded, for each A-module Af_i there exist a degree-preserving isomorphism to $A(d_i)$, where d_i is the degree of f_i, thus we write $M = \bigoplus_{i=1}^{m} A(d_i)$. Corresponding to these different types of modules we have different characterizations of resolutions.

Definition 5.1.2. *A free resolution, S, of a homogeneous ideal $I \trianglelefteq R_n$, is an exact sequence of free modules,*

$$... \longrightarrow S_{i+1} \xrightarrow{\partial_{i+1}} S_i \longrightarrow ... \longrightarrow S_1 \xrightarrow{\partial_1} S_0,$$

where $S_0/\operatorname{Im}(\partial_1) \simeq R_n/I$. It is a graded resolution if all the modules are graded and the homomorphisms are all degree-preserving.

When we introduced ideals in Chapter 1, we started with the Hilbert basis theorem. Here we also start with a result from Hilbert regarding the length of a resolution. We say that a resolution is finite if only a finite number of modules in the resolution are nonzero. If a resolution is finite we define the length of a resolution to be the largest index, i, such that S_i is nonzero.

Theorem 5.1.1 (Hilbert syzygy theorem)**.** *Every finitely generated graded R_n-module has a free graded resolution of length at most n.*

Proof. See Theorem 3.8 of Chapter 6 in [6]. □

This theorem guarantees the existence of the minimum length of a free graded resolution of an ideal, $I \triangleleft R_n$. Thus we give this minimum a name and call it the *projective dimension* of I.

It follows from our comment on free graded modules that a free graded resolution can be written on the form

$$... \longrightarrow \bigoplus_{j\in\mathbb{Z}} R_n(j)^{d_{i+1,j}} \xrightarrow{\partial_{i+1}} \bigoplus_{j\in\mathbb{Z}} R_n(j)^{d_{i,j}} \longrightarrow ...$$

$$... \longrightarrow \bigoplus_{j\in\mathbb{Z}} R_n(j)^{d_{1,j}} \xrightarrow{\partial_1} \bigoplus_{j\in\mathbb{Z}} R_n(j)^{d_{0,j}},$$

where, for each i, $\sum d_{i,j} = m_i$ is the number of minimal generators of the i^{th} module. Thus the map ∂_i, can be given by a matrix of the form

$$\partial_i = \begin{bmatrix} \alpha^i_{1,1} & \alpha^i_{1,2} & ... & \alpha^i_{1,m_i} \\ \alpha^i_{2,1} & ... & ... & ... \\ ... & ... & ... & ... \\ \alpha^i_{m_{i-1},1} & ... & ... & \alpha^i_{m_{i-1},m_i} \end{bmatrix}, \tag{2}$$

where each $\alpha^i_{k,j}$ is a homogeneous element of R_n. For more on this, see for instance chapter 2 and 4 of [19].

With such a description of a free graded resolution, we can define a free graded resolution to be *minimal* if in addition $\alpha^i_{k,j}$ is either zero or have positive degree. The name minimal comes from the following proposition:

Proposition 5.1.1. *A free graded resolution S is minimal if and only if the image of the standard basis of S_i is a minimal generating set of $\mathrm{Im}(\partial_i)$ for all i.*

Proof. See Proposition 3.10 of Chapter 6 in [6]. □

As the next result will show, minimal resolutions possess certain properties that make them very nice to consider.

Theorem 5.1.2. *Let I be a homogeneous ideal of R_n. Then*

- *there exist a minimal free graded resolution of I,*
- *a minimal free graded resolution of I is unique up to isomorphism, and*
- *the length of a minimal free graded resolution is minimal.*

Proof. See Theorem 7.5 in Chapter 7 of [19]. □

Thus we often speak of *the minimal free graded resolution* of I and could equivalently define the projective dimension of I to be the length of the minimal free graded resolution. Note that when we speak of an isomorphism of graded resolutions, S and F, we refer to a sequence of degree-preserving isomorphisms, $(\phi_i : S_i \to F_i)_i$, such that the following diagram commutes

$$\begin{array}{ccc} S_i & \xrightarrow{\partial_i} & S_{i-1} \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} \\ F_i & \xrightarrow{d_i} & F_{i-1} \end{array},$$

that is, $\phi_{i-1} \circ \partial_i = d_i \circ \phi_i$.

A more thorough description of a free graded resolution than just looking at the projective dimension, comes from considering the *(graded) Betti numbers*:

Definition 5.1.3. *The Betti number, $b^R_{i,j}(I)$, of an ideal $I \trianglelefteq R_n$ is the number of components of the form $R_n(j)$ in the i^{th} module of a minimal free resolution of I over R.*

Thus the Betti numbers describe the degree of each generator and the number of generators of each module in the minimal free resolution of I. Also, we can describe the projective dimension of I in terms of the Betti numbers, $\mathrm{pd}(I) = \max\{i | b^{R_n}_{i,j}(I) \neq 0, \text{ for some } j\}$.

Similarly, we define the *Castelnuovo-Mumford regularity* (or *regularity*) of I to be the number $\mathrm{reg}(I) = \max\{j | b^{R_n}_{i,i+j}(I) \neq 0, \text{ for some } i\}$. The reason we look at $\max\{j | b^R_{i,i+j}(I) \neq 0, \text{ for some } i\}$ instead of $\max\{j | b^{R_n}_{i,j}(I) \neq 0, \text{ for some } i\}$, is that $b^{R_n}_{i,j}(I) = 0$ whenever $j < i$. This is due to the fact that the elements $\alpha^i_{k,j}$, as in (2), have positive degree if they are nonzero. Thus, since the maps ∂_i are degree preserving, there will be at least one degree shift per module in the resolution.

So by removing these minimal shifts we can see that the regularity provides a bound for the degrees of the minimal generators of the modules in the minimal free resolution. We have the following bound of the regularity for a general homogeneous ideal:

Theorem 5.1.3. *If $I \trianglelefteq R_n$ is minimally generated by homogeneous elements of degrees at most d, then* $\mathrm{reg}(I) \leq (2d)^{2^{n-2}}$.

Proof. See Theorem 18.10 in Chapter 18 of [19]. □

5.2. Open questions: Specht ideals

So far we have been looking into Hilbert series and Gröbner bases of Specht ideals and the equivariant Gröbner bases and Hilbert series of their limiting ideals. And although we are by no means done, there are other interesting questions to consider with regards to these filtrations and symmetric filtrations in general.

For instance, in [23], they present a conjecture (Conjecture 1.1) that the regularity of an Inc($\mathbb{N}$)-filtration, $(I_n)_{n \in \mathbb{N}_0}$, is eventually a linear function in n and in [14] (Conjecture 5.1) they conjecture that the projective dimension is eventually linear in n. It would be interesting to see whether this is true in general, or in particular, whether it is true for the Specht ideals. Also, in [14], they present several open problems regarding Inc($\mathbb{N}$)-filtrations. Amongst them is to study how the *Betti tables* (tables of Betti numbers) develop with the increase in the number of variables.

We know by Theorem 5.1.1 that the projective dimension of a filtration is bounded by a linear function. Also, the standard specht polynomials of a partition, λ, is a minimal generating set is a minimal generating set of the corresponding Specht ideal (see Theorem 1.1 in [18]). Thus, according to Theorem 5.1.3, the regularity of a filtration of Specht ideals is bounded by $(2d)^{2^{n-2}}$, where d is the degree of any Specht polynomial of shape λ. The latter is not a very good bound, so it would be interesting to investigate this further.

In the article [14] they also want to determine the primary decomposition of ideals in a filtration. Related to this question, there is a conjecture that the Specht ideals are radical. Thus if this was shown to be true, then the primary decomposition of the Specht ideals would follow from [16], where they give the decomposition of the radical ideal of any Specht ideal. So far it has been confirmed that the Specht ideals are radical for the partitions $(n-k, 1^k)$ in Theorem 1.1 of [24], and also for the partitions $(n-k, k)$ and $(k, k, 1)$ in [25], Theorem 3.1 and 4.2, but not for Specht ideals in general.

The question of whether the Specht ideals are radical was contemplated in the process of writing this thesis, but nothing became of it. Although, one might hope that proving the Gröbner basis conjecture could be of help in such an endeavor. Then we would know that the filtrations would have the intersection property, which might be a good starting point for an inductive proof.

Lastly, it would be interesting to determine whether or not the filtrations of Specht ideals have the intersection property. This could be achieved by confirming Conjecture 4.3.1, or by finding an easier condition, than the one mentioned after Proposition 3.2.1, to determine which filtrations have the intersection property. Since the Specht ideals are so closely related to the symmetric group, it would be interesting to know if they have the intersection property. Especially since filtrations with the intersection property are in one-to-one correspondence with the symmetric ideals in the infinite polynomial ring.

Appendix

Polynomial sequences

In this section we go through the proof of Lemma 4.5.2 and we follow the same line of reasoning as in the proof of Corollary 4.1.7 in Chapter 4 of [3]. We will not go into all the details but focus mostly on the parts where the field in question is of particular relevance.

The difference between our situation and the situation in [3] is that $(p(n))_{n\in\mathbb{N}_0}$ is a sequence of complex polynomials and not just complex numbers.

The first thing they do in [3] is to establish the following as different bases of the vector space of complex polynomials in x of degree $\leq d$:

$$B_1 = \{x^m | 0 \leq m \leq d\},$$
$$B_2 = \{x^m(1-x)^{d-m} | 0 \leq m \leq d\},$$
$$B_3 = \{\tbinom{x}{m} | 0 \leq m \leq d\},$$
$$B_4 = \{\tbinom{x+m}{d} | 0 \leq m \leq d\}.$$

We are considering the vector space $\mathbb{C}[t,x]$ as a module over the ring $\mathbb{C}[t]$ and thus our analogous statement is that the sets above are generating sets where the generators are linearly independent (over $\mathbb{C}[t]$).

Proposition 6.0.1. *The sets, B_i, for $i = 1,..,4$, are linearly independent generating sets of $(\mathbb{C}[t])[x]_{\leq d}$ over $\mathbb{C}[t]$.*

Proof. To see that they are all generating sets, note that since they are bases of $\mathbb{C}[x]_{\leq d}$, and $T = \{1, t, t^2, ..\}$ is a generating set of $(\mathbb{C}[x]_{\leq d})[t]$ over $\mathbb{C}[x]_{\leq d}$, and thus the sets $T \times B_i$ is a generating set over $\mathbb{C}$. Lastly, that means that B_i is a generating set of $(\mathbb{C}[t])[x]_{\leq d}$ over $\mathbb{C}[t]$.

From Proposition 4.1.2 of [3] we know that the sets B_i are bases of $\mathbb{C}[x]_{\leq d}$. To see that they are linearly independent sets over $\mathbb{C}[t]$, let $B_i = \{b_{i,0}, ..., b_{i,d}\}$ and suppose there exist polynomials $p_0, ..., p_d \in \mathbb{C}[t]$ such that $\sum_{j=0}^{d} p_j b_{i,j} = 0$. Then $\frac{1}{k!}\sum_{j=0}^{d} p_j^{(k)}(0) b_{i,j} = 0$ for all $k \in \mathbb{N}_0$, where $p_j^{(k)}(0)$ denotes the evaluation at $t = 0$ of the k^{th} derivative of p_j with respect to t. $\mathbb{C}$. Since B_i is linearly independent over $\mathbb{C}$ then $\frac{1}{k!}p_j^{(k)}(0) = 0 \ \forall\ k, j$. That is, the coefficients of the polynomials are all zero.

$\square$

The next step is to introduce the linear operators I, Δ and S on the module of complex-polynomial-valued sequences $\{(p(n))_{n\in\mathbb{N}_0}\}$ defined as follows:

$$(Ip)(n) := p(n),$$

$$(\Delta p)(n) := p(n+1) - p(n),$$
$$(Sp)(n) := p(n+1).$$

Proposition 6.0.2. *A sequence $(p(n))_{n\in\mathbb{N}_0}$ is polynomial of degree $\leq d$ if and only if $(\Delta^m p)(0) = 0$ for all $m > d$.*

Proof. The proof follows exactly the same was as the proof of Proposition 4.1.3 in [3]. □

From the proof of Proposition 4.1.3 in [3] it also follows that the coefficients of the polynomial $p(n)$ are $p^{(m)} := (\Delta^m p)(0)$ in the B_3 basis. Thus we can write $p(n) = \sum_{m=0}^{d} p^{(m)} \binom{n}{m}$ when $p(n)$ is polynomial in degree d.

We can use this to rewrite the generating function:

$$\sum_{n\geq 0} p(n)x^n = \sum_{n\geq 0} \left(\sum_{m=0}^{d} p^{(m)} \binom{n}{m} \right) x^n = \sum_{m=0}^{d} p^{(m)} \sum_{n\geq 0} \binom{n}{m} x^n.$$

From Lemma 4.1.2 we know how to rewrite the last sum. Thus

$$\sum_{m=0}^{d} p^{(m)} \sum_{n\geq 0} \binom{n}{m} x^n = \sum_{m=0}^{d} p^{(m)} \frac{x^m}{(1-x)^{m+1}} =$$

$$\frac{\sum_{m=0}^{d} p^{(m)} x^m (1-x)^{d-m}}{(1-x)^{d+1}} = \frac{h(x)}{(1-x)^{d+1}},$$

where $h(x) = sum_{m=0}^{d} p^{(m)} x^m (1-x)^{d-m} \in (\mathbb{C}[t])[x]$ since $p^{(m)} \in \mathbb{C}[t]$ $\forall$ $m \in [d]$.

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