ON THE BOUNDEDNESS OF SUBSEQUENCES OF VILENKIN-FEJÉR MEANS ON THE MARTINGALE HARDY SPACES

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Abstract. In this paper we characterize subsequences of Fejér means with respect to Vilenkin systems, which are bounded from the Hardy space H_p to the Lebesgue space L_p, for all $0<p<1/2$. The result is in a sense sharp.

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1. Introduction

In the one-dimensional case the weak (1,1)-type inequality for the maximal operator of Fejér means

$$\sigma^{*}f:=\sup_{n\in\mathbb{N}}\left|\sigma_{n}f\right|$$

can be found in Schipp [12] for Walsh series and in Pál, Simon [10] for bounded Vilenkin series. Here, as usual, the symbol σ_n denotes the Fejér mean with respect to the Vilenkin system (and thus also called the Vilenkin-Fejér means, see Section 2).

Fujji [6] and Simon [14] verified that σ^* is bounded from H_1 to L_1. Weisz [23] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p>1/2$. Simon [13] gave a counterexample, which shows that boundedness does not hold for $0<p<1/2$. A counterexample for $p=1/2$ was given by Goginava [8] (see also [2] and [3]). Weisz [24] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak-L_{1/2}$. The boundedness of weighted maximal operators are considered in [9], [16] and [17].

Weisz [22] (see also [21]) also proved that the following theorem is true:

Theorem W: (Weisz) Let $p>0$. Then the maximal operator

$$\sigma^{\nabla,*}f=\sup_{n\in\mathbb{N}}\left|\sigma_{M_n}f\right| \tag{1}$$

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where $M_0 := 1,\ M_{n+1} := m_n M_n\ (n \in \mathbb{N})$ and $m := (m_0, m_1, \dots)$ be a sequences of the positive integers not less than 2, which generate Vilenkin systems, is bounded from the Hardy space H_p to the space L_p.

In [11] the result of Weisz was generalized and it was found the maximal subspace $S \subset \mathbb{N}$ of positive numbers, for which the restricted maximal operator on this subspace $\sup_{n \in S \subset \mathbb{N}} |\sigma_n f|$ of Fejér means is bounded from the Hardy space H_p to the space L_p for all $0 < p \leq 1/2$. The new theorem (Theorem 1) in this paper show in particular that this result is in a sense sharp. In particular, for every natural number $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}\ (k \in \mathbb{N}_+)$ we define numbers

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\}, \quad \rho\,(n) = |n| - \langle n \rangle$$

and prove that

$$S = \{n \in \mathbb{N} : \rho\,(n) \leq c < \infty.\}$$

Since $\rho(M_n) = 0$ for all $n \in \mathbb{N}$ we obtain that $\{M_n : n \in \mathbb{N}\} \subset S$ and that follows i.e. that result of Weisz [22] (see also [21]) that restricted maximal operator (1) is bounded from the Hardy space H_p to the space L_p.

The main aim of this paper is to generalize Theorem W and find the maximal subspace of positive numbers, for which the restricted maximal operator of Fejér means in this subspace is bounded from the Hardy space H_p to the space L_p for all $0 < p \leq 1/2$. As applications, both some well-known and new results are pointed out.

This paper is organized as follows: In order not to disturb our discussions later on some preliminaries (definitions, notations and lemmas) are presented in Section 2. The main result (Theorem 1) and some of its consequences can be found in Section 3. The detailed proof of Theorem 1 is given in Section 4.

2. Preliminaries

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_n} := \{0, 1, \dots, m_n - 1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups Z_{m_n} with the product of the discrete topologies of Z_{m_n}'s. In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_{n \in \mathbb{N}} m_n < \infty$.

The direct product μ of the measures $\mu_n\,(\{j\}) := 1/m_n,\ (j \in Z_{m_n})$ is the Haar measure on G_m with $\mu\,(G_m) = 1$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_n, \dots)\,,\ (x_n \in Z_{m_n})\,.$$

It is easy to give a base for the neighbourhood of G_m :

$$I_0\,(x) := G_m,\ I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}\ (x \in G_m,\ n \in \mathbb{N})\,.$$

Set $I_n := I_n\,(0)$, for $n \in \mathbb{N}_+$ and

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m \qquad (n \in \mathbb{N}).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1,\dots,} x_{N-1}), & k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0), & l = N. \end{cases}$$

It is easy to show that

$$\overline{I_N} = \left(\bigcup_{i=0}^{N-2} \bigcup_{j=i+1}^{N-1} I_N^{i,j} \right) \bigcup \left(\bigcup_{i=0}^{N-1} I_N^{i,N} \right), \quad n = 2, 3, \dots \tag{2}$$

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \ M_{n+1} := m_n M_n \quad (n \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}$ $(k \in \mathbb{N}_+)$ and only a finite number of n_k's differ from zero. Let

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\} \quad \text{and} \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},$$

that is $M_{|n|} \leq n \leq M_{|n|+1}$. Set $\rho(n) = |n| - \langle n \rangle$, for all $n \in \mathbb{N}$.

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k), \ \left(i^2 = -1, x \in G_m, \ k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system, when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $weak - L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \quad \|f\|_{L_{p,\infty}}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < \infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [20]).

If $f \in L_1(G_m)$ we can define Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \quad (k \in \mathbb{N}),$$

$$S_nf \ : \ =\sum_{k=0}^{n-1}\widehat{f}\left(k\right)\psi_k,\qquad D_n:=\sum_{k=0}^{n-1}\psi_k\qquad\left(\ n\in\mathbb{N}_+\ \right),$$

$$\sigma_nf \ : \ =\frac{1}{n}\sum_{k=0}^{n-1}S_kf,\qquad K_n:=\frac{1}{n}\sum_{k=0}^{n-1}D_k\quad\left(\ n\in\mathbb{N}_+\ \right).$$

Recall that (see e.g. [1])

$$D_{M_n}\left(x\right)=\left\{\begin{array}{ll} M_n, & \text{if } \ x\in I_n, \\ 0, & \text{if } \ \ x\notin I_n,\end{array}\right. \tag{3}$$

and

$$D_{s_nM_n}=D_{s_nM_n}\sum_{k=0}^{s_n-1}\psi_{kM_n}=D_{M_n}\sum_{k=0}^{s_n-1}r_n^k, \tag{4}$$

where $n\in\mathbb{N}$ and $1\leq s_n\leq m_n-1$.

The σ-algebra generated by the intervals $\left\{I_n\left(x\right):x\in G_m\right\}$ will be denoted by $F_n\ \left(n\in\mathbb{N}\right)$. Denote by $f=\left(f^{(n)},n\in\mathbb{N}\right)$ a martingale with respect to F_n $(n\in\mathbb{N})$ (for details see e.g. [21]). The maximal function of a martingale f is defined by

$$f^{*}=\sup_{n\in\mathbb{N}}\left|f^{(n)}\right|.$$

In the case $f\in L_1(G_m)$, the maximal functions are just also given by

$$f^{*}\left(x\right)=\sup_{n\in\mathbb{N}}\frac{1}{\left|I_n\left(x\right)\right|}\left|\int_{I_n(x)}f\left(u\right)\mu\left(u\right)\right|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p\left(G_m\right)$ consist of all martingales f, for which

$$\left\|f\right\|_{H_p}:=\left\|f^{*}\right\|_p<\infty.$$

If $f\in L_1(G_m)$, then it is easy to show that the sequence $\left(S_{M_n}\left(f\right):n\in\mathbb{N}\right)$ is a martingale. If $f=\left(f^{(n)},n\in\mathbb{N}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}\left(i\right):=\lim_{k\rightarrow\infty}\int_{G_m}f^{(k)}\left(x\right)\overline{\psi}_i\left(x\right)d\mu\left(x\right).$$

The Vilenkin-Fourier coefficients of $f\in L_1\left(G_m\right)$ are the same as those of the martingale $\left(S_{M_n}f:n\in\mathbb{N}\right)$ obtained from f.

A bounded measurable function a is said to be a p-atom if there exists an interval I, such that

$$\int_I ad\mu=0,\quad\left\|a\right\|_{\infty}\leq\mu\left(I\right)^{-1/p},\quad\text{supp}\left(a\right)\subset I.$$

For the proof of the main result (Theorem 1) we need the following Lemmas:

Lemma 1 (see e.g. [22]). *A martingale* $f=\left(f^{(n)},n\in\mathbb{N}\right)$ *is in* $H_p\,(0<p\leq 1)$ *if and only if there exist a sequence* $(a_k,k\in\mathbb{N})$ *of p-atoms and a sequence* $(\mu_k,k\in\mathbb{N})$ *of real numbers such that for every* $n\in\mathbb{N}:$

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=f^{(n)} \tag{5}$$

and

$$\sum_{k=0}^{\infty}\left|\mu_k\right|^p<\infty.$$

Moreover, $\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}\left|\mu_k\right|^p\right)^{1/p}$, *where the infimum is taken over all decomposition of* f *of the form (5).*

Lemma 2 (see e.g. [22]). *Suppose that an operator* T *is* σ*-linear and for some* $0<p\leq 1$

$$\int\limits_{\overline{I}}\left|Ta\right|^p d\mu\leq c_p<\infty,$$

for every p-atom a, where I *denotes the support of the atom. If* T *is bounded from* L_∞ *to* L_∞*, then*

$$\left\|Tf\right\|_p\leq c_p\left\|f\right\|_{H_p}.$$

Lemma 3 (see [7]). *Let* $n>t,\ t,n\in\mathbb{N},\ x\in I_t\backslash\ I_{t+1}$. *Then*

$$K_{M_n}(x)=\begin{cases}0, & \text{if } x-x_te_t\notin I_n,\\ \frac{M_t}{1-r_t(x)}, & \text{if } x-x_te_t\in I_n.\end{cases}$$

Lemma 4 (see [17]). *Let* $x\in I_N^{i,j},\ i=0,\dots,N-1,\ j=i+1,\dots,N$. *Then*

$$\int_{I_N}\left|K_n(x-t)\right|d\mu(t)\leq\frac{cM_iM_j}{M_N^2},\qquad \text{for}\ \ n\geq M_N.$$

Lemma 5 (see [11]). *Let* $n\in\mathbb{N}$. *Then*

$$\left|K_n(x)\right|\leq\frac{c}{n}\sum_{l=\langle n\rangle}^{|n|}M_l\left|K_{M_l}\right|\leq c\sum_{l=\langle n\rangle}^{|n|}\left|K_{M_l}\right| \tag{6}$$

and

$$\left|nK_n\right|\geq\frac{M_{\langle n\rangle}^2}{2\pi\lambda},\qquad x\in I_{\langle n\rangle+1}\left(e_{\langle n\rangle-1}+e_{\langle n\rangle}\right), \tag{7}$$

where $\lambda:=\sup m_n$.

3. The Main Result and applications

Our main result reads:

Theorem 1. *a) Let $0<p<1/2$, $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\left\|\sigma_{n_k}f\right\|_{H_p}\leq\frac{c_pM_{|n_k|}^{1/p-2}}{M_{\langle n_k\rangle}^{1/p-2}}\left\|f\right\|_{H_p}.$$

b) (sharpness) Let $0<p<1/2$ and $\Phi(n)$ be any nondecreasing function, such that

$$\sup_{k\in\mathbb{N}}\rho\left(n_k\right)=\infty,\quad\overline{\lim_{k\to\infty}}\frac{M_{|n_k|}^{1/p-2}}{M_{\langle n_k\rangle}^{1/p-2}\Phi\left(n_k\right)}=\infty. \tag{8}$$

Then there exists a martingale $f\in H_p$, such that

$$\sup_{k\in\mathbb{N}}\left\|\frac{\sigma_{n_k}f}{\Phi\left(n_k\right)}\right\|_{L_{p,\infty}}=\infty.$$

Corollary 1. *Let $0<p<1/2$, and $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\left\|\sigma_{n_k}f\right\|_{H_p}\leq c_p\left\|f\right\|_{H_p},\quad k\in\mathbb{N}$$

if and only if

$$\sup_{k\in\mathbb{N}}\rho\left(n_k\right)<c<\infty.$$

As an application we also obtain the previous mentioned result by Weisz [21], [22] (Theorem W).

Corollary 2. *Let $0<p<1/2$, $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\left\|\sigma_{M_n}f\right\|_{H_p}\leq c_p\left\|f\right\|_{H_p},\quad n\in\mathbb{N}.$$

On the other hand, the following unexpected result is true:

Corollary 3. *a) Let $0<p<1/2$, $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\left\|\sigma_{M_n+1}f\right\|_{H_p}\leq c_pM_n^{1/p-2}\left\|f\right\|_{H_p},\quad n\in\mathbb{N}.$$

b) Let $0<p<1/2$ and $\Phi(n)$ be any nondecreasing function, such that

$$\overline{\lim_{k\to\infty}}\frac{M_k^{1/p-2}}{\Phi\left(k\right)}=\infty.$$

Then there exists a martingale $f\in H_p$, such that

$$\sup_{k\in\mathbb{N}}\left\|\frac{\sigma_{M_k+1}f}{\Phi\left(k\right)}\right\|_{L_{p,\infty}}=\infty.$$

Remark 1. *From Corollary 2 we obtain that* σ_{M_n} *are bounded from* H_p *to* H_p*, but from Corollary 3 we conclude that* σ_{M_n+1} *are not bounded from* H_p *to* H_p*. The main reason is that Fourier coefficients of martingales* $f \in H_p$ *are not uniformly bounded (for details see e.g.* [18]*).*

In the next corollary we state some estimates for the Walsh system only to clearly see the difference of divergence rates for the various subsequences:

Corollary 4. *a)Let* $0 < p < 1/2, \quad f \in H_p$*. Then there exists an absolute constant* c_p*, depending only on* p*, such that*

$$\left\|\sigma_{2^n+1} f\right\|_{H_p} \leq c_p 2^{(1/p-2)n} \left\|f\right\|_{H_p}, \quad n \in \mathbb{N} \tag{9}$$

and

$$\left\|\sigma_{2^n+1} f\right\|_{H_p} \leq c_p 2^{\frac{(1/p-2)n}{2}} \left\|f\right\|_{H_p}, \quad n \in \mathbb{N}. \tag{10}$$

b) The rates $2^{(1/p-2)n}$ *and* $2^{\frac{(1/p-2)n}{2}}$ *in inequalities (9) and (10) are sharp in the same sense as in Theorem 1.*

4. Proof of Theorem 1

Proof. a) Since

$$\sup_{n \in \mathbb{N}} \int_{G_m} \left|K_n(x)\right| d\mu(x) \leq c < \infty, \tag{11}$$

we obtain that

$$\frac{M_{\langle n_k \rangle}^{1/p-2} \left|\sigma_{n_k} a(x)\right|}{M_{|n_k|}^{1/p-2}}$$

is bounded from L_∞ to L_∞. According to Lemma 2 we find that the proof of Theorem 1 will be complete, if we show that

$$\int_{\overline{I_N}} \left| \frac{M_{\langle n_k \rangle}^{1/p-2} \sigma_{n_k} a(x)}{M_{|n_k|}^{1/p-2}} \right|^p < c < \infty,$$

for every p-atom a, with support I and $\mu(I) = M_N^{-1}$. We may assume that $I = I_N$. It is easy to see that $\sigma_{n_k}(a) = 0$ when $n_k \leq M_N$. Therefore, we can suppose that $n_k > M_N$.

Since $\|a\|_{\infty}\leq M_N^{1/p}$ we find that

$$\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\leq\frac{M_{\langle n_k\rangle}^{1/p-2}}{M_{|n_k|}^{1/p-2}}\int_{I_N}\left|a\left(t\right)\right|\left|K_{n_k}\left(x-t\right)\right|d\mu\left(t\right) \tag{12}$$

$$\leq\frac{M_{\langle n_k\rangle}^{1/p-2}\left\|a\right\|_{\infty}}{M_{|n_k|}^{1/p-2}}\int_{I_N}\left|K_{n_k}\left(x-t\right)\right|d\mu\left(t\right)$$

$$\leq\frac{M_{\langle n_k\rangle}^{1/p-2}M_N^{1/p}}{M_{|n_k|}^{1/p-2}}\int_{I_N}\left|K_{n_k}\left(x-t\right)\right|d\mu\left(t\right)$$

$$\leq M_{\langle n_k\rangle}^{1/p-2}M_{|n_k|}^{2}\int_{I_N}\left|K_{n_k}\left(x-t\right)\right|d\mu\left(t\right).$$

Without loss the generality we may assume that $i<j$. Let $x\in I_N^{i,j}$ and $j<\langle n_k\rangle$. Then $x-t\in I_N^{i,j}$ for $t\in I_N$ and, according to Lemma 3, we obtain that

$$\left|K_{M_l}\left(x-t\right)\right|=0,\quad\text{for all }\ \langle n_k\rangle\leq\ l\leq|n_k|.$$

By applying (12) and (6) in Lemma 5, for $x\in I_N^{i,j},\quad 0\leq i<j<\langle n_k\rangle$ we get that

$$\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\leq M_{\langle n_k\rangle}^{1/p-2}M_{|n_k|}^{2}\sum_{l=\langle n_k\rangle}^{|n_k|}\int_{I_N}\left|K_{M_l}\left(x-t\right)\right|d\mu\left(t\right)=0. \tag{13}$$

Let $x\in I_N^{i,j}$, where $\langle n_k\rangle\leq j\leq N$. Then, in the view of Lemma 4, we have that

$$\int_{I_N}\left|K_{n_k}\left(x-t\right)\right|d\mu\left(t\right)\leq\frac{cM_iM_j}{M_N^2}.$$

By using again (12) we find that

$$\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\leq\frac{M_{\langle n_k\rangle}^{1/p-2}M_N^{1/p}}{M_{|n_k|}^{1/p-2}}\int_{I_N}\left|K_{n_k}\left(x-t\right)\right|d\mu\left(t\right) \tag{14}$$

$$\leq\frac{M_{\langle n_k\rangle}^{1/p-2}M_N^{1/p}}{M_{|n_k|}^{1/p-2}}\frac{M_iM_j}{M_N^2}\leq M_{\langle n_k\rangle}^{1/p-2}M_iM_j.$$

By combining (2) and (12)-(14) we get that

$$
\begin{aligned}
&\int_{\overline{I_N}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu \\
=& \sum_{i=0}^{N-2}\sum_{j=i+1}^{N-1}\int_{I_N^{i,j}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu \\
+& \sum_{i=0}^{N-1}\int_{I_N^{k,N}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu \\
\leq& \sum_{i=0}^{\langle n_k\rangle-1}\sum_{j=\langle n_k\rangle}^{N-1}\int_{I_N^{i,j}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu \\
&+\sum_{i=\langle n_k\rangle}^{N-2}\sum_{j=i+1}^{N-1}\int_{I_N^{i,j}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu \\
&+\sum_{i=0}^{N-1}\int_{I_N^{i,N}}\left|\frac{M_{\langle n_k\rangle}^{1/p-2}\left|\sigma_{n_k}a\left(x\right)\right|}{M_{|n_k|}^{1/p-2}}\right|^p d\mu
\end{aligned}
$$

$$
\begin{aligned}
\leq& \sum_{i=0}^{\langle n_k\rangle-1}\sum_{j=\langle n_k\rangle}^{N-1}\int_{I_N^{i,j}}\left|M_{\langle n_k\rangle}^{1/p-2}M_iM_j\right|^p d\mu+\sum_{i=\langle n_k\rangle}^{N-2}\sum_{j=i+1}^{N-1}\int_{I_N^{i,j}}\left|M_{\langle n_k\rangle}^{1/p-2}M_iM_j\right|^p d\mu \\
&+\sum_{i=0}^{N-1}\int_{I_N^{i,N}}\left|M_{\langle n_k\rangle}^{1/p-2}M_iM_N\right|^p d\mu \\
\leq& c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=0}^{\langle n_k\rangle-1}\sum_{j=\langle n_k\rangle}^{N-1}\frac{(M_iM_j)^p}{M_j}+c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=\langle n_k\rangle}^{N-2}\sum_{j=i+1}^{N-1}\frac{(M_iM_j)^p}{M_j} \\
&+c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=0}\frac{(M_iM_N)^p}{M_N} \\
\leq& c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=0}^{\langle n_k\rangle}M_i^p\sum_{j=\langle n_k\rangle+1}^{N-1}\frac{1}{M_j^{1-p}}+M_{\langle n_k\rangle}^{1-2p}\sum_{i=\langle n_k\rangle}^{N-2}M_i^p\sum_{j=i+1}^{N-1}\frac{1}{M_j^{1-p}} \\
&+c_p\sum_{i=0}^{N-1}\frac{M_i^p}{M_N^p} \\
\leq& c_pM_{\langle n_k\rangle}^{1-2p}M_{\langle n_k\rangle}^p\frac{1}{M_{\langle n_k\rangle}^{1-p}}+c_pM_{\langle n_k\rangle}^{1-2p}\sum_{i=\langle n_k\rangle}^{N-2}\frac{1}{M_i^{1-2p}}+c_p\leq c_p<\infty.
\end{aligned}
$$

The proof of the a) part is complete.

b) Let $\{n_k : k \geq 0\}$ be a sequence of positive numbers, satisfying condition (8). Then

$$\sup_{k\in\mathbb{N}} \frac{M_{|n_k|}}{M_{\langle n_k\rangle}} = \infty. \tag{15}$$

Under condition (15) there exists a sequence $\{\alpha_k : \ k \geq 0\} \subset \{n_k : \ k \geq 0\}$ such that $\alpha_0 \geq 3$ and

$$\sum_{k=0}^{\infty} \frac{M_{\langle \alpha_k\rangle}^{(1-2p)/2}\Phi^{p/2}(\alpha_k)}{M_{|\alpha_k|}^{(1-2p)/2}} < c < \infty. \tag{16}$$

Let

$$f^{(n)} = \sum_{\{k;\ |\alpha_k|<n\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{\lambda M_{\langle \alpha_k\rangle}^{(1/p-2)/2}\Phi^{1/2}(\alpha_k)}{M_{|\alpha_k|}^{(1/p-2)/2}}$$

and

$$a_k = \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda}\left(D_{M_{|\alpha_k|+1}} - D_{M_{|\alpha_k|}}\right).$$

B applying Lemma 1 we can conclude that $f \in H_p$.
It is evident that

$$\widehat{f}(j) = \begin{cases} M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k\rangle}^{(1/p-2)/2}\Phi^{1/2}(\alpha_k), \\ \text{if } \ j \in \left\{M_{|\alpha_k|}, ..., \ M_{|\alpha_k|+1}-1\right\}, \ k = 0,1,2..., \\ 0\ , \\ \ \text{if } \ \ j \notin \bigcup\limits_{k=0}^{\infty}\left\{M_{|\alpha_k|}, ..., \ M_{|\alpha_k|+1}-1\right\}. \end{cases} \tag{17}$$

Moreover,

$$\frac{\sigma_{\alpha_k} f}{\Phi(\alpha_k)} = \frac{1}{\alpha_k\Phi(\alpha_k)}\sum_{j=1}^{M_{|\alpha_k|}} S_j f + \frac{1}{\alpha_k\Phi(\alpha_k)}\sum_{j=M_{|\alpha_k|}+1}^{\alpha_k} S_j f := I + II.$$

Let $M_{|\alpha_k|} < j \leq \alpha_k$. Then, by applying (17) we get that

$$S_j f = S_{M_{|\alpha_k|}} f + M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k\rangle}^{(1/p-2)/2}\Phi^{1/2}(\alpha_k)\left(D_j - D_{M_{|\alpha_k|}}\right). \tag{18}$$

By using (18) we can rewrite II as

$$\begin{aligned} II &= \frac{\alpha_k - M_{|\alpha_k|}}{\alpha_k\Phi(\alpha_k)} S_{M_{|\alpha_k|}} f + \frac{M_{|\alpha_k|}^{1/2p} M_{\langle \alpha_k\rangle}^{(1/p-2)/2}}{\alpha_k\Phi^{1/2}(\alpha_k)}\sum_{j=M_{|\alpha_k|}}^{\alpha_k}\left(D_j - D_{M_{|\alpha_k|}}\right) \\ &:= II_1 + II_2. \end{aligned}$$

Since (for details see e.g. [5] and [19])

$$\left\|S_{M_{|\alpha_k|}}f\right\|_{weak-L_p}\leq c_p\left\|f\right\|_{H_p}$$

we obtain that

$$\begin{aligned}\left\|II_1\right\|^p_{weak-L_p} &\leq \left(\frac{\alpha_k-M_{|\alpha_k|}}{\alpha_k\Phi\left(\alpha_k\right)}\right)^p\left\|S_{M_{|\alpha_k|}}f\right\|^p_{weak-L_p}\\ \leq & \left\|S_{M_{|\alpha_k|}}f\right\|^p_{weak-L_p}\leq c_p\left\|f\right\|^p_{H_p}<\infty.\end{aligned}$$

By using part a) of Theorem 1 we find that

$$\left\|I\right\|^p_{weak-L_p}=\left(\frac{M_{|\alpha_k|}}{\alpha_k\Phi\left(\alpha_k\right)}\right)^p\left\|\sigma_{M_{|\alpha_k|}}f\right\|^p_{weak-L_p}\leq c_p\left\|f\right\|^p_{H_p}<\infty.$$

Let $x\in I^{\langle\alpha_k\rangle-1,\langle\alpha_k\rangle}_{\langle\alpha_k\rangle+1}$. Under condition (8) we can conclude that $\langle\alpha_k\rangle\neq|\alpha_k|$ and $\left\langle\alpha_k-M_{|\alpha_k|}\right\rangle=\langle\alpha_k\rangle$. Since

$$D_{j+M_n}=D_{M_n}+\psi_{M_n}D_j=D_{M_n}+r_nD_j,\ \text{ when }\ j<M_n \tag{19}$$

if we apply estimate (7) in Lemma 5 for II_2 we obtain that

$$\begin{aligned}\left|II_2\right| &= \frac{M^{1/2p}_{|\alpha_k|}M^{(1/p-2)/2}_{\langle\alpha_k\rangle}}{\alpha_k\Phi^{1/2}\left(\alpha_k\right)}\left|\sum_{j=1}^{\alpha_k-M_{|\alpha_k|}}\left(D_{j+M_{|\alpha_k|}}-D_{M_{|\alpha_k|}}\right)\right|\\ &= \frac{M^{1/2p}_{|\alpha_k|}M^{(1/p-2)/2}_{\langle\alpha_k\rangle}}{\alpha_k\Phi^{1/2}\left(\alpha_k\right)}\left|\psi_{M_{|\alpha_k|}}\sum_{j=1}^{\alpha_k-M_{|\alpha_k|}}D_j\right|\\ &\geq \frac{c_pM^{1/2p-1}_{|\alpha_k|}M^{(1/p-2)/2}_{\langle\alpha_k\rangle}}{\Phi^{1/2}\left(\alpha_k\right)}\left(\alpha_k-M_{|\alpha_k|}\right)\left|K_{\alpha_k-M_{|\alpha_k|}}\right|\\ &\geq \frac{c_pM^{1/2p-1}_{|\alpha_k|}M^{(1/p+2)/2}_{\langle\alpha_k\rangle}}{\Phi^{1/2}\left(\alpha_k\right)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\left\|II_2\right\|^p_{weak-L_p}\\ \geq\ & c_p\left(\frac{M^{(1/p-2)/2}_{|\alpha_k|}M^{(1/p+2)/2}_{\langle\alpha_k\rangle}}{\Phi^{1/2}\left(\alpha_k\right)}\right)^p\mu\left\{x\in G_m:\left|IV_2\right|\geq c_pM^{(1/p-2)/2}_{|\alpha_k|}M^{(1/p+2)/2}_{\langle\alpha_k\rangle}\right\}\\ \geq\ & c_p\frac{M^{1/2-p}_{|\alpha_k|}M^{1/2+p}_{\langle\alpha_k\rangle}\mu\left\{I^{\langle\alpha_k\rangle-1,\langle\alpha_k\rangle}_{\langle\alpha_k\rangle+1}\right\}}{\Phi^{p/2}\left(\alpha_k\right)}\geq\frac{c_pM^{1/2-p}_{|\alpha_k|}}{M^{1/2-p}_{\langle\alpha_k\rangle}\Phi^{p/2}\left(\alpha_k\right)}.\end{aligned}$$

Hence, for large k,

$$\begin{aligned}
& \left\| \sigma_{\alpha_k} f \right\|^p_{weak-L_p} \\
\geq\ & \left\| II_2 \right\|^p_{weak-L_p} - \left\| II_1 \right\|^p_{weak-L_p} - \left\| I \right\|^p_{weak-L_p} \\
\geq\ & \frac{1}{2} \left\| II_2 \right\|^p_{weak-L_p} \geq \frac{c_p M^{1/2-p}_{|\alpha_k|}}{2M^{1/2-p}_{\langle \alpha_k \rangle} \Phi^{p/2}\left(\alpha_k\right)} \to \infty, \text{ as } k \to \infty.
\end{aligned}$$

The proof is complete. □

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