Inverse and efficiency of heat transfer convex fin with multiple non-linearities

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Abstract

In this article, we first propose the novel semi-analytical technique, modified Adomian decomposition method (MADM) for a closed form solution of the nonlinear heat transfer equation of convex profile with singularity where all thermal parameters are functions of temperature. The longitudinal convex fin is subjected to different boiling regimes, which are defined by particular values of n (power index) of heat transfer coefficient. The energy balance equation of the convex fin with several temperatures dependent properties are solved separately using the MADM and the spectral quasi-linearization method (SQLM). Using the values obtained from the direct heat transfer method, the unknown parameters of the profile, such as, thermal conductivity, surface emissivity, heat generation number, conduction-convection parameter and radiation conduction parameter are inversely predicted by an inverse heat transfer analysis using the simplex search method (SSM). The effect of the measurement error and the number of measurement points has been presented. It is found that present measurement points and reconstruction of the exact temperature distribution of the convex fin are fairly in a good agreement.

Keywords: MADM; SQLM; inverse analysis; differential operator.

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Nomenclature

N_r Conduction-radiation parameter

C Constant which represents the temperature

k Temperature dependent thermal conductivity, $W/(mK)$

k_a Thermal conductivity corresponding to atmospheric condition, $W/(mK)$

ε_s The surface emissivity corresponding to radiation sinks temperature, T_s

T Temperature, K

P Fin perimeter, m

T_b Fin's base temperature, K

T_a Sink temperature corresponding to k_a, K

T_s Sink temperature for radiation, K

L Length of the fin, m

x Axial distance of fin, m

$A(x)$ Cross-sectional area at the location x, m^2

X Non-dimensional length

ε_G heat generation parameter

A thermal conductivity parameter

B surface emissivity parameter

G heat generation number

Greek symbols

α Parameter describe the variation of thermal conductivity, K^{-1}

β Parameter describe the variation of surface emissivity, K^{-1}

γ Parameter describe the variation of heat generation, K^{-1}

θ Non-dimensional temperature distribution of the fin,

θ_a Non-dimensional convection sink temperature,

θ_s Non-dimensional radiations sink temperature,

σ Stefan-Boltzmann constant

ε Emissivity

1. Introduction

Extended surfaces are commonly used to improve the rate of heat transfer from a solid boundary to the surrounding environment by conduction, convection and radiation. Kem and Kraus [1] have given a vast review on this topic. Lighter fins are the main choice in aerospace and rocket application. The variety of lighter fins include parabolic concave or convex, triangular profile etc. But some of the lighter fins are difficult to manufacture and pose safety hazard because of sharp edges. Thus convex fin profiles are the better alternative, with additional advantage of being simple and easy to fabricate.

Numerous research works have been conducted during the last decade for solving non-linear fin problems. Temperature dependent phenomena leads to nonlinearity in the energy equation. Chiu and Chen [2] employed classical Adomian decomposition method (ADM) to solve non-linear energy equation of rectangular longitudinal fin associated with variable thermal conductivity. The same method is extended to solve the non linear energy equation of convective-radiative longitudinal fin with non linear boundary condition [3]. The double decomposition method (DDM) [4] is used to obtain the stress field [5] of an isotropic convective-radiative circular fin by solving the nonlinear energy equation and the non linear boundary condition associated with the variable thermal parameter. The nonlinear energy equation of conductive-convective hyperbolic fin profile is solved by DDM with variable thermal conductivity [6]. Chang [7] employed again ADM to solve the non-linear energy equation of rectangular fin under different regimes of boiling heat transfer. Kundu and Wongwises [8] presented a decomposition solution of rectangular fin with variable thermal parameter and heat transfer coefficient with base wall maintained at high temperature. Singh [9] et al. presented a closed form solution of stepped fin with non-linear boundary condition. Other research works related to solving non linear heat transfer equation fin with classical methods are available in [10-16].

The nonlinear linear equation with the extension of singularity behavior gives a more mathematical complexity in the energy equation. The nonlinear equation is solved using classical ADM by defining a single operator containing single highest order derivative in the problem. But in the modified ADM, both the singularity and nonlinearity can be solved by a combined operator in the problem. Roy et al. [17] et al. presented decomposition method for solving non-linear heat transfer equation of rectangular, convex and triangular fin with temperature dependent properties. Further classical ADM is used for solving non-singular energy equation of rectangular fin and modified ADM is used for solving singular energy equation of convex and triangular fin problem [18].

There may be internal heat generation in the fin that is due to the flow of electric current as in electric filament or it can be an example of chemical reaction as in an atomic reactor. The heat transfers associated with boiling regimes are much higher than other forms of heat transfer process because of the phase change. For example, the particular power values of convective heat transfer coefficient, n, that describe different fluid boiling regimesare –0.25, 0.25, 0.33, 2 and 3. For constant heat transfer coefficient, the value of n is 0. Aziz and Fang [19] presented a comparative analytical study of rectangular, trapezoidal and concave profile without any variable thermal parameters. Torabi [20] et al employed the differential transform method (DTM) to obtain temperature distribution and efficiency of rectangular, trapezoidal and concave profile with multiple variable properties. The homotopy analysis method (HAM) used by Aziz and Khani [21]. Sun Y. et al [22] applied the spectral collocation method (CSM) of convective-radiative longitudinal fin parameters. Turkyilmazoglu M. [23] studied exponential moving fins with internal heat generation. S. Mosayebidorcheh et al [24] presented the optimization of longitudinal fins of different shapes and materials associated with temperature dependent phenomena. Darvishi M.T. et al [25] investigated numerically the hyperbolic annular fin with temperature dependent conductivity. Functionally graded longitudinal fin is analysed by Khan and Aziz [26]. However, most works predict the thermal performance in the form of either temperature distribution or fin efficiency from a priori knowledge of fin thermal parameters. These problems are forward or direct heat transfer problem and they are mathematically well-posed [27]. But the analysis becomes different when the parameters are predicted from known temperature satisfying the requirement. These problems are known as inverse heat transfer problems and they are mathematically ill-posed. The objective of this work is to explore both the accuracy and the convergence of the spectral quasilinearization method (SQLM) and modified Adomian decomposition method (MADM), for solving systems of nonlinear partial differential equations. The accuracy of all the spectral methods used is established through the evaluation of residual and solution error analysis. Further validation of present results is established by comparing with existing results from literature. In mathematical physics, many methods have been developed to solve differential equations, among which ADM is an efficient approximation technique used to solve initial boundary value problems. The advantage of the method is that it converges to exact solutions and needs less computation than the traditional discretization methods. The method is based on decomposition the solution of a nonlinear operator equation as a power series expansion of a function, resulting in a polynomial equation that is easier to solve. MADM has compared favorably with other numerical methods. In this study we do not implement MADM because MADM presents two main challenges; difficulties in the calculation of the Adomian polynomials and the successive application of the integral operator. These problems make it unsuitable for differential equations involving large systems of equations as it requires more computation time than spectral methods.

The aim of the work is to employ modified ADM to obtain the forward problem of convex fin and inverse heat transfer by simplex search method (SSM). The work is also aimed to introduce a numerical method that uses SQLM. The results of MADM are compared with SQLM solution and good agreement has been observed. The methodology of modified decomposition method was developed and introduced by Adomian G. [28].

2. Problem statement and mathematical formulation

The longitudinal convex fin is subjected to convection-radiation with an internal heat source. It has length L and t_b as the constant base thickness as shown in Fig 1. The fin dissipates heat at two different sink temperature. The fin width is very large as compared to its thickness so that the cross-sectional area is a function of fin thickness only and it changes parabolically according to law, $t(x) = t_b (x/L)^{0.5} = t_b X^{0.5}$.

The one dimensional steady state one dimensional energy balance equation can be expressed as,

$$\frac{d}{dx}\left[k(T) t_b \left(x/L \right)^{0.5} \frac{dT}{dx} \right] - h(T)(T - T_a) - \varepsilon(T)\sigma\left(T^4 - T_s^{\,4}\right) + q(T) t_b \left(x/L \right)^{0.5} = 0 \tag{1}$$

Thermal conductivity (k), surface emissivity (ε) and heat generation (q) are assumed to be a linear function of temperature, while the heat transfer coefficient (h) follows a power law of temperature as,

$$k(T) = \left[k_a + \alpha k_a (T - T_a) \right] \tag{2a}$$

$$h(T) = h_b \left(\frac{T - T_a}{T_b - T_a} \right)^n \tag{2b}$$

$$\varepsilon(T) = \left[\varepsilon_s + \beta \varepsilon_s (T - T_s) \right] \tag{2c}$$

$$q(T) = \left[q_0 + \gamma q_0 (T - T_a) \right] \tag{2d}$$

Considering an adiabatic tip, the associated boundary conditions are as follows:

$$x = 0, \qquad \frac{dT}{dx} = 0 \tag{3(a)}$$

$$x = L, \qquad T = T_b \tag{3(b)}$$

For simplicity and wide application, the following non-dimensional entities have been introduced,

$$N_c = \frac{h_b T_b^{\,n} P L^2}{k_a A_c (T_b - T_a)^n},\; N_r = \frac{\varepsilon_s \sigma T_b^{\,3} P L^2}{k_a A_c},\; G = \frac{q_0 L^2}{k_a T_b},\; A = \alpha T_b,\; B = \beta T_b,\; \varepsilon_G = \gamma T_b \tag{4}$$

where N_c is the conduction-convection parameter, N_r is the conduction-radiation parameter, and ε_G is the parameter describing the variation of heat generation.

Now, Eq. (1) can be rewritten as,

$$\frac{d}{dX}\left[\{1+(A\theta-A\theta_a)\}X^{0.5}\frac{d\theta}{dX}\right]-N_c(\theta-\theta_a)^{n+1}-N_r\begin{bmatrix}(\theta^4-\theta_s{}^4)+\\ B(\theta-\theta_s)(\theta^4-\theta_s{}^4)\end{bmatrix}+\begin{bmatrix}GX^{0.5}\\ +GX^{0.5}\varepsilon_G(\theta-\theta_a)\end{bmatrix}=0 \quad (5)$$

$$X=0,\ \frac{d\theta}{dX}=0 \quad (6a)$$

$$X=1,\ \ \theta=1 \quad (6b)$$

where $\theta=\frac{T}{T_b}$, $\theta_a=\frac{T}{T_a}$, $\theta_s=\frac{T}{T_s}$ are the non-dimensional temperatures and $X=\frac{x}{L}$ is the non-dimensional length of the fin.

3. Methodology of MADM

The mathematical model of convex fin with multiple nonlinearities is an equation with singularity. Since the non-dimensional length X is measured from the fin tip for X=0, the equation fails to have a unique solution at this point and the solution jumps to infinity i.e. singularity [29] occurs at the fin tip. Therefore, classical method ADM fails at this point and modification of the method is required is to address the singular points.

Consider singular boundary value equations of the form:

$$\begin{aligned}
&U^{(n_1+1)}+\frac{m_1}{X}U^{n_1}+NU=g(X)\\
&U(0)=a_0\\
&U'(0)=a_1\\
&U''(0)=a_2\\
&\vdots\\
&U^{n-1}(0)=a_{n-1}\\
&U(b_1)=c_1
\end{aligned} \quad (7)$$

Here U is a linear term, N is a nonlinear term of order less than n_1, m_1 is the remainder term of the linear operator U, g(X) is the given analytic function and $a_0, a_1, a_2 \cdots a_{n-1}, b_1$ and c_1 is constants. The equation cannot be solved directly by using the linear term as it is done in classical ADM because of the remainder term [30]. Therefore, in this method both linear as well as the remainder terms are taken together to form a new differential operator [31] and subsequently this new differential operator is proposed for solving that particular singular value equations. The generalized differential operator is a function of derivatives of order (n_1+1), remainder, and the linear terms of Eq. (7) and can be expressed mathematically as below,

$$L_X(\bullet) = X^{-1}\frac{d^{n_1-1}}{dX^{n_1-1}}X^{n_1-m_1}\frac{d}{dX}X^{m_1-n_1+1}\frac{d}{dX}(\bullet) \tag{8}$$

Where $m_1 \leq n_1 - 1$ and $n_1 \geq 1$.

The Eq. (7) can be written as simple form as below.

$$L_X U + NU = g(X) \tag{9}$$

Now the above Eq. (9) can be solved for the linear term and the generalized inverse differential operator $L_X^{-1}(\bullet)$ in (n_1 +1) fold integration, which can be written as,

$$L^{-1}(\bullet) = \int_0^X X^{n_1-m_1-1}\int_0^X X^{m_1-n_1}\underbrace{\int_0^X \cdots \int_0^X dX \cdots dX}_{n_1-1} \tag{10}$$

Applying the generalized differential of the Eq. (9)

$$L_X^{-1}LU + L_X^{-1}NU = L_X^{-1}g(X) \tag{11}$$

The above methodology can be applied to the governing equation Eq. (5) of the convex fin and can be written as follows,

$$\begin{aligned}\frac{d^2\theta}{dX^2} + \frac{1}{2X}\left(\frac{d\theta}{dX}\right) = -G + G\varepsilon_G\theta_a - G\varepsilon_G\theta + \frac{Nr\theta^4}{X^{1/2}} - \frac{Nr\theta_s^{\,4}}{X^{1/2}} + \frac{NrB\theta^5}{X^{1/2}} - \frac{NrB\theta_s\theta^4}{X^{1/2}} - \frac{NrB\theta_s^{\,4}\theta}{X^{1/2}} + \\ \frac{NrB\theta_s^{\,5}}{X^{1/2}} + \frac{N_c(\theta-\theta_a)^{n+1}}{X^{1/2}} + A\theta_a\frac{d^2\theta}{dX^2} - A\left(\frac{d\theta}{dX}\right)^2 - A\theta\frac{d^2\theta}{dX^2} + \frac{A\theta_a}{2X}\left(\frac{d\theta}{dX}\right) - \frac{A}{2X}\left(\theta\frac{d\theta}{dX}\right)\end{aligned} \tag{19}$$

The LHS of the Eq. (19) contains only linear term as well as remainder term. The coefficient of the remainder term is $\frac{1}{2}$. Therefore, the new forward and inverse operator for the above singular value equation are introduced as below,

$$L_X(\bullet) = X^{-0.5}\frac{d}{dX}X^{0.5}\frac{d}{dX}(\bullet), \qquad \text{and} \tag{20}$$

$$L_X^{-1}(\bullet) = \int_0^X X^{-0.5}\int_0^X X^{0.5}(\bullet)dXdX \tag{21}$$

The Eq. (19) of the convex fin in operator form can be expressed as,

$$
\begin{aligned}
L_X\theta = &-G + G\varepsilon_G\theta_a - G\varepsilon_G\theta + \frac{Nr\theta^4}{X^{1/2}} - \frac{Nr\theta_s{}^4}{X^{1/2}} + \frac{NrB\theta^5}{X^{1/2}} - \frac{NrB\theta_s\theta^4}{X^{1/2}} - \frac{NrB\theta_s{}^4\theta}{X^{1/2}} + \frac{NrB\theta_s{}^5}{X^{1/2}} \\
&+ \frac{N_c(\theta-\theta_a)^{n+1}}{X^{1/2}} + A\theta_a\frac{d^2\theta}{dX^2} - A\left(\frac{d\theta}{dX}\right)^2 - A\theta\frac{d^2\theta}{dX^2} + \frac{A\theta_a}{2X}\left(\frac{d\theta}{dX}\right) - \frac{A}{2X}\left(\theta\frac{d\theta}{dX}\right)
\end{aligned}
\tag{22}
$$

In Eq. (22), $\theta^4, (\theta-\theta_a)^{n+1}, \left(\frac{d\theta}{dX}\right)^2, \theta\frac{d\theta}{dX}$ and θ^5 are the nonlinear, and $\theta, \frac{d^2\theta}{dX^2}, \frac{d\theta}{dX}$ are the linear terms respectively. The nonlinear and linear terms are expressed by Adomian polynomials and equation can be written as follows,

$$
\begin{aligned}
L_X\theta = &-G + G\varepsilon_G\theta_a - G\varepsilon_G\theta + \frac{Nr(NA)}{X^{0.5}} - \frac{Nr\theta_s{}^4}{X^{0.5}} + \frac{NrB(NB)}{X^{0.5}} - \frac{NrB\theta_s(NC)}{X^{0.5}} - \frac{NrB\theta_s{}^4\theta}{X^{0.5}} + \frac{NrB\theta_s{}^5}{X^{0.5}} \\
&+ \frac{N_c(ND)}{X^{0.5}} + A\theta_a(NE) - A(NF) - A(NG) + \frac{A\theta_a}{2X}(NH) - \frac{A}{2X}(NI)
\end{aligned}
\tag{23}
$$

Applying the inverse differential operators $\left(L_X{}^{-1}\right)$ on both sides of the Eq. (23),

$$
\begin{aligned}
\theta = \theta_0 &- GL_X{}^{-1}(1) + \varepsilon_G\theta_a GL_X{}^{-1}(1) - \varepsilon_G GL_X{}^{-1}\left(\sum_0^\alpha \theta_m\right) + N_r L_X{}^{-1}\left(\frac{\sum_0^\alpha A_m}{X^{0.5}}\right) - \theta_s{}^4 N_r L_X{}^{-1}\left(\frac{1}{X^{0.5}}\right) \\
&+ BN_r L_X{}^{-1}\left(\frac{\sum_0^\alpha B_m}{X^{0.5}}\right) - N_r\theta_s BL_X{}^{-1}\left(\frac{\sum_0^\alpha C_m}{X^{0.5}}\right) - N_r\theta_s{}^4 BL_X{}^{-1}\left(\frac{\sum_0^\alpha \theta_m}{X^{0.5}}\right) + N_r\theta_s{}^5 BL_X{}^{-1}\left(\frac{1}{X^{0.5}}\right) \\
&+ N_C L_X{}^{-1}\left(\frac{\sum_0^\alpha D_m}{X^{0.5}}\right) + \theta_a AL_X{}^{-1}\left(\sum_0^\alpha E_m\right) - AL_X{}^{-1}\left(\sum_0^\alpha F_m\right) - AL_X{}^{-1}\left(\sum_0^\alpha G_m\right) + \theta_a AL_X{}^{-1}\left(\frac{\sum_0^\alpha H_m}{2X}\right) \\
&- AL_X{}^{-1}\left(\frac{\sum_0^\alpha I_m}{2X}\right)
\end{aligned}
\tag{24}
$$

where, $\theta_0 = \theta(0) + X\frac{d\theta(0)}{dX} = C + 0$ of Eq. (24) must be a constant in order to satisfy the boundary condition. Higher order terms are obtained recursively for the Eq. (24) for finite series of order, say, m.

$$\theta_{m+1} = -GL_X^{-1}(1) + G\varepsilon_G\theta_a L_X^{-1}(1) - \varepsilon_G GL_X^{-1}\left(\sum_0^\alpha \theta_m\right) + N_r L_X^{-1}\left(\frac{\sum_0^\alpha A_m}{X^{0.5}}\right) - N_r\theta_s^4 L_X^{-1}\left(\frac{1}{X^{0.5}}\right)$$

$$+ BN_r L_X^{-1}\left(\frac{\sum_0^\alpha B_m}{X^{0.5}}\right) - N_r\theta_s BL_X^{-1}\left(\frac{\sum_0^\alpha C_m}{X^{0.5}}\right) - N_r\theta_s^4 BL_X^{-1}\left(\frac{\sum_0^\alpha \theta_m}{X^{0.5}}\right) + N_r\theta_s^5 BL_X^{-1}\left(\frac{1}{X^{0.5}}\right)$$

$$+ N_C L_X^{-1}\left(\frac{\sum_0^\alpha D_m}{X^{0.5}}\right) + \theta_a AL_X^{-1}\left(\sum_0^\alpha E_m\right) - AL_X^{-1}\left(\sum_0^\alpha F_m\right) - AL_X^{-1}\left(\sum_0^\alpha G_m\right) + \theta_a AL_X^{-1}\left(\frac{\sum_0^\alpha H_m}{2X}\right)$$

$$- AL_X^{-1}\left(\frac{\sum_0^\alpha I_m}{2X}\right) \quad with \quad m \geq 0 \tag{26}$$

In the present analysis, we consider the first three significant terms, i.e., for *m* = 0, 1 and 2, second, third and fourth component of temperature fields are expressed as below

$$\begin{aligned}
\theta_1 &= -GL_X^{-1}(1) + \varepsilon_G\theta_a GL_X^{-1}(1) - \varepsilon_G GL_X^{-1}\theta_0 + N_r L_X^{-1}\left(\frac{A_0}{X^{0.5}}\right) - \theta_s^4 N_r L_X^{-1}\left(\frac{1}{X^{0.5}}\right) \\
&+ BN_r L_X^{-1}\left(\frac{B_0}{X^{0.5}}\right) - N_r\theta_s BL_X^{-1}\left(\frac{C_0}{X^{0.5}}\right) - N_r\theta_s^4 BL_X^{-1}\left(\frac{\theta_0}{X^{0.5}}\right) + N_r\theta_s^5 BL_X^{-1}\left(\frac{1}{X^{0.5}}\right) \\
&+ N_C L_X^{-1}\left(\frac{D_0}{X^{0.5}}\right) + \theta_a AL_X^{-1}E_0 - AL_X^{-1}F_0 - AL_X^{-1}G_0 + \theta_a AL_X^{-1}\left(\frac{H_0}{2X}\right) \\
&- AL_X^{-1}\left(\frac{I_0}{2X}\right)
\end{aligned} \tag{27}$$

$$\begin{aligned}
\theta_2 &= -\varepsilon_G GL_{XX}^{-1}\theta_1 + N_r L_{XX}^{-1}\left(\frac{A_1}{X^{0.5}}\right) + BN_r L_{XX}^{-1}\left(\frac{B_1}{X^{0.5}}\right) - N_r\theta_s BL_{XX}^{-1}\left(\frac{C_1}{X^{0.5}}\right) \\
&- N_r\theta_s^4 BL_{XX}^{-1}\left(\frac{\theta_1}{X^{0.5}}\right) + N_C L_{XX}^{-1}\left(\frac{D_1}{X^{0.5}}\right) + \theta_a AL_{XX}^{-1}E_1 - AL_{XX}^{-1}F_1 - AL_{XX}^{-1}G_1 \\
&+ \theta_a AL_{XX}^{-1}\left(\frac{H_1}{2X}\right) - AL_{XX}^{-1}\left(\frac{I_1}{2X}\right)
\end{aligned} \tag{28}$$

$$\theta_3 = -\varepsilon_G G L_{XX}{}^{-1}\theta_2 + N_r L_{XX}{}^{-1}\left(\frac{A_2}{X^{0.5}}\right) + BN_r L_{XX}{}^{-1}\left(\frac{B_2}{X^{0.5}}\right) - N_r\theta_s BL_{XX}{}^{-1}\left(\frac{C_2}{X^{0.5}}\right)$$

$$- N_r\theta_s{}^4 BL_{XX}{}^{-1}\left(\frac{\theta_2}{X^{0.5}}\right) + N_C L_{XX}{}^{-1}\left(\frac{D_2}{X^{0.5}}\right) + \theta_a AL_{XX}{}^{-1}E_2 - AL_{XX}{}^{-1}F_2 - AL_{XX}{}^{-1}G_2 \tag{29}$$

$$+ \theta_a AL_{XX}{}^{-1}\left(\frac{H_2}{2X}\right) - AL_{XX}{}^{-1}\left(\frac{I_2}{2X}\right)$$

Thus, the local temperature field for the convex profile is written in terms of the following finite series,

$$\theta = \sum_0^m \theta_m = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots \tag{30}$$

4. Spectral Quasilinearization Method (SQLM)

The governing energy equation of heat transfer, Eq. (5), is solved using spectral quasilinearization method (SQLM) [32]. Quasilinearisation [33] is used to linearize nonlinear singular/ordinary differential equations. When a spectral method is used to solve the linearised equations, the method is called the spectral quasi-linearization method (SQLM). The SQLM assumes that the difference between the approximation of the solution at the current iteration level and the previous iteration is small. Also, the difference between the derivatives at the subsequent iteration levels is assumed to be small. The energy equation θ for the convex fin and related boundary conditions can be written as,

$$\Theta \equiv \{1 + A(\theta - \theta_a)\}x^{\frac{1}{2}}\theta'' + Ax^{\frac{1}{2}}(\theta')^2 + \frac{1}{2}x^{-\frac{1}{2}}\{1 + A(\theta - \theta_a)\}\theta' - N_c(\theta - \theta_a)^{n+1}$$

$$- N_r[1 + B(\theta - \theta_s)](\theta^4 - \theta_s{}^4) + Gx^{\frac{1}{2}}[1 + \varepsilon_G(\theta - \theta_a)] = 0 \tag{31}$$

The interval, $X \in [a,b]$ is transformed to the region $\zeta \in [-1,1]$ using the linear transformation

$$\zeta = \frac{1}{2}(b-a)x + \frac{1}{2}(b+a) \tag{32}$$

where a and b represent the space domain interval. The solutions, can be approximated by a bivariate Lagrange interpolation polynomial of the form

$$\Theta(\zeta) \approx \sum_{i=0}^{N_x}\theta(\zeta_i)\ell_i(\eta) \tag{33}$$

The bivariate Lagrange interpolation polynomial interpolates $\Theta(\eta)$ at carefully chosen grid points in η directions. The selected grids points are defined by

$$\{\zeta_i\} = \left\{\cos\left(\frac{\pi i}{N_x}\right)\right\}_{i=0}^{N_x} \tag{34}$$

The function $\ell_i(\zeta)$ is the characteristic Lagrange cardinal polynomial based on the Chebyshev-Gauss-Lobatto points.

$$\ell_i(\zeta) = \prod_{\substack{i=0 \\ i\neq k}}^{N_x} \frac{\zeta - \zeta_k}{\zeta_i - \zeta_k} \tag{35}$$

The characteristics Lagrange cardinal polynomial has the following property,

$$\ell_i(\zeta_k) = \delta_{ik} = \begin{cases} 0 & if\ i \neq k \\ 1 & if\ i \neq k. \end{cases} \tag{36}$$

The next crucial procedure is the evaluation of the space derivatives at the grid points $x_i \;\; (i = 0,1,.........., N_x)$. This procedure is termed collocation. The values of the space derivatives are computated at the Chebyshev-Gauss-Lobatto points (ζ_i) for the n^th^ order derivative defined as

$$\left.\frac{\partial^n \Theta}{\partial \zeta^n}\right|_{(\eta_i)} = \sum_{\rho=0}^{N_\eta} D_{i\rho}^n \Theta(\zeta_\rho) = D^n \Theta_i, \;\; i = 0,1,2,.............., N_\zeta, \tag{37}$$

Where the vector Θ_i is defined as

$$\Theta_i = \left[\theta(\zeta_0), \theta(\zeta_1), ...\theta(\zeta_{N_x})\right]^T \tag{38}$$

The superscript T in the above equation denotes a matrix transpose. Applying the collocation technique gives the following system of equations.

$$a_{0,r}\theta''_{r+1} + a_{1,r}\theta'_{r+1} + a_{2,r}\theta_{r+1} = R_\theta \tag{39}$$

$$A\Theta = R_\theta \tag{40}$$

where the matrices are given by

$$A = a_{0,r}D^2 + a_{1,r}D^1 + a_{2,r}I, \tag{41}$$

and the vectors written in a general form are given by

$$R_\theta = a_{0,r}\theta''_r + a_{1,r}\theta'_r + a_{2,r}\theta_r - \Theta \tag{42}$$

and the variable coefficients are given by

$$a_{0,r} = \frac{\partial\Theta}{\partial\theta''} = \{1 + A(\theta_r - \theta_a)\}x^{\frac{1}{2}}, \tag{43}$$

$$a_{1,r} = \frac{\partial\Theta}{\partial\theta'} = 2Ax^{\frac{1}{2}}\theta_r' + \frac{1}{2}x^{-\frac{1}{2}}\{1 + A(\theta_r - \theta_a)\}, \tag{44}$$

$$a_{2,r} = \frac{\partial\Theta}{\partial\theta} = Ax^{\frac{1}{2}}\theta_r'' + \frac{1}{2}x^{-\frac{1}{2}}A\theta_r' - (n+1)N_c(\theta_r - \theta_a)^n$$
$$- 4N_r\theta_r^{\,3}[1 + B(\theta_r - \theta_a)] - N_rB(\theta_r^4 - \theta_s^4) + Gx^{\frac{1}{2}}\varepsilon_G \tag{45}$$

Thus applying the spectral method as described above gives the matrix, A of size $N_x \times N_x$. The approximate solutions for Θ are obtained by solving Eq. (42) after approximate boundary conditions,

$\theta_{r+1}(0) = 0 \quad and \quad \theta_{r+1}(N_x) = 1$

5. Efficiency

The thermal performance of a fin is measured by its efficiency. It is defined by the ratio of the actual heat dissipation to the maximum possible heat dissipation if the entire fin is considered to be at the base temperature. Thus, the efficiency for the proposed fin is formulated in the following form [34]:

$$\eta = \frac{Q_f}{Q_{\max}} = \frac{\int_{x=0}^{L}\left[h_b\left(\frac{T-T_b}{T_b-T_a}\right)^n(T-T_a) + \sigma\varepsilon_s\{1+\beta(T-T_s)\}(T^4-T_s^4)\right]dx}{\frac{Lh_b}{(T_b-T_a)^n}(T_b-T_a)^{n+1} + L\sigma\varepsilon_s\{1+\beta(T_b-T_s)\}(T_b^4-T_s^4)} \tag{46}$$

where Q_f and $Q_{\max}$ are the actual heat transfer and maximum possible heat transfer respectively. The above equation is non-dimensionalised as,

$$\eta = \frac{\int_{X=0}^{1}\left[N_c(\theta-\theta_a)^{n+1} + N_r\left\{(\theta^4-\theta_s^{\,4}) + B(\theta^4-\theta_s^{\,4})(\theta-\theta_s)\right\}\right]dX}{N_c(1-\theta_a)^{n+1} + N_r\left\{(1-\theta_s^{\,4}) + B(1-\theta_s)(1-\theta_s^{\,4})\right\}} \tag{47}$$

To obtain the fin efficiency, the numerator of Eq. (47) is solved numerically using trapezoidal rule.

6. Inverse modeling

An inverse study has a great significance for estimating the possible combination of key thermal parameters that can be used for an efficient fin design. The main purpose of the inverse study is to predict the various combinations of the unknown thermal parameters, merely from the known temperature field. The ~~predefined~~ temperature field is obtained either directly from experiment or from the theoretical estimation. Thus, the predefined temperature field is the direct solution of the problem and well-posed which can be measured directly using the known values of thermal parameters. Let the temperature field θ be measured from the temperatures θ_i at $x = x_i$ locations. Then using the data of the measured temperature, one can inversely estimate the associated thermal parameters over the specified temperature domain. The thermal parameters so obtained can possess multiple values within the prescribed limits or the solutions are non-unique. Thus, the problem turns ill-posed and the solution is obtained following predicted temperature field, θ_j^*, is closely matched with the direct temperature field:

$$\Re(x) = \sum_{i=1}^{\xi} \left[\theta(x_i) - \theta_j^*(x_i) \right]^2 \tag{48}$$

Here, ξ is the number of measurement points where the values of the objective function in each of iterations is to be evaluated. A direct search method, Nelder-Mead [35] based on simplex search optimization technique has been employed to optimize the inverse parameters. The key advantage of this method is that it is a derivative-free downhill optimization process which requires few function evolutions in each of iterations. In this method of optimization, only p+1 points are required at the initial simplex for p number of variables. It is to be noted that all the points should not lie along the same line. The algorithm is associated with the following possible operations:

(a) reflection, (b) expansion, (c) contraction, and (d) shrinking

Now, the initial simplex is constructed and the worst vertex is identified. At every iteration, the order of the vertices $\{x_j\}_{j=1}^{p+1}$ is in accordance to function,

$$g(x_1) \le g(x_2) \le ..... \le g(x_{p+1})$$

where x_1 is referred as the best vertex, and to x_{p+1} as the worst vertex. The worst vertex is replaced by a new vertex in every successive iteration. The simplex is updated by steering the search away from the worst point. For details pertaining the algorithm of the present simplex model, the readers is referred to the article [36].

7. Results and discussion

Results of parabolic convex fin are presented for multiple variable thermal parameters. A closed form solution for the nonlinear singular energy balance equation using the novel semi-analytical methods is investigated. The SQLM is a numerical technique that is used for solving both singular and non-singular differential equation. Further, the MADM results for finite values of thermo-physical parameters are compared with DTM results in Fig. 2. The quantitative values of Torabi and Zhang [37], presented in the figure, were extracted directly from their graphs using plot-digitizer software and higher order terms associated with heat generation in the present calculation are neglected. Close agreement is observed and the absolute error differences near the tip are found to be 0.0013 (n = 0, $G = \varepsilon_G$ = 0.1), 0.0046 (n = 2, $G = \varepsilon_G$ = 0.1) and 0.0012 (n = 2, $G = \varepsilon_G$ = 0). For completeness of the solution, the MADM results are further compared with the SQLM based numerical solution considering all the values of the power index. The results for the tip temperatures considering the parameters $A = B = 0.2$, $N_r = N_c = \theta_s = \theta_a = 0.25$, $G = 0.1$ and $\varepsilon_G = 0.2$ are presented in Table 1. For all the cases, the absolute decomposition errors near the tip are estimated to be below 0.8 %. Once the correctness of the present MADM solution is established, a parametric study is conducted. Unless mentioned otherwise, the results are presented for $A = 0.2$, $B = 0.2$, $G = 0.1$, $\varepsilon_G = 0.2$, $n = 2$, $N_r = 0.25$, $N_c = 0.25$, $\theta_s = 0.25$, $\theta_a = 0.25$. The effect of the variation of thermal conductivity parameter, A, on the temperature field for different values of power exponent of convective heat transfer coefficient, n, is illustrated in Fig.3. The temperature field increases with the increase in the parameter A for all the cases. The heat transfer is more through the material having higher values of A. The tip temperature increases with the increase of the value of n, and the maximum tip temperature is obtained at n =3. This result is in good agreement with the result of Sun and Xu [38]. Figure 4 depicts the effect of the surface emissivity parameter, B, on the temperature field for different values of n. The reduction in the tip temperature is due to continuous heat loss from the fin surface to the surroundings. When B =0, the heat transfer is attributed to the constant radiation-conduction parameter. The relative reduction in the tip temperature is further accelerated with a decrease in the parameter n. Figures 5 and 6 show the effects of the convection-conduction (N_c) and conduction-radiation (N_r) parameters on the local temperature distribution for different values of the power exponent n. It can be seen that in both the cases the higher temperature gradient is associated with the higher values of the parameters, N_c and N_r. As a result, the tip temperatures are reduced by the increase in the values of these parameters. This incident is due to heat loss from the surface to the surrounding which enhances with the increase of N_c and N_r. This can be alternatively explained that as the parameters N_c and N_r increase due to increase of conductive heat transfer resistance ($R_{cond} \propto 1/k_a$), the local temperature distribution reduces. Figure 7 illustrates the effect of the heat generation number (G) on the local temperature field for different values of the power exponent, n. It is interesting that for each value of the power exponent n, the temperature difference at the tip, $\Delta\theta_{tip} \approx 0.021$, was obtained when the value of G changes from 0.1 to 0.2. This result suggests that the variations in local temperature distribution are independently affected by the internal heat generation parameter and the power index. The heat transfer coefficient has obvious impact on the fin tip temperature, and film boiling is most affected as compared to other regimes of pool boiling curves or constant heat transfer coefficient.The performance of the fin is characterized by its efficiency. Figures 8 and 9 depict the variation of fin efficiency as a function of convection-conduction parameter (N_c) and conduction-radiation parameter (N_r). The physical interpretation of the parameter, N_c, is the ratio of axial conduction resistance to the surface convection resistance ($\approx R_{cond} / R_{conv}$). When the parameter N_c increases, the resistance is dominated by the conduction parameter, and hence reduces the fin efficiency as shown in Fig.8. On the other hand, the fin efficiency monotonically decreases with increase in N_r due to the radiative heat loss as shown in Fig.9. In both the figures, it is observed that the variation of efficiency is not consistent with the value of power exponent. Further, it can be seen that the fin efficiency increases with the increase of heat generation parameter in both the cases, as extra local temperature is produced in the fin. This result (efficiency as a function of heat generation) is consistent with the results of Sobamowo [39]. It is well understood that the heat generation leads to reduce the performance of fin. So the name of 'efficiency' is somehow misleading here – it implies that higher efficiency means better fin. But, this is not only the necessary condition. The primary purpose of fin is to remove heat from the surface. As the heat generation increases (i.e the local temperature increases) the efficiency also increases ($\eta \rightarrow \infty$ in the limit of $G \rightarrow \infty$) which defeats the purpose of fin. A similar argument is also applicable when the length of the fin is too small (i.e $L \rightarrow 0$). The real application of the fin efficiency is that, primarily it is a function of N_c and N_r. The thermal and geometric parameters play an important role in transferring heat, and thus, the overall performance of the fin. In this study, the key thermal parameters such as A, B, G, N_c, and N_r are regarded as unknown parameters, that are to be estimated inversely from a well-defined temperature field. Multiple combinations of those thermal parameters for a given temperature distribution are simultaneously optimized using the Neldar-Mead simplex search method (SSM) by minimizing the objective function. A fin designer can readily design the fin materials by choosing an appropriate combination of parameters from the various possible alternatives of the inverse solution. This may help to reduce manufacturing costs and the weight of the fin without compromising the performance. In the SSM inverse process, the initial guessed parameters, invariably different from the direct problem, are to be used in every simulation. The values of the parameters continuously change with the iteration until the required temperature field is obtained by minimizing the objective function. The parameters, A = 0.2, B = 0.2, G = 0.1, ε_G = 0.1, n = 2, N_r = 0.5, N_c = 0.5, θ_s = 0.5, θ_a = 0.5, were taken for the forward solution of inverse analysis. Table 2 shows the inverse estimation values of three parameters, A, B and G, considering error free measurement. The search starts with random initial guessed values for 100 and 50 iterations. Multiple combinations of unknown parameters are obtained, and the values of the parameters depend upon the initial values, as well as the iteration numbers. The convergence studies in each run are illustrated in Fig. 10. It is evident that the function values converge very fast (<50 iterations) and are continuously refined with an increase in the number of iterations (Fig. 10a). The fitness value as a function of function evaluation number is presented in Fig. 10b. Next, the challenge in this inverse study is to incorporate the additional two parameters, N_c and N_r, as unknown parameters. The iterative variations and the convergence histories of simultaneous estimation of these five parameters are reported in Table 3. The pictorial representations of the convergence histories of the evolution numbers is depicted in Fig. 11. Similar to three unknown parameters estimation, the result reveals that the fitness values gradually decrease with an increase in function evaluation numbers as shown in Fig. 11(a). The insect in Fig. 11(a) represents the convergence history of function value as a function of iteration number. On the other hand, the values of the inverse parameters are randomly adjusted with the function evaluation number, and finally become stable after reaching a particular function evaluation numbers as shown in Fig. 11(b) – (f). Slower convergence in function values is observed in five parameters estimation when compared with three parameters estimation. A comparison between the reconstructed temperature fields and the exact analytical solution [40] is represented in Fig. 12. As expected, the result demonstrates that the reconstructed temperature fields are in good agreement with the forward temperature field.

8. Conclusions

The results of MADM are verified with the results of SQLM and DTM. The results were found to be in good agreement for different thermal parameters of heat transfer processes. Results reveal that heat is transferred from the base to the tip of the fin and the local temperature gradually decreased towards the tip. For all the thermal parameters, the tip temperature monotonically increases with the increase of power index, n. The conductivity parameters, A, increase the relative heat transfer rate in the fin material. On the other hand, the increase in the parameters, B, N_c and N_r, reduces the local temperature due to heat loss from the surface to the surroundings. The fin efficiency was estimated for different values of G, N_c, N_r, and n. The fin efficiency reduces with an increase in parameters N_c and N_r. In addition, an inverse analysis for simultaneous estimation of the unknown thermal parameters has been performed using simplex search method. The reconstructed temperature field obtained from the inverse parameters agrees with the forward temperature field. The results and the technical approach of this study will serve as the development of mathematical modeling for improving the fin design in heat transfer equipment.

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