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RESEARCH PAPER

WEIGHTED FRACTIONAL HARDY OPERATORS AND THEIR COMMUTATORS ON GENERALIZED MORREY SPACES OVER QUASI-METRIC MEASURE SPACES

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Abstract

We study commutators of weighted fractional Hardy-type operators within the frameworks of local generalized Morrey spaces over quasi-metric measure spaces for a certain class of "radial" weights. Quasi-metric measure spaces may include, in particular, sets of fractional dimentsions. We prove theorems on the boundedness of commutators with CMO coefficients of these operators.

Given a domain Morrey space $\mathcal{L}^{p,\varphi}(X)$ for the fractional Hardy operators or their commutators, we pay a special attention to the study of the range of the exponent q of the target space $\mathcal{L}^{q,\psi}(X)$. In particular, in the case of classical Morrey spaces, we provide the upper bound of this range which is greater than the known Adams exponent.

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1. Introduction

We study commutators of weighted fractional Hardy-type operators $wH^\alpha\frac{1}{w}$ and $w\mathcal{H}^\alpha\frac{1}{w}$ within the frameworks of local Morrey spaces $\mathcal{L}^{p,\varphi}(X,w)$ over quasi-metric measure spaces (X,d,μ) for a certain class of "radial"

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weights w. The precise definition of the Hardy operators H^α and $\mathcal{H}^\alpha$ is given in Section 3.

In the Euclidian case they have the form

$$H^\alpha f(x) = |x|^{\alpha-n} \int\limits_{|y|<|x|} f(y)dy \text{ and } \mathcal{H}^\alpha f(x) = \int\limits_{|y|>|x|} |y|^{\alpha-n} f(y)dy, \ x \in \mathbb{R}^n, \tag{1.1}$$

$\alpha \in \mathbb{R}$, and their one-dimensional versions are usually considered in the form

$$H^\alpha f(x) = x^{\alpha-1} \int\limits_{y<x} f(y)dy \ \text{ and } \ \mathcal{H}^\alpha f(x) = \int\limits_{y>x} y^{\alpha-1} f(y)dy, \ x \in \mathbb{R}_+, \tag{1.2}$$

Such operators are called fractional Hardy operators.

The operators (1.1) and (1.2) do not satisfy the semigroup properties but possess a certain substitution of such property. In the case of the one-dimensional operator H^α this substitution has the form

$$H^\alpha H^\beta f(x) = \frac{1}{\beta}\left[H^{\alpha+\beta} f(x) - H_\beta^{\alpha+\beta} f(x)\right], \ \beta \neq 0,$$

where $H_\beta^{\alpha+\beta} f(x) = x^{\alpha+\beta-1} \int\limits_0^x \left(\frac{t}{x}\right)^\beta f(t)dt$.

The first operator with $\alpha = 0$ in (1.2) is known in the literature as the Cesàro operator.

Our interest to the study of operators in Morrey spaces is caused both by their wide use in applications in PDEs and by the fact that Morrey spaces provide more possibilities and flexibility for obtaining conditions for the boundedness of operators. Note that recently in [30] there was shown that Morrey spaces and the so called complementary Morrey spaces provide an effective language for describing of ingrability properties of integral transforms such as Laplace, Hankel and others.

Note that in contrast to Lebesgue spaces, we can admit negative values of α when considering operators H^α and $\mathcal{H}^\alpha$ in Morrey spaces.

There exists vast literature on operators H^α in L^p spaces, we refer only to the book [18] and references therein. The operators $\mathcal{H}^\alpha$ in L^p spaces are not so much studied in the literature by a natural reason: they may be treated by duality arguments. This does not work in case of Morrey spaces.

The multi-dimensional Hardy operators H^α and $\mathcal{H}^\alpha$ in Morrey spaces in weighted setting were studied in [21], [24], [26], [27], [18, Ch.7]. Commutators with CMO coefficients, of the weighted fractional Hardy operator H^α in weighted Morrey spaces in the Euclidian case were studied in [28].

Note also that Hausdorff-type generalization of the one-dimensional Cesàro operator and its commutators were studied in [10]. We refer, for instance to [19] for the notion of Hausdorff operators.

In general, commutators of many operators of harmonic analysis are known to be widely investigated in various function spaces due to their applications, in particular, in theory of PDE, see for instance, the books [11, 17, 32] and for papers [3, 4, 5, 12, 23, 34].

We prove the weighted $\mathcal{L}^{p,\varphi}(X) \to \mathcal{L}^{q,\psi}(X)$-boundedness of commutators with CMO coefficients of the "adjoint" operator $\mathcal{H}^\alpha$ in the frameworks of generalized Morrey spaces (recall that duality arguments do not work in the case of Morrey spaces). Moreover, we prove this in the general setting of quasi-metric measure spaces and admit both the situations $q \geq p$ and $q < p$. We also prove estimates for commutators of the operator H^α in Morrey spaces over (X, d, μ), improving and generalizing a result from [28].

We admit "radial" weights $w(d(x, x_0)), x_0 \in X$, depending on the distance to the point x_0, the singular point of the Hardy operators. Note that a characterization of weights for various operators in generalized Morrey spaces is mostly an open problem and even admission of power weights is often a subject of essential research, see, for instance, [7] and [8] and references therein. Admission of such radial-type weights allows to use Matuszewska-Orlicz indices for obtaining effective conditions on admissible weights.

Given a domain space $\mathcal{L}^{p,\varphi}(X), \ \ 1 < p < \infty$, for the fractional Hardy operators or their commutators, we pay a special attention to the study of the range of the exponent q of the target space $\mathcal{L}^{q,\psi}(X)$ in dependence on p, α and φ. In particular, in the case of classical Morrey spaces, i.e. $\varphi(r) = r^\lambda$ we show that the upper bound $q_{\sup}$ of that range is greater than the Adams exponent $q^\sharp$ defined by $\dfrac{1}{q^\sharp} = \dfrac{1}{p} - \dfrac{\alpha}{\nu - \lambda}$:

$$q_{\sup} = \frac{\nu}{\nu - \lambda} q^\sharp,$$

where ν comes from the growth condition and $0 < \lambda < \nu$. In the case of generalized Morrey spaces, a similar formula for $q_{\sup}$ holds with λ replaced by $\min\{m_0(\varphi), m_\infty(\varphi)\}$, where $m_0(\varphi), m_\infty(\varphi)$ are Matuszewska-Orlicz indices of φ at the origin and infinity.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on quasi-metric measure spaces, generalized Morrey spaces, CMO_p-spaces, quasi-monotone functions and their Matuszewska-Orlicz indices and prove some technical lemmas. Section 3 contains our main results. In Section 3.1 for the $\mathcal{L}^{p,\lambda}(X) \to \mathcal{L}^{q,\gamma}(X)$-boundedness of the Hardy operators H^α and $\mathcal{H}^\alpha$ themselves, we investigate the range of admissible values of the exponents q of the target space. In Section 3.1 the space (X, d, μ) is not assumed to be homogeneous but is supposed to satisfy the growth condition. In Section 3.2 we study a similar boundedness of commutators of the operator $w\mathcal{H}^\alpha \frac{1}{w}$. In Section 3.3 we give a similar result for commutators of the operator $wH^\alpha \frac{1}{w}$. In Sections 3.2 and 3.3 we assume that (X, d, μ) is homogeneous and satisfies the growth condition.

2. Preliminaries

2.1. **On quasi-metric measure spaces.** For the basics of quasi-metric measure spaces we refer e.g. to [6], [9] and [14]. Below we provide necessary definitions which we use in the paper.

Let (X, d, μ) be a quasi-metric measure space with Borel regular measure μ and quasi-distance d :

$$d(x, y) \leq k[d(x, z) + d(y, z)], \quad k \geq 1 \tag{2.1}$$

$d(x, y) = 0 \Longleftrightarrow x = y, \quad d(x, y) = d(y, x)$ and $\ell = \operatorname{diam} X, \ 0 < \ell \leq \infty$, $B(x, r) = \{y \in X : d(x, y) < r\}$. Everywhere in the sequel we suppose that the following properties of (X, d, μ) hold:

1) all balls are open sets;

2) the spheres $S(x, r) := \{y \in X : d(y, x) = r\}$ have zero measure for all x and r;

3) $\mu B(x, r)$ is continuous in $r \in [0, \ell)$ for every $x \in X$.

The set (X, d, μ) is said to satisfy the *growth condition* if there exist a constant $A > 0$ and exponent $\nu > 0$, which is fractional in general, such that

$$\mu B(x, r) \leq A r^\nu, \tag{2.2}$$

where $x \in X$ and $r \in (0, \ell)$.

The space (X, d, μ) is called *homogeneous*, if the measure is doubling: $\mu B(x, 2r) \leq c \mu B(x, r), \ x \in X, \ 0 < r < \frac{\ell}{2}$.

Estimates of the type provided by the lemma below are known, see for instance [9].

Lemma 2.1. *Let (X, d, μ) satisfy the growth condition* (2.2) *and $\beta > 0$. Then*

$$\int\limits_{B(x,r)} \frac{d\mu(y)}{d(x, y)^{\nu-\beta}} \leq C\, r^\beta. \tag{2.3}$$

2.2. **Morrey spaces on (X, d, μ).** Let E be an arbitrary measurable set in (X, d, μ). We define the generalized Morrey space $\mathcal{L}^{p,\varphi}_E(X)$, related to the set E as the set of measurable functions defind on X, such that

$$\|f\|_{\mathcal{L}_E^{p,\varphi}}(X) := \sup_{x\in E}\sup_{r>0}\left(\frac{1}{\varphi(r)}\int\limits_{B(x,r)}|f(y)|^p\,d\mu(y)\right)^{\frac{1}{p}} < \infty, \tag{2.4}$$

where $\varphi(r)$ is a non-negative measurable function on $[0,\ell]$, satisfying certain assumptions.

The space $\mathcal{L}_E^{p,\varphi}(X)$ is called *local* or *global* in the cases $E = \{x_0\}$, where x_0 is a fixed point in X, or $E = X$, respectively, $\mathcal{L}_E^{p,\varphi}(X) \hookrightarrow \mathcal{L}_{\{x_0\}}^{p,\varphi}(X)$ for any set $E \ni x_0$. Omission of E in writing $\mathcal{L}^{p,\varphi}(X)$ will just mean that the corresponding statement and arguments there do not depend on choice of the set E.

We refer for Morrey spaces on quasi-metric measure spaces, for instance, to [21], [29], [33], [35] and also the survey [25]. More on Morrey spaces and their applications can be found in the two-volume book [32] of Y. Sawano et al.

When (X, d, μ) satisfies the growth condition, then the condition

$$\sup_{0<r<\ell}\frac{r^\nu}{\varphi(r)} < \infty. \tag{2.5}$$

is sufficient for bounded functions with compact support to belong to the space $\mathcal{L}_E^{p,\varphi}(X)$.

When $\varphi(r) = r^\lambda$, we also use the notation

$$\mathcal{L}_E^{p,\lambda}(X) = \mathcal{L}_E^{p,\varphi}(X)\big|_{\varphi(r)=r^\lambda}$$

and this space is the classical Morrey space on E.

In the case $\varphi(r) = r^\lambda$, $X = \mathbb{R}^n$ and $x_0 = 0$, local Morrey spaces are also known ([2]) as *central classical Morrey spaces*.

With regard to the weighted operators, note that, given an operator A and a weight w, the boundedness of its weighted version, i.e. $wA\frac{1}{w}$ in the Morrey space $\mathcal{L}_E^{p,\varphi}$, is equivalent to the boundedness of the operator A itself in *the weighted Morrey space* $\mathcal{L}^{p,\varphi}(X, w)$, defined by the norm

$$\|f\|_{\mathcal{L}^{p,\varphi}(X,w)} := \sup_{x\in E}\sup_{r>0}\left(\frac{1}{\varphi(r)}\int\limits_{B(x,r)}|w(y)f(y)|^p\,dy\right)^{\frac{1}{p}}, \tag{2.6}$$

i.e in the form where the weight w and the function φ are independent of each other. We went into these details to avoid a misunderstanding in terminology: sometimes weighted Morrey spaces are introduced in a specific way, with the function φ depending on the weight w.

Lemma 2.2. *Let the space (X, d, μ) satisfy the growth condition* (2.2) *and let φ and ψ be positive functions on $(0, \ell)$. The embedding*

$$\mathcal{L}_E^{q,\psi}(X) \hookrightarrow \mathcal{L}_E^{p,\varphi}(X), \ 1 \le p \le q < \infty, \tag{2.7}$$

holds if

$$\sup_{r \in (0,\ell)} r^{\nu\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{\psi(r)^{\frac{1}{q}}}{\varphi(r)^{\frac{1}{p}}} < \infty.$$

In particular,

$$\mathcal{L}_E^{q,\gamma}(X) \hookrightarrow \mathcal{L}_E^{p,\lambda}(X), \ 1 \le p \le q < \infty, \ 0 < \gamma \le \lambda \le \nu, \tag{2.8}$$

if $\frac{\nu-\gamma}{q} \le \frac{\nu-\lambda}{p}$, when X is bounded and $\frac{\nu-\gamma}{q} = \frac{\nu-\lambda}{p}$, when X is unbounded.

P r o o f. The proof of the inequality $\|f\|_{p,\varphi;E} \le \|f\|_{q,\psi;E}$ is straightforward via the Hölder inequality and growth condition. □

Everywhere in the sequel we assume that the function φ defining Morrey space, is continuous in a neighborhood of the origin, almost increasing and satisfies the conditions

$$\varphi(0) = 0 \text{ and } \inf_{\delta < r < \ell} \varphi(r) > 0 \quad \text{for every} \quad \delta > 0. \tag{2.9}$$

Recall that a non-negative function φ on $(0, \ell)$ is called *almost increasing (almost decreasing)*, if there exists a constant $C (\ge 1)$ such that $\varphi(t) \le C\varphi(\tau)$ for all $t \le \tau$ ($t \ge \tau$, respectively). We use the abbreviation a.i for "almost increasing" and a.d. for "almost decreasing".

2.3. **On CMO$_p(X, x_0)$-spaces.** The BMO(X)-space, as is well known, is defined by the quasi-norm

$$\|a\|^* = \sup_{x \in X} \sup_{r > 0} \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |a(z) - a_{B(x,r)}| \, d\mu(z), \tag{2.10}$$

where $a_{B(x,r)} := \frac{1}{\mu B(x,r)} \int_{B(x,r)} a(z) d\mu(z)$. The BMO($X$) is an appropriate class of coefficients for commutators of many classical operators.

For the Hardy-type operators (3.1) and (3.2) with singular points only at x_0 and infinity (the latter in the case X is unbounded), a wider class of coefficients, with BMO-type-behavior only at $x_0 \in X$, is more appropriate. Such a local version of BMO(X), the space CMO(X, x_0) (*central mean oscillation*) is defined by the norm

$$\|a\|^* := \sup_{r > 0} \frac{1}{\mu B(x_0,r)} \int\limits_{B(x_0,r)} |a(z) - a_{B(x_0,r)}| \, d\mu(z), \tag{2.11}$$

We also need its generalization, the space $\mathrm{CMO}_p(X, x_0)$ depending on p, defined by the norm

$$\|a\|_p^* := \sup_{r>0} \left(\frac{1}{\mu B(x_0, r)} \int\limits_{B(x_0,r)} |a(z) - a_{B(x_0,r)}|^p \, d\mu(z) \right)^{\frac{1}{p}}, \qquad (2.12)$$

so that $\mathrm{CMO}(X, x_0) = \mathrm{CMO}_p(X, x_0)\big|_{p=1}$. However, contrast to the "global" $\mathrm{BMO}(X)$-space, such local $\mathrm{CMO}_p(X, x_0)$-spaces no more are independent of p. By Jensen inequality we have

$$\|a\|^* \leq \|a\|_p^* \leq \|a\|_q^*$$

and $\mathrm{BMO}(X) \subset \mathrm{CMO}_q(X, x_0) \subset \mathrm{CMO}_p(X, x_0) \subset \mathrm{CMO}(X, x_0)$, $1 < p < q < \infty$.

We refer to [2], [13] and [20] for the study of the classes $\mathrm{CMO}_p(X, x_0)$ in the case $X = \mathbb{R}^n$.

Lemma 2.3. *Let (X, d, μ) be a homogeneous space, $a \in CMO\ (X, x_0)$. Then*

$$\left| a_{B(x_0,r)} - a_{B(x_0,t)} \right| \leq C\|a\|^* \left(1 + \left| \ln \frac{t}{r} \right| \right) \quad \textit{for} \quad r, t \in \mathbb{R}_+. \qquad (2.13)$$

The statement of Lemma 2.3 is known for BMO-functions in the case $X = \mathbb{R}^n$ and goes back to [15]. In [28, Lemma 2.1] it was shown that the statement of this lemma is true also in the local setting of $\mathrm{CMO}(X, x_0)$-functions on $\mathbb{R}^n$. The proof in [28] remains the same for homogeneous quasi-metric measure spaces: the only difference from the Euclidian case is that one has to use the property that $\frac{\mu B(x,\lambda r)}{\mu B(x,r)} \leq c\lambda^\sigma$, $\lambda \geq 1$, for some $\sigma > 0$, which is valid for homogeneous spaces.

2.4. **Quasi-monotone functions and their indices.** Let $0 < \ell \leq \infty$.

Definition 2.1. By $\overline{W} = \overline{W}(0, \ell)$ we denote the class of non-negative functions on $(0, \ell)$ satisfying the conditions

1) $w(t)$ is continuous in neighborhoods of the origin and infinity (the latter in the case $\ell = \infty$), and $0 < \inf\limits_{\delta<t<\ell} w(t) \leq \sup\limits_{\delta<t<\ell} w(t) < \infty$ if $\ell < \infty$ and $0 < \inf\limits_{\delta<t<N} w(t) \leq \sup\limits_{\delta<t<N} w(t) < \infty$ if $\ell = \infty$, for some $0 < \delta < N$.

2) there exists $\beta \in \mathbb{R}$ such that $t^\beta w(t)$ is a.i. on $(0, \ell)$.

Similarly by $\underline{W} = \underline{W}(0,\ell)$ we denote the class of non-negative functions on $(0,\ell)$ satisfying the above condition 1) and the condition that there exists $\beta \in \mathbb{R}$ such $t^\beta w(t)$ is a.d. on $(0,\ell)$.

We will also use the notation

$$W = W(0,\ell) := \overline{W} \cup \underline{W}, \ 0 < \ell \le \infty.$$

Matuszewska-Orlicz indices ([22]) of functions $\varphi \in W(0,\ell)$ related to the origin and infinity, are defined by

$$m_0(\varphi) = \sup_{0<r<1} \frac{\ln \left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)} \right)}{\ln r} \quad \text{and} \quad M_0(\varphi) = \inf_{r>1} \frac{\ln \left(\limsup\limits_{h\to 0} \frac{\varphi(rh)}{\varphi(h)} \right)}{\ln r} \tag{2.14}$$

and (in the case $\ell = \infty$)

$$m_\infty(\varphi) = \sup_{r>1} \frac{\ln \left[\liminf\limits_{h\to\infty} \frac{\varphi(rh)}{\varphi(h)} \right]}{\ln \ r}, \quad M_\infty(\varphi) = \inf_{r>1} \frac{\ln \left[\limsup\limits_{h\to\infty} \frac{\varphi(rh)}{\varphi(h)} \right]}{\ln \ r}. \tag{2.15}$$

We refer to the properties of these indices to [27, Section 6], where they are presented in a form convenient for our goals. Note, in particular, that

$$m_0(t^a\varphi(t)^b) = a + bm_0(\varphi) \ \text{ and } \ M_0(t^a\varphi(t)^b) = a + bM_0(\varphi), \ a \in \mathbb{R}, \ b \in \mathbb{R}_+, \tag{2.16}$$

and similarly for the indices related to infinity. Also

$$\int_0^r \frac{w(t)}{t} dt \le cw(r), \ 0 < r < \ell, \Leftrightarrow \begin{cases} m_0(w) > 0, & \text{if } \ell < \infty, \\ \min\{m_0(w), m_\infty(w)\} > 0, & \text{if } \ell = \infty, \end{cases} \tag{2.17}$$

and

$$\int_r^\infty \frac{w(t)}{t} dt \le Cw(r), \ r \in \mathbb{R}_+ \ \Leftrightarrow \max\{M_0(w), M_\infty(w)\} < 0. \tag{2.18}$$

We refer also to the paper [16] where such indices were used to describe mapping properties of fractional integrals in weighted generalized Hölder spaces.

2.5. **Some technical lemmas.** The inequalities of the following "sum-to-integral" lemma are known been dispersed in the literature. Their proof is straightforward by using monotonicity properties of functions in the class $\mathcal{W}$, see for instance [28, Lemma 2.4].

LEMMA 2.4. *Let $g_1 \in \overline{W} \cap \Delta_2$ and $g_2 \in \underline{W}$. Then*

$$\sum_{k=0}^{\infty} g_1(2^{-k}r)\ g_2(2^{-k}r) \le C \int_0^r g_1(t)\ g_2(t) \frac{dt}{t},\ 0 < r < \ell \tag{2.19}$$

and

$$\sum_{0 \le k < \ln_2 \frac{\ell}{r}} g_1(2^k r)\ g_2(2^k r) \le C \int_r^{\ell} g_1(t)\ g_2(t) \frac{dt}{t},\ 0 < r < \frac{\ell}{2}. \tag{2.20}$$

The following lemma of similar type is slightly more general and is given with the proof.

LEMMA 2.5. *Let $g_1 \in \overline{W} \cap \Delta_2$ and $g_2 \in \underline{W}$. Then*

$$\sum_{k=0}^{\infty} g_1(2^{-k}r)\ g_2(2^{-k}r) \left(1 + \ln \frac{t}{2^{-k}r}\right) \le C \int_0^r g_1(s)\ g_2(s) \left(1 + \ln \frac{t}{s}\right) \frac{ds}{s} \tag{2.21}$$

for $0 < r \le t < \ell$, and

$$\sum_{0 \le k < \ln_2 \frac{\ell}{r}} g_1(2^k r)\ g_2(2^k r) \left(1 + \left|\ln \frac{t}{2^k r}\right|\right) \le C \int_r^{\ell} g_1(s)\ g_2(s) \left(1 + \left|\ln \frac{t}{s}\right|\right) \frac{ds}{s} \tag{2.22}$$

for $0 < r \le t < \frac{\ell}{2}$.

P r o o f. The inequality (2.21) follows from (2.19), but (2.22) requires the proof because the function $1 + \left|\ln \frac{t}{s}\right|$ changes monotonicity at the point $s = t \in (r, \ell)$. Let $\ell = \infty$ for simplicity. We have

$$\sum_{k=0}^{\infty} g_1(2^k r)\ g_2(2^k r) \left(1 + \left|\ln \frac{t}{2^k r}\right|\right)$$

$$= \frac{1}{\ln 2} \sum_{k=0}^{\infty} g_1(2^k r)\ g_2(2^k r) \left(1 + \left|\ln \frac{t}{2^k r}\right|\right) \int_{2^k r}^{2^{k+1} r} \frac{ds}{s}.$$

Denote $\xi = \frac{2^k r}{t}$ and $\sigma = \frac{s}{t}$. So that

$$\frac{2^k r}{t} < s < \frac{2^{k+1} r}{t} \Leftrightarrow \frac{\sigma}{2} < \xi < \sigma.$$

It is easy to show that $1+|\ln\xi|\le c(1+|\ln\sigma|)$ with $c=1+\ln 2$, when $\frac{1}{2}\le\frac{\xi}{\sigma}\le 1$. Consequently,

$$\sum_{k=0}^{\infty} g_1(2^k r)\ g_2(2^k r)\left(1+\left|\ln\frac{t}{2^k r}\right|\right)$$

$$\le c\sum_{k=0}^{\infty} g_1(2^k r)\ g_2(2^k r)\int\limits_{2^k r}^{2^{k+1}r}\left(1+\left|\ln\frac{t}{s}\right|\right)\frac{ds}{s},$$

after which it remains to use the monotonicity properties of the functions g_1 and g_2. □

Estimate (2.23) in the lemma below was proved in [28, Lemma 3.5.]. The proof of (2.24) follows the same lines.

Lemma 2.6. *If $g\in\overline{W}(\mathbb{R}_+)$ and $\min\{m_0(g),m_\infty(g)\}>0$, then*

$$\int_0^t g(s)\left(1+\ln\frac{r}{s}\right)^q\frac{ds}{s}\le C\left(1+\ln\frac{r}{t}\right)^q g(t),\ q\ge 0,\ 0<t\le r<\ell. \tag{2.23}$$

If $g\in\underline{W}(\mathbb{R}_+)$ and $\max\{M_0(g),M_\infty(g)\}<0$, then

$$\int_t^\ell g(s)\left(1+\left|\ln\frac{r}{s}\right|\right)^q\frac{ds}{s}\le C\left(1+\ln\frac{r}{t}\right)^q g(t),\ q\ge 0,\ 0<t\le r<\ell. \tag{2.24}$$

Quasi metric-measure spaces with growth condition admit a kind of analog of the passage to polar coordinates in case of functions depending on distance, see for instance, such analogs in [31, Lemmas 2.5 and 2.8]. We will need such an analog in the following form.

Lemma 2.7. *Let (X,d,μ) satisfy the growth condition (2.2). Let $g_1\in\overline{W}\cap\Delta_2$, $g_2\in\underline{W}$. Then for $0<r<\ell$, $x_0\in X$ there holds:*

$$\int\limits_{B(x_0,r)} g_1(d(x_0,z))\ g_2(d(x_0,z))d\mu(z)\le C\int\limits_0^r t^\nu g_1(t)\ g_2(t)\frac{dt}{t}. \tag{2.25}$$

P r o o f. Let $\ell=\infty$ for simplicity. By definition of $\overline{W}$ and $\underline{W}$, the functions $\frac{g_1(t)}{t^{\beta_1}}$ and $\frac{g_2(t)}{t^{\beta_2}}$ are a.i. and a.d., respectively, for some β_1 and β_2.

We have

$$\int\limits_{B(x_0,r)} g_1(d(x_0,z))\ g_2(d(x_0,z))d\mu(z)$$

$$\leq C\sum_{k=0}^{\infty}\frac{g_1(2^{-k}r)\ g_2(2^{-k}r)}{(2^{-k}r)^{\beta_1+\beta_2}}\int\limits_{2^{-k-1}r<d(x_0,z)<2^{-k}r} d(x_0,z)^{\beta_1+\beta_2}d\mu(z)$$

$$\leq c\sum_{k=0}^{\infty}g_1(2^{-k}r)\ g_2(2^{-k}r)(2^{-k}r)^{\nu}.$$

It remains to apply Lemma 2.4. □

3. **Main results**

In this section we study the action of the weighted fractional Hardy operators

$$H_w^{\alpha}f(x)=d(x,x_0)^{\alpha-n}w(d(x,x_0))\int\limits_{d(y,x_0)<d(x,x_0)}\frac{f(y)dy}{w(d(y,x_0))} \tag{3.1}$$

and

$$\mathcal{H}_w^{\alpha}f(x)=w(d(x,x_0))\int\limits_{d(y,x_0)>d(x,x_0)}\frac{f(y)dy}{d(y,x_0)^{n-\alpha}w(d(y,x_0))},\ x\in X, \tag{3.2}$$

where w is a weight, and their commutators in the frameworks of Morrey spaces on quasi-metric measure spaces (X,d,μ).

3.1. **On admission of exponents, better than Adams exponent, for Hardy operators in Morrey spaces.** Let first (X,d,μ) be $\mathbb{R}^n$ with Euclidean distance and Lebesgue measure. In this case as is known, the Riesz fractional operator

$$I^{\alpha}f(x):=\int\limits_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|^{n-\alpha}},\ 0<\alpha<n$$

acts from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}^{q^\sharp,\lambda}(\mathbb{R}^n)$ when $1<p<\frac{n-\lambda}{\alpha}$, $0\leq\lambda<n$, with $\frac{1}{q^\sharp}=\frac{1}{p}-\frac{\alpha}{n-\lambda}$, see [1]. We refer to $q^\sharp$ as *Adams exponent.*

The same is true for Hardy operators since they are dominated by the Riesz fractional operator:

$$|x|^{\alpha-n}\int\limits_{|y|<|x|}f(y)dy\leq 2^{n-\alpha}I^{\alpha}f(x) \text{ and } \int\limits_{|y|>|x|}\frac{f(y)dy}{|y|^{n-\alpha}}\leq 2^{n-\alpha}I^{\alpha}f(x),\ f\geq 0.$$

Hardy operators H^{α} and $\mathcal{H}^{\alpha}$ were studied in Morrey spaces $\mathcal{L}^{p,\lambda}$ also in weighted setting, see [26], where the Adams exponent was also used.

The main message of this subsection is the following. We show that for the Hardy operators, the boundedness from $\mathcal{L}^{p,\lambda}$ to $\mathcal{L}^{q,\gamma}$ may hold for exponents $q>q^\sharp$. First we note that both Hardy operators and Riesz fractional operator have kernels homogeneous of degree $\alpha - n$. For any integral operator with a homogeneous kernel of such degree it is easily checked, by means of the known trick via delation, see [36] (where the Riesz fractional operator and Lebesgue spaces were under consideration), that if such an operator is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\gamma}(\mathbb{R}^n)$ then there necessarily holds the following relation between the parameters

$$\frac{n-\gamma}{q} = \frac{n-\lambda}{p} - \alpha. \tag{3.3}$$

Thus, given a domain space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, we can consider the exponents q, γ for the target space $\mathcal{L}^{q,\gamma}(\mathbb{R}^n)$ only satisfying the necessary relation (3.3). According to the result of Adams, the choice $q = q^\sharp$ and $\gamma = \lambda$ is sufficient for the boundedness. Below we show that for the Hardy operators the boundedness is valid not only when $1 < q \leq q^\sharp$, but also for $q^\sharp \leq q < q_{\sup}$, keeping (3.3) for finding γ, where $q_{\sup}$ is defined by

$$q_{\sup} := \frac{n}{n-\lambda} q^\sharp \,. \tag{3.4}$$

This will be shown in a general setting of quasi-metric measure spaces with growth condition, see Theorem 3.1. The same possibility for the choice of q will be provided for commutators of the Hardy operators on homogeneous quasi-metric measure spaces, see Corollary 3.2 and Remark 3.6. in Sections 3.2 and 3.3. On the other hand in Sections 3.2 and 3.3 we consider a more general situation in the sense that we admit weighted Morrey spaces. Moreover, this will be obtained as a corollary from a more general statement for generalized Morrey spaces, both for the Hardy operators and their commutators.

Given a space $\mathcal{L}^{p,\varphi}(X)$ we use the notation

$$\Phi_{p,\varphi}(r) = \left(\frac{\varphi(r)}{r^\nu}\right)^{\frac{1}{p}}, \; r \in (0, \ell), \; \ell = \text{ diam } X,$$

which plays an important role in the study of Morrey spaces as can be seen from Lemmas 3.1, 3.2 and 3.3 below.

Lemma 3.1. *Let* (X, d, μ) *satisfy the growth condition* (2.2), $x_0 \in X$, $1 \leq p \leq q < \infty$, φ, $\psi \in \overline{W} \cap \underline{W}$ *and* $\alpha \in \mathbb{R}$.

I. *If* $0 < \min\{m_0(\varphi), m_\infty(\varphi)\} \leq \max\{M_0(\varphi), M_\infty(\varphi)\} < \nu$, *then*

$$\Phi_{p,\varphi} \in \mathcal{L}^{p,\varphi}_E(X), \; E \ni x_0.$$

II. *If*

$$r^\alpha \Phi_{p,\varphi}(r) \leq c\Phi_{q,\psi}(r),\ r \in (0,\ell) \ \text{ and}$$

$$\nu - \alpha p - \frac{\nu p}{q} < \min\{m_0(\varphi), m_\infty(\varphi)\} \leq \max\{M_0(\varphi), M_\infty(\varphi)\} < \nu - \alpha p, \tag{3.5}$$

then

$$r^\alpha \Phi_{p,\varphi}(r)\big|_{r=d(x,x_0)} \in \mathcal{L}_E^{q,\psi}(X),\ E \ni x_0.$$

The statements of Lemma 3.1 are derived from [21, Lemma 3.1, part *iii)*].

The following lemma is nothing else but a reformulation of Lemma 2.2.

Lemma 3.2. *Let the space* (X,d,μ) *satisfy the growth condition* (2.2) *and* ψ_1 *and* ψ_2 *be positive functions on* $(0,\ell)$ *and* $1 \leq q_1 \leq q_2 < \infty$. *If*

$$\Phi_{q_2,\psi_2}(r) \leq C\Phi_{q_1,\psi_1}(r),$$

then $\mathcal{L}^{q_2,\psi_2}(X) \hookrightarrow \mathcal{L}^{q_1,\psi_1}(X)$.

In this section and Sections 3.2 and 3.3 we will essentially use point-wise estimates of the Hardy operators provided in the following lemma.

Note that in Lemma 3.3 we use $\min\{m_0(\varphi), m_\infty(\varphi)\}$ and $\max\{M_0(\varphi), M_\infty(\varphi)\}$, which preassumes that $\ell = \infty$. If $\ell < \infty$, the information about $m_\infty(\varphi)$ and $M_\infty(\varphi)$ should be everywhere omitted.

Lemma 3.3. *Let* (X,d,μ) *satisfy the growth condition,* φ *satisfy the assumptions in* (2.5) *and* (2.9)*, and* $f \in \mathcal{L}_E^{p,\varphi}(X)$.

$$\min\{m_0(\varphi), m_\infty(\varphi)\} \geq 0 \text{ for } p > 1 \text{ and } \min\{m_0(\varphi), m_\infty(\varphi)\} > 0 \text{ for } p = 1. \tag{3.6}$$

Then

$$|H^\alpha f(x)| \leq C d(x,x_0)^\alpha \Phi_{p,\varphi}(d(x,x_0)) \|f\|_{\mathcal{L}_E^{p,\varphi}(X)},\ E \ni x_0. \tag{3.7}$$

Let $\max\{M_0(\varphi), M_\infty(w)\} < \nu - \alpha p$. *Then*

$$|\mathcal{H}^\alpha f(x)| \leq C d(x,x_0)^\alpha \Phi_{p,\varphi}(d(x,x_0)) \|f\|_{\mathcal{L}_E^{p,\varphi}(X)},\ E \ni x_0. \tag{3.8}$$

P r o o f. Estimates (3.7) and (3.8) are derived from more general estimates in Theorems 5.1 and 5.2 in [21]. (Note that in the estimates in Theorems 5.1 and 5.2 in [21] were given for simplicity fror the case $\ell = \infty$, but they hold under the same proof for $\ell < \infty$. We also use this opportunity to note a misprint in [21]: in the formula (1.2) there should stand $\frac{u(x)}{d(x,x_0)N}$ instead of $u(x)$).

After the choice $u(x) \equiv d(x,x_0)^\alpha$ and $w(y) \equiv 1$ in Theorem 5.1 in [21] and $u(x) \equiv 1$ and $w(y) \equiv d(y,x_0)^{-\alpha}$ in Theorem 5.2 in [21], from those theorems we have

$$|H^\alpha f(x)| \leq \frac{C}{r^{n-\alpha}} \left[V(r) + \int\limits_0^r V(t)\frac{dt}{t} \right] \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)}, \ r = d(x,x_0), \tag{3.9}$$

and

$$|\mathcal{H}^\alpha f(x)| \leq C \int\limits_r^\ell \mathcal{V}(t)\frac{dt}{t} \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)}, \ r = d(x,x_0), \tag{3.10}$$

where $V(t) = t^{\frac{\nu}{p'}}\varphi(t)^{\frac{1}{p}} = t^\nu \Phi_{p,\varphi}(t)$, $\mathcal{V}(t) = t^{-\frac{\nu}{p}-\alpha}\varphi(t)^{\frac{1}{p}} = t^{-\alpha}\Phi_{p,\varphi}(t)$. Recall that $\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)} \leq \|f\|_{\mathcal{L}^{p,\varphi}_E(X)}$.

By (2.16) we have $\min\{m_0(V), m_\infty(V)\} = \frac{\nu}{p'} + \frac{1}{p}\min\{m_0(\varphi), m_\infty(\varphi)\} > 0$. Hence, $\int_0^r V(t)\frac{dt}{t} \leq cV(r)$ by (2.17) and (3.6) then (3.9) turns into (3.7).

By similar arguments via the properties (2.16) and (2.18) it easy to check that (3.10) turns into (3.8). □

We define the exponent $q_{\sup}$ by

$$\frac{1}{q_{\sup}} := \frac{1}{p} - \frac{\lambda + \alpha p}{\nu p} = \frac{\nu - \lambda}{\nu} \cdot \frac{1}{q^\sharp},$$

where $\frac{1}{q^\sharp} = \frac{1}{p} - \frac{\alpha}{\nu - \lambda}$; $q^\sharp$ may be referred to as the Adams exponent like in the Euclidian case.

In Theorem 3.1 below we assume that $\ell \leq \infty$ for the operator H^α and $\ell = \infty$ for the operator $\mathcal{H}^\alpha$. The upper bound $q < q_{\sup}$ for admissible values of the exponent q of the target space, given in Theorem 3.1, in general cannot be replaced by $q \leq q_{\sup}$, see Remark 3.3.

THEOREM 3.1. *Let (X,d,μ) satisfy the growth condition (2.2), $x_0 \in E \subseteq X$ and $1 \leq p < \infty$. Let the space $\mathcal{L}^{p,\varphi}_E(X)$ satisfy the assumptions*

$$\varphi \in \overline{W} \cap \underline{W}, \ \varphi(0) = 0, \ \min\{m_0(\varphi), m_\infty(\varphi)\} \geq 0, \ \sup_{0<r<\ell} \frac{r^\nu}{\varphi(r)} < \infty$$

and

$$\max\{M_0(\varphi), M_\infty(\varphi)\} < \nu - \alpha p. \tag{3.11}$$

Then the Hardy operators H^α and $\mathcal{H}^\alpha$, $\alpha \geq 0$, map the space $\mathcal{L}^{p,\varphi}_E(X)$, with $\min\{m_0(\varphi), m_\infty(\varphi)\} > 0$ in the case $p = 1$ for the Hardy operator

H^α, into any space $\mathcal{L}_E^{q,\psi}(X)$, satisfying the conditions

$$p \le q < \frac{\nu p}{\nu - \alpha p - \min\{m_0(\varphi), m_\infty(\varphi)\}}, \ \psi \in \overline{W} \cap \underline{W}, \ \psi(0) = 0$$

and

$$\psi(r) \ge c r^{\alpha q + \nu(1 - \frac{q}{p})} \varphi(r), \ r \in (0, \ell). \tag{3.12}$$

P r o o f. We consider the operator H^α, the arguments for $\mathcal{H}^\alpha$ being similar in view of estimates (3.7) and (3.8).

First we note that the condition (3.12) is nothing else but the inequality

$$r^\alpha \Phi_{p,\varphi}(r) \le c \Phi_{q,\psi}(r), \ r \in (0, \ell). \tag{3.13}$$

By (3.7) we obtain

$$\|H^\alpha f\|_{\mathcal{L}^{q,\psi}(X)} \le C \|d(\cdot, x_0)^\alpha \Phi_{p,\varphi}(d(\cdot, x_0))\|_{\mathcal{L}^{q,\psi}(X)} \|f\|_{\mathcal{L}^{p,\varphi}(X)},$$

provided $\Phi_{p,\varphi}(d(x, x_0)) \in \mathcal{L}^{q,\psi}(X)$. The latter inclusion holds by part II of Lemma 3.1, Since all the assumptions of Lemma 3.1 are satisfied by (3.11) and the condition $p \le q < \frac{\nu p}{\nu - \alpha p - \min\{m_0(\varphi), m_\infty(\varphi)\}}$. □

REMARK 3.1. The statements of Theorem 3.1 remain valid if the condition (3.11) on the domain space is omitted but replaced by the condition $0 < \min\{m_0(\psi), m_\infty(\psi)\} \le \max\{M_0(\psi), M_\infty(\psi)\} < \nu$ on the target space. To show this in the proof, it suffices to observe that

$$\|d(\cdot, x_0)^\alpha \Phi_{p,\varphi}(d(\cdot, x_0))\|_{\mathcal{L}^{q,\psi}(X)} \le c \|\Phi_{q,\psi}(d(\cdot, x_0))\|_{\mathcal{L}^{q,\psi}(X)}$$

and apply the part I of Lemma 3.1.

REMARK 3.2. The target space $\mathcal{L}^{q,\psi}(X)$ diminishes when q increases, say $q_1 < q_2$, and the choice of ψ is subject to the condition $\Phi_{q_2,\psi_2}(r) \le C\Phi_{q_1,\psi_1}(r)$, see Lemma 3.2.

COROLLARY 3.1. *Let the space (X, d, μ) satisfy the growth condition (2.2) and $x_0 \in E \subseteq X$. Let $1 \le p \le q < q_{\sup}$, $0 < \lambda < \nu$ and $0 \le \alpha < \frac{\nu - \lambda}{p}$ and*

$$\frac{\nu - \gamma}{q} = \frac{\nu - \lambda}{p} - \alpha. \tag{3.14}$$

Then the operators H^α and $\mathcal{H}^\alpha$ are bounded from $\mathcal{L}^{p,\lambda}(X)$ to $\mathcal{L}^{q,\gamma}(X)$

REMARK 3.3. Suppose that the space (X, d, μ) satisfies the regularity condition of the form

$$c_1 r^\nu \le \mu(B(x_0, 2r) \setminus (B(x_0, r)) \le c_2 r^\nu, \ 0 < r < \delta \tag{3.15}$$

for some $\delta > 0$. Then, under preservation of the relation (3.14), the operators H^α and $\mathcal{H}^\alpha$ are not bounded from $\mathcal{L}^{p,\lambda}(X)$ to $\mathcal{L}^{q,\gamma}(X)$ when $q = q_{\sup}$. Recall that the relation (3.14) is necessary in the Euclidian case.

P r o o f. It is easy to check that the relation (3.14) with $q = q_{\sup}$ turns into $\gamma(\frac{\nu-\lambda}{p} - \alpha) = 0$, i.e. $\gamma = 0$, and then $\mathcal{L}^{q_{\sup},\gamma}(X) = \mathcal{L}^{q_{\sup}}(X)$. Choose $f_0(x) = \frac{1}{d(x,x_0)^{\frac{\nu-\lambda}{p}}}$ for $d(x,x_0) < \delta$ and equal to zero otherwise. Then $f_0 \in \mathcal{L}^{p,\lambda}(X)$ by Lemma 3.1. However, it is easy to show, via the diadic decomposition with the regularity property (3.15) taken into account, that

$$H^\alpha f(x) \geq \frac{c}{d(x,x_0)^{\frac{\nu-\lambda}{p}-\alpha}} = \frac{c}{d(x,x_0)^{\frac{\nu}{q_{\sup}}}}$$

for $0 < r < \delta$. Similar estimate for $\mathcal{H}^\alpha f_0$ is also obtained in the same way. It remains to observe that $\frac{1}{d(x,x_0)^{\frac{\nu}{q_{\sup}}}} \notin \mathcal{L}^{q_{\sup}}(X)$ under the regularity condition (3.15). $\square$

3.2. **Estimation of the commutator of the weighted operator $\mathcal{H}^\alpha$.** For the commutator of an operator A we use the notation

$$[a, Af] := aAf - A(af).$$

We define the class $\mathcal{W} = \mathcal{W}(o,\ell)$ as

$$\mathcal{W} := \overline{W} \cup (\underline{W} \cap \Delta_2^r),$$

where Δ_2^r stands for the class of non-negative functions w on $(0,\ell)$, satisfying the reverse doubling condition $w(t) \leq cw(2t), 0 < t < \frac{\ell}{2}$. Since $\overline{W} \subset \Delta_2^r$, we have $\mathcal{W} = (\underline{W} \cup \overline{W}) \cap \Delta_2^r$.

Remark 3.4. From the definition of the class $\mathcal{W}$, for $w \in \mathcal{W}$ there exists a number $\beta \in \mathbb{R}$, $\beta = \beta(w)$, such that $\frac{w(r)}{r^\beta}$ is either a.i or a.d. One can take any β less than $\min\{m_0(w), m_\infty(w)\}$ in the first case and any β greater than $\max\{M_0(w), M_\infty(w)\}$ in the second case. It is easily checked that

$$w \in \mathcal{W} \Rightarrow w(t) \leq cw(r) \text{ for } 0 < r < t < 2r < \ell, \tag{3.16}$$

where $c = c(\beta)$ depends only on $\beta = \beta(w)$.

In the following theorem we use the notation

$$\mathcal{A}_{pq}^s(r) := r^{\frac{\nu}{qs'}} \left(\int_0^r \varrho^\nu w(\varrho)^{qs} \left(\int_\varrho^\ell \frac{t^\alpha}{w(t)} \Phi_{p,\varphi}(t) \frac{dt}{t} \right)^{qs} \frac{d\varrho}{\varrho} \right)^{\frac{1}{qs}}, \tag{3.17}$$

where $1 < s < \infty,\ \frac{1}{s} + \frac{1}{s'} = 1,$

$$\mathcal{B}_{pq}(r) := \left(\int_0^r \varrho^{\nu} w(\varrho)^q \left(\int_{\varrho}^{\ell} \frac{t^{\alpha}}{w(t)} \Phi_{p,\varphi}(t) \left(1 + \left|\ln \frac{r}{t}\right|\right) \frac{dt}{t} \right)^q \frac{d\varrho}{\varrho} \right)^{\frac{1}{q}}, \quad (3.18)$$

where $\beta = \beta(w)$ for $w \in \mathcal{W}$, is any number defined in Remark 3.4.

Theorem 3.2. *Let (X, d, μ) be homogeneous and satisfy the growth condition (2.2) and $\alpha \in \mathbb{R}$. Let p, q, φ and the weight w satisfy the following à priori assumptions*:

i) $1 < p < \infty,\ 1 < q < \infty,\ w \in \mathcal{W}(\mathbb{R}_+)$,

ii) φ and ψ fulfil the conditions (2.9), $\varphi(2r) \le C\varphi(r),\ r \in \mathbb{R}_+$.

Assuming that $\mathcal{A}^s_{pq}(r) < \infty$ and $\mathcal{B}_{pq}(r) < \infty$ for $r \in \mathbb{R}_+$, suppose that

$$k_{1,s} := \sup_{r>0} \frac{\mathcal{A}^s_{pq}(r)}{\psi^{\frac{1}{q}}(r)} < \infty,\ s > 1, \quad \text{and} \quad k_2 := \sup_{r>0} \frac{\mathcal{B}_{pq}(r)}{\psi^{\frac{1}{q}}(r)} < \infty \qquad (3.19)$$

for some $s > 1$ and $a \in CMO_{P_s}(X, x_0),\ P_s := \max\{p', qs'\}$. Then

$$\left\| \left[a, w\mathcal{H}^{\alpha} \frac{1}{w} \right] f \right\|_{\mathcal{L}^{q,\psi}_{\{x_0\}}(X)} \le K \|a\|^*_{P_s} \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)},\ \ x_0 \in E \subseteq X,$$

where $K = C_0 \max\{k_{1,s}, k_2\}$ and C_0, not depending on f and y, depends only on constants involved in the assumptions in i) and ii).

P r o o f. We take $\ell = \infty$ for simplicity. We split the proof into two steps.

Pointwise estimate via Morrey norm. We first prove the estimate

$$\left| \left[a, w\mathcal{H}^{\alpha} \frac{1}{w} \right] f(y) \right| \le C \left(\|a\|^*_{p'} g_1(d(y, x_0)) + g_2(y) \right) \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)},\ \ y \in X, \tag{3.20}$$

where

$$g_1(d(y, x_0)) := w(d(y, x_0)) \int_{d(y,x_0)}^{\infty} \frac{t^{\alpha}}{w(t)} \Phi_{p,\varphi}(t) \frac{dt}{t},$$

and

$$g_2(y) := w(d(y, x_0)) \sum_{k=0}^{\infty} \frac{(d_{k,y})^{\alpha}}{w(d_{k,y})} \Phi_{p,\varphi}(d_{k,y}) \left| a(y) - a_{B(x_0, d_{k,y})} \right| .$$

To this end we use diadic decomposition and proceed as follows:

$$\left|\left[a, w\mathcal{H}^\alpha \frac{1}{w}\right] f(y)\right|$$

$$\leq w(d(y,x_0)) \sum_{k=1}^{\infty} \int\limits_{d_{k,y}<d(z,x_0)<d_{k+1,y}} \frac{|a(z)-a(y)|\ |f(z)|}{d(z,x_0)^{\nu-\alpha} w(d(z,x_0))} d\mu(z).$$

Hence, by (3.16) we obtain

$$\left|\left[a, w\mathcal{H}^\alpha \frac{1}{w}\right] f(y)\right|$$

$$\leq C \frac{w(d(y,x_0))}{d(y,x_0)^{\nu-\alpha}} \sum_{k=1}^{\infty} \frac{2^{k(\alpha-\nu)}}{w(d_{k,y})} \int\limits_{B(x_0,d_{k+1,y})} \left|a(z) - a_{B(x_0,d_{k+1,y})}\right| |f(z)| d\mu(z)$$

$$+ C \frac{w(d(y,x_0))}{d(y,x_0)^{\nu-\alpha}} \sum_{k=1}^{\infty} \frac{2^{k(\alpha-\nu)}}{w(d_{k,y})} \int\limits_{B(x_0,d_{k+1,y})} \left|a(y) - a_{B(x_0,d_{k+1,y})}\right| |f(z)| d\mu(z)$$

$$=: C d(y,x_0)^{\alpha-\nu} w(d(y,x_0))(s_1(d(y,x_0)) + s_2(y)).$$

For s_1 by Hölder's inequality we have

$$s_1 \leq C \sum_{k=1}^{\infty} \frac{1}{2^{k(\nu-\alpha)} w(d_{k,y})} \left(\int\limits_{B(x_0,d_{k+1,y})} \left|a(z) - a_{B(x_0,d_{k+1,y})}\right|^{p'} d\mu(z) \right)^{\frac{1}{p'}}$$

$$\times \left(\int\limits_{B(x_0,d_{k+1,y})} |f(z)|^p d\mu(z) \right)^{\frac{1}{p}} \leq C \|a\|^*_{p'} \sum_{k=1}^{\infty} \frac{\varphi(d_{k,y})^{\frac{1}{p}} d(y,x_0)^{\frac{\nu}{p'}}}{2^{k\left(\frac{\nu}{p}-\alpha\right)} w(d_{k,y})} \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)}.$$

Applying the inequality (2.20) of Lemma 2.4, we obtain that

$$s_1(d(y,x_0)) \leq C d(y,x_0)^{\nu-\alpha} \|a\|^*_{p'} \int\limits_{d(y,x_0)}^{\infty} \frac{t^\alpha}{w(t)} \Phi_{p,\varphi}(t) \frac{dt}{t} \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)}.$$

For s_2 we have

$$s_2 \leq C \sum_{k=1}^{\infty} \frac{\left|a(y) - a_{B(x_0,d_{k+1,y})}\right|}{2^{k(\nu-\alpha)} w(d_{k,y})} \int\limits_{d(z,x_0)<d_{k+1,y}} |f(z)| d\mu(z)$$

By Hölder's inequality we have

$$s_2 \le C \sum_{k=1}^{\infty} \frac{\left|a(y) - a_{B(x_0,d_{k+1,y})}\right| (d_{k+1,y})^{\frac{\nu}{p'}}}{2^{k(\nu-\alpha)} w(d_{k,y})} \left(\int\limits_{d(z,x_0)<d_{k+1,y}} |f(z)|^p d\mu(z) \right)^{\frac{1}{p}}$$

$$\le C \sum_{k=1}^{\infty} \frac{\varphi(d_{k+1,y})^{\frac{1}{p}} (d_{k+1,y})^{\frac{\nu}{p'}}}{2^{k(\nu-\alpha)} w(d_{k,y})} \left|a(y) - a_{B(x_0,d_{k+1,y})}\right| \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)},$$

so that

$$s_2 \le C d(y,x_0)^{\nu-\alpha} \sum_{k=1}^{\infty} \frac{(d_{k,y})^{\alpha}}{w(d_{k,y})} \Phi_{p,\varphi}(d_{k,y}) \left|a(y) - a_{B(x_0,d_{k+1,y})}\right| \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)}.$$

It remains to gather the estimates for s_1 and s_2.

Estimation of $L^q(B(x_0,r))$-norms of $g_1(d(y,x_0))$ and $g_2(y)$. By Lemma 2.7 we have

$$\|g_1\|_{\mathcal{L}^q(B(x_0,r))} \le C \left(\int_0^r \varrho^{\nu-1} \left(w(\varrho) \int_{\varrho}^{\infty} \frac{t^{\alpha}}{w(t)} \Phi_{p,\varphi}(t) \frac{dt}{t} \right)^q d\varrho \right)^{\frac{1}{q}} =: \mathcal{A}_{pq}. \tag{3.21}$$

For g_2 we obtain

$$g_2 \le C(\mathcal{E}(y) + \mathcal{F}(y)),$$

where

$$\mathcal{E}(y) := \left|a(y) - a_{B(x_0,r)}\right| w(d(y,x_0)) \sum_{k=1}^{\infty} \frac{(d_{k,y})^{\alpha}}{w(d_{k,y})} \Phi_{p,\varphi}(d_{k,y})$$

and

$$\mathcal{F}(y) := w(d(y,x_0)) \sum_{k=1}^{\infty} \left|a_{B(x_0,d_{k+1,y})} - a_{B(x_0,r)}\right| \frac{\left(2^k d(y,x_0)\right)^{\alpha}}{w(d_{k,y})} \Phi_{p,\varphi}(d_{k,y}).$$

By Lemma 2.4

$$\mathcal{E}(y) \le \left|a(y) - a_{B(x_0,r)}\right| G(r), \quad G(r) = w(r) \int_r^{\infty} \frac{t^{\alpha}}{w(t)} \Phi_{p,\varphi}(t) \frac{dt}{t}.$$

Hence by Hölder's inequality with $s > 1$ we have

$$\|\mathcal{E}\|_{\mathcal{L}^q(B(x_0,r)}$$

$$\le \left(\int\limits_{B(x_0,r)} \left|a(y) - a_{B(x_0,r)}\right|^{qs'} d\mu(y) \right)^{\frac{1}{qs'}} \left(\int\limits_{B(x_0,r)} G(d(y,x_0))^{qs} d\mu(y) \right)^{\frac{1}{qs}}.$$

Applying Lemma 2.7 to the second factor, we obtain

$$\|\mathcal{E}\|_{\mathcal{L}^q(B(x_0,r)} \le c\|a\|^*_{qs'}\mathcal{A}^s_{pq}. \tag{3.22}$$

For the function $\mathcal{F}(y)$, applying Lemma 2.3 and then Lemma 2.4, we obtain

$$\mathcal{F}(y) \le C\|a\|^* w(d(y,x_0)) \int\limits_{d(y,x_0)}^{\infty} \frac{t^\alpha}{w(t)}\Phi_{p,\varphi}(t)\left(1+\left|\ln\frac{r}{t}\right|\right)\frac{dt}{t}.$$

Hence

$$\|\mathcal{F}\|_{L^q(B(x_0,r))} \le C\|a\|^*\mathcal{B}_{pq}(r). \tag{3.23}$$

Collecting the results in (3.21), (3.22) and (3.23) and observing that $\mathcal{A}_{pq} \le \mathcal{B}_{pq}$, we obtain:

$$\left\|\left[a, w\mathcal{H}^\alpha\frac{1}{w}\right]f\right\|_{\mathcal{L}^q(B(x_0,r))} \le C\|a\|^*_{P_s}\max\{\mathcal{A}^s_{pq},\mathcal{B}_{pq}\}\|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)},$$

where we took into account that the norm $\|a\|^*_s$ is increasing in s. The proof is complete. □

Theorem 3.2 provides rather general conditions for the $\mathcal{L}^{p,\varphi}_{\{x_0\}} \to \mathcal{L}^{q,\psi}_{\{x_0\}}$-boundedness of the commutator $[a, w\mathcal{H}^\alpha\frac{1}{w}]$ for arbitrary values $1 < p < \infty$, $1 < q < \infty$, with the relation (3.19) between the functions φ and ψ. Note that though the domain space is a generalized Morrey space with a non-necessarily power function φ, the target space is again generalized Morrey space, i.e. it does not use the language of Orlicz-Morrey space.

In the next theorem imposing some slight additional assumptions on the function $\frac{1}{w}\Phi_{p,\varphi}$, involved in (3.17) and (3.18), and on the function φ, we obtain an essentially more constructive result on the relation between the functions φ and ψ defining the domain and target spaces. In particular, it will allow to obtain the norm estimation of the commutator for classical Morrey spaces $\mathcal{L}^{p,\lambda}_{\{x_0\}}(X)$ for the exponent q greater than the Adams exponent. These assumptions will be formulated in terms of the upper Matuszewska-Orlicz index of the function $\frac{1}{w}\Phi_{p,\varphi}$, and the lower Matuszewska-Orlicz index of the function φ. We use the notation

$$m(f) := \min\{m_0(f), m_\infty(f)\} \quad \text{and} \quad M(f) := \max\{M_0(f), M_\infty(f)\}.$$

Theorem 3.3. *Let (X,d,μ) be homogeneous and satisfy the growth condition* (2.2) *and $\alpha \in \mathbb{R}$. Let $a \in CMO_{P_s}(X,x_0)$, $P_s := \max\{p', qs'\}$ for some $s > 1$, and p, q, φ and the weight w satisfy the à priori assumptions i) and ii) of Theorem 3.2. Assume that*

$$M\left(\frac{1}{w}\Phi_{p,\varphi}\right) < -\alpha \text{ and } m(\varphi) > \nu p\left(\frac{1}{p}-\frac{1}{q}\right) - \alpha p. \tag{3.24}$$

Then

$$\left\| \left[a, w\mathcal{H}^{\alpha} \frac{1}{w} \right] f \right\|_{\mathcal{L}^{q,\psi}_{\{x_0\}}(X)} \leq C \|a\|^{*}_{P_s} \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)}, \tag{3.25}$$

where

$$\psi(r) \geq \left(r^{\alpha+\nu\left(\frac{1}{q}-\frac{1}{p}\right)} \varphi(r)^{\frac{1}{p}} \right)^{q} \tag{3.26}$$

REMARK 3.5. The assumption $M\left(\frac{1}{w}\Phi_{p,\varphi}\right) < -\alpha$, equivalent to $M\left(\frac{\varphi}{w^p}\right) < \nu - \alpha p$, imposes a condition on the weight w, depending on the function φ. As regards the assumption on $m(\varphi)$, we always have $m(\varphi) \geq 0$ since φ is almost increasing. Consequently, the assumption on $m(\varphi)$ in (3.24) may be omitted when $\alpha > \nu\left(\frac{1}{p} - \frac{1}{q}\right)$, and reduces just to $m(\varphi) > 0$ when $\alpha = \nu\left(\frac{1}{p} - \frac{1}{q}\right)$.

P r o o f. We will show that, when (3.24) holds, then the conditions (3.19) are satisfied with ψ defined in (3.26), under some choice of $s > 1$. First we use the condition $M\left(\frac{1}{w}\Phi_{p,\varphi}\right) < -\alpha$ and get $\int\limits_{\varrho}^{\ell} \frac{t^{\alpha}}{w(t)}\Phi_{p,\varphi}(t)\frac{dt}{t} \leq c\frac{\varrho^{\alpha}}{w(\varrho)}\Phi_{p,\varphi}(\varrho)$ by (2.18). Then

$$\mathcal{A}^{s}_{pq} \leq r^{\frac{\nu}{qs'}} \left(\int\limits_{0}^{r} \varrho^{\nu+\left(\alpha-\frac{\nu}{p}\right)qs} \varphi(\varrho)^{\frac{qs}{p}} \frac{d\varrho}{\varrho} \right)^{\frac{1}{qs}}.$$

Let us choose the parameter $s > 1$ so that the lower Matuszewska-Orlicz index of the function $h(\varrho) := \varrho^{\nu+\left(\alpha-\frac{\nu}{p}\right)qs} \varphi(\varrho)^{\frac{qs}{p}}$) is possible. Since $m(h) = \nu + \left(\alpha - \frac{\nu}{p}\right)qs + \frac{qs}{p}m(\varphi)$, such a choice means that $\left[\frac{\nu-m(\varphi)}{p} - \alpha\right] s < \frac{\nu}{q}$. Hence, s may be chosen arbitrary in $(1,\infty)$ if $\frac{\nu-m(\varphi)}{p} \leq \alpha$, and $1 < s < \frac{\nu/q}{\frac{\nu-m(\varphi)}{p}-\alpha}$ otherwise. The latter interval is non-empty by the assumption on $m(\varphi)$ in (3.24).

Under this choice of s we have $\int\limits_{0}^{r} h(t)\frac{dt}{t} \leq ch(r)$ by (2.17). Then we obtain that

$$\frac{\mathcal{A}^{s}_{pq}(r)}{\psi(r)^{\frac{1}{q}}} \leq C, \ 0 < r < \ell,$$

under the choice (3.26) of the function ψ.

For $\mathcal{B}_{pq}(r)$ we observe that the condition $M\left(\frac{\varrho^\alpha}{w(\varrho)}\Phi_{p,\varphi}(\varrho)\right) < 0$ allows to use estimate (2.24) of Lemma 2.6, so that we get

$$\mathcal{B}_{pq}(r) \le C\left(\int\limits_0^r \varrho^{\nu-1+\left(\alpha-\frac{\nu}{p}\right)q}\varphi(\varrho)^{\frac{q}{p}}\left(1+\ln\frac{r}{\varrho}\right)^q d\varrho\right)^{\frac{1}{q}}.$$

The condition on $m(\varphi)$ in (3.24) allows to use the estimate (2.23) of Lemma 2.6, so that we get

$$\mathcal{B}_{pq}(r) \le Cr^{\nu\left(\frac{1}{q}-\frac{1}{p}\right)+\alpha}\varphi(r)^{\frac{1}{p}}.$$

Hence, with $\psi(r)$ defined in (3.26) we have

$$\frac{\mathcal{B}_{pq}(r)}{\psi(r)^{\frac{1}{q}}} \le C, \ 0 < r < \ell,$$

which completes the proof (note that the restriction on the choice of the parameter s used in the proof of the theorem, is not of importance, since $\|a\|_s^*$ is increasing in s). $\square$

COROLLARY 3.2. *Let (X,d,μ) be homogeneous and satisfy the growth condition* (2.2) *and $\alpha \in \mathbb{R}$. Let $1 < p < \infty$, $1 < q < \infty$, $0 < \lambda < \nu$, $w \in \mathcal{W}$, $m(w) > \frac{\lambda-\nu}{p}$ and $a \in CMO_{P_s}(X,x_0)$, where $P_s = \max\{p', qs'\}$, $s > 1$. If*

$$\frac{\nu-\lambda}{p} - \frac{\nu}{q} < \alpha < \frac{\nu-\lambda}{p}, \tag{3.27}$$

then

$$\left\|\left[a, w\mathcal{H}^\alpha\frac{1}{w}\right]f\right\|_{\mathcal{L}^{q,\gamma}_{\{x_0\}}(X)} \le C\|a\|^*_{P_s}\|f\|_{\mathcal{L}^{p,\lambda}_{\{x_0\}}(X)}, \tag{3.28}$$

where

$$\gamma = q\left(\alpha + \frac{\lambda}{p} - \nu\left(\frac{1}{p}-\frac{1}{q}\right)\right). \tag{3.29}$$

In particular, $\gamma = \lambda$ in the case we chose the Adams exponent, i.e. $\frac{1}{q} = \frac{1}{q^\sharp} = \frac{1}{p} - \frac{\alpha}{\nu-\lambda}$.

P r o o f. The proof is a matter of direct verification of conditions of Theorem 3.3. Note that (3.27) after the substitution $\frac{1}{q}$ from (3.29) to (3.27) is nothing else but $0 < \gamma < \nu$. $\square$

REMARK 3.6. The upper bound $q_{\sup}$ for the choice of q according to (3.27) is given by

$$\frac{1}{q_{\sup}} = \frac{\nu-\lambda}{\nu p} - \frac{\alpha}{\nu}, \quad \text{i.e.} \quad q_{\sup} = \frac{\nu}{\nu-\lambda} q^{\sharp} \tag{3.30}$$

as in Section 3.1 for the Hardy operators themselves.

Similar bound $q_{\sup}$ for the generalized Morrey spaces is also valid: $\frac{1}{q_{\sup}} = \frac{\nu - m(\varphi)}{\nu p} - \frac{\alpha}{\nu}$, which follows from the condition $m(\varphi) > \nu p \left(\frac{1}{p} - \frac{1}{q}\right) - \alpha p$ of Theorem 3.3.

3.3. **Estimation of the commutator of the weighted operator H^α.** Below we provide estimates for the commutators of the Hardy operator H^α. Such estimates in the Euclidian case were obtained in [28]. The statements given below not only generalize results of [28] from Euclidian case to the case of quasi-metric measure spaces, but essentially improve them. We omit proofs in this section because they are obtained following the same arguments as in Section 3.2 for commutators of the operator $\mathcal{H}^\alpha$.

In the following theorem we use the notation

$$\mathbb{A}^s_{pq}(r) := r^{\alpha + \frac{\nu}{qs'}} \left(\int_0^r \varrho^\nu w(\varrho)^{qs} \left(\int_0^\varrho \frac{t^\nu}{w(t)} \Phi_{p,\varphi}(t) \frac{dt}{t} \right)^{qs} \frac{d\varrho}{\varrho} \right)^{\frac{1}{qs}}, \tag{3.31}$$

where $1 < s < \infty$, $\frac{1}{s} + \frac{1}{s'} = 1$,

$$\mathbb{B}_{pq}(r) := r^\alpha \left(\int_0^r \varrho^\nu w(\varrho)^q \left(\int_0^\varrho \frac{t^\nu}{w(t)} \Phi_{p,\varphi}(t) \left(1 + \ln \frac{r}{t}\right) \frac{dt}{t} \right)^q \frac{d\varrho}{\varrho} \right)^{\frac{1}{q}}. \tag{3.32}$$

THEOREM 3.4. *Let (X, d, μ), p, q, α, φ and w satisfy the assumptions of Theorem 3.2*

Assuming that $\mathbb{A}^s_{pq}(r) < \infty$ and $\mathbb{B}_{pq}(r) < \infty$ for $r \in \mathbb{R}_+$, suppose that

$$\varkappa_{1,s} := \sup_{r>0} \frac{\mathbb{A}^s_{pq}(r)}{\psi^{\frac{1}{q}}(r)} < \infty, \ s > 1, \quad \text{and} \quad \varkappa_2 := \sup_{r>0} \frac{\mathbb{B}_{pq}(r)}{\psi^{\frac{1}{q}}(r)} < \infty$$

for some $s > 1$ and $a \in CMO_{P_s}(X, x_0)$, $P_s := \max\{p', qs'\}$. Then

$$\left\| \left[a, wH^\alpha \frac{1}{w} \right] f \right\|_{\mathcal{L}^{q,\psi}_{\{x_0\}}(X)} \le K \|a\|^*_{P_s} \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)},$$

where $K = C_0 \max\{\varkappa_{1,s}, \varkappa_2\}$ and C_0, not depending on f and y, depends only on constants involved in the assumptions in i) and ii) of Theorem 3.2.

The following theorem is derived from Theorem 3.4 similarly to obtaining Theorem 3.3 from Theorem 3.2.

THEOREM 3.5. *Let (X, d, μ), p, q, α, φ and w satisfy the assumptions of Theorem 3.3 and $a \in CMO_{P_s}(X, x_0)$, $P_s := \max\{p', qs'\}$ for some $s > 1$. If*

$$m\left(\frac{\varphi}{w^p}\right) > -\nu(p-1),$$

then

$$\left\| \left[a, wH^\alpha \frac{1}{w}\right] f \right\|_{\mathcal{L}^{q,\psi}_{\{x_0\}}(X)} \leq C\|a\|^*_{P_s} \|f\|_{\mathcal{L}^{p,\varphi}_{\{x_0\}}(X)}, \tag{3.33}$$

for any

$$\psi(r) \geq c\left(r^{\alpha+\nu\left(\frac{1}{q}-\frac{1}{p}\right)} \varphi(r)^{\frac{1}{p}}\right)^q. \tag{3.34}$$

COROLLARY 3.3. *Let (X, d, μ) be homogeneous and satisfy the growth condition (2.2) and $\alpha \in \mathbb{R}$. Let $1 < p < \infty$, $1 < q < \infty$, $0 < \lambda < \nu$, $w \in \mathcal{W}$ and $a \in CMO_{P_s}(X, x_0)$, where $P_s = \max\{p', qs'\}$ for some $s > 1$. Then*

$$\left\| \left[a, wH^\alpha \frac{1}{w}\right] f \right\|_{\mathcal{L}^{q,\gamma}_{\{x_0\}}(X)} \leq C\|a\|^*_{P_s} \|f\|_{\mathcal{L}^{p,\lambda}_{\{x_0\}}(X)}$$

under the conditions

$$\frac{\nu-\lambda}{p} - \frac{\nu}{q} < \alpha < \frac{\nu-\lambda}{p}, \tag{3.35}$$

$$\frac{\nu-\gamma}{q} + \alpha = \frac{\nu-\lambda}{p} \quad \text{and} \quad M(w) < \frac{\lambda}{p} + \frac{\nu}{p'}. \tag{3.36}$$

Recall that the condition (3.35) is nothing else but the inequality $0 < \gamma < \nu$ under the choice of γ provided by (3.36), see Corollary 3.2.

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