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Some higher order Hardy inequalities

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Abstract

We investigate the k-th order Hardy inequality (1.1) for functions satisfying rather general boundary conditions (1.2), show which of these conditions are admissible and derive sufficient, and necessary and sufficient, conditions (for $0 < q < \infty$, $p > 1$) on u, v for (1.1) to hold.

1 Introduction

We will consider the k-th order Hardy inequality

$$\left(\int_a^b |f(x)|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_a^b \left|f^{(k)}(x)\right|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}} \tag{1.1}$$

with k a positive integer, where $-\infty < a < b < +\infty$, p and q are real parameters, $p > 1$, $q > 1$, and u, v are weight functions, i.e., functions measurable and positive a.e. in (a, b). For some early contributions concerning such inequalities see [1] and the references given there. For some later results we refer to the book [2, Chapter 3] and the PhD thesis by Nassyrova [3] and the references given there. In this article we assume that the functions $f \in C^{k-1}[a, b]$, $f^{(k-1)} \in AC(a, b)$ satisfy the "boundary conditions"

$$\sum_{j=1}^{k}\left[\alpha_{ij} f^{(j-1)}(a) + \beta_{ij} f^{(j-1)}(b)\right] = 0 \quad \text{for } i = 1, \dots, k, \tag{1.2}$$

with $\{\alpha_{i,j}\}_{i,j=1}^k$ and $\{\beta_{i,j}\}_{i,j=1}^k$ given real numbers.

The conditions (1.2) are reasonable since they allow to exclude, e.g., polynomials of order $\le k-1$, for which the right hand side in (1.1) vanishes while the left hand side can be positive. On the other hand, not every choice of $\alpha_{i,j}$, $\beta_{i,j}$ is admissible, which can be illustrated by the following simple example.

Example 1.1. We choose $k = 1$; then (1.2) has the form

$$\alpha f(a) + \beta f(b) = 0. \tag{1.3}$$

For $\alpha = -\beta \ne 0$, any non-zero *constant* function f satisfies (1.3), while the right hand side in (1.1) (with $k = 1$!) equals zero. Hence, the choice $\alpha + \beta = 0$ is not allowed.

Let us consider the boundary value problem (BVP) consisting of the ordinary differential equation

$$f^{(k)}(x) = g(x) \quad \text{on} \quad (a, b) \tag{1.4}$$

and of the boundary conditions (1.2).

If we denote by $G(x, y)$ the *Green function* of this BVP, then we have that

$$f(x) = \int_a^b G(x,t)g(t)\mathrm{d}t \tag{1.5}$$

and we can rewrite (1.1) as the *weighted norm inequality*

$$\left(\int_a^b \left|\int_a^b G(x,t)g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_a^b |g(x)|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}. \tag{1.6}$$

Consequently, we have to solve two problems:

Problem A. To find the Green function of the BVP (1.4) & (1.2), i.e., to determine the values $\alpha_{i,j}$, $\beta_{i,j}$ for which this BVP is uniquely solvable, and to determine the form of $G(x, t)$.

Problem B. With $G(x, t)$ given, to find conditions (sufficient or necessary and sufficient) on the weight functions u, v, for which (1.6) holds for every function g.

2 Problem A: to find the Green function

The general solution of Equation (1.4) has the following form:

$$f(x) = \sum_{m=1}^{k} c_m x^{m-1} - \int_x^b \frac{(x-t)^{k-1}}{(k-1)!} g(t)\mathrm{d}t \tag{2.1}$$

with arbitrary coefficients $c_1, c_2, \dots, c_k$. Then conditions (1.2) lead to the following system of linear equations for the unknown c_i's:

$$\sum_{m=1}^{k} c_m \left[\sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!}\left[\alpha_{i,j}a^{m-j} + \beta_{i,j}b^{m-j}\right]\right] = \sum_{j=1}^{k} \alpha_{i,j} \int_a^b \frac{(a-t)^{k-j}}{(k-j)!} g(t)\mathrm{d}t \tag{2.2}$$

for $i = 1, \dots, k$.

The determinant of this system has the following form:

$$\Delta = \begin{vmatrix} \alpha_{1,1} + \beta_{1,1} & \cdots & \sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!}\left[\alpha_{1,j}a^{m-j} + \beta_{1,j}b^{m-j}\right] & \cdots & \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!}\left[\alpha_{1,j}a^{k-j} + \beta_{1,j}b^{k-j}\right] \\ \downarrow i\vdots & \cdots & \vdots & \cdots & \downarrow i\vdots \\ \alpha_{i,1} + \beta_{i,1} & \cdots & \sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!}\left[\alpha_{i,j}a^{m-j} + \beta_{i,j}b^{m-j}\right] & \cdots & \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!}\left[\alpha_{1,j}a^{m-j} + \beta_{i,j}b^{k-j}\right] \\ \downarrow i\vdots & \cdots & \vdots & \cdots & \downarrow i\vdots \\ \alpha_{k,1} + \beta_{k,1} & \cdots & \sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!}\left[\alpha_{k,j}a^{m-j} + \beta_{k,j}b^{m-j}\right] & \cdots & \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!}\left[\alpha_{k,j}a^{k-j} + \beta_{k,j}b^{k-j}\right] \end{vmatrix}. \tag{2.3}$$

The system (2.2) has a unique solution if and only if its determinant Δ is not equal to zero, and hence, we have immediately the following result:

Theorem 2.1. *The k-th order Hardy inequality* (1.1) *is meaningful for functions f satisfying* (1.2) *if and only if* $\Delta \neq 0$, *where* Δ *is given by* (2.3).

Example 2.2. Let us consider the case mentioned in Example 1.1, i.e., $k = 1$ and the condition (1.3). Condition (1.3) is then condition (1.2) with $\alpha_{1,1} = \alpha$, $\beta_{1,1} = \beta$ and we have $\Delta = \alpha + \beta$. Hence the Hardy inequality (1.1) (for $k = 1$!) is meaningful for functions f satisfying (1.3) if and only if $\alpha \neq -\beta$.

Example 2.3. Some particular cases of the conditions (1.2) have been investigated earlier. In the book [2], the k-th order Hardy inequality (1.1) was considered under the boundary conditions

$$\begin{aligned} f^{(i)}(a) &= 0 \quad \text{for } i \in M_0, \\ f^{(j)}(b) &= 0 \quad \text{for } j \in M_1, \end{aligned} \tag{2.4}$$

where M_0, M_1 are subsets of the set $\mathbb{N}_k = \{0, 1, \ldots, k - 1\}$. In [2, Chapter 4], it was shown that the Hardy inequality (1.1) is meaningful if and only if the sets M_0, M_1 satisfy the so-called *Pólya condition*, i.e., that

$$\sum_{i=0}^{r} \left(e_{0,i} + e_{1,i}\right) \geq r + 1, \quad r = 0, 1, \ldots, k - 1,$$

where

$$e_{\alpha,i} = \begin{cases} 1 \text{ if } i \in M_\alpha \\ 0 \text{ if } i \notin M_\alpha. \end{cases}$$

Hence, the condition $\Delta \neq 0$ can be called the generalized Polya condition appropriate for the general case (1.2).

Assuming that $\Delta \neq 0$ with Δ given by (2.3) and solving the system (2.2) we see that the components of its solution $[c_1, c_2, \ldots, c_k]$ are linear combinations of the integrals on the right hand side of (2.2). Hence we have the solution f of our BVP due to (2.1) in the form (1.5), i.e.,

$$f(x) = \int_a^b G(x, t) g(t) \mathrm{d}t,$$

where the Green function is given by the formula

$$G(x, t) = \sum_{n=1}^{k} P_n(x) t^{n-1} - \frac{(x - t)^{k-1}}{(k - 1)!} \chi_{(x,b)}(t), \tag{2.5}$$

where $P_n(x) = \sum_{m=1}^{k} a_{n,m} x^{m-1}$, $n = 1, \ldots, k$, are polynomials of order $\leq k - 1$. More precisely,

$$G(x, t) = \begin{cases} G_1(x, t) \text{ for } a < t \leq x < b, \\ G_2(x, t) \text{ for } a < x < t < b, \end{cases} \tag{2.6}$$

where

$$\begin{aligned} G_1(x, t) &= \sum_{n=1}^{k} P_n(x) t^{n-1} \\ G_2(x, t) &= \sum_{n=1}^{k} P_n(x) t^{n-1} - \frac{(x - t)^{k-1}}{(k - 1)!}, \end{aligned} \tag{2.7}$$

i.e.,

$$G_2(x,t) = \sum_{n=1}^{k} Q_n(x)t^{n-1} \tag{2.8}$$

with

$$Q_n(x) = P_n(x) + \frac{(-1)^n}{(k-1)!}\binom{k-1}{n-1}x^{k-n}.$$

Consequently, the Green function is fully described and the problem A is solved.

3 Problem B: to characterize the corresponding higher order Hardy inequality

In the sequel, we will suppose that $\Delta \neq 0$ with Δ defined by (2.3).

3.1 Sufficient conditions

Since, due to (2.6), we have that

$$\int_a^b G(x,t)g(t)\mathrm{d}t = \int_a^x G_1(x,t)g(t)\mathrm{d}t + \int_x^b G_2(x,t)g(t)\mathrm{d}t,$$

it yields that

$$\begin{aligned}&\int_a^b \left|\int_a^b G(x,t)g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x \\ &\leq 2^{q-1}\left[\int_a^b \left|\int_a^x G_1(x,t)g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x + \int_a^b \left|\int_x^b G_2(x,t)g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right].\end{aligned}$$

Hence: if we derive sufficient conditions for the *two* Hardy-type inequalities

$$\left(\int_a^b \left|\int_a^x G_1(x,t)g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \leq C_1\left(\int_a^b |g(x)|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.1}$$

$$\left(\int_a^b \left|\int_x^b G_2(x,t)g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \leq C_2\left(\int_a^b |g(x)|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.2}$$

we obviously obtain also sufficient conditions for the inequality (1.6) to hold.

Let us first consider (3.1). Due to (2.7), inequality (3.1) will be satisfied if there will be

$$\int_a^b \left|\int_a^x P_n(x)t^{n-1}g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x \leq C_{1,n}\left(\int_a^b |g(x)|^p v(x)\mathrm{d}x\right)^{\frac{q}{p}} \tag{3.3}$$

for $n = 1, 2, \ldots, k$. If we denote $h(t) = g(t)t^{n-1}$, we can rewrite (3.3) as

$$\int_a^b \left|\int_a^x h(t)\mathrm{d}t\right|^q |P_n(x)|^q u(x)\mathrm{d}x \le C_{1,n}\left(\int_a^b |g(x)|^p v(x)\mathrm{d}x\right)^{\frac{q}{p}}. \tag{3.4}$$

But this is just the Hardy inequality for the function h with weight functions $U(x) = |P_n(x)|^q u(x)$, $V(x) = x^{-(n-1)p}v(x)$, and it is well-known that this inequality holds for $1 < p \le q < \infty$ if and only if the function

$$A_{M,n}(x) = \left(\int_x^b u(t)|P_n(t)|^q \mathrm{d}t\right)^{\frac{1}{q}}\left(\int_a^x v^{1-p'}(t)t^{(n-1)p'}\mathrm{d}t\right)^{\frac{1}{p'}} \tag{3.5}$$

is bounded, while for the case $1 < q < p < \infty$, the necessary and sufficient condition reads

$$B_{M,n} = \left(\int_a^b \left(\int_x^b u(t)|P_n(t)|^q \mathrm{d}t\right)^{\frac{r}{q}}\left(\int_a^x v^{1-p'}(t)t^{(n-1)p'}\mathrm{d}t\right)^{\frac{r}{q'}} v^{1-p'}(x)x^{(n-1)p'}\mathrm{d}x\right)^{\frac{1}{r}} < \infty; \tag{3.6}$$

here and in the sequel $p' = \frac{p}{p-1}$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ (for details, see, e.g., [2].)

Now, let us consider (3.2). Analogously as in the foregoing case, (3.2) will be satisfied, if–due to (2.8)–the following Hardy-type inequality for the function h with weight functions $U(x) = |Q_n(x)|^q u(x)$, $V(x) = x^{-(n-1)p}v(x)$ will be satisfied:

$$\int_a^b \left|\int_x^b h(t)\mathrm{d}t\right|^q |Q_n(x)|^q u(x)\mathrm{d}x \le C_{2,n}\left(\int_a^b |h(x)|^p x^{-(n-1)p}v(x)\mathrm{d}x\right)^{\frac{q}{p}}. \tag{3.7}$$

In this case, it is well-known (see, e.g., [4]) that the boundedness of the function

$$\tilde{A}_{M,n}(x) = \left(\int_a^x u(t)|Q_n(t)|^q \mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^b v^{1-p'}(t)t^{(n-1)p'}\mathrm{d}t\right)^{\frac{1}{p'}} \tag{3.8}$$

for $1 < p \le q < \infty$ or the finiteness of the number

$$\tilde{B}_{M,n} = \left(\int_a^b \left(\int_a^x u(t)|Q_n(t)|^q \mathrm{d}t\right)^{\frac{r}{q}}\left(\int_x^b v^{1-p'}(t)t^{(n-1)p'}\mathrm{d}t\right)^{\frac{r}{q'}} v^{1-p'}(x)x^{(n-1)p'}\mathrm{d}x\right)^{\frac{1}{r}} \tag{3.9}$$

for $1 < q < p < \infty$ is necessary and sufficient for (3.7) to hold.

Consequently, we have found *sufficient* conditions of the validity of the k-th order Hardy inequality (1.1):

Theorem 3.1. *Let $1 < p, q < \infty$ and for $k \in \mathbb{N}$, let $n = 1, 2, \ldots, k$. Let $P_n(x)$ and $Q_n(x)$ be the polynomials from* (2.7) *and* (2.8), *respectively. Let $A_{M,n}(x)$ and $\tilde{A}_{M,n}(x)$ be defined by* (3.5) *and* (3.8), *respectively, and $B_{M,n}$ and $\tilde{B}_{M,n}$ by* (3.6) *and* (3.9), *respectively. Then the k-th order Hardy inequality* (1.1) *holds for functions f satisfying the boundary conditions* (1.2) *if the weight functions u, v satisfy for $n = 1, 2, \ldots, k$ the conditions*

$$\sup_{x\in(a,b)} A_{M,n}(x) < \infty, \qquad \sup_{x\in(a,b)} \tilde{A}_{M,n}(x) < \infty \tag{3.10}$$

in the case $1 < p \le q < \infty$, *and the conditions*

$$B_{M,n} < \infty, \quad \tilde{B}_{M,n} < \infty \tag{3.11}$$

in the case $1 < q < p < \infty$.

3.2 Necessary and sufficient conditions

The Hardy inequality of higher order is, as we have seen, closely connected with the weighted norm inequality (1.6). This inequality with rather general kernels $K(x, t)$ was investigated by many authors, see e.g. [2,5]. Here, we use the fact that $K(x, t)$ is a Green function and we assume that $1 < p < \infty$, $q > 0$ and that

$$u, v^{1-p'} \in L^1_{\mathrm{loc}}(a,b). \tag{3.12}$$

Let us denote Δ_1 and Δ_2 the closed triangles $\{(x, t): a \le t \le x \le b\}$ and $\{(x, t): a \le x \le t \le b\}$, respectively. Due to (2.6), (2.7), and (2.8), we have that

$$G_i \in C(\Delta_i), \quad i = 1, 2. \tag{3.13}$$

Furthermore, suppose that

$$\left.\begin{array}{l} G_1(x,a), \quad G_1(b,t), \quad G_2(a,t), \quad G_2(x,b) \\ \text{do not vanish } \textit{identically} \text{ in (a,b).} \end{array}\right\} \tag{3.14}$$

Theorem 3.2. *Let* $1 < p < \infty$, $q > 0$ *and suppose that* (3.12), (3.13) *and* (3.14) *hold. Then the Hardy-type inequality* (1.6) *holds if and only if*

$$u, v^{1-p'} \in L^1(a,b). \tag{3.15}$$

Proof. **Necessity**: Suppose that (1.6) holds.

(i) Due to (3.14), there exists a point $t_a \in (a, b)$ such that $G_2(a, t_a) \neq 0$. Consequently, there exists $\varepsilon > 0$ such that $|G(x, t)| = |G_2(x, t)| \ge C_a > 0$ for all $(x, t) \in (a, a+\varepsilon) \times (t_a - \varepsilon, t_a+\varepsilon)$. Here we suppose that $[t_a - \varepsilon, t_a + \varepsilon] \subset (a, b)$. If we choose the test function as $f(t) = \chi_{(t_a-\varepsilon,t_a+\varepsilon)}(t)v^{1-p'}(t)$, we get from (1.6) that

$$\begin{aligned} C\left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p}} &= C\left(\int_a^b |f(t)|^p v(t)\mathrm{d}t\right)^{\frac{1}{p}} \\ &\ge \left(\int_a^b \left|\int_a^b G(x,t)f(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\ &= \left(\int_a^b \left|\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)G(x,t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\ &\ge \left(\int_a^{a+\varepsilon} \left|\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)G_2(x,t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\ &\ge C_a\left(\int_a^{a+\varepsilon} \left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\ &= C_a\left(\int_a^{a+\varepsilon} u(x)\mathrm{d}x\right)^{\frac{1}{q}} \int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\mathrm{d}t, \end{aligned}$$

i.e.,

$$\left(\int_a^{a+\varepsilon} u(x)\mathrm{d}x\right)^{\frac{1}{q}} \le \frac{C}{C_a}\left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^{-\frac{1}{p'}} < \infty$$

due to (3.12). Together with (3.12), the last inequality implies that

$$\int_a^c u(x)\mathrm{d}x < \infty \quad \text{for every} \quad c < b,$$

which means that

$$u \in L^1_{\mathrm{loc}}\left([a, b]\right). \tag{3.16}$$

(ii) Due to (3.14), there exists a point $t_b \in (a, b)$ such that $G_1(b, t_b) \neq 0$ and $|G(x, t)| = |G_2(x, t)| \ge C_b > 0$ for all $(x, t) \in (b\text{-}\varepsilon, b)\times(t_b\text{-}\varepsilon, t_b{+}\varepsilon)$. The choice $f(t) = v^{1-p'}(t)\chi_{(t_b-\varepsilon, t_b+\varepsilon)}(t)$ leads analogously as in (i) to the estimate

$$\left(\int_{b-\varepsilon}^{b} u(x)\mathrm{d}x\right)^{\frac{1}{q}} \le \frac{C}{C_b}\left(\int_{t_b-\varepsilon}^{t_b+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^{-\frac{1}{p'}} < \infty,$$

i.e.

$$\int_d^b u(x)\mathrm{d}x < \infty \quad \text{for every } d > a$$

so that

$$u \in L^1_{\mathrm{loc}}\left([a, b]\right).$$

This together with (3.16) and (3.12) gives that $u \in L^1(a, b)$.

(iii) Due to (3.14), there exists a point $x_a \in (a, b)$ such that $G_1(x_a, a) \neq 0$ and $\left|G(x, t)\right| = \left|G_1(x, t)\right| \ge \hat{C}_a > 0$ for all $(x, t) \in (x_a - \varepsilon, x_a + \varepsilon) \times (a, a + \varepsilon)$. Let us choose a test function in (1.6) as

$$f(t) = \chi_{(a+\delta, a+\varepsilon)}(t)v^{1-p'}(t),$$

where $\delta \in (0, \varepsilon)$ is a parameter. Then we get that

$$\begin{aligned}
C\left(\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p}} &= C\left(\int_a^b |f(t)|^p v(t)\mathrm{d}t\right)^{\frac{1}{p}} \\
&\ge \left(\int_a^b \left|\int_a^b \mathrm{G}(x, t)f(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \left(\int_a^b \left|\int_{a-\delta}^{a+\varepsilon} v^{1-p'}(t)\mathrm{G}(x, t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\
&\ge \left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} \left|\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\mathrm{G}_1(x, t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\
&\ge \hat{C}_a\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} \left(\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \\
&= \hat{C}_a\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} u(x)\mathrm{d}x\right)^{\frac{1}{q}} \int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\mathrm{d}t,
\end{aligned}$$

i.e., that

$$\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} u(x)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{a+\delta}^{a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p'}} < \frac{C}{\hat{C}_a}.$$

This estimate holds for all $\delta \in (0, \varepsilon)$, and with δ tending to zero on the left hand side of the estimate we obtain that

$$\left(\int_{x_a-\varepsilon}^{x_a+\varepsilon} u(x)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{a}^{a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p'}} \le \frac{C}{\hat{C}_a}$$

which implies that $v^{1-p'} \in L^1_{\mathrm{loc}}\left(\left[a, b\right]\right)$.

(iv) Finally, we obtain analogously from $G_2(x_b, b) \neq 0$ that $v^{1-p'} \in L^1_{\mathrm{loc}}\left(\left[a, b\right]\right)$. Hence $v^{1-p'} \in L^1(a, b)$ and the necessity is proved.

Sufficiency: Using the boundedness of the function $G(x, t)$ (which follows from (3.13)), Holder's inequality and (3.15), we can estimate the left hand side of (1.6) as follows:

$$\begin{aligned}
&\left(\int_a^b \left|\int_a^b G(x,t)g(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le \left(\int_a^b \left(\int_a^b \left|G(x,t)\right|\left|g(t)\right|\mathrm{d}t\right)^q u(x)\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le C_1\left(\int_a^b \left(\int_a^b \left|g(t)\right|\mathrm{d}t\right)^q u(x)\mathrm{d}x\right)^{\frac{1}{q}}\\
&= C_1\left(\int_a^b u(x)\mathrm{d}x\right)^{\frac{1}{q}}\int_a^b \left|g(t)\right|\mathrm{d}t\\
&= C_1\left(\int_a^b u(x)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_a^b \left|g(t)\right| v^{\frac{1}{p}}(t)v^{-\frac{1}{p}}(t)\mathrm{d}t\right)\\
&\le C_1\left(\int_a^b u(x)\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_a^b v^{1-p'}(x)\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_a^b \left|g(x)\right|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le C\left(\int_a^b \left|g(x)\right|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

The proof is complete.

Remark 3.3. We have considered the Hardy-type inequality (1.6) for the case that $G(x, t)$ was a Green function, i.e., $G_i(x, t)$ have been *polynomials*. It is obvious that we can repeat our approach for *any* function $G(x, t)$, which satisfies (3.13) and (3.14). Hence, our approach gives some new criteria for the validity of (1.6) for rather general kernels G.

Example 3.4. In Example 1.1, the first order Hardy inequality with boundary condition (1.3) was considered. It can be easily shown that in this case the Green function has the form

$$G(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta} & \text{for } a < t < x < b, \\ -\frac{\beta}{\alpha+\beta} & \text{for } a < x \le t < b, \end{cases}$$

where $\alpha + \beta \neq 0$. If $\alpha \neq 0$ and $\beta \neq 0$, and then the conditions (3.14) are satisfied and we can use Theorem 3.2. According to this theorem, the Hardy inequality (1.6) holds if and only if

$$u, v^{1-p'} \in L^1(a,b).$$

Example 3.5. For simplicity let us assume for (a, b) the interval (0, 1) and consider the second order Hardy inequality

$$\left(\int_0^1 |f(x)|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^1 |f''(x)|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}. \tag{3.17}$$

Then boundary conditions (1.2) take the following form:

$$\begin{cases} \alpha_{1,1}f(0) + \alpha_{1,2}f'(0) + \beta_{1,1}f(1) + \beta_{1,2}f'(1) = 0 \\ \alpha_{2,1}f(0) + \alpha_{2,2}f'(0) + \beta_{2,1}f(1) + \beta_{2,2}f'(1) = 0. \end{cases} \tag{3.18}$$

This inequality was considered in [4] and the corresponding Green function has the following form:

$$G(x,t) = \begin{cases} \frac{1}{\Delta}(a + bx + ct + dxt) & \text{for } 0 < t < x < 1; \\ \frac{1}{\Delta}\left(a + (b-\Delta)x + (c+\Delta)t + dxt\right) & \text{for } 0 < x \le t < 1, \end{cases}$$

where

$$\Delta = \begin{vmatrix} \lambda_1 \mu_1 \\ \lambda_2 \mu_2 \end{vmatrix}, \quad a = \begin{vmatrix} \mu_1 \nu_1 \\ \mu_2 \nu_2 \end{vmatrix}, \quad b = \begin{vmatrix} \lambda_1 \alpha_{1,2} \\ \lambda_2 \alpha_{2,2} \end{vmatrix}, \quad c = \begin{vmatrix} \mu_1 \alpha_{1,1} \\ \mu_2 \alpha_{2,1} \end{vmatrix}, \quad d = \begin{vmatrix} \lambda_1 \beta_{1,1} \\ \lambda_2 \beta_{2,1} \end{vmatrix}$$

with $\lambda_i := \alpha_{i,1} + \beta_{i,1}$, $\mu_i := \alpha_{i,1} + \beta_{i,1} + \beta_{i,2}$, $\nu_i := \beta_{i,1} + \beta_{i,2}$, $i = 1, 2$. Notice, that Δ is the corresponding determinant from (2.3).

Let us use Theorem 3.2; for this aim we consider the polynomials:

$$G(x,0) = \frac{a+bx}{\Delta}, \quad G(1,t) = \frac{(a+b)+(c+d)t}{\Delta},$$
$$G(x,1) = \frac{(a+c+\Delta)+(b+d-\Delta)x}{\Delta}, \quad G(0,t) = \frac{a+(c+\Delta)t}{\Delta}.$$

These polynomials satisfy conditions (3.14) if and only if

$$|a| + |b| \neq 0, \quad |a+b| + |c+d| \neq 0, \quad |a+c+\Delta| + |b+d-\Delta| \neq 0, \quad |a| + |c+\Delta| \neq 0, \tag{3.19}$$

and these conditions imply that *the second order Hardy inequality holds if and only if* $u, v^{1-p'} \in L^1(0,1)$.

If the condition (3.14) is violated, then Theorem 3.2 cannot be used. Nevertheless, in some cases, it is possible to use the following generalization:

Theorem 3.6. *Suppose that* $1 < p < \infty$, $q > 0$ *and the functions* $G_i(x, t)$ $(i = 1, 2)$ *are not identically equal to zero.*

(i) If the Hardy-type inequality (1.6) *holds, then there exist polynomials* $P_i(x)$, $Q_i(t)$ $(i = 1, 2)$ *on* (a, b) *such that*

$$|Q_1|^{p'}v^{1-p'}, |P_2|^q u \in L^1_{\text{loc}}\left([a,b]\right) \quad \textit{and} \quad |Q_2|^{p'}v^{1-p'}, |P_1|^q u \in L^1_{\text{loc}}\left([a,b]\right), \tag{3.20}$$

and that the corresponding Green function $G(x, t)$ *can be written as*

$$G_i(x,t) = P_i(x)Q_i(t)\hat{G}_i(x,t), \quad i = 1,2, \tag{3.21}$$

where the functions $\hat{G}_1(x,t)$, $\hat{G}_2(x,t)$ *satisfy* (3.14).

If, moreover, $\hat{G}_i(a,a) \neq 0$, $\hat{G}_i(b,b) \neq 0$, *then*

(i-1) for $p \le q$

$$\sup_{x\in(a,b)} A_i(a,b;x) < \infty, \quad i = 1,2, \tag{3.22}$$

where

$$A_1(a,b;x) := \left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{1}{q}}, \tag{3.23}$$

$$A_2(a,b;x) := \left(\int_x^b |Q_2|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_a^x |P_2|^q u\mathrm{d}t\right)^{\frac{1}{q}}; \tag{3.24}$$

(i-2) for $q < p$

$$B_i(a,b) < \infty, \quad i = 1,2, \tag{3.25}$$

where

$$B_1(a,b) := \left(\int_a^b \left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}} \left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x\right)^{\frac{1}{r}}, \tag{3.26}$$

$$B_2(a,b) := \left(\int_a^b \left(\int_x^b |Q_2|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_a^x |P_2|^q u\mathrm{d}t\right)^{\frac{r}{q}} \left|Q_2(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x\right)^{\frac{1}{r}}. \tag{3.27}$$

(ii) If there exist polynomials $P_i(x)$, $Q_i(t)$ *on* (a, b) $(i = 1, 2)$ *such that* (3.21) *holds and the conditions* (3.22) *(for* $p \le q$*),* (3.25) *(for* $q < p$*) are satisfied, then the Hardy-type inequality* (1.6) *holds.*

Proof. **(i)** Let the Hardy-type inequality (1.6) hold, then the following inequality

$$\left(\int_a^{a+\varepsilon}\left|\int_a^{a+\varepsilon} G(x,t)f(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_a^{a+\varepsilon}|f(t)|^p v(t)\mathrm{d}t\right)^{\frac{1}{p}} \tag{3.28}$$

also holds for arbitrary function $f \in L^p(v)$, which follows from (1.6) considered for the function $g(t) = f(t)\chi_{(a,a+\varepsilon)}(t)$ and then from the monotonicity of the outer integral on the left hand side of (1.6).

(i.1) If $G_2(a, t)$ does not vanish identically on (a, b), then the proof of the existence of the polynomial $P_1(t)$ follows from point (i) of the proof of Theorem 3.2, i.e., in this case $u \in L^1_{\mathrm{loc}}([a,b])$ and the polynomial can be chosen as $P_2(x) \equiv 1$.

(i.2) If $G_2(a, t)$ vanishes on (a, b), then there exists a positive integer α_2 such that $G_2(x,t)(x-a)^{\alpha_2}\hat{G}_2(x,t)$, where $\hat{G}_2(a,t)$ does not vanish on (a, b). Choosing $\varepsilon > 0$ in inequality (3.28) sufficiently small and repeating the calculations in point (i) of the proof of Theorem 3.2 we obtain that

$$\left(\int_a^{a+\varepsilon}(x-a)^{\alpha_2 q}u(x)\mathrm{d}x\right)^{\frac{1}{q}} \le \frac{C}{C_a}\left(\int_{t_a-\varepsilon}^{t_a+\varepsilon} v^{1-p'}(t)\mathrm{d}t\right)^{-\frac{1}{p'}} < \infty$$

which implies that $(x-a)^{\alpha_2 q}u \in L^1_{\mathrm{loc}}[a,b)$ and the polynomial can be chosen as $P_2(x) \equiv (x-a)^{\alpha_2}$.

(i.3) Similarly, we can prove that there exist nonnegative integers α_1, β_1, β_2 such that

$$(b-x)^{\beta_2 q}u \in L^1_{\mathrm{loc}}([a,b]), \quad (b-t)^{\beta_1 p'}v^{1-p'}(t) \in L^1_{\mathrm{loc}}([a,b],), \quad (t-a)^{\alpha_1 p'} \in L^1_{\mathrm{loc}}([a,b])$$

and the polynomials can be chosen as

$$P_1(x) \equiv (b-x)^{\beta_1}, \quad Q_1(t) \equiv (t-a)^{\alpha_1}, \quad Q_2(t) \equiv (b-t)^{\beta_2}.$$

Moreover, it can be easily shown that the weight functions with these polynomials satisfy (3.20) and (3.21).

(i.4) Now we show that the conditions (3.22) and (3.25) are satisfied. Using (3.21) we rewrite (3.28) in the form

$$\left(\int_a^{a+\varepsilon}\left|\int_a^x P_1(x)Q_1(t)\hat{G}_1(x,t)f(t)\mathrm{d}t + \int_x^{a+\varepsilon} P_2(x)Q_2(t)\hat{G}_2(x,t)f(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}}$$
$$\le C\left(\int_a^{a+\varepsilon}|f(t)|^p v(t)\mathrm{d}t\right)^{\frac{1}{p}}$$

and taking into account that $\hat{G}_i(a,a) \neq 0$ $(i = 1,2)$ we obtain the following equivalent inequality

$$\left(\int_a^{a+\varepsilon}\left|P_1(x)\hat{G}_1(a,a)\int_a^x Q_1(t)f(t)\mathrm{d}t+P_2(x)\hat{G}_2(a,a)+\int_x^{a+\varepsilon} Q_2(t)f(t)\mathrm{d}t\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}}$$
$$\leq C\left(\int_a^{a+\varepsilon}|f(t)|^p v(t)\mathrm{d}t\right)^{\frac{1}{p}}$$

for all $f \in L^p(v)$ and for sufficiently small $\varepsilon > 0$. Using Theorem 2.3 in [2] we obtain the following equivalent conditions on the interval $(a, a+\varepsilon)$

(for $p \leq q$)

$$\sup_{x\in(a,a+\varepsilon)} A_i(a,a+\varepsilon) < \infty, \quad i = 1, 2; \tag{3.29}$$

(for $q < p$)

$$B_i(a,a+\varepsilon) < \infty, \quad i = 1, 2. \tag{3.30}$$

Similarly, we obtain the following conditions on the interval $(b - \varepsilon, b)$:

(for $p \leq q$)

$$\sup_{x\in(a,a+\varepsilon)} A_i(b-\varepsilon,b) < \infty, \quad i = 1, 2; \tag{3.31}$$

(for $q < p$)

$$B_i(b-\varepsilon,b) < \infty, \quad i = 1, 2. \tag{3.32}$$

All these conditions together with (3.20) imply that conditions (3.22) and (3.25) are satisfied:

Let us prove (i-1). Using (3.20) it is easy to show that the condition is satisfied if and only if there exist the limits

$$\limsup_{x\to a+} A_i(a,b;x) \quad \text{and} \quad \limsup_{x\to b-} A_i(a,b;x) \quad i = 1, 2.$$

Otherwise, the existence of these limits is equivalent to the existence of

$$\limsup_{x\to a+} A_i(a,a+\varepsilon;x) \quad \text{and} \quad \limsup_{x\to b-} A_i(b-\varepsilon,b;x) \quad i = 1, 2.$$

For the proof of this assertion, we only show the following equality, since the others can be proved analogously:

$$\limsup_{x\to a+} A_1(a,b;x)$$
$$= \limsup_{x\to a+}\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_x^{a+\varepsilon}|P_1|^q u\mathrm{d}t+\int_{a+\varepsilon}^b |P_1|^q u\mathrm{d}t\right)^{\frac{1}{q}}$$
$$= \limsup_{x\to a+}\left[\left[A_1\,(a,a+\varepsilon l x)\right]^q+\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{q}{p'}}\int_{a+\varepsilon}^b |P_1|^q u\mathrm{d}t\right]^{\frac{1}{q}}$$
$$= \limsup_{x\to a+} A_1(a,a+\varepsilon;x).$$

The existence of the limits follows from (3.29) and (3.31) and (i-1) is obtained.

To prove (i-2) is enough to show $B_1(a, b) < \infty$, since the case $B_2(a, b) < \infty$ can be proved analogously. First we rewrite $B_1(a, b)$ in the form

$$\begin{aligned} B_1(a,b)^r &= \int_a^{a+\varepsilon}\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x \\ &\quad+\int_{a+\varepsilon}^{b-\varepsilon}\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x \\ &\quad+\int_{b-\varepsilon}^{b}\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

The boundedness of I_2 follows from (3.20). Moreover, (3.20) together with (3.30) implies that

$$\begin{aligned} I_1 &\le 2^{\frac{r}{q}-1}\left[\int_a^{a+\varepsilon}\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_x^{a+\varepsilon} |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x\right. \\ &\quad\left.+\int_a^{a+\varepsilon}\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_{a+\varepsilon}^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x\right] \\ &= 2^{\frac{r}{p}}\left[\left[B_1\left(a,a+\varepsilon\right)\right]^r+\left(\int_{a+\varepsilon}^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\int_a^{a+\varepsilon}\left(\int_a^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x\right] \\ &= 2^{\frac{r}{p}}\left[\left[B_1(a,a+\varepsilon)\right]^r+\frac{p'}{r}\left(\int_{a+\varepsilon}^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_a^{a+\varepsilon} |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{p'}}\right] < \infty \end{aligned}$$

and

$$\begin{aligned} I_3 &\le \int_{b-\varepsilon}^{b}\left(\int_{b-\varepsilon}^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x \\ &\quad+\left(\int_a^{b-\varepsilon} |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\int_{b-\varepsilon}^{b}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x \\ &\le C_\varepsilon\int_{b-\varepsilon}^{b}\left(\int_{b-\varepsilon}^x |Q_1|^{p'}v^{1-p'}\mathrm{d}t\right)^{\frac{r}{q'}}\left(\int_x^b |P_1|^q u\mathrm{d}t\right)^{\frac{r}{q}}\left|Q_1(x)\right|^{p'}v^{1-p'}(x)\mathrm{d}x \\ &= C_\varepsilon\left[B_1\left(a,a+\varepsilon\right)\right]^r < \infty. \end{aligned}$$

To get the last estimate we used that

$$\left(\int_a^{b-\varepsilon} |Q_1|^{p'} v^{1-p'} \mathrm{d}t\right)^{\frac{r}{q'}} \int_{b-\varepsilon}^{b} \left(\int_x^b |P_1|^q u \mathrm{d}t\right)^{\frac{r}{q}} \left|Q_1(x)\right|^{p'} v^{1-p'}(x)\mathrm{d}x$$

$$\le C_\varepsilon \int_{b-\varepsilon}^{b} \left(\int_{b-\varepsilon}^{x} |Q_1|^{p'} v^{1-p'} \mathrm{d}t\right)^{\frac{r}{q'}} \left(\int_x^b |P_1|^q u \mathrm{d}t\right)^{\frac{r}{q}} \left|Q_1(x)\right|^{p'} v^{1-p'}(x)\mathrm{d}x,$$

which follows from (3.20).

Finally, we obtain **(ii)** only using boundedness of the polynomials $\hat{G}_i$ $(\mathrm{i} = 1, 2)$ and [2, Theorem 2.3]. The proof is, then, complete.

Example 3.7. Let us go back to Example 3.5. If, e.g., $|a| + |\mathrm{b}| = 0$, then one of the conditions (3.19) is violated. In this case we proceed according to Theorem 3.6 where $G_1(x, t) = \frac{1}{\Delta}t(c + dx) = t(c + dx)\hat{G}_1(x, t), \hat{G}_1(x, t) = \frac{1}{\Delta}$, and $\hat{G}_1(0, 0) \neq 0,\ \ \hat{G}_1(1, 1) \neq 0$; if, moreover, $c + \Delta = 0$, then $G_2(x, t) = \dfrac{x(dt - \Delta)}{\Delta}\hat{G}_2(x, t)$ where $\hat{G}_2(x, t) \equiv \frac{1}{\Delta}$, and the Hardy inequality (3.17) holds for functions satisfying (3.18) if and only if (3.22) (for $p \le q$) or (3.25) (for $q < p$) hold with $P_1(x) = c + dx$, $Q_1(t) = t$; $P_2(x) = x$, $Q_2(t) = dt - \Delta$. The other cases of violation of (3.19) can be considered analogously.

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Authors' contributions
All authors contributed in all parts in equal extent, and read and approved the final manuscript.

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