



Symmetries of Supergeometries Related to Nonholonomic Superdistributions

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Received: 19 April 2021 / Accepted: 14 February 2022
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Abstract

We extend Tanaka theory to the context of supergeometry and obtain an upper bound on the supersymmetry dimension of geometric structures related to strongly regular bracket-generating distributions on supermanifolds and their structure reductions.

1 Introduction and the Main Results

A theorem of Kobayashi states that G -structures of finite type have finite-dimensional symmetry algebras and automorphism groups, and that the dimension of both is bounded via Sternberg prolongation [19, 37]. This also applies to the class of Cartan geometries that allow higher order reductions of the structure group. Similarly, the Tanaka prolongation dimension bounds the symmetry dimension in the case of strongly regular bracket-generating nonholonomic distributions and related geometric structures [38, 41].

An analog of the Tanaka prolongation in the super-setting is well-defined and was used by Kac in his classification [18], based on the ideas of Weisfeiler filtration [40]. We use this algebraic *Tanaka–Weisfeiler prolongation* to construct a super-analog of the Cartan–Tanaka frame bundle, with normalization conditions induced from the generalized Spencer complexes. This in turn implies a bound on the dimension of the symmetry superalgebra.

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As we will recall in Section 3.3, to every distribution¹ \mathcal{D} on a supermanifold $M = (M_o, \mathcal{A}_M)$, one can associate a sheaf of negatively graded Lie superalgebras $\text{gr}(\mathcal{T}M)$ over \mathcal{A}_M , which are fundamental for a bracket-generating \mathcal{D} . Under a strong regularity assumption, the sheaf is associated to a classical bundle over the reduced manifold M_o with fiber given by the symbol \mathfrak{m} of \mathcal{D} (also called the Carnot algebra of \mathcal{D}). Assume the Tanaka–Weisfeiler prolongation $\mathfrak{g} = \text{pr}(\mathfrak{m})$ of $\mathfrak{m} = \mathfrak{g}_-$ is finite-dimensional and let G be a Lie supergroup with Lie superalgebra $\text{Lie}(G) = \mathfrak{g}$. We will then construct the Cartan–Tanaka prolongation of the structure, which is a fiber bundle over M with typical fiber diffeomorphic to a subsupergroup $P \subset G$ having $\text{Lie}(P) = \mathfrak{g}_{\geq 0}$.

Sometimes the geometric structure (M, \mathcal{D}) allows reductions, resulting in a reduction of \mathfrak{g} . In this paper, we will mainly consider reductions of the (graded) automorphism supergroup $\text{Aut}_{gr}(\mathfrak{m})$ to a smaller G_0 , which implies a reduction of $\text{pr}_0(\mathfrak{m}) = \text{det}_{gr}(\mathfrak{m})$ to $\text{Lie}(G_0) = \mathfrak{g}_0$, but we will also briefly discuss higher order reductions. In many important cases, the reduction is given by a choice of auxiliary geometric structure q on the superdistribution \mathcal{D} , which corresponds to \mathfrak{m}_{-1} . (For instance a tensor or a span of those. For higher order reductions q is not tensorial.) We will refer to (M, \mathcal{D}, q) as a filtered G_0 -structure. In this case the Tanaka–Weisfeiler prolongation is denoted by $\mathfrak{g} = \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$ and pure prolongation $\mathfrak{g}_0 = \text{pr}_0(\mathfrak{m})$ corresponds to the structure q void.

Theorem 1.1 *Let \mathfrak{s} be the symmetry superalgebra of a bracket-generating, strongly regular filtered G_0 -structure (M, \mathcal{D}, q) , with Tanaka–Weisfeiler prolongation $\mathfrak{g} = \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$ of $(\mathfrak{m}, \mathfrak{g}_0)$. Assume the reduced manifold M_o is connected. Then $\dim \mathfrak{s} \leq \dim \mathfrak{g}$ in the strong sense: the inequality applies to the dimensions of the even and odd parts respectively.*

The Lie superalgebra \mathfrak{s} can be considered as a superalgebra of supervector fields localized in a fixed neighborhood $U_o \subset M_o$ or as germs of those — the result holds in both cases.

Assuming $\dim \mathfrak{g}$ is finite, the above bound is sharp, meaning that there exists a standard model with symmetry superalgebra \mathfrak{s} equal to \mathfrak{g} : the homogeneous supermanifold G/P gives a geometric structure of type (\mathcal{D}, q) with a maximal space of automorphisms, meaning that G is the automorphism supergroup (or differs from it by a discrete quotient) and $\dim G$ is the maximal possible dimension of such a supergroup.

Theorem 1.2 *Let (M, \mathcal{D}, q) be a bracket-generating, strongly regular filtered G_0 -structure with a finite-dimensional Tanaka–Weisfeiler prolongation $\mathfrak{g} = \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$. If M_o has finitely many connected components, then $\text{Aut}(M, \mathcal{D}, q)$ is a Lie supergroup. If M_o is connected, then $\dim \text{Aut}(M, \mathcal{D}, q) \leq \dim \mathfrak{g}$ in the strong sense as above.*

¹We will sometimes refer to a “distribution on a supermanifold” as simply a “superdistribution” for short.

As noted above the dimension bound is sharp. In fact we have $\dim \text{Aut}(M, \mathcal{D}, q) \leq \dim \mathfrak{s}$ and the Lie superalgebra $\text{Lie}(\text{Aut}(M, \mathcal{D}, q))$ is the subalgebra of \mathfrak{s} consisting of the complete supervector fields. (We recall that any supervector field possesses a local flow in a suitable sense and it is called complete if its maximal flow domain is $\mathbb{R}^{1|1} \times M$, cf. [14, 29]. Moreover, it is complete if and only if the associated vector field on the reduced manifold M_o is so.) Thus in many cases the inequality is strict.

The structure of the paper is as follows. After introducing the main tools for working with geometric structures on supermanifolds in Section 2, we will show that the main ideas behind the classical results can be carried over to the super-setting. However, special care should be taken with the reduction of the structure group and usage of superpoints in frame bundles. We manage this through a geometric-algebraic correspondence, elaborated for principal bundles. In Section 3, we recall the algebraic prolongation following the ideas of Tanaka–Weisfeiler and construct the prolonged frame bundle with an absolute parallelism. Introduction of normalization conditions via the generalized Spencer complex is inspired by a previous work by Zelenko [42]. One of our main technical features is the geometric realization as supermanifolds of the sheaves of frames introduced in [2] (in the context of G -structures). This is crucial to carry out the inductive geometric prolongation argument.

In Section 4, we give the proof of the main theorems, using the constructed frame bundles, and discuss supersymmetry dimension bounds. Furthermore, we exploit a relation of the prolongation to the Lie equation and note that the symmetry algebra \mathfrak{s} of a filtered geometric structure can be obtained by a filtered subdeformation of \mathfrak{g} , i.e., by passing to a graded Lie sub-superalgebra and changing its filtered structure while preserving its associated-graded. We also discuss the maximal supersymmetry models there. Some applications, in particular new symmetry bounds, are given in Section 5. This covers holonomic supermanifolds, equipped with affine, metric, symplectic, periplectic and projective structures, as well as nonholonomic ones such as exceptional $G(3)$ -contact structures, equations of super Hilbert–Cartan type, super-Poincaré structures and some scalar odd ODE.

The automorphism supergroup in the case of G -structures was studied by Ostermayr [31] though this reference does not contain the supersymmetry dimension bounds. Our class of geometries is considerably larger, and in addition we consider the infinitesimal symmetry superalgebra that gives a finer dimension bound. We can also vary smoothness in the real case, to which we restrict for simplicity, and our results hold true in the complex analytic or algebraic cases too, as well as in the mixed case (cs manifolds, allowing for real bodies and complex odd directions) considered in [31]. Indeed, our arguments do not rely on any Batchelor realization of $M = (M_o, \mathcal{A}_M)$ [5], which is well-known to fail for most classes of supermanifolds.

For algebraic computations of prolongations, related to certain geometric structures, we refer to the works of Leites et al. [26, 32] (see also references therein).

Finally, we remark that while Theorems 1.1 and 1.2 are formulated for strongly regular distributions (with possible reductions), we expect them to hold in the general case, allowing singularities. This would superize the result of [21]. One should only

require the existence of a dense set of localizations where the derived sheaves give rise to distributions.

2 Bundles on Supermanifolds and Geometric-Algebraic Correspondence

For details on the background material on supermanifolds we refer to [7, 11, 26, 34, 39]. Here we elaborate the geometric-algebraic correspondence for the description of fiber, vector and principal bundles. To illustrate it, here are three definitions of tangent bundles:

- (i) Tangent bundle: This is the datum of an appropriate supermanifold TM with a surjective submersion $\pi : TM \rightarrow M$ of supermanifolds and typical fiber isomorphic to $T_x M$, $x \in M_o$.
- (ii) Reduced tangent bundle: $\iota^* TM = TM|_{M_o} = \bigcup_{x \in M_o} T_x M$ is a classical \mathbb{Z}_2 -graded vector bundle over $\iota : M_o \hookrightarrow M$. The supervector space $T_x M$ is the fiber over $x \in M_o$.
- (iii) Tangent sheaf: superderivations $\mathcal{T}M = \text{Der}(\mathcal{A}_M)$ form a sheaf on M_o of \mathcal{A}_M -modules, whose global sections $\mathfrak{X}(M) = \mathcal{T}M(M_o)$ consist of the supervector fields on M .

Approaches (ii) and (iii) appear, e.g., in [2, 15]; we will elaborate upon (i) below. It will be shown that the geometric approach (i) and the algebraic approach (iii) are equivalent, while the reduced tangent bundle $TM|_{M_o}$ encodes less information than TM or $\mathcal{T}M$. We will also establish a similar geometric-algebraic correspondence for principal bundles, crucial for our developments.

2.1 Supermanifolds

A *supermanifold* is understood in the sense of Berezin–Kostant–Leites, i.e., a ringed space $M = (M_o, \mathcal{A}_M)$ such that $\mathcal{A}_M|_{\mathcal{U}_o} \cong \mathcal{C}_{M_o}^\infty|_{\mathcal{U}_o} \otimes \Lambda^\bullet \mathbb{S}^*$ as sheaves of superalgebras for any sufficiently small open subset $\mathcal{U}_o \subset M_o$. Here \mathbb{S} is a vector space of fixed dimension. We set $\dim(M) = (m|n) = (\dim M_o | \dim \mathbb{S})$, call M_o the reduced manifold and \mathcal{A}_M the structure sheaf, which is \mathbb{Z}_2 -graded: $\mathcal{A}_M = (\mathcal{A}_M)_0 \oplus (\mathcal{A}_M)_1$. We shortly call *superdomain* the supermanifold $\mathcal{U} = (\mathcal{U}_o, \mathcal{A}_M|_{\mathcal{U}_o})$ associated to any open subset $\mathcal{U}_o \subset M_o$, even if \mathcal{U}_o is not connected. This terminology is also used for $\varphi^{-1}(\mathcal{U}) = (\varphi_o^{-1}(\mathcal{U}_o), \mathcal{A}_N|_{\varphi_o^{-1}(\mathcal{U}_o)})$, where $\varphi = (\varphi_o, \varphi^*) : N = (N_o, \mathcal{A}_N) \rightarrow M = (M_o, \mathcal{A}_M)$ is a morphism of supermanifolds. Despite its notation, the superdomain $\varphi^{-1}(\mathcal{U})$ is defined only in terms of \mathcal{U}_o and φ_o . Finally, an open cover of M is a family of superdomains $\{\mathcal{U}_i : i \in I\}$ such that $\bigcup_{i \in I} (\mathcal{U}_i)_o = M_o$ and $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ is the superdomain with reduced manifold $(\mathcal{U}_{ij})_o = (\mathcal{U}_i)_o \cap (\mathcal{U}_j)_o$, for all $i, j \in I$.

For any sheaf \mathcal{E} over M_o , its restriction to an open subset $\mathcal{U}_o \subset M_o$ will be denoted by $\mathcal{E}|_{\mathcal{U}_o}$, the space of its sections on \mathcal{U}_o simply by $\mathcal{E}(\mathcal{U})$ and the stalk at $x \in M_o$ by $\mathcal{E}_x = \varinjlim_{\mathcal{U}_o \ni x} \mathcal{E}(\mathcal{U})$. In particular, we set $\mathcal{A}(\mathcal{U}) := \mathcal{A}_M(\mathcal{U}_o)$ and $\mathcal{A}_{M,x} := (\mathcal{A}_M)_x$ for the structure sheaf.

Let $\mathcal{J} = \langle \mathcal{A}_{\bar{1}} \rangle$ be the subsheaf generated by nilpotents: $\mathcal{J} = \mathcal{J}_{\bar{0}} \oplus \mathcal{J}_{\bar{1}}$ with $\mathcal{J}_{\bar{1}} = \mathcal{A}_{\bar{1}}$ and $\mathcal{J}_{\bar{0}} = \mathcal{A}_{\bar{1}}^2$. For any sheaf \mathcal{E} of \mathcal{A}_M -modules on M_o we consider the evaluation $\text{ev} : \mathcal{E} \rightarrow \mathcal{E}/(\mathcal{J} \cdot \mathcal{E})$. In particular we get the reduction of superfunctions $\text{ev} : \mathcal{A}_M \rightarrow C_{M_o}^\infty$, $f \mapsto \text{ev}(f)$, and in turn the canonical morphism of supermanifolds $\iota = (\mathbb{1}_{M_o}, \text{ev}) : M_o = (M_o, C_{M_o}^\infty) \hookrightarrow M = (M_o, \mathcal{A}_M)$. Evaluation of the classical function $\text{ev}(f)$ at $x \in M_o$ is denoted by $\text{ev}_x(f)$. We stress, however, that there is no *canonical* morphism from M to M_o — this is a key feature of supergeometry.

For any supermanifold S , we will denote the *set of S -points* of M by $M[S] = \text{Hom}(S, M)_{\bar{0}}$, the set of all morphisms of supermanifolds from S to M . (By definition morphisms are even, so the subscript “ $\bar{0}$ ” might look redundant. However, we reserve symbols like Hom and Aut for *superspaces* of morphisms, see Section 2.2 below.) The functor of points $M[-] : \text{SMan}^{op} \rightarrow \text{Set}$ from the category of supermanifolds to the category of sets is a (contravariant) functor that fully determines M . However, there exist functors that are not representable, i.e., do not necessarily arise as the functor of points of a supermanifold. We refer to them as *superspaces* (also known as generalized supermanifolds, and not to be confused with the superspaces introduced by Manin [28]).

A very useful criterion to check identities involving morphisms is the *Yoneda lemma*: each morphism $\varphi : M \rightarrow N$ defines a natural transformation $\varphi[-] : M[-] \rightarrow N[-]$ (i.e., a family of maps between sets $\varphi[S] : M[S] \rightarrow N[S]$ that depends functorially on S) and *any* natural transformation between $M[-]$ and $N[-]$ arises from a unique morphism in this way.

For any point $x \in M_o$ and supermanifold S , we let $\hat{x} = (\hat{x}_o, \hat{x}^*) : S \rightarrow M$ be the unique morphism such that $\hat{x}^*(f) = \text{ev}_x(f) \cdot 1 \in \mathcal{A}(S)$ for all $f \in \mathcal{A}(M)$. This gives $\hat{x}_o(S_o) = x \in M_o$.

2.2 Lie Supergroups and Their Actions

A Lie supergroup is a supermanifold $G = (G_o, \mathcal{A}_G)$ endowed with a multiplication morphism $m : G \times G \rightarrow G$, an inverse morphism $i : G \rightarrow G$ and a unit morphism $e : \mathbb{R}^{0|0} \rightarrow G$ with usual compatibilities, which make G a group object in the category of supermanifolds. The reduced manifold G_o is a classical Lie group.

The associated functor of points $G[-] : \text{SMan}^{op} \rightarrow \text{Group}$ is particularly useful in the case of linear Lie supergroups. For example, consider the general linear Lie supergroup $G = \text{GL}(V)$ associated to a supervector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ of $\dim V = (p|q)$. The set of S -points $G[S] = \text{Hom}(S, G)_{\bar{0}}$ of G is the group

$$G[S] = \left\{ \text{invertible} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A = (a_j^i), B = (b_j^\beta), C = (c_j^\alpha), D = (d_\beta^\alpha) \right. \\ \left. \text{with } a_j^i, d_\beta^\alpha \in \mathcal{A}_{\bar{0}}(S), b_j^\beta, c_j^\alpha \in \mathcal{A}_{\bar{1}}(S) \text{ for all } 1 \leq i, j \leq p, 1 \leq \alpha, \beta \leq q \right\} \quad (2.1)$$

of the even invertible $(p|q) \times (p|q)$ matrices with entries in $\mathcal{A}(S)$. This group acts on the set of S -points of V

$$V[S] \cong (V \otimes \mathcal{A}(S))_{\bar{0}} = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_{p+q} \end{pmatrix} \mid v_1, \dots, v_p \in \mathcal{A}_{\bar{0}}(S), v_{p+1}, \dots, v_{p+q} \in \mathcal{A}_{\bar{1}}(S) \right\}, \quad (2.2)$$

where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is thought as the linear supermanifold $V = (V_{\bar{0}}, \mathcal{C}_{V_{\bar{0}}}^{\infty} \otimes \Lambda^{\bullet} V_{\bar{1}}^*)$. By Yoneda, we then have an action morphism of supermanifolds $\alpha : G \times V \rightarrow V$. We note that G is a superdomain of the linear supermanifold $\mathfrak{gl}(V) = \mathfrak{gl}(V)_{\bar{0}} \oplus \mathfrak{gl}(V)_{\bar{1}}$, with extended morphism of supermanifolds $\alpha : \mathfrak{gl}(V) \times V \rightarrow V$.

One may similarly define a linear supergroup $G \subset \text{GL}(V)$ with a morphism $\alpha : G \times V \rightarrow V$ that satisfies the usual properties of a linear action. See [7] for more details.

Remark 2.1 If $\Pi V = V \otimes \mathbb{R}^{0|1}$ is the parity change supervector space, then $\text{GL}(V) \cong \text{GL}(\Pi V)$ as Lie supergroups via the natural transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} D & C \\ B & A \end{pmatrix}$, however V and ΠV are not equivalent as representations. We also note for later use that the action of $G[S]$ on $V[S]$ defined above extends to the whole $\mathcal{A}(S)^{p|q} \cong V \otimes \mathcal{A}(S)$, whence $G[S] \cong \text{Aut}_{\mathcal{A}(S)}(\mathcal{A}(S)^{p|q})_{\bar{0}}$.

We may define an action of a supergroup on a supermanifold in the same vein, namely via the functor of points $G[S] \times M[S] \rightarrow M[S]$, which by Yoneda yields a morphism $G \times M \rightarrow M$ with the usual properties of a (nonlinear) action. If the action is effective, this leads to an embedding $G \subset \text{Aut}(M)$ to the ‘‘supergroup of diffeomorphisms,’’ but the sheaf approach is not sufficient to define the superstructure of an infinite-dimensional supermanifold [11, §2.6], so we treat $\text{Aut}(M)$ as a super-space via the functor of points. More precisely, given two supermanifolds M and N , the *superspace* of morphisms $\text{Hom}(M, N)$ is required to satisfy the usual adjunction formula

$$\text{Hom}(M, N)[S] = \text{Hom}(S, \text{Hom}(M, N))_{\bar{0}} \cong \text{Hom}(S \times M, N)_{\bar{0}}$$

for all supermanifolds S . See [33, §5.2] for an explicit description. If $M = N$, the supergroup of diffeomorphisms $\text{Aut}(M)$ is defined as a subfunctor of $\text{Hom}(M, M)$, and its ‘‘reduced space’’ $\text{Aut}(M)_{\bar{0}} \subset \text{Hom}(M, M)_{\bar{0}}$ is the group of all diffeomorphisms of M [33, §5.1, §6.1]. It is then a straightforward task to define, e.g., the supergroup $\text{Aut}(M, M')$ of diffeomorphisms of M preserving a subsupermanifold $M' \subset M$ in terms of commutative diagrams.

The chart approach developed below is better adapted, but we will mainly be interested in the structures of finite type, where the automorphisms form a genuine Lie supergroup.

2.3 Fiber Bundles and Sections

Recall that a morphism $\pi : E \rightarrow M$ is a submersion if $d\pi|_{E_o} : TE|_{E_o} \rightarrow TM|_{M_o}$ is surjective [25]. Locally, this is a product of supermanifolds, and this is the basis of the following.

Definition 2.2 A morphism $\pi : E \rightarrow M$ is a *fiber bundle* with typical fiber F if $\forall x \in M_o \exists$ a superdomain $\mathcal{U} \subset M$, $x \in \mathcal{U}_o$, and a diffeomorphism (local trivialization)

$\varphi : \pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U} \times F$ such that $\text{pr}_{\mathcal{U}} \circ \varphi = \pi$. A morphism of fiber bundles $\pi_1 : E_1 \rightarrow M_1$ and $\pi_2 : E_2 \rightarrow M_2$ is defined via the commutative diagram

$$\begin{CD} E_1 @>\psi>> E_2 \\ @V\pi_1VV @VV\pi_2V \\ M_1 @>\psi_b>> M_2 \end{CD}$$

Locally this means $\psi = (\psi_b \circ \pi_1, \varphi) : \mathcal{U}_1 \times F_1 \rightarrow \mathcal{U}_2 \times F_2$ for a morphism $\varphi : \mathcal{U}_1 \times F_1 \rightarrow F_2$.

Let $\{\mathcal{U}_i : i \in I\}$ be an open cover of M . A family $\{\varphi_{ij} : \mathcal{U}_{ij} \times F \rightarrow \mathcal{U}_{ij} \times F\}_{i,j \in I}$ of isomorphisms of trivial fiber bundles over the identity is called a *cocycle* if $\varphi_{ii} = \mathbb{1}_{\mathcal{U}_{ii} \times F}$ and $\varphi_{ij} = \varphi_{ik} \circ \varphi_{kj}$ where all three are defined. Since the φ_{ij} cover the identity in the first component, they can be equivalently written as $\varphi_{ij} : \mathcal{U}_{ij} \times F \rightarrow F$, by abusing the notation, or even as morphisms $\tilde{\varphi}_{ij} : \mathcal{U}_{ij} \rightarrow \text{Aut}(F)$. We refer to the latter as “transition morphisms.”

Proposition 2.3 *Let $\pi : E \rightarrow M$ be a fiber bundle with local trivializations $(\mathcal{U}_i, \varphi_i)$, $i \in I$. Then the family $\{\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}\}_{i,j \in I}$ is a cocycle. Conversely any cocycle determines a unique fiber bundle.*

Proof This is [20, Prop 4.1.2] for the case of vector bundles, see also [4, Prop. 4.9]. The idea is to glue E from the local data $E_i = \mathcal{U}_i \times F$ through the cocycles φ_{ij} on $E_{ij} = E_i \cap E_j$. First, glue E_o from the reduced data $(E_i)_o$ and $(\varphi_{ij})_o$ as in the classical theory with $\pi_o : E_o \rightarrow M_o$. Then glue the sheaves of superfunctions into a sheaf $\mathcal{A}_E : \mathcal{V}_o \mapsto \mathcal{A}_E(\mathcal{V}_o)$ over E_o defined as

$$\mathcal{A}_E(\mathcal{V}_o) = \left\{ (s_i \in \mathcal{A}_{E_i}((E_i)_o \cap \mathcal{V}_o))_{i \in I} : \varphi_{ij}^*(s_i|_{(E_{ij})_o \cap \mathcal{V}_o}) = s_j|_{(E_{ij})_o \cap \mathcal{V}_o} \forall i, j \in I \right\}.$$

We note that $\mathcal{A}_E|_{\pi_o^{-1}(\mathcal{U}_i)_o} \cong \mathcal{A}_{E_i}$ by virtue of the cocycle conditions, hence the ringed space $E = (E_o, \mathcal{A}_E)$ is a supermanifold. □

Corollary 2.4 *Let $\pi : E \rightarrow M$ be a fiber bundle with cocycle $\{\varphi_{ij}\}_{i,j \in I}$ and $\psi : N \rightarrow M$ a morphism of supermanifolds. Then, the pullback fiber bundle $p : \psi^*E \rightarrow N$ exists and is determined by the cocycle $\varphi_{ij} \circ (\psi \times \mathbb{1}_F) : \mathcal{V}_{ij} \times F \rightarrow F$, where $\mathcal{V}_i = \psi^{-1}(\mathcal{U}_i)$ and $\mathcal{V}_{ij} = \mathcal{V}_i \cap \mathcal{V}_j$ for all $i, j \in I$.*

Pullback fits in a commutative diagram, where the fiber bundle morphism ψ^\sharp is determined by ψ :

$$\begin{CD} \psi^*E @>\psi^\sharp>> E \\ @VpVV @VV\pi V \\ N @>\psi>> M \end{CD}$$

Recall that a subsupermanifold is called *closed* if its reduction is a closed submanifold.

Definition 2.5 The fiber $E_x = \pi^{-1}(x) \hookrightarrow E$ at $x \in M_o$ is the closed subsupermanifold given as the pullback

$$\begin{CD} \pi^{-1}(x) @>>> E \\ @VVV @VV\pi V \\ \mathbb{R}^{0|0} @>\hat{x}>> M \end{CD}$$

Remark 2.6 It can be specified more concretely via [4, Prop. 3.4]: the algebra of global superfunctions of the fiber is $\mathcal{A}(E_x) \cong \mathcal{A}(E)/\mathcal{A}(E)\pi^*(\mu_x)$, where $\mu_x = \{f \in \mathcal{A}(M) \mid \text{ev}_x(f) = 0\}$ is the maximal ideal. The fiber E_x is (non-canonically) diffeomorphic to the typical fiber F .

The reduced fiber bundle is defined by

$$i^*E = E|_{M_o} = \bigcup_{x \in M_o} E_x$$

and it is a fiber bundle over M_o with typical fiber F , according to Definition 2.2. Its defining cocycle $(\tilde{\varphi}_{ij})_o : (\mathcal{U}_{ij})_o \rightarrow \text{Aut}(F)_{\bar{0}}$ is the reduced morphism of the cocycle of E .

Definition 2.7 An even section of a fiber bundle $\pi : E \rightarrow M$ is a morphism of supermanifolds $\sigma : M \rightarrow E$ such that $\pi \circ \sigma = \mathbb{1}_M$. Locally $\sigma = (\mathbb{1}_{\mathcal{U}}, s) : \mathcal{U} \rightarrow \pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times F$ with $s : \mathcal{U} \rightarrow F$.

The same notion applies to superdomains $\mathcal{U} \subset M$ and we denote the set of all even sections by $\Gamma_E(\mathcal{U})_{\bar{0}} = \{\sigma : \mathcal{U} \rightarrow \pi^{-1}(\mathcal{U}) \mid \pi \circ \sigma = \mathbb{1}_{\mathcal{U}}\}$. (A superspace of sections can also be introduced using the functor of points [33, §4.3], but we won't need that for our arguments.)

2.4 Vector Bundles on Supermanifolds

Let V be a finite-dimensional supervector space.

Definition 2.8 (Geometric approach) A geometric vector bundle is a fiber bundle $\pi : E \rightarrow M$ with typical fiber V such that the transition morphisms are fiberwise linear, i.e., they take values in a linear supergroup: $\tilde{\varphi}_{ij} : \mathcal{U}_{ij} \rightarrow G \subseteq \text{GL}(V)$. If G is a proper subsupergroup, then π is called a G -vector bundle.

More concretely, any cocycle $\varphi_{ij} : \mathcal{U}_{ij} \times V \rightarrow V$ acts on linear coordinates (x^a, θ^α) of V by

$$\begin{aligned} \varphi_{ij}^*(x^a) &= \tilde{\varphi}_{ij}^*(A_b^a) \cdot x^b + \tilde{\varphi}_{ij}^*(B_\beta^a) \cdot \theta^\beta, \\ \varphi_{ij}^*(\theta^\alpha) &= \tilde{\varphi}_{ij}^*(C_b^\alpha) \cdot x^b + \tilde{\varphi}_{ij}^*(D_\beta^\alpha) \cdot \theta^\beta, \end{aligned} \tag{2.3}$$

where $(A_b^a, B_\beta^a, C_b^\alpha, D_\beta^\alpha)$ are coordinates on $GL(V)$, so that $\begin{pmatrix} \tilde{\varphi}_{ij}^*(A_b^a) & \tilde{\varphi}_{ij}^*(B_\beta^a) \\ \tilde{\varphi}_{ij}^*(C_b^\alpha) & \tilde{\varphi}_{ij}^*(D_\beta^\alpha) \end{pmatrix} \in GL(V)[\mathcal{U}_{ij}]$. In other words, $\varphi_{ij} = \alpha \circ (\tilde{\varphi}_{ij} \times \mathbb{1}_V) : \mathcal{U}_{ij} \times V \rightarrow V$. Note also that fiberwise linearity for the reduced bundle is a weaker condition than that for the geometric vector bundle because the off-diagonal blocks of the above matrix are odd, hence vanish upon evaluation [4, Ex. 4.15].

A morphism of geometric vector bundles E_1, E_2 is a fiber bundle morphism $\psi : E_1 \rightarrow E_2$ that is fiberwise linear. More concretely, let $\pi_1 : E_1 \rightarrow M_1$ and $\pi_2 : E_2 \rightarrow M_2$ be geometric vector bundles with typical fibers V and W , respectively. Denote by $\alpha : \text{Hom}(V, W) \times V \rightarrow W$ the natural composition morphism of supermanifolds. Then, similar to [4, Def. 4.12], we define a morphism of vector bundles in a local component $\varphi : \mathcal{U}_1 \times V \rightarrow W$ as $\varphi = \alpha \circ (\rho \times \mathbb{1}_V) : \mathcal{U}_1 \times V \rightarrow W$ for some morphism $\rho : \mathcal{U}_1 \rightarrow \text{Hom}(V, W)$.

Proposition 2.3 and Corollary 2.4 specialize straightforwardly to geometric vector bundles. Next, formula (2.3) shows that the assignment $\mathcal{U}_o \rightarrow \Gamma_E(\mathcal{U})_{\bar{0}}$ describing local even sections, gives a sheaf of right $(\mathcal{A}_M)_{\bar{0}}$ -modules on M_o . (This can be converted to a left module with the usual rule of signs.) This sheaf however is *not* locally free (e.g., for $E = TM, M = \mathbb{R}^{0|2}(\theta^1, \theta^2)$, the module of even supervector fields $\langle \theta^\alpha \partial_{\theta^\beta} \rangle$ is not free over $(\mathcal{A}_M)_{\bar{0}} = \langle 1, \theta^1 \theta^2 \rangle$), but it can be enlarged to a locally free sheaf $\mathcal{U}_o \rightarrow \Gamma_E(\mathcal{U}) = \Gamma_E(\mathcal{U})_{\bar{0}} \oplus \Gamma_E(\mathcal{U})_{\bar{1}}$ of (right) \mathcal{A}_M -modules of rank $(p|q)$ as follows.

First we define the parity change vector bundle $\pi : \Pi E \rightarrow M$ as the geometric vector bundle with typical fiber ΠV determined by the transition morphisms $\Pi \tilde{\varphi}_{ij} : \mathcal{U}_{ij} \rightarrow GL(V) \cong GL(\Pi V)$. Then we let $\Gamma_E(\mathcal{U})_{\bar{1}} = \Gamma_{\Pi E}(\mathcal{U})_{\bar{0}}$ and henceforth $\Gamma_E(\mathcal{U}) = \Gamma_E(\mathcal{U})_{\bar{0}} \oplus \Gamma_E(\mathcal{U})_{\bar{1}}$.

Definition 2.9 The sheaf Γ_E on M_o is called the *sheaf of sections* of $\pi : E \rightarrow M$.

This leads to an alternative definition of a vector bundle. For simplicity of exposition, we assume that M_o is connected for the remaining part of Section 2.

Definition 2.10 (Algebraic approach)

- (1) A locally free sheaf \mathcal{E} on M_o of (right) \mathcal{A}_M -modules of finite rank is called an *algebraic vector bundle* over M .
- (2) A morphism $\psi : \mathcal{G} \rightarrow \mathcal{F}$ of sheaves on M_o of \mathcal{A}_M -modules consists of a family of morphisms $\psi_{\mathcal{U}_o} : \mathcal{G}(U_o) \rightarrow \mathcal{F}(U_o)$ of $\mathcal{A}(\mathcal{U})$ -modules for each open subset $\mathcal{U}_o \subset M_o$, subject to the natural compatibility conditions with restrictions to $\mathcal{V}_o \subset \mathcal{U}_o$.

Proposition 2.11 Assume M_o is connected. Then the category of geometric vector bundles on M with morphisms of vector bundles covering the identity $\mathbb{1}_M : M \rightarrow M$ is equivalent to the category of algebraic vector bundles over M with morphisms of sheaves of \mathcal{A}_M -modules.

Proof This is [4, Prop. 4.22], to which we refer the reader. □

Recall that a *coherent sheaf* on $M = (M_o, \mathcal{A}_M)$ is a sheaf \mathcal{F} on M_o of \mathcal{A}_M -modules that has a *finite local presentation*: every point $x \in M_o$ has an open neighborhood \mathcal{U}_o in which there is an exact sequence $\mathcal{A}_M^r|_{\mathcal{U}_o} \rightarrow \mathcal{A}_M^s|_{\mathcal{U}_o} \rightarrow \mathcal{F}|_{\mathcal{U}_o} \rightarrow 0$ for some $r, s \in \mathbb{N}$. The algebraic vector bundles are therefore the locally free coherent sheaves.

Let $\varphi : M \rightarrow N$ be a morphism and \mathcal{F} a sheaf on M_o of \mathcal{A}_M -modules. Then the *direct image sheaf* $\varphi_*\mathcal{F}$ is the sheaf of \mathcal{A}_N -modules over N_o given by the law $\varphi_*\mathcal{F} : N_o \supset \mathcal{U}_o \mapsto \mathcal{F}(\varphi_o^{-1}(\mathcal{U}_o))$ and the homomorphism $\varphi^* : \mathcal{A}_N \rightarrow (\varphi_o)_*\mathcal{A}_M$. The kernel, cokernel and direct image of a morphism of coherent sheaves are coherent sheaves. The inverse image sheaf $\varphi_o^{-1}\mathcal{G}$ of a sheaf \mathcal{G} on N_o exhibits some relatively subtle features and it is easier to define directly in terms of stalks: given $x \in M_o$, one has $(\varphi_o^{-1}\mathcal{G})_x \cong \mathcal{G}_{\varphi_o(x)}$. If \mathcal{G} is a sheaf of \mathcal{A}_N -modules, then $\varphi_o^{-1}\mathcal{G}$ is only a sheaf of $\varphi_o^{-1}\mathcal{A}_N$ -modules. In this situation, the *inverse image sheaf* is defined as the sheaf of \mathcal{A}_M -modules by the formula $\varphi^*\mathcal{G} = \varphi_o^{-1}\mathcal{G} \otimes_{\varphi_o^{-1}\mathcal{A}_N} \mathcal{A}_M$, where the left action of $\varphi_o^{-1}\mathcal{A}_N$ on \mathcal{A}_M is defined by the map $\varphi_o^{-1}\mathcal{A}_N \rightarrow \varphi_o^{-1}(\varphi_o)_*\mathcal{A}_M \rightarrow \mathcal{A}_M$ induced by $\varphi^* : \mathcal{A}_N \rightarrow (\varphi_o)_*\mathcal{A}_M$. If \mathcal{G} is locally free, then the sheaf $\varphi^*\mathcal{G}$ is locally free as well.

There is a natural adjunction correspondence between morphisms of sheaves of modules: $\text{Hom}_{\mathcal{A}_M}(\varphi^*\mathcal{G}, \mathcal{F})_{\bar{0}} \cong \text{Hom}_{\mathcal{A}_N}(\mathcal{G}, \varphi_*\mathcal{F})_{\bar{0}}$.

Definition 2.12 A morphism $\mathcal{E}_1 \rightarrow \mathcal{E}_2$ of locally free coherent sheaves \mathcal{E}_i of \mathcal{A}_{M_i} -modules, $i = 1, 2$, is a pair (ψ, ψ_b) , where $\psi_b : M_1 \rightarrow M_2$ is a morphism of supermanifolds and $\psi : \mathcal{E}_1 \rightarrow \psi_b^*\mathcal{E}_2$ is a morphism of sheaves of \mathcal{A}_{M_1} -modules.

Now Proposition 2.11 extends to morphisms of vector bundles over different bases:

Theorem 2.13 *The category of the geometric vector bundles with morphisms of vector bundles is equivalent to the category of algebraic vector bundles with morphisms of locally free coherent sheaves, provided the reduced manifolds of the bases of the bundles are connected.*

Proof Only the part of the proof on morphisms has to be modified and this is based on the following observations.

- (i) The pullback of geometric vector bundles correspond to the inverse image of locally free coherent sheaves.
- (ii) A morphism of geometric vector bundles from E_1 to E_2 always factorizes through the pullback bundle as

$$\begin{array}{ccccc}
 E_1 & \xrightarrow{\psi} & \psi_b^*E_2 & \xrightarrow{(\psi_b)^\sharp} & E_2 \\
 \downarrow \pi_1 & & \downarrow p_2 & & \downarrow \pi_2 \\
 M_1 & \xrightarrow{\mathbb{1}_{M_1}} & M_1 & \xrightarrow{\psi_b} & M_2
 \end{array}$$

Therefore it can be viewed as a morphism of supermanifolds $\psi_b : M_1 \rightarrow M_2$ paired with a morphism of geometric vector bundles $\psi : E_1 \rightarrow \psi_b^* E_2$ covering the identity $\mathbb{1}_{M_1}$. The claim then follows from the claim on morphisms of Proposition 2.11. \square

Definitions 2.8 and 2.10 can therefore be used interchangeably, but care must be given to distinguish between morphisms of vector bundles and sheaves. For instance the kernel and the direct image of a morphism of vector bundles can only be interpreted as coherent sheaves in general. Henceforth we will use the nomenclature vector bundle without specification.

2.5 Principal Bundles on Supermanifolds

Let $G = (G_o, \mathcal{A}_G)$ be a Lie supergroup.

Definition 2.14 (*Geometric approach*) A *geometric principal bundle* with structure group G is a fiber bundle $\pi : P \rightarrow M$ with typical fiber G such that the transition morphisms take values in the Lie supergroup acting on itself by left multiplication: $\tilde{\varphi}_{ij} : \mathcal{U}_{ij} \rightarrow G \subset \text{Aut}(G)$.

The right action of G on a local trivialization $\mathcal{U}_i \times G$ is given by $\mathbb{1}_{\mathcal{U}_i} \times m : \mathcal{U}_i \times G \times G \rightarrow \mathcal{U}_i \times G$, and it extends to a well-defined action morphism of supermanifolds $\alpha : P \times G \rightarrow P$ satisfying $\pi \circ \alpha = \pi \circ \text{pr}_P$. Moreover the φ_{ij} are G -equivariant, i.e., we have the commutative diagram

$$\begin{CD} \mathcal{U}_{ij} \times G \times G @>\varphi_{ij} \times \mathbb{1}_G>> G \times G \\ @V\mathbb{1}_{\mathcal{U}_{ij}} \times mVV @VVmV \\ \mathcal{U}_{ij} \times G @>\varphi_{ij}>> G \end{CD}$$

and by the Yoneda lemma this is equivalent to the identity $\varphi_{ij} = m \circ (\tilde{\varphi}_{ij} \times \mathbb{1}_G)$.

Definition 2.15 The *fundamental vector field* $\zeta_X \in \mathfrak{X}(P)$ associated to $X \in \mathfrak{g}$ is the supervector field defined by $(\mathbb{1}_P \otimes X) \circ \alpha^* = \alpha^* \circ \zeta_X$. Equivalently, given any local trivialization $\pi^{-1}(U) \cong \mathcal{U} \times G$, it is the left-invariant supervector field on G corresponding to X .

Let $\pi_1 : P_1 \rightarrow M_1, \pi_2 : P_2 \rightarrow M_2$ be geometric principal bundles with structure groups G_1 and G_2 , respectively, and let $\gamma : G_1 \rightarrow G_2$ be a homomorphism of Lie supergroups. We define a γ -*morphism of principal bundles* to be a fiber bundle morphism $\psi : P_1 \rightarrow P_2$ that is γ -equivariant. More concretely, this is expressed as the commutative diagram

$$\begin{CD} P_1 \times G_1 @>\psi \times \gamma>> P_2 \times G_2 \\ @V\alpha_1VV @VV\alpha_2V \\ P_1 @>\psi>> P_2 \end{CD}$$

Equivalently a local component $\varphi : \mathcal{U}_1 \times G_1 \rightarrow G_2$ of a γ -morphism has the following form $\varphi = m_2 \circ (g \times \gamma) : \mathcal{U}_1 \times G_1 \rightarrow G_2$, for some morphism $g : \mathcal{U}_1 \rightarrow G_2$.

If $G = G_1 = G_2$ and $\gamma = \mathbb{1}_G$, we simply say that ψ is a morphism of G -principal bundles.

Example 2.16 Let $\gamma : G_1 \hookrightarrow G_2$ be a subsupergroup. A γ -morphism $\psi : P_1 \rightarrow P_2$ of principal bundles over the same base M with $\psi_b = \mathbb{1}_M$ is called a *reduction* of the structure group.

Proposition 2.3 and Corollary 2.4 specialize straightforwardly for principal bundles.

Now we consider the algebraic approach. Restricting the functor of points of G to superdomains of M , we get a sheaf of classical groups $\mathcal{G}_M : \mathcal{U}_o \mapsto \mathcal{G}_M(\mathcal{U}_o) := G[\mathcal{U}]$ over M_o .

Definition 2.17 A *sheaf of right \mathcal{G}_M -sets* is a sheaf of sets \mathcal{P} on M_o on which the sheaf \mathcal{G}_M acts on the right: \forall open subset $\mathcal{U}_o \subset M_o$ we have an action $\alpha_{\mathcal{U}_o}^{\mathcal{P}} : \mathcal{P}(\mathcal{U}_o) \times \mathcal{G}_M(\mathcal{U}_o) \rightarrow \mathcal{P}(\mathcal{U}_o)$ and these actions are compatible with restrictions to open subsets $\mathcal{V}_o \subset \mathcal{U}_o$.

Let \mathcal{P} be a sheaf of right \mathcal{G}_M -sets and \mathcal{Q} a sheaf of right \mathcal{H}_M -sets on M_o , and let $\gamma : G \rightarrow H$ be a homomorphism of Lie supergroups. A γ -morphism $\psi : \mathcal{P} \rightarrow \mathcal{Q}$ associates morphisms of sets $\psi_{\mathcal{U}_o} : \mathcal{P}(\mathcal{U}_o) \rightarrow \mathcal{Q}(\mathcal{U}_o)$ compatible with restrictions to open subsets $\mathcal{V}_o \subset \mathcal{U}_o$ and that are γ -equivariant:

$$\begin{CD} \mathcal{P}(\mathcal{U}_o) \times \mathcal{G}_M(\mathcal{U}_o) @>\psi_{\mathcal{U}_o \times \gamma[\mathcal{U}]}>> \mathcal{Q}(\mathcal{U}_o) \times \mathcal{H}_M(\mathcal{U}_o) \\ @V\alpha_{\mathcal{U}_o}^{\mathcal{P}}VV @VV\alpha_{\mathcal{U}_o}^{\mathcal{Q}}V \\ \mathcal{P}(\mathcal{U}_o) @>\psi_{\mathcal{U}_o}>> \mathcal{Q}(\mathcal{U}_o) \end{CD}$$

Definition 2.18 (*Algebraic approach*) An *algebraic principal bundle* over M with structure group G is a sheaf \mathcal{P} of right \mathcal{G}_M -sets that is *locally simply transitive* in the following sense: $\forall x \in M_o \exists$ open neighborhood \mathcal{U}_o for which $\mathcal{G}_M(\mathcal{U}_o)$ acts simply transitively on $\mathcal{P}(\mathcal{U}_o)$.

Let $\varphi : M \rightarrow N$ be a morphism of supermanifolds and \mathcal{P} a sheaf on N_o of right \mathcal{H}_N -sets. Similar to the case of algebraic vector bundles we note that the inverse image sheaf $\varphi_o^{-1}\mathcal{P}$ is only a sheaf of $\varphi_o^{-1}\mathcal{H}_N$ -sets. In this situation, the *inverse image sheaf* is defined as the sheaf of \mathcal{H}_M -sets by the formula $\varphi^*\mathcal{P} = \varphi_o^{-1}\mathcal{P} \times_{\varphi_o^{-1}\mathcal{H}_N} \mathcal{H}_M$, where the left action of $\varphi_o^{-1}\mathcal{H}_N$ on \mathcal{H}_M is defined via $\varphi_o^{-1}\mathcal{H}_N \rightarrow \varphi_o^{-1}(\varphi_o)_*\mathcal{H}_M \rightarrow \mathcal{H}_M$. If \mathcal{P} is locally simply transitive, then $\varphi^*\mathcal{P}$ is locally simply transitive as well.

Definition 2.19 A γ -morphism of algebraic principal bundle $\mathcal{P}_1 \rightarrow \mathcal{P}_2$ is a pair (ψ, ψ_b) where $\psi_b : M_1 \rightarrow M_2$ is a morphism of supermanifolds and $\psi : \mathcal{P}_1 \rightarrow \psi_b^* \mathcal{P}_2$ a γ -morphism.

To link geometric principal bundles with algebraic principal bundles, it is sufficient to consider even sections as defined in Definition 2.7. For any morphism $g : \mathcal{U} \rightarrow G$ and any even section $\sigma \in \Gamma_P(\mathcal{U})_{\bar{0}}$ we define another section by $\sigma \cdot g = \alpha \circ (\sigma, g)$, so the assignment $\mathcal{U}_o \mapsto \Gamma_P(\mathcal{U})_{\bar{0}}$ is sheaf of right \mathcal{G}_M -sets. By the Yoneda lemma, one easily sees the following:

Lemma 2.20 For any $\sigma, \tau \in \Gamma_P(\mathcal{U})_{\bar{0}}$ there exists a unique morphism $g : \mathcal{U} \rightarrow G$ such that $\tau = \sigma \cdot g$.

In other words $\mathcal{G}_M(\mathcal{U}_o) = G[\mathcal{U}]$ acts simply transitively on $\Gamma_P(\mathcal{U})_{\bar{0}}$ if it is nonempty. Since this condition is always satisfied for small superdomains $\mathcal{U} \subset M$ we conclude the following.

Proposition 2.21 The sheaf of even sections of a geometric principal bundle $\pi : P \rightarrow M$ is an algebraic principal bundle.

Conversely, given an algebraic principal bundle \mathcal{P} , there is an open cover $\{\mathcal{U}_i\}_{i \in I}$ of M such that $\mathcal{G}_M((\mathcal{U}_i)_o)$ acts simply transitively on $\mathcal{P}((\mathcal{U}_i)_o)$ for all $i \in I$. In other words, we have an identification $t_i : \mathcal{P}((\mathcal{U}_i)_o) \rightarrow \mathcal{G}_M((\mathcal{U}_i)_o)$ of right $\mathcal{G}_M((\mathcal{U}_i)_o)$ -sets for all $i \in I$. On intersections \mathcal{U}_{ij} we get isomorphisms $\varphi_{ij} = t_i \circ t_j^{-1} : \mathcal{G}_M((\mathcal{U}_{ij})_o) \rightarrow \mathcal{G}_M((\mathcal{U}_{ij})_o)$ of simply transitive right $\mathcal{G}_M((\mathcal{U}_{ij})_o)$ -sets. These are left multiplication by some $g_{ij} \in \mathcal{G}_M((\mathcal{U}_{ij})_o) = G[\mathcal{U}_{ij}]$, whence the morphisms $\tilde{\varphi}_{ij} = g_{ij} : \mathcal{U}_{ij} \rightarrow G$ that satisfy the cocycle conditions.

Therefore we obtain a geometric principal bundle $\pi : P \rightarrow M$.

Theorem 2.22 The category of geometric principal bundles with γ -morphisms is equivalent to the category of algebraic principal bundles with γ -morphisms, for homomorphisms of supergroups γ .

Proof We already proved the correspondence between objects. The correspondence between morphisms is similar to that of the proof of Theorem 2.13:

- (i) The pullback of a geometric principal bundle corresponds to the inverse image sheaf.
- (ii) A γ -morphism of geometric principal bundles from P_1 to P_2 always factorizes through the pullback bundle as

$$\begin{array}{ccccc}
 P_1 & \xrightarrow{\psi} & \psi_b^* P_2 & \xrightarrow{(\psi_b)^\sharp} & P_2 \\
 \downarrow \pi_1 & & \downarrow p_2 & & \downarrow \pi_2 \\
 M_1 & \xrightarrow{\mathbb{1}_{M_1}} & M_1 & \xrightarrow{\psi_b} & M_2
 \end{array}$$

where $(\psi_b)^\sharp : \psi_b^* P_2 \rightarrow P_2$ is a morphism of G_2 -principal bundles. Therefore it can be viewed as a morphism of supermanifolds $\psi_b : M_1 \rightarrow M_2$ paired with a γ -morphism of geometric principal bundles $\psi : P_1 \rightarrow \psi_b^* P_2$ covering the identity $\mathbb{1}_{M_1} : M_1 \rightarrow M_1$. One then concludes as in Theorem 2.13. \square

2.6 Subbundles

Let $F' \subset F$ be a subsupermanifold.

Definition 2.23 Let $\pi : E \rightarrow M$ be a fiber bundle with typical fiber F . A *subbundle* with typical fiber F' is a subsupermanifold $E' \subset E$ such that for some open cover $\{\mathcal{U}_i : i \in I\}$ of M :

- (1) the transition morphisms $\tilde{\varphi}_{ij} : \mathcal{U}_{ij} \rightarrow \text{Aut}(F)$ of E factor through $\text{Aut}(F, F')$,
- (2) $\pi : E' \rightarrow M$ is itself a fiber bundle with typical fiber F' , whose transition morphisms are obtained by postcomposing with the restriction map $\text{Aut}(F, F') \rightarrow \text{Aut}(F')$.

This has a clear specification for vector and principal bundles:

- For vector subbundles the typical fibers $V' \subset V$ are supervector spaces and we consider $\text{GL}(V) \supset \text{GL}(V, V') \rightarrow \text{GL}(V')$ instead of general automorphisms;
- For principal subbundles we have instead a Lie subgroup $G' \subset G$.

These are geometric subbundles. There are also algebraic subbundles, of which we specify only the vector version, leaving algebraic principal subbundles to the reader.

Definition 2.24 An *algebraic vector subbundle* is subsheaf \mathcal{E}' of \mathcal{E} that is locally a direct factor, i.e., $\forall x \in M_o \exists$ a neighborhood \mathcal{U}_o and a subsheaf $\mathcal{E}'' \subset \mathcal{E}|_{\mathcal{U}_o}$ such that $\mathcal{E}_y = \mathcal{E}'_y \oplus \mathcal{E}''_y \forall y \in \mathcal{U}_o$.

An algebraic vector subbundle is itself an algebraic vector bundle. Indeed (cf. [39, §4.7]) by Nakayama’s lemma $\forall x \in \mathcal{U}_o$ the stalk \mathcal{E}'_x is a free module over the local ring $\mathcal{A}_{M,x}$ of dimension $\dim_{\mathcal{A}_{M,x}}(\mathcal{E}'_x) = \dim_{\mathbb{R}}(\mathcal{E}'_x/\mu_x \mathcal{E}'_x)$, where $\mu_x \subset \mathcal{A}_{M,x}$ is the maximal ideal. Since \mathcal{E}' is locally a direct factor, we have:

- (i) the fiber $E'_x \cong \mathcal{E}'_x/\mu_x \mathcal{E}'_x$ embeds into the fiber $E_x \cong \mathcal{E}_x/\mu_x \mathcal{E}_x$;
- (ii) the rank $\dim_{\mathbb{R}} E'_x = \dim_{\mathcal{A}_{M,x}}(\mathcal{E}'_x)$ is locally constant, thus constant.

Consequently \mathcal{E}' is a locally free coherent sheaf, proving our claim.

Note that a vector subbundle is not the same as an injective morphism of vector bundles. Indeed, the latter may not be locally a direct factor. Similarly to the previous subsections one can prove the equivalence of algebraic and geometric definitions of subbundles.

We can also define associated bundles. Let $\pi : P \rightarrow M$ be a geometric principal bundle determined by transition morphisms $g_{ij} : \mathcal{U}_{ij} \rightarrow G$. For any representation $\rho : G \rightarrow \text{GL}(V)$ the associated geometric vector bundle $E = P \times_{\rho} V$ is defined

through the transition morphisms $\rho_{ij} = \rho \circ g_{ij} : \mathcal{U}_{ij} \rightarrow \text{GL}(V)$. Its sections are in bijective correspondence with G -equivariant morphisms $f : P \rightarrow V \oplus \Pi V$.

3 Prolongation Structures

We begin by revising prolongations of G -structures on supermanifolds using our setup and then will pass to the filtered version.

3.1 Frame Bundles

Let $V_M = M \times V \rightarrow M$ be the trivial vector bundle over M with the fiber $V = \mathbb{R}^{m|n}$, where $\dim(M) = (m|n)$, and \mathcal{V}_M is the associated locally free sheaf on M_o . The frame bundle $\pi : Fr_M \rightarrow M$ is defined via the geometric-algebraic correspondence as the sheaf $\mathcal{F}r_M : M_o \supset \mathcal{U}_o \mapsto \mathcal{F}r_M(\mathcal{U}_o)$ of \mathcal{A}_M -linear isomorphisms from \mathcal{V}_M to $\mathcal{T}M$:

$$\mathcal{F}r_M(\mathcal{U}_o) = \{ \mathcal{A}|_{\mathcal{U}_o}\text{-linear isomorphism } F : \mathcal{V}_M|_{\mathcal{U}_o} \rightarrow \mathcal{T}M|_{\mathcal{U}_o} \}. \tag{3.1}$$

We prefer to think of \mathcal{V}_M and $\mathcal{T}M$ as locally free sheaves of *left* \mathcal{A}_M -modules and define the linearity of F accordingly; such a linear isomorphism still yields, via the sign rule, a local isomorphism $V_M \rightarrow TM$ of vector bundles covering the identity of M .

Setting $\mathcal{T}M^{m|n} = \mathcal{T}M^{\oplus m} \oplus \Pi \mathcal{T}M^{\oplus n}$, we have an embedding of sheaves $\mathcal{F}r_M \hookrightarrow (\mathcal{T}M^{m|n})_{\bar{0}}$ whose image, by the Nakayama lemma, consists of $(m|n)$ -tuples of even and odd supervector fields such that their reductions to M_o give a basis of $T_x M = (T_x M)_{\bar{0}} \oplus (T_x M)_{\bar{1}}$ at each point $x \in M_o$. In other words, by passing to the reduced bundle $Fr_M|_{M_o} \rightarrow M_o$ we get the usual frame bundle for classical \mathbb{Z}_2 -graded vector bundles over M_o . More concretely, $F \in \mathcal{F}r_M(\mathcal{U}_o)$ is a frame $(X_1, \dots, X_m | Y_1, \dots, Y_n)$ with $X_i \in \mathfrak{X}(\mathcal{U}_{\bar{0}})$ and $Y_j \in \mathfrak{X}(\mathcal{U}_{\bar{1}})$ for $1 \leq i \leq m$, $1 \leq j \leq n$, considered as an isomorphism

$$F : \mathcal{V}_M|_{\mathcal{U}_o} \rightarrow \mathcal{T}M|_{\mathcal{U}_o}, \quad (a_i | b_j) \mapsto \sum_{i=1}^m a_i X_i + \sum_{j=1}^n b_j Y_j. \tag{3.2}$$

The sheaf of groups $\mathcal{G}\mathcal{L}_M : \mathcal{U}_o \mapsto \text{GL}(V)[\mathcal{U}]$ acts naturally on the right on the set of frame fields via

$$(X_1, \dots, X_m | Y_1, \dots, Y_n) \cdot g = \left(\sum_{i=1}^m a_i^i X_i - \sum_{\alpha=1}^n c_1^\alpha Y_\alpha, \dots \mid \dots, \sum_{i=1}^m b_n^i X_i + \sum_{\alpha=1}^n d_n^\alpha Y_\alpha \right). \tag{3.3}$$

where $g \in \text{GL}(V)[\mathcal{U}]$ is parametrized as in Section 2.1. This action is locally simply transitive, hence $\mathcal{F}r_M$ is an algebraic principal bundle over M with structure group $\text{GL}(V)$. (The minus sign in (3.3) follows from the sign rule, so that (3.3) is indeed an action.)

Higher order frame bundles $Fr_M^k \rightarrow M$ ($k \geq 1$, with $Fr_M^1 = Fr_M$) can be introduced via the description of jet superbundles of [17], which does not rely either on (topological) points or the functor of points. Let $J^k(\mathbb{R}^{m|n}, M)$ be the vector bundle

of jets from $\mathbb{R}^{m|n}$ to M , which is defined as a supermanifold of homomorphisms of appropriate algebras in [17, §6]. This is a geometric vector bundle over the product $\mathbb{R}^{m|n} \times M$. The open subsupermanifold $J_{inv}^k(\mathbb{R}^{m|n}, M)$ of $J^k(\mathbb{R}^{m|n}, M)$ of invertible jets is easily defined via local coordinates and its pullback to M by the natural injection $M \cong \{0\} \times M \hookrightarrow \mathbb{R}^{m|n} \times M$ is the higher order frame bundle $Fr_M^k \rightarrow M$. The corresponding sheaf is obtained via the geometric-algebraic correspondence. Below we adapt a different approach (which is though similar in spirit) to introduce higher frame bundles in the nonholonomic situation.

3.2 G-Structures on Supermanifolds

In geometric language, a G -structure for $G \subset GL(V)$ is a reduction of the frame bundle as in Example 2.16; following Section 2.6, this corresponds to a subbundle $F_G \subset Fr_M$. In algebraic language this is a subsheaf $\mathcal{F}_G \subset \mathcal{F}_M$ on which the subsheaf $\mathcal{G}_M \subset \mathcal{GL}_M$ acts locally simply transitively from the right.

The soldering form $\vartheta \in \Omega^1(F_G, V)$ is given by

$$\vartheta(\xi) = F^{-1}(\pi_*\xi),$$

where $\xi \in \mathfrak{X}(F_G)$, $\pi = (\pi_o, \pi^*) : F_G \rightarrow M$ is the natural projection and F a local field of frames. More precisely, the R.H.S. is a short-hand for the operation detailed in the following.

Lemma 3.1 *The soldering form is a well-defined even G -equivariant horizontal form on F_G .*

Proof We work by localizing at a point $p \in (F_G)_o$, that is, we consider $\xi \in (\mathcal{T}F_G)_p$ and its image via the push-forward $\Xi = \pi_*\xi = \xi \circ \pi^* : (\mathcal{A}_M)_{\pi_o(p)} \rightarrow (\mathcal{A}_{F_G})_p$. The push-forward Ξ is a π -superderivation, i.e., it satisfies $\Xi(fg) = \Xi(f)\pi^*(g) + (-1)^{|\Xi||f|}\pi^*(f)\Xi(g)$ for all $f, g \in (\mathcal{A}_M)_{\pi_o(p)}$.

Now, the sheaf of π -superderivations is isomorphic to the inverse image sheaf

$$\pi^*\mathcal{T}M = \mathcal{A}_{F_G} \otimes_{\pi_o^{-1}\mathcal{A}_M} \pi_o^{-1}\mathcal{T}M$$

whose stalk at p is $(\mathcal{A}_{F_G})_p \otimes_{(\mathcal{A}_M)_{\pi_o(p)}} \mathcal{T}M_{\pi_o(p)}$, so we may express Ξ as follows:

$$\Xi = \sum_{i=1}^q f^i(\pi^* \circ X_i), \tag{3.4}$$

with $f^i \in (\mathcal{A}_{F_G})_p$ and $X_i \in \mathcal{T}M_{\pi_o(p)}$. We then have

$$\vartheta(\xi) = F^{-1}(\Xi) = \sum_{i=1}^q f^i(F^{-1}X_i)_p, \tag{3.5}$$

where we identified X_i with the associated G -equivariant morphism $F^{-1}X_i : F_G \rightarrow V \oplus \Pi V$, i.e., with an element of $V \otimes \mathcal{A}(F_G) \cong \mathcal{A}(F_G) \otimes V$. Therefore, (3.5) is an element of $(\mathcal{A}_{F_G})_p \otimes V$, as expected, and it is easy to see that it does not depend on the fixed expression (3.4).

The other claims are obvious — e.g., G -equivariancy can be checked as in the classical case thanks to Yoneda lemma. \square

The following exact sequence defines the first prolongation of $\mathfrak{g} \subset \mathfrak{gl}(V)$:

$$0 \rightarrow \mathfrak{g}^{(1)} \longrightarrow \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 V^* \rightarrow 0, \tag{3.6}$$

with $\delta : V \otimes V^* \otimes V^* \rightarrow V \otimes \Lambda^2 V^*$ the Spencer skew-symmetrization operator (in the super-sense), that is $\delta(w \otimes \alpha \otimes \beta) = w \otimes (\alpha \otimes \beta - (-1)^{|\alpha||\beta|} \beta \otimes \alpha)$, and $\mathfrak{g}^{(1)} = \text{Ker}(\delta) = \mathfrak{g} \otimes V^* \cap V \otimes S^2 V^*$. If $\mathfrak{g}_{F_G} = F_G \times \mathfrak{g} \rightarrow F_G$ is the trivial vector bundle over F_G with fiber the Lie superalgebra \mathfrak{g} and, by abuse of notation, we denote the corresponding locally free sheaf on $(F_G)_o$ with the same symbol, then we have the exact sequence of sheaves

$$0 \rightarrow \mathfrak{g}_{F_G}^{(1)} \longrightarrow \mathfrak{g}_{F_G} \otimes \mathcal{V}_{F_G}^* \xrightarrow{\delta} \mathcal{V}_{F_G} \otimes \Lambda^2 \mathcal{V}_{F_G}^* \rightarrow 0. \tag{3.7}$$

Let $\pi_* : \mathcal{J}F_G \rightarrow \pi^* \mathcal{J}M$ be the differential of $\pi : F_G \rightarrow M$, where $\pi^* \mathcal{J}M = \mathcal{A}_{F_G} \otimes_{\pi_o^{-1} \mathcal{A}_M} \pi_o^{-1} \mathcal{J}M$ is the sheaf of π -superderivations.

Definition 3.2

- (1) A *horizontal distribution* is a subsheaf $\mathcal{H} \subset \mathcal{J}F_G$ on $(F_G)_o$ of \mathcal{A}_{F_G} -modules that is complementary to the subsheaf of \mathcal{A}_{F_G} -modules $\text{Ker}(\pi_*) \subset \mathcal{J}F_G$.
- (2) A *normalization* is a supervector space $N \subset V \otimes \Lambda^2 V^*$ that is complementary to $\text{Im}(\delta)$ in (3.6). A similar terminology is used for the associated subsheaf $\mathcal{N} \subset \mathcal{V}_{F_G} \otimes \Lambda^2 \mathcal{V}_{F_G}^*$ of \mathcal{A}_{F_G} -modules on $(F_G)_o$.

Since $\text{Ker}(\pi_*)$ and $\text{Im}(\delta)$ are locally free sheaves, all normalizations and horizontal distributions are as well, see Section 2.6. Any horizontal distribution gives an isomorphism $\mathcal{H} \cong \pi^* \mathcal{J}M$.

As $\pi_o^{-1} \mathcal{J}M \subset \pi^* \mathcal{J}M$ naturally via $X \mapsto \pi^* \circ X$, we have a morphism $\phi_{\mathcal{H}} : \pi_o^{-1} \mathcal{J}M \rightarrow \mathcal{J}F_G$: the horizontal lift $X \mapsto \phi_{\mathcal{H}} X \in \Gamma(\mathcal{H})$ is the right-inverse to the projection π_* . The torsion of the horizontal distribution \mathcal{H} is then defined by

$$T_{\mathcal{H}}(X_1, X_2) = d\vartheta(\phi_{\mathcal{H}} X_1, \phi_{\mathcal{H}} X_2)$$

for all $X_1, X_2 \in \pi_o^{-1} \mathcal{J}M$, where the differential $d\vartheta \in \Omega^2(F_G, V)$ can be computed by the Cartan formula. In other words, we have a morphism of sheaves from $\Lambda^2 \pi_o^{-1} \mathcal{J}M$ to \mathcal{V}_{F_G} , which clearly extends to $\Lambda^2 \pi^* \mathcal{J}M$. Since $\mathcal{H} \cong \pi^* \mathcal{J}M$ and the soldering form $\vartheta \in \Omega^1(F_G, V)$ induces an isomorphism of sheaves $\vartheta|_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{V}_{F_G}$, the torsion can be in turn identified with a global even section of the trivial vector bundle over F_G with the fiber $V \otimes \Lambda^2 V^*$.

Let $F_{r_0} = F_G$, $\mathcal{F}_{r_0} = \mathcal{F}_G$, and define $\mathcal{F}_{r_1} : (F_G)_o \supset \mathcal{V}_o \mapsto \mathcal{F}_{r_1}(\mathcal{V}_o)$ to be the sheaf on $(F_{r_0})_o$ given by

$$\mathcal{F}_{r_1}(\mathcal{V}_o) = \{ \mathcal{H}(\mathcal{V}_o) \mid \mathcal{H} = \text{horiz. distrib. contained in } \mathcal{J}F_{r_0}|_{\mathcal{V}_o} \text{ such that } T_{\mathcal{H}} \in \mathcal{N}|_{\mathcal{V}_o} \}, \tag{3.8}$$

for any open subset \mathcal{V}_o of $(F_{r_0})_o$. The sheaf of Abelian groups $\mathcal{G}_{F_G}^{(1)} : \mathcal{V}_o \mapsto \mathfrak{g}^{(1)}[\mathcal{V}]$ on $(F_{r_0})_o$ acts simply transitively on \mathcal{F}_{r_1} from the right, so this gives an affine bundle $F_{r_1} \rightarrow F_{r_0}$ by the geometric-algebraic correspondence.

Further prolongations follow the same scheme (literally as in [37]) and yield the tower of prolongations

$$M \leftarrow Fr_0 \leftarrow Fr_1 \leftarrow Fr_2 \leftarrow \dots \tag{3.9}$$

A G -structure F_G is called of *finite type* if this tower stabilizes.

Remark 3.3 Alternatively, a geometric structure can be defined via its Lie equations [22]: instead of frames one considers the supermanifold $J^1(V, M)$ of 1-jets of maps $V \rightarrow M$, and the defining equation is a subsupermanifold $\mathcal{E}_1 \subset J^1(V, M)$ which is in bijective correspondence with F_G through V -translations. Prolongations are defined as differential ideals $\mathcal{E}_k = \{D_\sigma f_i = 0 : |\sigma| < k\}$, where the f_i are defining equations of \mathcal{E}_1 , and D_σ are iterated total derivatives.

Symmetries of G -structures may be introduced via automorphism supergroups as in Section 2.2, but a more concrete description is in terms of super Harish-Chandra pairs. To this, we recall that the differential $\varphi_* : \mathcal{T}M \rightarrow (\varphi_o)_*^{-1}\mathcal{T}M$ of any automorphism $\varphi = (\varphi_o, \varphi^*) \in \text{Aut}(M)_{\bar{0}}$ induces an isomorphism $\varphi_* : \mathcal{F}r_M \rightarrow (\varphi_o)_*^{-1}\mathcal{F}r_M$ and we note that the Lie superalgebra $\mathfrak{g} \otimes \mathcal{A}(\mathcal{U}_o)$ acts from the right on $\mathcal{T}_M^{m|n}(\mathcal{U})$.

Definition 3.4

- (i) An *automorphism* of \mathcal{F}_G is a $\varphi \in \text{Aut}(M)_{\bar{0}}$ such that $\varphi_*(\mathcal{F}_G) \subset (\varphi_o)_*^{-1}\mathcal{F}_G$;
- (ii) An *infinitesimal automorphism* of \mathcal{F}_G on a superdomain $\mathcal{U} \subset M$ is a super-vector field $X \in \mathfrak{X}(\mathcal{U})$ such that $\mathcal{L}_X(\mathcal{F}_G(\mathcal{U}_o)) \subset \mathcal{F}_G(\mathcal{U}_o) \cdot (\mathfrak{g} \otimes \mathcal{A}_M(\mathcal{U}_o)) \subset \mathcal{T}_M^{m|n}(\mathcal{U}_o)$.

The symmetries of G -structures are majorized by the tower of principal bundles (3.9) by the classical construction of Sternberg [37] (see also [16]), extended to the supercase in [31]: it is proven there that automorphisms of a finite type G -structure F_G on a supermanifold M form a Lie supergroup $\text{Aut}(M, F_G)$. We will generalize this in what follows.

3.3 Superdistributions and Algebraic Prolongations

A distribution on a supermanifold M is a graded \mathcal{A}_M -subsheaf $\mathcal{D} = \mathcal{D}_{\bar{0}} \oplus \mathcal{D}_{\bar{1}}$ of the tangent sheaf $\mathcal{T}M$ that is locally a direct factor. As explained in Section 2.6, any such sheaf is locally free, so we may consider the associated vector bundle D over M . The latter induces a reduced subbundle $D|_{M_o} \subset TM|_{M_o}$, but as usual with evaluations, $D|_{M_o}$ does not determine \mathcal{D} . We focus here on the algebraic perspective.

The weak derived flag of \mathcal{D} is defined as follows:

$$\mathcal{D}^1 = \mathcal{D} \subset \mathcal{D}^2 \subset \dots \subset \mathcal{D}^i \subset \dots, \quad \mathcal{D}^i = [\mathcal{D}, \mathcal{D}^{i-1}], \tag{3.10}$$

where each term is a graded \mathcal{A}_M -subsheaf of $\mathcal{T}M$. We assume the *bracket-generating* property $\mathcal{D}^\mu = \mathcal{T}M$ for some $\mu > 0$, and also that \mathcal{D} is *regular*, i.e., all subsheaves \mathcal{D}^i are locally direct factors in $\mathcal{T}M$.

Example 3.5 For many examples of (strongly) regular superdistributions, see [23]. We give here a superdistribution that is *not* regular. It is a superextension of the Hilbert–Cartan equation depending on two odd variables. (In [23], we discussed a more general extension with $G(3)$ -symmetry.)

Consider the supermanifold $\mathbb{R}^{5|2}$ with coordinates $(x, u, p, q, z | \theta, v)$, endowed with the following superdistribution of rank $(2|1)$:

$$\mathcal{D} = \langle D_x = \partial_x + p\partial_u + q\partial_p + q^2\partial_z, \partial_q | D_\theta = \partial_\theta + q\partial_v + \theta\partial_p + 2v\partial_z \rangle.$$

We directly compute

$$\begin{aligned} [\partial_q, D_x] &= \partial_p + 2q\partial_z, \quad [\partial_q, D_\theta] = \partial_v, \\ [D_x, D_\theta] &= -\theta\partial_u, \quad [D_\theta, D_\theta] = 2(\partial_p + 2q\partial_z), \end{aligned}$$

so that

$$\mathcal{D}^2 = \langle D_x, \partial_q, \partial_p + 2q\partial_z | D_\theta, \partial_v, \theta\partial_u \rangle.$$

The latter is clearly not a superdistribution, due to the presence of the supervector field $\theta\partial_u$ with trivial evaluation.

In the case of a regular \mathcal{D} we get an increasing filtration \mathcal{D}^i of $\mathcal{T}M$ by superdistributions, which is compatible with brackets of supervector fields: for each superdomain $U \subset M$ we have

$$[\mathcal{D}^i(U), \mathcal{D}^j(U)] \subset \mathcal{D}^{i+j}(U) \quad \forall i, j > 0,$$

as follows easily by induction from the Jacobi superidentities. Note that the bracket is only \mathbb{R} -linear and it satisfies the Leibniz superidentity as a module over \mathcal{A}_M . Clearly, we also have the increasing filtration of $TM|_{M_o}$ given by the classical \mathbb{Z}_2 -graded vector bundles $D^i|_{M_o}$.

Setting $\text{gr}(\mathcal{T}M)_{-i} = \mathcal{D}^i/\mathcal{D}^{i-1}$ for any $i > 0$, we get a locally free sheaf of \mathcal{A}_M -modules $\text{gr}(\mathcal{T}M) = \bigoplus_{i < 0} \text{gr}(\mathcal{T}M)_i$ over M_o . It has a natural structure of a sheaf of negatively graded Lie superalgebras over \mathcal{A}_M : if $v_k \in \mathcal{D}^k$ with associated quotient $\bar{v}_k \in \mathcal{D}^k/\mathcal{D}^{k-1}$ for any $k > 0$, then $[\bar{v}_i, \bar{v}_j] \in \mathcal{D}^{i+j}/\mathcal{D}^{i+j-1}$ and $[\bar{v}_i, f\bar{v}_j] = f[\bar{v}_i, \bar{v}_j]$ for all $f \in \mathcal{A}_M$.

In particular, the bracket on $\text{gr}(\mathcal{T}M)$ is \mathcal{A}_M -linear and thus descends to a Lie superalgebra bracket on the supervector space $\mathfrak{m}(x) = \bigoplus_{i < 0} \mathfrak{g}_i(x)$, $\mathfrak{g}_i(x) = D^{-i}|_x/D^{-i-1}|_x$ for any $x \in M_o$. We shall set $\text{gr}(TM|_{M_o}) = \bigoplus_{x \in M_o} \mathfrak{m}(x)$, which is a classical vector bundle over M_o . (Indeed, this is nothing but the reduction of the vector bundle over M associated to the sheaf $\text{gr}(\mathcal{T}M)$.) Since supervector fields are not determined by their values at points of M_o , the reduction map $\text{ev} : \text{gr}(\mathcal{T}M) \rightarrow \text{gr}(TM|_{M_o})$ can lose information. However the entire information is recoverable in the case of strongly regular distributions, whose correct generalization to the supercase is given in terms of the stalks:

Definition 3.6 Let \mathcal{D} be a regular distribution on a supermanifold $M = (M_o, \mathcal{A}_M)$ that is bracket-generating of depth μ . Then \mathcal{D} is *strongly regular* if there exists a negatively graded Lie superalgebra $\mathfrak{m} = \bigoplus_{-\mu \leq i < 0} \mathfrak{g}_i$ such that $\text{gr}(\mathcal{T}_x M) \cong (\mathcal{A}_M)_x \otimes \mathfrak{m}$ at any $x \in M_o$, as graded Lie superalgebras over $(\mathcal{A}_M)_x$. In this case, \mathfrak{m} is called the *symbol* of \mathcal{D} .

Concretely, a regular superdistribution is strongly regular if it has a local basis of supervector fields adapted to the weak derived flag and whose brackets, after the appropriate quotients, are given by the structure constants of \mathfrak{m} (which are real constants).

From now on, we assume all arising superdistributions to be strongly regular. Note that by construction \mathfrak{m} is *fundamental*, i.e., generated by \mathfrak{m}_{-1} . We will also assume that \mathfrak{m} is *non-degenerate*, i.e., \mathfrak{g}_{-1} contains no central elements of \mathfrak{m} if $\mu > 1$. (Typically, one has $\mathfrak{z}(\mathfrak{m}) = \mathfrak{g}_{-\mu}$.) The *Tanaka–Weisfeiler prolongation* of \mathfrak{m} is the maximal \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ such that

- (i) $\mathfrak{g}_- = \mathfrak{m}$,
- (ii) $\text{Ker}(\text{ad}(\mathfrak{g}_{-1})|_{\mathfrak{g}_i}) = 0 \forall i \geq 0$.

It is denoted $\mathfrak{g} = \text{pr}(\mathfrak{m})$. The proof of the existence and uniqueness of $\text{pr}(\mathfrak{m})$ from [38, 40] extends verbatim to the Lie superalgebra case. Concretely $\mathfrak{g}_0 = \text{der}_{gr}(\mathfrak{m})$ and \mathfrak{g}_i for $i > 0$ are defined recursively by the condition (applies also for $i = 0$)

$$\mathfrak{g}_i = \left\{ u : \bigoplus_{j>0} \mathfrak{g}_{-j} \rightarrow \bigoplus_{j>0} \mathfrak{g}_{i-j} \text{ of } \mathbb{Z}\text{-degree } i \text{ (identified with } \text{ad}_u = [u, \cdot]) \right. \\ \left. \text{such that } [u, [v, w]] = [[u, v], w] + (-1)^{|u||v|} [v, [u, w]] \forall v, w \in \mathfrak{m} \right\}. \tag{3.11}$$

It is easy to verify that $\text{pr}(\mathfrak{m}) = \bigoplus_{i \geq -\mu} \mathfrak{g}_i$ is a Lie superalgebra.

There are several variations on this construction. The most popular one is related to a reduction to a subalgebra $\mathfrak{g}_0 \subset \text{der}_{gr}(\mathfrak{m})$. Then (i) in the definition of the prolongation is changed to $\mathfrak{g}_{\leq 0} = \mathfrak{m} \oplus \mathfrak{g}_0$ and (ii) remains with the same formula but $\forall i > 0$. The resulting prolongation superalgebra is denoted by $\mathfrak{g} = \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$. A more sophisticated reduction is as follows. Assume we have already computed the prolongation to the level $\ell > 0$ and let \mathfrak{g}_ℓ as \mathfrak{g}_0 -module be reducible: $\mathfrak{g}_\ell = \mathfrak{g}'_\ell \oplus \mathfrak{g}''_\ell$. Let also $[\mathfrak{g}_j, \mathfrak{g}_{\ell-j}] \subset \mathfrak{g}'_\ell$ for all $1 \leq j \leq \ell - 1$. Then we can reduce $\mathfrak{g}_{-\mu} \oplus \dots \oplus \mathfrak{g}_{\ell-1} \oplus \mathfrak{g}_\ell$ to $\mathfrak{g}_{-\mu} \oplus \dots \oplus \mathfrak{g}_{\ell-1} \oplus \mathfrak{g}'_\ell$ and prolong for $i > \ell$ by adapting the range of the map u in (3.11). The result will be denoted by $\text{pr}(\mathfrak{m}, \dots, \mathfrak{g}'_\ell, \dots)$, where we list all reductions, or simply $\mathfrak{g} = \bigoplus_{i=-\mu}^\infty \mathfrak{g}_i$ if no confusion arises. An example of this higher order reduction is projective geometry, cf. the classical case in [22, Example 3], which we will also discuss in the super-setting in Section 5.1.5.

The generalized Spencer complex of a reduced prolongation algebra $\mathfrak{g} = \bigoplus_{i=-\mu}^\infty \mathfrak{g}_i$ is the Lie superalgebra cohomology complex $\Lambda^\bullet \mathfrak{m}^* \otimes \mathfrak{g}$ with the Chevalley–Eilenberg differential δ :

$$H^j(\mathfrak{m}, \mathfrak{g}) = H^\bullet(\Lambda^{j-1} \mathfrak{m}^* \otimes \mathfrak{g} \xrightarrow{\delta} \Lambda^j \mathfrak{m}^* \otimes \mathfrak{g} \xrightarrow{\delta} \Lambda^{j+1} \mathfrak{m}^* \otimes \mathfrak{g}).$$

It is naturally bi-graded $H^\bullet(\mathfrak{m}, \mathfrak{g}) = \bigoplus_d H^{d, \bullet}(\mathfrak{m}, \mathfrak{g})$, where d is the \mathbb{Z} -degree of a cochain, and it also admits a parity decomposition into even and odd parts as a supervector space. It follows from definitions that $H^{i,1}(\mathfrak{m}, \mathfrak{g}) = 0$ if and only if \mathfrak{g}_i is the full prolongation of $\mathfrak{m} \oplus \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_{i-1}$, therefore $H^{\geq 0,1}(\mathfrak{m}, \mathfrak{g}) = \bigoplus_{i \geq 0} H^{i,1}(\mathfrak{m}, \mathfrak{g})$ encodes all possible reductions.

3.4 Filtered Geometric Structures

Now we superize the notion of filtered geometric structure as developed in [22, 30, 38]. Let \mathcal{D} be a strongly regular, fundamental, non-degenerate distribution on a supermanifold M . The corresponding zero-order frame bundle is a principal bundle $\pi : Fr_0 = Pr_0(M, \mathcal{D}) \rightarrow M$ defined via the geometric-algebraic correspondence as the sheaf $\mathcal{F}r_0 : M_o \supset \mathcal{U}_o \mapsto \mathcal{F}r_0(\mathcal{U}_o)$ of \mathcal{A}_M -linear Lie superalgebra isomorphisms from \mathfrak{m}_M to $\text{gr}(\mathcal{T}M)$:

$$\mathcal{F}r_0(\mathcal{U}_o) = \left\{ \mathcal{A}|_{\mathcal{U}_o}\text{-linear Lie superalgebra isomorphism } F : \mathfrak{m}_M|_{\mathcal{U}_o} \rightarrow \text{gr}(\mathcal{T}M)|_{\mathcal{U}_o} \right\}. \tag{3.12}$$

Here we denoted by $\mathfrak{m}_M = M \times \mathfrak{m} \rightarrow M$ the trivial vector bundle over M with the fiber \mathfrak{m} and, by abuse of notation, the associated locally free sheaf with the same symbol. The structure group of the bundle is the Lie supergroup $G_0 = \text{Aut}_{gr}(\mathfrak{m})$ which, by the Harish-Chandra construction, can be identified with the pair $(\text{Aut}_{gr}(\mathfrak{m})_{\bar{0}}, \text{der}_{gr}(\mathfrak{m}))$ formed by the Lie group of degree zero automorphisms of \mathfrak{m} and the Lie superalgebra of degree zero superderivations of \mathfrak{m} . Since \mathfrak{m} is fundamental, the structure group G_0 embeds into the Lie supergroup $\text{GL}(\mathfrak{g}_{-1})$ and $\mathcal{F}r_0$ can be realized as a sheaf of special \mathcal{A}_M -linear isomorphisms from $(\mathfrak{g}_{-1})_M$ to \mathcal{D} .

More generally a first-order reduction is given by a G_0 -reduction $F_0 \subset Pr_0(M, \mathcal{D})$ with structure group a Lie subsupergroup $G_0 \subset \text{Aut}_{gr}(\mathfrak{m})$, which again can be thought of as an inclusion of super Harish-Chandra pairs $((G_0)_{\bar{0}}, \mathfrak{g}_0) \subset (\text{Aut}_{gr}(\mathfrak{m})_{\bar{0}}, \text{der}_{gr}(\mathfrak{m}))$. (Often such reductions are given by order 1 invariants, e.g., tensors or their spans. As first example, an $\text{OSp}(m|2n)$ -reduction in the case $\mathcal{D} = \mathcal{T}M$ corresponds to an even supermetric $q \in S^2\mathcal{T}^*M$.)

In the next section we will construct higher order frame bundles $Fr_i = Pr_i(M, \mathcal{D})$, which fit into a tower of principal bundles with projections

$$M \leftarrow Fr_0 \leftarrow Fr_1 \leftarrow Fr_2 \leftarrow \dots, \tag{3.13}$$

where the principal bundle $Fr_i \rightarrow Fr_{i-1}$ has Abelian structure group \mathfrak{g}_i , for all $i > 0$.

The bottom projections have the structure of fiber bundles over M : $Fr_1 \rightarrow M$ with fiber $G^1 = G_0 \times \mathfrak{g}_1$, $Fr_2 \rightarrow M$ with fiber $G^2 = G^1 \times \mathfrak{g}_2$, etc, but in general these are not principal bundles. Similiar to the embedding $\mathcal{F}r_M \subset \mathcal{T}_M^{m|n}$ described in Section 3.1, the higher order frame bundle $\mathcal{F}r_i$ successively embeds into a locally free sheaf $\widetilde{\mathcal{F}r}_i$ of $\mathcal{A}_{\mathcal{F}r_{i-1}}$ -modules

$$\widetilde{\mathcal{F}r}_i(\mathcal{U}_o) = \left\{ \text{even and odd } \mathcal{A}|_{\mathcal{U}_o}\text{-linear maps } u : \mathfrak{m}_{\mathcal{U}_o} \rightarrow \mathcal{T}\mathcal{F}r_{i-1}|_{\mathcal{U}_o} \right\} \tag{3.14}$$

for every superdomain $\mathcal{U} \subset Fr_{i-1}$. This corresponds to a vector bundle, whose fiber can be further reduced but it is not relevant here.

For any first-order reduction $F_0 \subset Pr_0(M, \mathcal{D})$ we denote the prolongation bundles by

$$F_0^{(i)} = Pr_i(M, \mathcal{D}, F_0) \subset Fr_i,$$

for all $i > 0$. They also fit into a tower of principal bundles analogous to (3.13). For higher order reductions, we restrict to a subbundle $F_i \subset F_0^{(i)}$ for some $i > 0$, and the geometric object q responsible for this reduction will have higher order. (E.g., a

projective superstructure is given by an equivalence class of superconnections. The associated Lie equations for symmetry, cf. Remark 3.3, are of second order.) Further reductions can be imposed in a similar way on the prolongations $F_j^{(i)}$. In the rest of the paper, in order not to overload notations, we will mostly concentrate on pure prolongations or first-order reductions. However, the results apply in the general situation.

Definition 3.7 A filtered geometric structure (M, \mathcal{D}, F) on a supermanifold M consists of a strongly regular, fundamental, non-degenerate distribution \mathcal{D} on M and possibly some reductions F of the tower (3.13). If F are encoded by a tensorial or higher order structure q , we will also use the notation (M, \mathcal{D}, q) .

3.5 Geometric Prolongation

Now we shall construct the higher (super) frame bundles partially following the revision by Zelenko [42] of the constructions by Sternberg [37] and Tanaka [38] (beware: our notations differ from theirs). Our approach is novel in the following: we construct the tower of bundles $\mathcal{F}_\ell, \ell \geq 0$, and the frames $\varphi_{\mathcal{F}_\ell}$ on them, using the entire Spencer differential (instead of a reduced one) and recognize the choices of complements as the space of 0- and 1-cochains therein (with freedom being co-boundaries).

3.5.1 First Prolongation

Thanks to Section 3.4, we assume that the bundle $\pi_0 : F_0 \rightarrow M$ is already constructed. Via pullback by $d\pi_0$, the filtration on $\mathcal{T}M$ induces a filtration on $\mathcal{T}F_0$:

$$\begin{array}{ccccccc} \mathcal{T}F_0 = & \mathcal{T}^{-\mu}F_0 & \supset & \dots & \supset & \mathcal{T}^{-1}F_0 & \supset & \mathcal{T}^0F_0 := \ker(d\pi_0) \\ & \downarrow & & & & \downarrow & & \\ \mathcal{T}M = & \mathcal{T}^{-\mu}M & \supset & \dots & \supset & \mathcal{T}^{-1}M & & \end{array} \tag{3.15}$$

where we also set $\mathcal{T}^k F_0 = \mathcal{T}F_0$ for all $k < -\mu$, $\mathcal{T}^k F_0 = 0_{F_0}$ for all $k > 0$ and, for simplicity, omit the inverse image symbol π_0^* in front of each of the sheaves on M in the bottom row of (3.15). Via $d\pi_0$, we then have

$$d\pi_0 : \text{gr}_-(\mathcal{T}F_0) \xrightarrow{\cong} \pi_0^* \text{gr}(\mathcal{T}M)$$

as sheaves of negatively \mathbb{Z} -graded Lie superalgebras on F_0 , with $\pi_0^* \text{gr}(\mathcal{T}M)$ referring to the inverse image sheaf. There is the canonical (\mathcal{A}_{F_0} -linear, even) vertical 0-trivialization

$$\gamma_0 : \mathcal{T}^0 F_0 \xrightarrow{\cong} \mathcal{A}_{F_0} \otimes \mathfrak{g}_0 \tag{3.16}$$

given on fundamental vector fields by $\gamma_0(\zeta_X) = X$ for all $X \in \mathfrak{g}_0$.

Let $\mathcal{U} \subset M$ be a superdomain and consider a section of F_0 on \mathcal{U} , which is identified with an $\mathcal{A}_M|_{\mathcal{U}_o}$ -linear isomorphism $\varphi_0 = \{\varphi_0^i\}_{i < 0} : \mathfrak{m}_M|_{\mathcal{U}_o} \rightarrow \text{gr}(\mathcal{T}M)|_{\mathcal{U}_o}$ of zero \mathbb{Z} -degree, cf. (3.12). We also call it a horizontal 0-frame. Working with stalks of inverse image sheaves as in the proof of Lemma 3.1, one easily checks that the following is well-defined.

Definition 3.8 The tuple $\vartheta_0 = \{\vartheta_0^i\}_{i < 0} \in \text{Hom}(\text{gr}_-(\mathcal{T}F_0), \mathfrak{m}_{F_0})$ of morphisms of \mathcal{A}_{F_0} -modules defined by $\vartheta_0 = (\varphi_0)^{-1} \circ d\pi_0$ is called the *soldering form* of F_0 .

Concretely, one may compute ϑ_0 by restricting to a superdomain $\mathcal{U} \times G_0 \cong \pi_0^{-1}(\mathcal{U}) \subset F_0$ trivialized by a fixed frame φ_0 and recalling that all other frames are obtained by the action of G_0 : $\varphi_0 \cdot g := \alpha \circ (\varphi_0, g)$ for any morphism $g : \mathcal{U} \rightarrow G_0$, with $\alpha : F_0 \times G_0 \rightarrow F_0$ the right action. The soldering form is G_0 -equivariant. We also note that

$$\vartheta_0^i \in \text{Hom}(\mathcal{T}^i F_0 / \mathcal{T}^{i+1} F_0, (\mathfrak{m}_i)_{F_0}) \tag{3.17}$$

for $i < 0$, is invertible, and that $\vartheta_0 = \{\vartheta_0^i\}_{i < 0}$ is an isomorphism of sheaves of \mathbb{Z} -graded Lie superalgebras over \mathcal{A}_{F_0} . In particular, using ϑ_0 and γ_0 , we obtain a full frame of $\text{gr}(\mathcal{T}F_0)$. (We caution the reader that this does not identify $\text{gr}(\mathcal{T}F_0)$ with $\mathcal{A}_{F_0} \otimes (\mathfrak{m} \oplus \mathfrak{g}_0)$ as Lie superalgebras, as the bracket on $\text{gr}(\mathcal{T}F_0)$ is not \mathcal{A}_{F_0} -linear if at least one entry is a vertical supervector field.)

For each $k \in \mathbb{Z}$, we set $\mathcal{T}_0^k := \mathcal{T}^k F_0$. Let $i < 0$ and consider the following exact sequence of sheaves on F_0 :

$$0 \rightarrow \mathcal{T}_0^{i+1} / \mathcal{T}_0^{i+2} \xrightarrow{t_0^i} \mathcal{T}_0^i / \mathcal{T}_0^{i+2} \xrightarrow{j_0^i} \mathcal{T}_0^i / \mathcal{T}_0^{i+1} \rightarrow 0. \tag{3.18}$$

Because all \mathcal{T}_0^i are superdistributions, the image of t_0^i in (3.18) is a direct factor, i.e., there exists a complementary subsheaf $\mathcal{H}_0^i \subset \mathcal{T}_0^i / \mathcal{T}_0^{i+2}$ so that we have a splitting

$$\mathcal{T}_0^i / \mathcal{T}_0^{i+2} = \mathcal{H}_0^i \oplus \mathcal{T}_0^{i+1} / \mathcal{T}_0^{i+2}. \tag{3.19}$$

The sequence (3.18) makes sense for $i = 0$ as well, in which case it simply says that $\mathcal{H}_0^0 := \mathcal{T}_0^0$. By the splitting lemma, \mathcal{H}_0^i is the kernel of a left-inverse h_0^i to t_0^i or, equivalently, the image of the right-inverse $k_0^i := (\text{id} - t_0^i \circ h_0^i) \circ (j_0^i)^{-1}$ to j_0^i . We will write $h_0 = \{h_0^i\}_{i \leq 0}$ and $k_0 = \{k_0^i\}_{i \leq 0}$.

Set now $\text{gr}^{[1]}(\mathcal{T}F_0) = \bigoplus_{i \leq 0} \mathcal{T}_0^i / \mathcal{T}_0^{i+2}$. Given a fixed choice of complements $\mathcal{H}_0 = \{\mathcal{H}_0^i\}_{i \leq 0}$ as above, we define a *1-frame*

$$\varphi_{\mathcal{H}_0} : (\mathfrak{g}_{\leq 0})_{F_0} \rightarrow \text{gr}^{[1]}(\mathcal{T}F_0) \tag{3.20}$$

as the map of zero \mathbb{Z} -degree with the components $\varphi_{\mathcal{H}_0}^i : (\mathfrak{g}_i)_{F_0} \rightarrow \mathcal{T}_0^i / \mathcal{T}_0^{i+2}$ determined by the soldering form via the equation $\varphi_{\mathcal{H}_0}^i \circ \vartheta_0^i = k_0^i$ for all $i < 0$, and $\varphi_{\mathcal{H}_0}^i \circ \gamma_0 = k_0^i = \mathbb{1}_{\mathcal{T}_0^0}$ for $i = 0$. Note that $\varphi_{\mathcal{H}_0}$ is an isomorphism with image $\text{Im}(\varphi_{\mathcal{H}_0}^i) = \text{Im}(k_0^i) = \mathcal{H}_0^i$ for all $i \leq 0$.

In terms of the maps $h_0 = \{h_0^i\}_{i \leq 0}$ determined by \mathcal{H}_0 , we define the *1st structure function*

$$c_{\mathcal{H}_0} \in (\Lambda^2 \mathfrak{m}_{F_0}^* \otimes_{\mathcal{A}_{F_0}} \mathfrak{m}_{F_0}) = \mathcal{A}_{F_0} \otimes (\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{m})$$

on the entries $v_k \in \mathfrak{g}_k, k < 0$, by

$$c_{\mathcal{H}_0}(v_i, v_j) = \vartheta_0 \left(h_0 \left([\varphi_{\mathcal{H}_0}(v_i), \varphi_{\mathcal{H}_0}(v_j)] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \right) \tag{3.21}$$

and then extend by \mathcal{A}_{F_0} -linearity to $(\mathfrak{g}_k)_{F_0} = \mathcal{A}_{F_0} \otimes \mathfrak{g}_k$. Since the filtration on $\mathcal{T}F_0$ is respected by the Lie bracket, then the input of ϑ_0 above lies in $\text{gr}_{i+j+1}(\mathcal{T}F_0) =$

$\mathcal{T}_0^{i+j+1}/\mathcal{T}_0^{i+j+2}$, which is mapped by ϑ_0 to \mathfrak{g}_{i+j+1} . In particular, $c_{\mathcal{H}_0}$ is well-defined, it has even parity and \mathbb{Z} -degree 1, i.e., it maps $\mathfrak{g}_i \otimes \mathfrak{g}_j$ to \mathfrak{g}_{i+j+1} . We let $\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} (\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g})_k$ be the natural decomposition of $\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}$ into \mathbb{Z} -graded components, so that $c_{\mathcal{H}_0} \in \mathcal{A}_{F_0} \otimes (\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g})_1$. The space $\mathfrak{m}^* \otimes \mathfrak{g}$ has an analogous decomposition and clearly $(\mathfrak{m}^* \otimes \mathfrak{g})_1 \subsetneq \mathfrak{m}^* \otimes (\mathfrak{m} \oplus \mathfrak{g}_0)$.

Let us take another complement $\mathcal{H}_0 = \{\mathcal{H}_0^i\}_{i \leq 0}$ and the 1-frame $\varphi_{\widetilde{\mathcal{H}}_0}$. By construction, for any $v_i \in (\mathfrak{g}_i)_{F_0}$ with $i < 0$, we have that $\varphi_{\widetilde{\mathcal{H}}_0}(v_i) - \varphi_{\mathcal{H}_0}(v_i)$ is an element of $\mathcal{T}_0^{i+1}/\mathcal{T}_0^{i+2}$, hence

$$\vartheta_0^{i+1} \left(\varphi_{\widetilde{\mathcal{H}}_0}(v_i) - \varphi_{\mathcal{H}_0}(v_i) \right) = \psi(v_i) \text{ for } i < -1, \tag{3.22}$$

$$\gamma_0 \left(\varphi_{\widetilde{\mathcal{H}}_0}(v_i) - \varphi_{\mathcal{H}_0}(v_i) \right) = \psi(v_i) \text{ for } i = -1, \tag{3.23}$$

for some morphism $\psi : \mathfrak{m}_{F_0} \rightarrow (\mathfrak{m} \oplus \mathfrak{g}_0)_{F_0}$ of sheaves of \mathcal{A}_{F_0} -modules. It is clear that ψ has \mathbb{Z} -degree 1, in other words, it is an element of even parity of $\mathcal{A}_{F_0} \otimes (\mathfrak{m}^* \otimes \mathfrak{g})_1$. Conversely, given any such ψ , there is a unique complement $\widetilde{\mathcal{H}}_0 = \{\widetilde{\mathcal{H}}_0^i\}_{i \leq 0}$ for which (3.22)–(3.23) hold.

Lemma 3.9 *Under a change of the complement, the structure function transforms as $c_{\widetilde{\mathcal{H}}_0} = c_{\mathcal{H}_0} + \delta\psi$, where δ is the Chevalley–Eilenberg differential from $C^{1,1}(\mathfrak{m}, \mathfrak{g})_{F_0}$ to $C^{1,2}(\mathfrak{m}, \mathfrak{g})_{F_0}$.*

Proof One directly infers from (3.22) that $\vartheta_0^{k+1} \circ \widetilde{h}_0 = \vartheta_0^{k+1} \circ h_0 - \psi \circ \vartheta_0^k \circ j_0^k$ for all $k < -1$. Suppressing upper indices for simplicity and denoting by $\Psi : \mathfrak{m}_{F_0} \rightarrow \text{gr}(\mathcal{T}F_0)$ the morphism obtained composing ψ with the identifications (3.16)–(3.17), we get for all $v_l \in \mathfrak{g}_l, l \leq -1$:

$$\begin{aligned} c_{\widetilde{\mathcal{H}}_0}(v_i, v_j) &= \vartheta_0 \left(\widetilde{h}_0 \left(\left[\varphi_{\widetilde{\mathcal{H}}_0}(v_i), \varphi_{\widetilde{\mathcal{H}}_0}(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \right) \\ &= \vartheta_0 \left(h_0 \left(\left[\varphi_{\mathcal{H}_0}(v_i), \varphi_{\mathcal{H}_0}(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \right) \\ &\quad - \psi \circ \vartheta_0 \circ j_0 \left(\left[\varphi_{\widetilde{\mathcal{H}}_0}(v_i), \varphi_{\widetilde{\mathcal{H}}_0}(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \\ &= \vartheta_0 \left(h_0 \left(\left[\varphi_{\mathcal{H}_0}(v_i) + \Psi(v_i), \varphi_{\mathcal{H}_0}(v_j) + \Psi(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \right) \\ &\quad - \psi \circ \vartheta_0 \circ j_0 \left(\left[\varphi_{\mathcal{H}_0}(v_i) + \Psi(v_i), \varphi_{\mathcal{H}_0}(v_j) + \Psi(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \\ &= c_{\mathcal{H}_0}(v_i, v_j) + \vartheta_0 \left(\left[\varphi_{\mathcal{H}_0}(v_i), \Psi(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \\ &\quad + \vartheta_0 \left(\left[\Psi(v_i), \varphi_{\mathcal{H}_0}(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+2} \right) \\ &\quad - \psi \circ \vartheta_0 \left(\left[\varphi_{\mathcal{H}_0}(v_i), \varphi_{\mathcal{H}_0}(v_j) \right] \text{ mod } \mathcal{T}_0^{i+j+1} \right) \\ &= c_{\mathcal{H}_0}(v_i, v_j) + \underbrace{[v_i, \psi(v_j)] - (-1)^{|v_i||v_j|} [v_j, \psi(v_i)] - \psi([v_i, v_j])}_{\delta\psi(v_i, v_j)}, \end{aligned}$$

where the last equality follows from the definition of structure function and the fact that the soldering form ϑ_0 is a G_0 -equivariant morphism of Lie superalgebras. \square

This gives the following method to restrict the \mathcal{H}_0 's. Take a complement $N_1 \subset (\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g})_1$ to $\delta(\mathfrak{m}^* \otimes \mathfrak{g})_1$ and denote the corresponding sheaf over F_0 by $\mathcal{N}_1 = \mathcal{A}_{F_0} \otimes N_1$. Then we define the sheaf $\text{Pr}_1(M, \mathcal{D}, F_0)$ over F_0 by

$$\text{Pr}_1(M, \mathcal{D}, F_0)(\mathcal{V}_o) = \left\{ \mathcal{H}_0(\mathcal{V}_o) \mid \mathcal{H}_0 = \{\mathcal{H}_0^i\}_{i \leq 0} \text{ on } \mathcal{V}_0 \text{ such that } c_{\mathcal{H}_0} \in \mathcal{N}_1|_{\mathcal{V}_o} \right\}, \tag{3.24}$$

equivalently the collection of the associated 1-frames (3.20). By (3.22)–(3.23) and Lemma 3.9, this is a principal bundle $\pi_1 : \text{Pr}_1(M, \mathcal{D}, F_0) \rightarrow F_0$ over F_0 with Abelian structure group $G_1 = \exp(\mathfrak{g}_1)$ consisting of all $\psi \in (\mathfrak{m}^* \otimes \mathfrak{g})_1$ in the kernel of the Spencer operator δ , i.e., of all elements of the first prolongation $\mathfrak{g}_1 = \mathfrak{g}_0^{(1)}$.

The affine bundle $\text{Pr}_1(M, \mathcal{D}, F_0)$ may have a further reduction resulting in the first frame bundle $F_1 \subseteq \text{Pr}_1(M, \mathcal{D}, F_0)$. By (3.20), a section φ_1 of F_1 over $\mathcal{V} \subset F_0$ can be equivalently thought as an element $\varphi_1 : (\mathfrak{g}_{\leq 0})_{F_0} \rightarrow \text{gr}^{[1]}(\mathcal{T}F_0)$ such that $\mathcal{H}_0 = \text{Im}(\varphi_1)$.

3.5.2 Higher Frame Bundles

The higher frame bundles are constructed similarly. We will not specify structure reductions anymore, denoting (reduced or non-reduced) frame bundles by the same symbol F_i .

The construction is inductive. For $\ell \geq 1$, suppose that we have constructed:

- (1) the affine bundle $\pi_\ell : F_\ell \rightarrow F_{\ell-1}$ with Abelian structure group G_ℓ with associated Lie superalgebra $\mathfrak{g}_\ell = \mathfrak{g}_{\ell-1}^{(1)}$;
- (2) a decreasing filtration

$$\mathcal{T}F_{\ell-1} = \mathcal{T}^{-\mu} F_{\ell-1} \supset \dots \supset \mathcal{T}^{\ell-1} F_{\ell-1} = \text{Ker}(d\pi_{\ell-1}) \tag{3.25}$$

on $F_{\ell-1}$ with associated soldering form and vertical $(\ell - 1)$ -trivialization

$$\begin{aligned} \vartheta_{\ell-1} &= \{\vartheta_{\ell-1}^i\}_{i < \ell-1} \in \text{Hom}(\text{gr}_{< \ell-1}(\mathcal{T}F_{\ell-1}), (\mathfrak{g}_{< \ell-1})_{F_{\ell-1}}), \\ \gamma_{\ell-1} &: \mathcal{T}^{\ell-1} F_{\ell-1} \xrightarrow{\cong} \mathcal{A}_{F_{\ell-1}} \otimes \mathfrak{g}_{\ell-1}. \end{aligned} \tag{3.26}$$

Henceforth, we write $\mathcal{T}^i_j := \mathcal{T}^i F_j$, with the understanding that $\mathcal{T}^i F_j = \mathcal{T} F_j$ for $i < -\mu$ and $\mathcal{T}^i F_j = 0_{F_j}$ for $i > j$;

- (3) an ℓ -frame, which is an injective morphism of sheaves of $\mathcal{A}_{F_{\ell-1}}$ -modules

$$\varphi_\ell : (\mathfrak{g}_{\leq \ell-1})_{F_{\ell-1}} \rightarrow \text{gr}^{[\ell]}(\mathcal{T}F_{\ell-1}), \tag{3.27}$$

where $\text{gr}^{[\ell]}(\mathcal{T}F_{\ell-1}) := \bigoplus_{i \leq \ell-1} \mathcal{T}^i_{\ell-1} / \mathcal{T}^{i+\ell+1}_{\ell-1}$. (Equivalently, we have a section φ_ℓ of $\pi_\ell : F_\ell \rightarrow F_{\ell-1}$ over a superdomain $\mathcal{V} \subset F_{\ell-1}$.) The ℓ -frame $\varphi_\ell = \{\varphi_\ell^i\}_{i \leq \ell-1}$ selects horizontal subspaces $\mathcal{H}_{\ell-1} = \{\mathcal{H}_{\ell-1}^i\}_{i \leq \ell-1} = \text{Im}(\varphi_\ell)$, with

$$\mathcal{H}_{\ell-1}^i \subset \mathcal{T}^i_{\ell-1} / \mathcal{T}^{i+\ell+1}_{\ell-1} \tag{3.28}$$

for $i < 0$, and

$$\mathcal{H}_{\ell-1}^i \subset \mathcal{T}^i_{\ell-1} \tag{3.29}$$

for $0 \leq i \leq \ell - 1$, with $\mathcal{H}_{\ell-1}^{\ell-1} = \mathcal{T}_{\ell-1}^{\ell-1}$. The component $\varphi_{\ell}^{\ell-1}$ is the identification of $(\mathfrak{g}_{\ell-1})_{F_{\ell-1}}$ with $\mathcal{T}_{\ell-1}^{\ell-1}$ given by the vertical $(\ell - 1)$ -trivialization and the component φ_{ℓ}^i does not vary upon the action of the structure group G_{ℓ} , for any $i \geq 0$.

Since the framework of supermanifolds does not allow to work at a fixed point, what we will really need is the pullback bundle $\pi_{\ell}^* F_{\ell} \rightarrow F_{\ell}$ with its canonical section. In other words $\varphi_{\ell} : (\mathfrak{g}_{\leq \ell-1})_{F_{\ell}} \rightarrow \pi_{\ell}^* \text{gr}^{[\ell]}(\mathcal{T}F_{\ell-1})$ and the sheaf $\mathcal{H}_{\ell-1}^i$ is a subsheaf of $\pi_{\ell}^* \left(\mathcal{T}_{\ell-1}^i / \mathcal{T}_{\ell-1}^{i+\ell+1} \right)$ and $\pi_{\ell}^* \mathcal{T}_{\ell-1}^i$ for, respectively, negative and non-negative indices.

In this subsection, we construct the new horizontal subspaces $\mathcal{H}_{\ell} = \{\mathcal{H}_{\ell}^i\}_{i \leq \ell}$, which includes the construction for the non-negative indices $0 \leq i \leq \ell$.

Via pullback by $d\pi_{\ell}$, the filtration (3.25) on $\mathcal{T}F_{\ell-1}$ induces a filtration on $\mathcal{T}F_{\ell}$:

$$\begin{array}{ccccccc} \mathcal{T}F_{\ell} = & \mathcal{T}_{\ell}^{-\mu} & \supset & \dots & \supset & \mathcal{T}_{\ell}^{\ell-1} & \supset & \mathcal{T}_{\ell}^{\ell} = \ker(d\pi_{\ell}) \\ & \downarrow & & & & \downarrow & & \\ \mathcal{T}F_{\ell-1} = & \mathcal{T}_{\ell-1}^{-\mu} & \supset & \dots & \supset & \mathcal{T}_{\ell-1}^{\ell-1} & & \end{array}$$

where, as usual, we omit inverse image sheaf symbol π_{ℓ}^* for the sheaves in the bottom row. It is important to note for later use that the filtration on $\mathcal{T}F_{\ell}$ is respected by the Lie bracket only for *non-positive* filtration indices (because of the Leibniz rule). For instance, the vertical subbundle $\mathcal{T}_{\ell}^{\ell}$ is integrable and it also preserves all \mathcal{T}_{ℓ}^i for $-\mu \leq i \leq \ell - 1$, since the latter bundle is induced via pullback. Similarly one has

$$[\mathcal{T}_{\ell}^m, \mathcal{T}_{\ell}^n] \subset \mathcal{T}_{\ell}^n \tag{3.30}$$

for all $0 \leq m \leq \ell$ and $-\mu \leq n \leq m$.

Note the isomorphism $\text{gr}_{< \ell}(\mathcal{T}F_{\ell}) \xrightarrow{\cong} \pi_{\ell}^* \text{gr}(\mathcal{T}F_{\ell-1})$ as sheaves of $\mathcal{A}_{F_{\ell}}$ -modules. The *soldering form* $\vartheta_{\ell} = \{\vartheta_{\ell}^i\}_{i < \ell} \in \text{Hom}(\text{gr}_{< \ell}(\mathcal{T}F_{\ell}), (\mathfrak{g}_{< \ell})_{F_{\ell}})$ on F_{ℓ} is defined by composing this isomorphism with the soldering form and the vertical trivialization on $F_{\ell-1}$, i.e., it is the pullback via π_{ℓ} of the forms (3.26). We also have a canonical vertical ℓ -trivialization $\gamma_{\ell} : \mathcal{T}_{\ell}^{\ell} \rightarrow \mathcal{A}_{F_{\ell}} \otimes \mathfrak{g}_{\ell}$.

Consider the following two exact sequences of sheaves over F_{ℓ} (the sequence over $F_{\ell-1}$ lifts via the inverse image operation, the notation of which we suppress again), with $i < 0$:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{T}_{\ell}^{i+\ell+1} / \mathcal{T}_{\ell}^{i+\ell+2} & \xrightarrow{\iota_{\ell}^i} & \mathcal{T}_{\ell}^i / \mathcal{T}_{\ell}^{i+\ell+2} & \xleftarrow{\jmath_{\ell}^i} & \mathcal{T}_{\ell}^i / \mathcal{T}_{\ell}^{i+\ell+1} & \longrightarrow 0 \\ & \vdots & & \downarrow b_{\ell}^i & \swarrow a_{\ell}^i & & \\ 0 \longrightarrow & \mathcal{T}_{\ell-1}^{i+\ell} / \mathcal{T}_{\ell-1}^{i+\ell+1} & \xrightarrow{\iota_{\ell-1}^i} & \mathcal{T}_{\ell-1}^i / \mathcal{T}_{\ell-1}^{i+\ell+1} & \xleftarrow{\jmath_{\ell-1}^i} & \mathcal{T}_{\ell-1}^i / \mathcal{T}_{\ell-1}^{i+\ell} & \longrightarrow 0 \end{array}$$

For all $i < 0$, the differential $d\pi_{\ell}$ induces the map $a_{\ell}^i : \mathcal{T}_{\ell}^i / \mathcal{T}_{\ell}^{i+\ell+1} \rightarrow \pi_{\ell}^* \left(\mathcal{T}_{\ell-1}^i / \mathcal{T}_{\ell-1}^{i+\ell+1} \right)$, which is an isomorphism. We then define the middle vertical map by $b_{\ell}^i := a_{\ell}^i \circ \jmath_{\ell}^i$. As in Section 3.5.1, we will later see in Lemma 3.11 that these

sequences split, with dashed lines indicating left-inverses h_j^i to l_j^i and right-inverses $k_j^i := (\text{id} - l_j^i \circ h_j^i) \circ (J_j^i)^{-1}$ to J_j^i . We consider \mathcal{H}_ℓ^i satisfying

$$\mathcal{T}_\ell^i / \mathcal{T}_\ell^{i+\ell+2} \supset (b_\ell^i)^{-1} (\mathcal{H}_{\ell-1}^i) = \mathcal{H}_\ell^i \oplus \text{Im}(t_\ell^i), \tag{3.31}$$

so that the restriction of b_ℓ^i to \mathcal{H}_ℓ^i defines an isomorphism $\mathcal{H}_\ell^i \xrightarrow{\cong} \mathcal{H}_{\ell-1}^i$. (We recall for reader's convenience that by definition $\mathcal{H}_{\ell-1}^i \subset \pi_\ell^* (\mathcal{T}_{\ell-1}^i / \mathcal{T}_{\ell-1}^{i+\ell+1})$ and that $\text{Ker}(b_\ell^i) = \text{Ker}(J_\ell^i) = \text{Im}(t_\ell^i)$.)

For all $0 \leq i \leq \ell - 1$, we let $b_\ell^i : \mathcal{T}_\ell^i \rightarrow \pi_\ell^* \mathcal{T}_{\ell-1}^i$ be the projection induced by the differential, whose kernel is \mathcal{T}_ℓ^i . We then have the following exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{T}_\ell^i & \xrightarrow{\quad t_\ell^i \quad} & \mathcal{T}_\ell^i & \xrightarrow{\quad J_\ell^i \quad} & \mathcal{T}_\ell^i / \mathcal{T}_\ell^i & \longrightarrow & 0 \\ & & \vdots & & \downarrow b_\ell^i & & \swarrow a_\ell^i & & \\ 0 & \longrightarrow & \mathcal{T}_{\ell-1}^{i-1} & \xrightarrow{\quad t_{\ell-1}^{i-1} \quad} & \mathcal{T}_{\ell-1}^{i-1} & \xrightarrow{\quad J_{\ell-1}^{i-1} \quad} & \mathcal{T}_{\ell-1}^{i-1} / \mathcal{T}_{\ell-1}^{i-1} & \longrightarrow & 0 \end{array}$$

with a_ℓ^i the isomorphism induced by b_ℓ^i on the quotient. We choose a complement \mathcal{H}_ℓ^i to \mathcal{T}_ℓ^i in $(b_\ell^i)^{-1} (\mathcal{H}_{\ell-1}^i)$ for every $0 \leq i \leq \ell - 1$ as before in (3.31), namely

$$\mathcal{T}_\ell^i \supset (b_\ell^i)^{-1} (\mathcal{H}_{\ell-1}^i) = \mathcal{H}_\ell^i \oplus \mathcal{T}_\ell^i, \tag{3.32}$$

and set $\mathcal{H}_\ell^\ell = \mathcal{T}_\ell^\ell$. Dashed lines indicate the respective inverses h_j^i and k_j^i to l_j^i and J_j^i .

Note that $\mathcal{H}_\ell^i \xrightarrow{\cong} \mathcal{H}_{\ell-1}^i$ via b_ℓ^i for all $i \leq \ell - 1$, the inverse of which we denote by $(b_\ell^i)^{-1}$. We set

$$\text{gr}^{[\ell+1]}(\mathcal{T}F_\ell) := \bigoplus_{i \leq \ell} \mathcal{T}_\ell^i / \mathcal{T}_\ell^{i+\ell+2}$$

and define an $(\ell + 1)$ -frame $\varphi_{\ell+1} : (\mathfrak{g}_{\leq \ell})_{F_\ell} \rightarrow \text{gr}^{[\ell+1]}(\mathcal{T}F_\ell)$ by $\varphi_{\ell+1}^i := (b_\ell^i)^{-1} \circ \varphi_\ell^i$ for $i \leq \ell - 1$ and using the principal bundle structure via $\varphi_{\ell+1}^\ell := (\gamma_\ell)^{-1}$ for $i = \ell$. We also note that $\mathcal{H}_\ell = \{\mathcal{H}_\ell^i\}_{i \leq \ell} = \text{Im}(\varphi_{\ell+1})$ and that each component

$$\varphi_{\ell+1}^i : \mathcal{A}_{F_\ell} \otimes \mathfrak{g}_i \rightarrow \mathcal{T}_\ell^i / \mathcal{T}_\ell^{i+\ell+2}$$

is an embedding that projects to an isomorphism $\mathcal{A}_{F_\ell} \otimes \mathfrak{g}_i \cong \mathcal{T}_\ell^i / \mathcal{T}_\ell^{i+1}$. (Because $\mathcal{T}_\ell^k = 0_{F_\ell}$ for $k > \ell$, there is no truncation for $i \geq -1$, that is $\varphi_{\ell+1}^i(v)$ is a vector field on F_ℓ for any $v \in \mathfrak{g}_i$ with $i \geq -1$.)

Lemma 3.10 *Given $\mathcal{H}_\ell = \{\mathcal{H}_\ell^i\}_{i \leq \ell}$, we have*

- (i) $\mathcal{T}_\ell^i / \mathcal{T}_\ell^{i+\ell+2} = \mathcal{T}_\ell^{i+1} / \mathcal{T}_\ell^{i+\ell+2} \oplus \mathcal{H}_\ell^i$ for all $i \leq \ell$,
- (ii) $\mathcal{T}_\ell^{i+s} / \mathcal{T}_\ell^{i+\ell+2} = \mathcal{T}_\ell^{i+s+1} / \mathcal{T}_\ell^{i+\ell+2} \oplus \pi_\ell^s (\mathcal{H}_\ell^{i+s})$ for all $0 \leq s \leq \ell+1$ and $i+s \leq \ell$,

where $\pi_\ell^s : \mathcal{T}_\ell^{i+s} / \mathcal{T}_\ell^{i+s+\ell+2} \rightarrow \mathcal{T}_\ell^{i+s} / \mathcal{T}_\ell^{i+\ell+2}$ is the natural projection.

We omit the proof by induction of (i) for the sake of brevity. Claim (ii) follows from (i) considering $i + s$ instead of i and taking the quotient by $\mathcal{T}_\ell^{i+\ell+2} / \mathcal{T}_\ell^{i+s+\ell+2}$. We note that $\pi_\ell^s(\mathcal{H}_\ell^{i+s}) \cong \mathcal{H}_\ell^{i+s}$ in (ii). The following result is then a straightforward consequence of (ii).

Proposition 3.11 *Given $\mathcal{H}_\ell = \{\mathcal{H}_\ell^i\}_{i \leq \ell}$, we have*

$$\begin{aligned} \mathcal{T}_\ell^i / \mathcal{T}_\ell^{i+\ell+2} &= \bigoplus_{0 \leq s \leq \ell} \pi_\ell^s(\mathcal{H}_\ell^{i+s}) \oplus \mathcal{T}_\ell^{i+\ell+1} / \mathcal{T}_\ell^{i+\ell+2} \\ &\cong \bigoplus_{0 \leq s \leq \ell} \mathcal{H}_\ell^{i+s} \oplus \mathcal{T}_\ell^{i+\ell+1} / \mathcal{T}_\ell^{i+\ell+2} \end{aligned} \tag{3.33}$$

for all $i \leq \ell - 1$, where $\pi_\ell^s : \mathcal{T}_\ell^{i+s} / \mathcal{T}_\ell^{i+s+\ell+2} \rightarrow \mathcal{T}_\ell^{i+s} / \mathcal{T}_\ell^{i+\ell+2}$ is the natural projection. In particular

$$\text{Ker}(h_\ell^i) = \text{Im}(k_\ell^i) \cong \begin{cases} \bigoplus_{i \leq s \leq \ell+i} \mathcal{H}_\ell^s & \text{if } i < 0, \\ \bigoplus_{i \leq s \leq \ell-1} \mathcal{H}_\ell^s & \text{if } 0 \leq i \leq \ell - 1, \end{cases} \tag{3.34}$$

are the complements to $\mathcal{T}_\ell^{i+\ell+1} / \mathcal{T}_\ell^{i+\ell+2}$ and \mathcal{T}_ℓ^ℓ , respectively.

Let us take another complement $\tilde{\mathcal{H}}_\ell = \{\tilde{\mathcal{H}}_\ell^i\}_{i \leq \ell}$ constructed as before and the associated $(\ell + 1)$ -frame $\tilde{\varphi}_{\ell+1}$. By construction, for any $v_i \in (\mathfrak{g}_i)_{F_\ell}$, we have that $\tilde{\varphi}_{\ell+1}(v_i) - \varphi_{\ell+1}(v_i)$ is an element of $\mathcal{T}_\ell^{i+\ell+1} / \mathcal{T}_\ell^{i+\ell+2}$ if $i < 0$ and \mathcal{T}_ℓ^ℓ if $0 \leq i \leq \ell - 1$. Hence

$$\vartheta_\ell^{i+\ell+1}(\tilde{\varphi}_{\ell+1}(v_i) - \varphi_{\ell+1}(v_i)) = \psi(v_i) \text{ for } i < -1, \tag{3.35}$$

$$\gamma_\ell(\tilde{\varphi}_{\ell+1}(v_i) - \varphi_{\ell+1}(v_i)) = \psi(v_i) \text{ for } -1 \leq i \leq \ell - 1, \tag{3.36}$$

for some morphism $\psi : (\mathfrak{g}_{\leq \ell-1})_{F_\ell} \rightarrow (\mathfrak{g}_{\leq \ell})_{F_\ell}$ of sheaves of \mathcal{A}_{F_ℓ} -modules. It is clear that the components

$$\psi_- : \mathfrak{m}_{F_\ell} \rightarrow \mathfrak{g}_{F_\ell}, \tag{3.37}$$

$$\psi_+ : (\mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_{\ell-1})_{F_\ell} \rightarrow (\mathfrak{g}_\ell)_{F_\ell} \tag{3.38}$$

are elements of even parity, with the first component having \mathbb{Z} -degree $(\ell + 1)$. In other words ψ_- is an even element of $\mathcal{A}_{F_\ell} \otimes (\mathfrak{m}^* \otimes \mathfrak{g})_{\ell+1}$ and ψ_+ of $\mathcal{A}_{F_\ell} \otimes ((\mathfrak{g}_{\leq \ell-1}^+)^* \otimes \mathfrak{g}_\ell)$. Conversely, given any such

$$\psi = \psi_- + \psi_+$$

there is a unique complement $\tilde{\mathcal{H}}_\ell = \{\tilde{\mathcal{H}}_\ell^i\}_{i \leq \ell}$ with the required properties.

3.5.3 Normalization Conditions

In this section, we detail the normalization conditions to be enforced on the $(\ell + 1)$ -frames. Since the Lie bracket is compatible with the filtration on \mathcal{T}_{F_ℓ} only for non-positive filtration indices, we first need to collect some finer properties satisfied by the frames.

Lemma 3.12 *Let $\zeta \in \mathcal{T}_\ell^k$ with $0 \leq k \leq \ell$ and $\varphi_{\ell+1} : (\mathfrak{g}_{\leq \ell})_{F_\ell} \rightarrow \text{gr}^{[\ell+1]}(\mathcal{T}F_\ell)$ be an $(\ell + 1)$ -frame. Then, for $v_i \in \mathfrak{g}_i, i < 0$:*

$$[\zeta, \varphi_{\ell+1}(v_i)] \in \begin{cases} \mathcal{T}_\ell^{k-1} & \text{if } i = -1; \\ \mathcal{T}_\ell^{i+k} / \mathcal{T}_\ell^k & \text{if } k - \ell - 2 < i \leq -2; \\ \mathcal{T}_\ell^{i+k} / \mathcal{T}_\ell^{i+\ell+2} & \text{if } -\mu \leq i \leq k - \ell - 2. \end{cases} \tag{3.39}$$

Given a choice of complements $\mathcal{H}_\ell = \{\mathcal{H}_\ell^i\}_{i \leq \ell}$, we define the $(\ell + 1)$ -th horizontal structure function $c_{\mathcal{H}_\ell}^- \in \mathcal{A}_{F_\ell} \otimes (\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}_{< \ell})$ on the entries $v_k \in \mathfrak{g}_k, k < 0$, by

$$c_{\mathcal{H}_\ell}^-(v_i, v_j) = \vartheta_\ell^{i+j+\ell+1} \left(h_\ell^{i+j} \left([\varphi_{\ell+1}(v_i), \varphi_{\ell+1}(v_j)] \text{ mod } \mathcal{T}_\ell^{i+j+\ell+2} \right) \right) \tag{3.40}$$

and extending by \mathcal{A}_{F_ℓ} -linearity to the entries from $(\mathfrak{g}_k)_{F_\ell} = \mathcal{A}_{F_\ell} \otimes \mathfrak{g}_k$. Evidently $[\mathcal{T}_\ell^i, \mathcal{T}_\ell^j] \subset \mathcal{T}_\ell^{i+j}$ as $i, j < 0$. However, the Lie bracket is compatible with the filtration on $\mathcal{T}F_\ell$ only for the non-positive filtration indices, so the fact that (3.40) is well-defined deserves an additional explanation: we show that the input of $\vartheta_\ell \circ h_\ell$ above is a well-defined element in $\mathcal{T}_\ell^{i+j} / \mathcal{T}_\ell^{i+j+\ell+2}$.

Lemma 3.13 *The horizontal structure function $c_{\mathcal{H}_\ell}^-$ is well-defined.*

Proof Recall $i, j < 0$. If both $i + \ell + 2$ and $j + \ell + 2$ are non-positive, the claim follows immediately from the general properties of the Lie bracket. Otherwise we may assume, say, $i + \ell + 2 > 0, j \leq i$. Now $[\mathcal{T}_\ell^{i+\ell+2}, \mathcal{T}_\ell^{j+\ell+2}] \subset \mathcal{T}_\ell^{j+\ell+2} \subset \mathcal{T}_\ell^{i+j+\ell+2}$ by (3.30) and $[\mathcal{T}_\ell^{i+\ell+2}, \varphi_{\ell+1}(v_j)] \equiv 0 \text{ mod } \mathcal{T}_\ell^{i+j+\ell+2}$ by Lemma 3.12, so we are left to deal with $[\mathcal{T}_\ell^{j+\ell+2}, \varphi_{\ell+1}(v_i)]$.

If $j + \ell + 2 \leq 0$, then $[\mathcal{T}_\ell^{j+\ell+2}, \varphi_{\ell+1}(v_i)] \equiv 0 \text{ mod } \mathcal{T}_\ell^{i+j+\ell+2}$ by the general property of the Lie bracket, and the same result follows from Lemma 3.12 if $j + \ell + 2 > 0$. □

Note that $c_{\mathcal{H}_\ell}^-$ has \mathbb{Z} -degree $(\ell + 1)$, i.e., it is an element of $C^{\ell+1,2}(\mathfrak{m}, \mathfrak{g})_{F_\ell}$. As in Lemma 3.9:

Lemma 3.14 *Under a change of complement, the structure function transforms as $\widetilde{c_{\mathcal{H}_\ell}^-} = c_{\mathcal{H}_\ell}^- + \delta\psi_-$, where δ is the Chevalley–Eilenberg differential from $C^{\ell+1,1}(\mathfrak{m}, \mathfrak{g})_{F_\ell}$ to $C^{\ell+1,2}(\mathfrak{m}, \mathfrak{g})_{F_\ell}$.*

We know from Proposition 3.11 that for $0 \leq k \leq \ell - 1$, a complement of \mathcal{T}_ℓ^ℓ in \mathcal{T}_ℓ^k is $\oplus_{s=k}^{\ell-1} \mathcal{H}_\ell^s$, so there is a projection pr_s^k from \mathcal{T}_ℓ^k to $\mathcal{H}_\ell^s \cong \mathcal{A}_{F_\ell} \otimes \mathfrak{g}_s$ for any $k \leq s \leq \ell - 1$, where the last isomorphism is given by the soldering form. The analogous projection from \mathcal{T}_ℓ^k to $\mathcal{T}_\ell^k / \mathcal{T}_\ell^{k+\ell+2}$ and then to \mathcal{H}_ℓ^s for any $k \leq s \leq \ell + k$ is defined for all $k < 0$. We note that $\text{Ker}(\text{pr}_s^k) \supset \mathcal{T}_\ell^{s+1}$.

Again we need some finer properties of the frames.

Lemma 3.15 *Let $\varphi_{\ell+1} : (\mathfrak{g}_{\leq \ell})_{F_\ell} \rightarrow \text{gr}^{[\ell+1]}(\mathcal{T}F_\ell)$ be an $(\ell + 1)$ -frame and $i \geq 0$. We then have $[\zeta, \varphi_{\ell+1}(v_i)] \in \mathcal{T}_\ell^k$ for all $\zeta \in \mathcal{T}_\ell^k$ with $i < k \leq \ell$.*

Note that the claim of Lemma 3.15 is automatically satisfied also for $k \leq i$, due to (3.30). Let $\mathfrak{g}_{\leq \ell-1}^+ = \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_{\ell-1}$ as before. The $(\ell + 1)$ -th vertical structure function

$$c_{\mathcal{H}_\ell}^+ \in \mathcal{A}_{F_\ell} \otimes (\mathfrak{g}_{\leq \ell-1}^+)^* \otimes (\mathfrak{m}^* \otimes \mathfrak{g})_\ell \subset \mathcal{A}_{F_\ell} \otimes \text{Hom}(\mathfrak{m} \otimes \mathfrak{g}_{\leq \ell-1}^+, \mathfrak{g}_{\leq \ell-1})$$

is defined as the \mathcal{A}_{F_ℓ} -linear extension of the following formula

$$c_{\mathcal{H}_\ell}^+(v_i, v_j) = \vartheta_\ell^{i+\ell} \left(\text{pr}_{i+\ell}^{i+j} [\varphi_{\ell+1}(v_i), \varphi_{\ell+1}(v_j)] \right)$$

where $v_i \in \mathfrak{g}_i$ with $i < 0$, and $v_j \in \mathfrak{g}_j$ with $0 \leq j \leq \ell - 1$. By Lemma 3.12 one sees that the input of $\vartheta_\ell^{i+\ell} \circ \text{pr}_{i+\ell}^{i+j}$ is in \mathcal{J}_ℓ^{i+j} with some ambiguity, which in this case lies in $[\mathcal{J}_\ell^{i+\ell+2}, \varphi_{\ell+1}(v_j)]$. By Lemma 3.15 and (3.30), the input lies then in $\mathcal{J}_\ell^{i+j} / \mathcal{J}_\ell^{i+\ell+2}$, so that $c_{\mathcal{H}_\ell}^+$ is well-defined.

Lemma 3.16 *Changing complement, the structure function transforms as $\widetilde{c_{\mathcal{H}_\ell}^+} = c_{\mathcal{H}_\ell}^+ + \bar{\delta}\psi_+$, where*

$$\bar{\delta} = \delta \otimes \text{id} : C^{\ell,0}(\mathfrak{m}, \mathfrak{g})_{F_\ell} \otimes (\mathfrak{g}_{\leq \ell-1}^+)^*_{F_\ell} \rightarrow C^{\ell,1}(\mathfrak{m}, \mathfrak{g})_{F_\ell} \otimes (\mathfrak{g}_{\leq \ell-1}^+)^*_{F_\ell} \quad (3.41)$$

is the tensor product of the Chevalley–Eilenberg differential from $C^{\ell,0}(\mathfrak{m}, \mathfrak{g})_{F_\ell}$ to $C^{\ell,1}(\mathfrak{m}, \mathfrak{g})_{F_\ell}$ with the identity of $(\mathfrak{g}_{\leq \ell-1}^+)^*_{F_\ell}$.

Proof If $0 \leq k \leq \ell - 1$, it is clear from Proposition 3.11 and (3.36) that $\widetilde{\text{pr}_s^k(Y^k)} \equiv \text{pr}_s^k(Y^k) \pmod{\mathcal{J}_\ell^k}$, for all $Y^k \in \mathcal{J}_\ell^k$. Similarly $\widetilde{\text{pr}_s^k(Y^k)} \equiv \text{pr}_s^k(Y^k) \pmod{\mathcal{J}_\ell^{s+\ell+1} / \mathcal{J}_\ell^{s+\ell+2}}$ if $k < 0$. Since $\text{Ker}(\vartheta_\ell^{i+\ell}) = \mathcal{J}_\ell^{i+\ell+1} \supset \mathcal{J}_\ell^i + \mathcal{J}_\ell^{i+2\ell+1}$, one directly infers that $\vartheta_\ell^{i+\ell} \circ \widetilde{\text{pr}_{i+\ell}^{i+j}} = \vartheta_\ell^{i+\ell} \circ \text{pr}_{i+\ell}^{i+j}$.

Denoting by $\Psi : (\mathfrak{g}_{\leq \ell-1})_{F_\ell} \rightarrow \text{gr}(\mathcal{J}F_\ell)$ the morphism obtained by composing ψ with the inverses of the soldering form and vertical ℓ -trivialization, we then have

$$\begin{aligned} [\Psi(v_i), \varphi_{\ell+1}(v_j)] &\equiv 0 \pmod{\mathcal{J}_\ell^{i+\ell+1}} \\ [\Psi(v_i), \Psi(v_j)] &\equiv 0 \pmod{\mathcal{J}_\ell^{i+\ell+1}} \end{aligned} \quad (3.42)$$

by Lemma 3.15 and since bracketing with \mathcal{J}_ℓ^ℓ preserves all other bundles in the filtration. On the other hand $[\varphi_{\ell+1}(v_i), \Psi(v_j)] \in \mathcal{J}_\ell^{i+\ell}$ by Lemma 3.12, up to elements in $\mathcal{J}_\ell^{i+\ell+1}$.

Since $\mathcal{J}_\ell^{i+\ell+1} \subset \text{Ker}(\text{pr}_{i+\ell}^{i+j})$, we then have

$$\begin{aligned} \widetilde{c_{\mathcal{H}_\ell}^+}(v_i, v_j) &= \vartheta_\ell^{i+\ell} \left(\text{pr}_{i+\ell}^{i+j} [\varphi_{\ell+1}(v_i) + \Psi(v_i), \varphi_{\ell+1}(v_j) + \Psi(v_j)] \right) \\ &= c_{\mathcal{H}_\ell}^+(v_i, v_j) + \vartheta_\ell^{i+\ell} \left(\text{pr}_{i+\ell}^{i+j} [\varphi_{\ell+1}(v_i), \Psi(v_j)] \right) \\ &= c_{\mathcal{H}_\ell}^+(v_i, v_j) + \bar{\delta}\psi(v_i, v_j), \end{aligned} \quad (3.43)$$

where $\bar{\delta}\psi(v_i, v_j) = [v_i, \psi(v_j)]$. □

Choose complements $N_{\ell+1}^- \subset C^{\ell+1,2}(\mathfrak{m}, \mathfrak{g})$ to $\delta C^{\ell+1,1}(\mathfrak{m}, \mathfrak{g})$, $N_{\ell+1}^+ \subset C^{\ell,1}(\mathfrak{m}, \mathfrak{g})$ to $\delta C^{\ell,0}(\mathfrak{m}, \mathfrak{g})$, and consider the sheaves $\mathcal{N}_{\ell+1}^- = \mathcal{A}_{F_\ell} \otimes N_{\ell+1}^-$ and $\mathcal{N}_{\ell+1}^+ = \mathcal{A}_{F_\ell} \otimes N_{\ell+1}^+ \otimes (\mathfrak{g}_{\leq \ell-1}^+)^*$ over F_ℓ . We then require that

$$c_{\mathcal{H}_\ell}^\pm \in \mathcal{N}_{\ell+1}^\pm, \tag{3.44}$$

and define the sheaf $\text{Pr}_{\ell+1}(M, \mathcal{D}, F_0)$ over F_ℓ by

$$\text{Pr}_{\ell+1}(M, \mathcal{D}, F_0)(\mathcal{V}_o) = \left\{ \mathcal{H}_\ell(\mathcal{V}_o) \mid \mathcal{H}_\ell = \{\mathcal{H}_\ell^i\}_{i \leq \ell} \text{ on } \mathcal{V}_0 \text{ such that } c_{\mathcal{H}_\ell}^\pm \in \mathcal{N}_{\ell+1}^\pm|_{\mathcal{V}_o} \right\}, \tag{3.45}$$

equivalently the collection of the associated $(\ell + 1)$ -frames. Since the Chevalley–Eilenberg differential $0 \rightarrow \mathfrak{g} \xrightarrow{\delta} \mathfrak{m}^* \otimes \mathfrak{g}$ is injective on $\mathfrak{g}_{\geq 0}$, hence on \mathfrak{g}_ℓ , also the differential (3.41) is injective and (3.45) is a principal bundle $\pi_{\ell+1} : \text{Pr}_{\ell+1}(M, \mathcal{D}, F_0) \rightarrow F_\ell$ over F_ℓ . It has Abelian structure group $G_{\ell+1} = \exp(\mathfrak{g}_{\ell+1})$ consisting of all elements of $C^{\ell+1,1}(\mathfrak{m}, \mathfrak{g})$ in the kernel of the Spencer operator δ , i.e., of all elements of the prolongation $\mathfrak{g}_{\ell+1}$.

3.6 Canonical Parallelism and Comparison with Zelenko’s Approach

Theorem 3.17 *Let (M, \mathcal{D}, q) be a filtered structure of finite type, i.e., the Tanaka prolongation stabilizes: $\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \dots \oplus \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_d$ (with $\mathfrak{g}_d \neq 0$, but $\mathfrak{g}_{d+1} = 0$). Then there exists a fiber bundle $\pi : P \rightarrow M$ of $\dim P = \dim \mathfrak{g}$ and an absolute parallelism $\Phi \in \Omega_0^1(P, \mathfrak{g})$, which is natural in the sense that any equivalence transformation $f : M \rightarrow M'$ lifts to a unique map $F : P \rightarrow P'$ preserving the parallelisms Φ and Φ' .*

Proof Let $P = F_d$ and consider the structure Φ_d consisting of $\varphi_{d+1}(v_i)$, which are supervector fields for $i \geq -1$ and truncated supervector fields for $i < 0$. Let us prolong further, but first note that the map $F_k \rightarrow F_{k-1}$ is a principal bundle with trivial fiber $\mathfrak{g}_k = 0$ for any $k \geq d + 1$, hence a diffeomorphism. Take $j = d + \mu - 1$. Then $\varphi_{j+1}|_{\mathfrak{g}_i} : \mathcal{A}_{F_j} \otimes \mathfrak{g}_i \rightarrow \mathcal{T}_j^i / \mathcal{T}_j^{i+j+2} = \mathcal{T}_j^i$ because $\mathcal{T}_j^{i+j+2} \cong \mathcal{T}_d^{i+j+2} = 0$. Thus we get the required non-truncated supervector fields.

The frames $\varphi_{j+1} : \mathcal{A}_{F_j} \otimes \mathfrak{g} \rightarrow \mathcal{T}_j$ give an absolute parallelism Φ on $P := F_j \cong F_d$. Indeed a basis on \mathfrak{g} , respecting the parity and \mathbb{Z} -grading, gives a basis of supervector fields on P . □

Remark 3.18 In general, the fiber bundle $\pi : P \rightarrow M$ is not principal, so the parallelism Φ lacks equivariancy and it is not a Cartan superconnection, but it suffices for dimensional bounds. If the normalizations can be chosen invariantly w.r.t. the Lie supergroup $G_0 \times \exp(\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_d)$, then we expect that $\pi : P \rightarrow M$ is a principal bundle and a Cartan superconnection exists. (We have verified that this is true in the $d = 0$ case.) In this case, the step of our construction involving vertical structure functions and normalizations would not be required: one may simply take the fundamental vector fields of the principal action.

An essential difference with the argument in [42] is that this reference uses the reduced differential $\partial : \mathfrak{m}^* \otimes \mathfrak{g} \rightarrow \mathfrak{A} := (\mathfrak{g}_{-1}^* \wedge \mathfrak{m}^*) \otimes \mathfrak{g}$ while we are using the Spencer

differential² $\delta : \mathfrak{m}^* \otimes \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}$. They are related through the restriction map $p : \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g} \rightarrow \mathfrak{A}$, $\partial = p \circ \delta$, or equivalently $\partial \alpha = \delta \alpha|_{\mathfrak{g}_{-1} \wedge \mathfrak{m}}$. Our normalizations agree as follows.

Lemma 3.19 *The map p is injective when restricted to the kernel of δ on $\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}$. In particular, it is injective when restricted to $\delta(\mathfrak{m}^* \otimes \mathfrak{g})$ and $\text{Ker}(\partial) = \text{Ker}(\delta)$.*

Proof We need to show that if $\omega \in \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}$ is δ -closed and $p(\omega) = 0$, i.e., $\omega(\mathfrak{g}_{-1}, \cdot) = 0$, then $\omega = 0$. Let $u, v \in \mathfrak{g}_{-1}$, $w \in \mathfrak{m}$. Then (up to signs, which are not essential here)

$$\begin{aligned} 0 = \delta(\omega)(u, v, w) &= [u, \omega(v, w)] - [v, \omega(u, w)] + [w, \omega(u, v)] \\ &\quad - \omega([u, v], w) + \omega([u, w], v) - \omega([v, w], u) \\ &= -\omega([u, v], w). \end{aligned}$$

Thus $\omega(\Pi_2, \cdot) = 0$ for $\Pi_2 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Applying the above formula for $v \in \Pi_2$ we now obtain $\omega(\Pi_3, \cdot) = 0$ for $\Pi_3 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and iterating yields our first claim. The rest is clear. □

Since p is also the projection to \mathfrak{A} along the \mathbb{Z} -graded complement $\mathfrak{B} = \bigoplus_{i,j \leq -2} (\mathfrak{g}_i^* \wedge \mathfrak{g}_j^*) \otimes \mathfrak{g}$ in $\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}$, the following result follows straightforwardly.

Corollary 3.20 *If Z is a complement to $\text{Im}(\partial)$ in \mathfrak{A} then $N = Z \oplus \mathfrak{B}$ is a complement to $\text{Im}(\delta)$ in $\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g}$.*

Consequently, any complement Z is obtained as $Z = p(N)$ for some complement N . However, it is not true that any N is of the indicated type $N = Z \oplus \mathfrak{B}$, yet for our purposes this distinction plays no role. In concrete cases, when the prolongation $\mathfrak{g}_{\geq 0}$ acts completely reducibly this may turn important in finding an invariant complement.

4 Dimension Bounds for Symmetry

4.1 The Automorphism Supergroup and Dimension Bounds

4.1.1 Basic Definitions

Symmetries of filtered supergeometries are defined as follows.

Definition 4.1

- (i) An automorphism $\varphi \in \text{Aut}(M, \mathcal{D}, F)_{\bar{0}}$ of a filtered structure is an element $\varphi \in \text{Aut}(M)_{\bar{0}}$ such that:

²In a private discussion with BK, Igor Zelenko confirmed that he was also considering this approach.

- it preserves the distribution: $\varphi_*(\mathcal{D}) \subset (\varphi_o)_*^{-1}\mathcal{D}$, so it induces an isomorphism of $\text{Pr}_0(M, \mathcal{D})$ (which, by abuse of notation, we denote by φ_*);
 - in the case of a first-order reduction $F = F_0 \subset \text{Pr}_0(M, \mathcal{D})$, we also require that $\varphi_*(\mathcal{F}_0) \subset (\varphi_o)_*^{-1}\mathcal{F}_0$;
 - then it induces an isomorphism of $F_0^{(i)}$ and, if there are higher order reductions, we also require that it preserves them: $\varphi_*(\mathcal{F}_i) \subset (\varphi_o)_*^{-1}\mathcal{F}_i$.
- (ii) An *infinitesimal automorphism* $X \in \text{inf}(M, \mathcal{D}, F)$ on M is a supervector field $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X(\mathcal{D}) \subset \mathcal{D}$, and it successively preserves the structure reductions, namely:

$$\mathcal{L}_X(\mathcal{F}_i(\mathcal{V}_o)) \subset \mathcal{F}_i(\mathcal{V}_o) \cdot (\mathfrak{g}_i \otimes \mathcal{A}_{F_{i-1}}(\mathcal{V})) \subset \widetilde{\mathcal{F}}r_i(\mathcal{V}_o),$$

- for any open subset $\mathcal{V}_o \subset (F_{i-1})_o$. (See (3.14) for the definition of $\widetilde{\mathcal{F}}r_i$.)
- (iii) An infinitesimal automorphism $X \in \text{inf}(M, \mathcal{D}, F)$ is *complete* if it is so as a supervector field, i.e., its local flow (in the sense of [14, 29]) has maximal flow domain $\mathbb{R}^{1|1} \times M$. We denote the collection of all complete infinitesimal automorphisms by $\text{aut}(M, \mathcal{D}, F)$.

We recall that a supervector field on $M = (M_o, \mathcal{A}_M)$ is complete if and only if the associated vector field on M_o is complete [14, 29]. The automorphism supergroup is in this way defined as the super Harish-Chandra pair $\text{Aut}(M, \mathcal{D}, F) := (\text{Aut}(M, \mathcal{D}, F)_{\bar{0}}, \text{aut}(M, \mathcal{D}, F))$.

Assume the filtered structure is of finite type. By the naturality of the constructions, any automorphism of the filtered structure on M lifts to a symmetry of the bundle $\pi : P \rightarrow M$ constructed in Theorem 3.17 and it preserves the absolute parallelism Φ on it. Likewise, any infinitesimal symmetry of the filtered structure on M lifts to an infinitesimal symmetry of the bundle $\pi : P \rightarrow M$ preserving the absolute parallelism Φ . In particular, the dimension of the symmetry superalgebra \mathfrak{s} is bounded by the dimension of the infinitesimal symmetries of Φ . To prove $\dim \mathfrak{s} \leq \dim P$ (in the strong sense) and complete the proof of Theorems 1.1 and 1.2, it is sufficient to establish a bound on the dimension of the symmetries of the absolute parallelism.

4.1.2 Dimension of the Symmetry Superalgebra

By fixing a basis of \mathfrak{g} , the absolute parallelism Φ corresponds to a coframe field $\{\omega^\beta\}$ on P , where the index β run over both the even and odd indices. Let $\{e_\alpha\}$ be the dual frame, i.e., $\langle e_\alpha, \omega^\beta \rangle = (-1)^{|\alpha||\beta|} \omega^\beta(e_\alpha) = \delta_\alpha^\beta$. Here α runs through both even and odd indices as well. The following result was originally established in [31, Lemma 13]. We give here a simplified proof that does not use the concept of flow for supermanifolds.

Lemma 4.2 *Let $\{e_\alpha\}$ be a frame on a supermanifold $P = (P_o, \mathcal{A}_P)$ with connected reduced manifold. Fix a point $x \in P_o$. Then any infinitesimal automorphism $v \in \mathfrak{X}(P)$ of the frame is determined by its value at x .*

Proof The statement is equivalent to the claim that $\text{ev}_x(v) = 0$ implies $v = 0$.

Consider the ideal $\mathcal{J} = (\mathcal{A}_P)_1^2 \oplus (\mathcal{A}_P)_1$ of \mathcal{A}_P generated by nilpotents. Then for any $k > 0$ the map

$$\mathcal{J}^k / \mathcal{J}^{k+1} \rightarrow \mathcal{T}^*P \otimes \mathcal{J}^{k-1} / \mathcal{J}^k, \quad f \bmod \mathcal{J}^{k+1} \mapsto \sum_{\text{odd } \alpha} \omega^\alpha \otimes e_\alpha(f) \bmod \mathcal{J}^k \quad (f \in \mathcal{J}^k)$$

is injective. In other words, if $f \notin \mathcal{J}^{k+1}$, then there exists an odd α such that $e_\alpha(f) \notin \mathcal{J}^k$. Now if v is in $\mathcal{J}^k \otimes \mathcal{TP}$ but not in $\mathcal{J}^{k+1} \otimes \mathcal{TP}$ for some $k > 0$, then the Lie equation $L_v e_\alpha = 0$ cannot hold for all odd α because there exists one for which it is wrong already modulo $\mathcal{J}^k \otimes \mathcal{TP}$. This tells us that the evaluation map $\text{ev} : \mathfrak{X}(P) \rightarrow \Gamma(TP|_{P_o})$ is injective on the symmetries.

Set $\tilde{v} = \text{ev}v \in \Gamma(TP|_{P_o})$. Taking the Lie equation $L_v e_\alpha = 0$ modulo $\mathcal{J} \otimes \mathcal{TP}$ for an even α we get a pair of reduced Lie equations

$$\begin{aligned} \text{ev}(L_{v_0} e_\alpha) &= 0, \\ \text{ev}(L_{e_\alpha} v_1) &= 0, \end{aligned} \tag{4.1}$$

which depend only on $\tilde{v} = \tilde{v}_0 + \tilde{v}_1$. More precisely \tilde{v}_0 is a classical vector field on P_o and the first reduced Lie equation is $L_{\tilde{v}_0} \tilde{e}_\alpha = 0$, where $\{\tilde{e}_\alpha = \text{ev}e_\alpha\}_{\alpha \text{ even}}$ is the induced absolute parallelism on P_o . The infinitesimal version of Lemma 1 from the proof of [19, Thm. 3.2] applies: the set of critical points of \tilde{v}_0 is simultaneously closed and open, so \tilde{v}_0 is determined by its value at $x \in P_o$. On the other hand, \tilde{v}_1 is a section of the bundle $(TP|_{P_o})_1$ with the natural flat connection defined by

$$\nabla_{f^\alpha \tilde{e}_\alpha} \tilde{v}_1 := f^\alpha \text{ev}(L_{e_\alpha} v_1),$$

where $f^\alpha \in C^\infty_{P_o}$ for any even α . Hence the value of a parallel section v at one point determines the section everywhere. In summary, the map $v \mapsto \text{ev}_x v$ is injective. \square

Let us now observe how the dimension of the solution space is constrained. The structure equations

$$[e_\alpha, e_\beta] = c_{\alpha\beta}^\gamma e_\gamma \quad \Leftrightarrow \quad d\omega^\gamma = -\frac{1}{2}(-1)^{|\alpha||\beta|} (\omega^\alpha \wedge \omega^\beta) c_{\alpha\beta}^\gamma$$

involve structure superfunctions $c_{\alpha\beta}^\gamma \in \mathcal{A}_P$ of parity $|\alpha| + |\beta| + |\gamma|$. An infinitesimal symmetry is a supervector field $v = a^\delta e_\delta \in \mathfrak{X}(P)$ such that $L_v e_\alpha = 0$ for all α . Equivalently it must preserve the coframe, so we get

$$0 = L_v \omega^\gamma = d\iota_v \omega^\gamma + \iota_v d\omega^\gamma = da^\gamma - \frac{1}{2} a^\delta (-1)^{|\alpha||\beta|} \iota_{e_\delta} (\omega^\alpha \wedge \omega^\beta) c_{\alpha\beta}^\gamma,$$

for all γ , which we rewrite as

$$da^\gamma = (-1)^{|\beta||v|} (\omega^\beta) a^\alpha c_{\alpha\beta}^\gamma. \tag{4.2}$$

This is a complete PDE on the superfunctions a^γ 's and the dimension bound $\dim P$ is achieved if and only if the compatibility conditions $d^2 a^\gamma = 0$ holds. We can see

this explicitly in local coordinates x^α on P . Let $e_\alpha = \chi_\alpha^\delta \partial_\delta$, where $\partial_\beta = \frac{\partial}{\partial x^\beta}$, with the dual coframe $\omega^\beta = (dx^\delta) \tilde{\chi}_\delta^\beta$, where $\chi_\alpha^\delta \tilde{\chi}_\delta^\beta = \delta_\alpha^\beta$. Then, formula (4.2) becomes

$$\begin{aligned} \partial_\epsilon a^\gamma &= (-1)^{|v||\beta|} \tilde{\chi}_\epsilon^\beta a^\alpha c_{\alpha\beta}^\gamma \\ &= (-1)^{|v||\epsilon|+|\alpha||\beta|+|\alpha||\epsilon|} a^\alpha \tilde{\chi}_\epsilon^\beta c_{\alpha\beta}^\gamma \\ &= a^\alpha \sigma_{\epsilon\alpha}^\gamma, \end{aligned}$$

where we denoted $\sigma_{\epsilon\alpha}^\gamma = (-1)^{|v||\epsilon|+|\alpha||\beta|+|\alpha||\epsilon|} \tilde{\chi}_\epsilon^\beta c_{\alpha\beta}^\gamma$. The compatibility conditions given by the vanishing of the supercommutator of $\partial_{\epsilon'}$ and $\partial_{\epsilon''}$ on a^γ are

$$\begin{aligned} a^\alpha \left((-1)^{|\alpha||\epsilon'|+|v||\epsilon'|} \partial_{\epsilon'} \sigma_{\epsilon''\alpha}^\gamma - (-1)^{|\alpha||\epsilon''|+|v||\epsilon''|+|\epsilon'|\epsilon''} \partial_{\epsilon''} \sigma_{\epsilon'\alpha}^\gamma \right. \\ \left. + \sigma_{\epsilon'\alpha}^\nu \sigma_{\epsilon''\nu}^\gamma - (-1)^{|\epsilon'|\epsilon''} \sigma_{\epsilon''\alpha}^\nu \sigma_{\epsilon'\nu}^\gamma \right) = 0. \end{aligned} \tag{4.3}$$

If the parenthetical expression vanishes for all indices, then any initial value for the a^α 's produces a unique solution v , and the dimension of the solution space is $\dim P$. If not, then we have to differentiate the L.H.S. of (4.3), substitute (4.2) and study the 0th-order linear equations on the a^α 's. When the system stabilizes, the corank of the resulting matrix (i.e., the matrix size minus the size of the largest invertible minor), gives the dimension of the solution space.

4.1.3 Dimension of the Automorphism Supergroup

By the results in [31, §4.2], the group of automorphisms $\text{Aut}(\Phi)_{\bar{0}}$ of an absolute parallelism Φ on a supermanifold $P = (P_o, \mathcal{A}_P)$ with connected reduced manifold P_o is a (finite-dimensional) Lie group. At first, if we denote by $\Phi_{\bar{0}}$ the induced absolute parallelism on P_o (in the notation of Section 4.1.2, this is $\Phi_{\bar{0}} = \{eve_\alpha\}_{\alpha \text{ even}}$), then the classical argument of [19] proves that $\text{Aut}(\Phi_{\bar{0}})$ is a Lie group: $\text{Aut}(\Phi_{\bar{0}})$ is mapped to P_o as the orbit through $x \in P_o$ and the stabilizer of a classical absolute parallelism at any point is trivial. Then, the forgetful map $\text{Aut}(\Phi)_{\bar{0}} \rightarrow \text{Aut}(\Phi_{\bar{0}})$ is injective with closed image, cf. [31, Lemmas 10 and 11].

It follows from this and Lemma 4.2 that the automorphism supergroup $(\text{Aut}(\Phi)_{\bar{0}}, \text{aut}(\Phi))$ is a finite-dimensional super Harish-Chandra pair, in other words a Lie supergroup. Here $\text{aut}(\Phi)$ is the Lie superalgebra of complete infinitesimal automorphisms. We remark that for a pair of complete supervector fields neither their linear combination nor their commutator is complete in general. (This holds also in the classical case.) However the set of complete supervector fields that are infinitesimal automorphisms of an absolute parallelism form a supervector space, and moreover a Lie superalgebra. This is because the sum and Lie bracket of two infinitesimal automorphisms of $\Phi_{\bar{0}}$ is still complete by the classical result of [19] and a supervector field $v \in \mathfrak{X}(P)$ is complete if and only if the associated vector field $\text{ev}(v_{\bar{0}}) \in \mathfrak{X}(P_o)$ on P_o is so [14, 29]. This shows that the ‘‘representability issue’’ of [31, Thm 15] can be amended: completeness of the infinitesimal automorphisms (i.e., the requirement that $\text{inf}(\Phi) = \text{aut}(\Phi)$) is *not* an obstruction for the representability of the automorphism supergroup.

By the construction of the absolute parallelism Φ on P , it is not difficult to see that the automorphism supergroup $\text{Aut}(M, \mathcal{D}, q) = (\text{Aut}(M, \mathcal{D}, q)_{\bar{0}}, \text{aut}(M, \mathcal{D}, q))$ of a nonholonomic geometric structure (\mathcal{D}, q) on M or, more generally, of a filtered structure, is a closed subsupergroup of $\text{Aut}(\Phi) = (\text{Aut}(\Phi)_{\bar{0}}, \text{aut}(\Phi))$. Therefore it is a Lie supergroup, whose dimension is bounded by $\dim P = \dim \mathfrak{g}$, and this finishes the proof of Theorem 1.2.

A more careful analysis shows that $\text{Aut}(M, \mathcal{D}, q)$ is a discrete quotient of $\text{Aut}(\Phi)$. Indeed, by Theorem 3.17, automorphisms in the base lift to the frame bundle. On the other hand, for any k , automorphisms of the frame bundle F_k preserve the fundamental fields from \mathfrak{g}_k and therefore they project to automorphisms of a cover of the frame bundle F_{k-1} , namely to the quotient of F_k by the connected component of unity in the structure group G_k . We apply this for k descending from d to 0 and conclude the claim. If the structure groups G_k are connected (this is usually the case for $k > 0$, if no higher order reductions are imposed, because the fibers are affine), then we have the equality $\text{Aut}(M, \mathcal{D}, q) = \text{Aut}(\Phi)$.

Remark 4.3 In the case M_o has finitely many connected components, say $n \in \mathbb{N}$, one easily modifies the above arguments to get the following dimension bound:

$$\dim \text{Aut}(M, \mathcal{D}, q) \leq \dim \mathfrak{s} \leq n \cdot \dim \mathfrak{g}.$$

Indeed, enumerate the components $1, \dots, n$ of M_o and let $\sigma \in S_n$ encodes a (possibly trivial) permutation of components. Then all automorphisms are parametrized as follows: no more than $\dim \mathfrak{g}$ parameters for maps of the 1st component to that number $\sigma(1)$, no more than $\dim \mathfrak{g}$ parameters for maps of the 2nd component to that number $\sigma(2)$, etc.

4.2 Structure of the Symmetry Superalgebra and Maximally Symmetric Spaces

We now discuss the following statement, which is not primary for the purposes of this paper. We therefore will only sketch the proof, referring the reader for details to the original papers.

Let $\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \dots \oplus \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_d$ be the Tanaka algebra associated to the filtered structure (M, \mathcal{D}, q) . Its natural (decreasing) filtration is given by subspaces $\mathfrak{g}^i = \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_d$, $i \geq -\mu$, which for $\mu = 1$ is the so-called filtration by stabilizers and for $\mu > 1$ is the weighted (or Weisfeiler) filtration.

Theorem 4.4 *The symmetry algebra \mathfrak{s} of (M, \mathcal{D}, q) embeds into \mathfrak{g} as a filtered subspace $\iota : \mathfrak{s} \rightarrow \mathfrak{g}$ in such a way that the corresponding graded map $gr(\iota) : gr(\mathfrak{s}) \rightarrow \mathfrak{g}$ is an injection of Lie algebras.*

Proof Fix a point $x \in M_o$ and consider the weighted filtration of the stalk $\mathcal{T}M_x$ that refines the filtration by the maximal ideal in $(\mathcal{A}_M)_x$ using the weighted filtration induced by the distribution \mathcal{D} . This generalizes to the super-setting the second filtration on the symmetry Lie algebra sheaf from [21] and gives the required embedding $\iota : \mathfrak{s} \rightarrow \mathfrak{g}$.

Alternatively, consider the Lie equation governing infinitesimal symmetries of (M, \mathcal{D}, q) as a subsupermanifold embedded into the space of weighted super-jets. This provides the solution space \mathfrak{s} of the equation with the desired filtration, see [22]. □

This theorem serves as a base to obtain submaximally symmetric models via filtered deformations of large graded subalgebras of \mathfrak{g} , see [24] for applications in the classical case and [23] for examples in the super case.

Remark 4.5 Spaces (M, \mathcal{D}, q) with $\dim \text{Aut}(M, \mathcal{D}, q) = \dim \text{aut}(M, \mathcal{D}, q) = \dim \mathfrak{g}$ as well as spaces with $\dim \text{inf}(M, \mathcal{D}, q) = \dim \mathfrak{g}$ are non-unique but there are always two cases when the maximal symmetry dimension is attained.

The so-called *flat model* is the homogeneous space G/H , with G a Lie supergroup with $\text{Lie}(G) = \mathfrak{g}$ and H its closed subsupergroup with $\text{Lie}(H) = \mathfrak{g}^0$. (One can impose simply connectedness of G/H though this is not necessary.) The filtration \mathfrak{g}^i on \mathfrak{g} induces a left-invariant filtration \mathcal{F}^i on G and therefore the distribution $\mathcal{D} = \mathcal{F}^{-1}/\mathcal{F}^0$ on G/H with the desired derived flag. In addition, all the reductions are invariant w.r.t. G , hence the induced filtered structure is invariant. If q encodes the filtered structure, then $\text{inf}(G/H, \mathcal{D}, q) = \mathfrak{g}$ and $\text{Aut}(G/H, \mathcal{D}, q)$ coincides with the supergroup G or its discrete factor.

The so-called *standard model* is obtained through a left-invariant structure (\mathcal{D}, q) , or more generally a filtered structure, on the nilpotent Lie supergroup $M = \exp(\mathfrak{m})$. This usually does not have the maximal automorphism group, but it is locally isomorphic to the flat model and hence $\text{inf}(M, \mathcal{D}, q) = \mathfrak{g}$. Complete description of other models with maximal symmetry dimension can be obtained via the technique of filtered deformations of $\mathfrak{s} = \mathfrak{g}$.

5 Examples and Applications

Here we demonstrate how our dimensional bounds work. We emphasize that all our main results are applicable to both real smooth and complex analytic cases, so some examples will be stated over \mathbb{R} and some over \mathbb{C} .

5.1 Holonomic Structures

Let us first illustrate the symmetry bounds with some particular geometric structures on a supermanifold $M = (M_o, \mathcal{A}_M)$ of $\dim M = (m|n)$ in the holonomic case $\mathcal{D} = \mathcal{T}M$ (thus $\mathfrak{m} = \mathfrak{g}_{-1}$ in this subsection).

5.1.1 Affine Superconnections

An affine superconnection is an even map $\nabla : \mathfrak{X}(M) \otimes_{\mathbb{R}} \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, $(X, Y) \mapsto \nabla_X Y$, which is \mathcal{A}_M -linear in X and satisfies $\nabla_X(fY) = (-1)^{|f||X|} f \nabla_X Y + X(f)Y$. In local coordinates it is given via the Christoffel symbols $\nabla_{\partial_\alpha} \partial_\beta = \Gamma_{\alpha\beta}^\gamma \partial_\gamma$, where

$|\Gamma_{\alpha\beta}^\gamma| = |\alpha| + |\beta| + |\gamma|$. From the viewpoint of G -structures, an affine superconnection is a reduction of the second order, i.e., $F_0 = Fr_M \cong F_1$ or equivalently $\mathfrak{g}_0 = \mathfrak{gl}(V)$ and $\mathfrak{g}_1 = 0$. A choice of \mathcal{H} as in Section 3.2 that is equivariant under $GL(V)$ is equivalent to the choice of a connection 1-form $\omega \in \Omega^1(Fr_M, \mathfrak{gl}(V))_{\bar{0}}$, $\mathcal{H} = \text{Ker}(\omega)$, which in turn is equivalent to ∇ .

Thus for the symmetry algebra of ∇ we have:

$$\dim \mathfrak{s} \leq \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_0 = (m + n^2 + m^2 \mid n + 2mn).$$

The Lie algebra of the even part $\text{Aut}(M, \nabla)_{\bar{0}}$ of the Lie supergroup of affine transformations $\text{Aut}(M, \nabla)$ consists of supervector fields that are complete. Therefore, its dimension might be smaller than $\dim \mathfrak{s}$ in general.

5.1.2 Super-Riemannian Structures

A super-Riemannian structure on M is given by a non-degenerate even supersymmetric \mathcal{A}_M -bilinear form q on $\mathcal{T}M$. (In the real case, the even part of q can have any signature.) It is a G_0 -structure with $G_0 = \text{OSp}(m|n)$, $n \in 2\mathbb{Z}$. For $\mathfrak{g}_0 = \text{Lie}(G_0)$ it is known that $\mathfrak{g}_1 = \mathfrak{g}_0^{(1)} = 0$. The argument straightforwardly generalizes the classical one [37], see [31], which corresponds to the analog of the Levi-Civita connection [15]. Thus (M, q) determines an affine structure.

The Lie superalgebra of Killing supervector fields satisfies

$$\dim \mathfrak{s} \leq \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_0 = \left(\binom{m+1}{2} + \binom{n+1}{2} \mid n + mn \right).$$

The above remark about completeness for affine structures applies to super-Riemannian structures and the isometry supergroup as well.

5.1.3 Almost Super-symplectic Structures

An almost super-symplectic structure on M is given by a non-degenerate even super-skew-symmetric bilinear form q on TM . It is a G_0 -structure with $G_0 = \text{SpO}(m|n)$, $m \in 2\mathbb{Z}$. In this case $\mathfrak{g}_0 = \text{Lie}(G_0) = \mathfrak{spo}(m|n) \cong \mathfrak{osp}(n|m)$ but as representations on $V = \mathbb{R}^{m|n} \cong \Pi\mathbb{R}^{n|m}$ these Lie superalgebras are quite different, cf. Remark 2.1. In particular $\mathfrak{g} \subset \mathfrak{gl}(V)$ is of infinite type unless V is purely odd.

Lemma 5.1 *We have: $\mathfrak{g}_i = \mathfrak{g}_0^{(i)} = S^{i+2}V^*$ (in the super-sense), which is nonzero $\forall i \geq 0$ if $m > 0$.*

The proof of this claim mimics the proof of the classical computation for almost symplectic structure [37] and will be omitted. We note that $(S^i V^*)_{\bar{0}} = \bigoplus_{j=0}^{\max(i,n)} S^{i-j} V_0^* \otimes \Lambda^j V_1^*$, where the symmetric and exterior powers in the R.H.S. are meant in the classical sense.

Also $\mathcal{T}M^* \cong \mathcal{T}M$ via $q \in \Omega^2(M)$ and, provided $dq = 0$, the local symmetries are all of the form $q^{-1}dH$ for $H \in \mathcal{A}_M$. Thus in this case we may have $\dim \mathfrak{s} = \infty$ and $\text{Aut}(M, q)$ is not necessarily a Lie supergroup. This may happen even when $dq \neq 0$.

In the case M is purely odd ($m = 0$), we have:

$$\dim \mathfrak{s} \leq \sum_{i=-1}^{n-2} \dim \mathfrak{g}_i = \sum_{k=1}^n \dim \Lambda^k \Pi(V^*) = 2^n - 1.$$

5.1.4 Periplectic-Related Structures

Let \mathfrak{P} be non-degenerate bilinear form on $V = \mathbb{R}^{n|n}$ that is odd, i.e., $\mathfrak{P}(V_0, V_0) = \mathfrak{P}(V_1, V_1) = 0$. When \mathfrak{P} is supersymmetric, i.e., $\mathfrak{P}(x, y) = (-1)^{|x||y|}\mathfrak{P}(y, x)$ for all pure parity x, y , we define the *periplectic* Lie superalgebra by

$$\mathfrak{pe}(n) := \{X \in \mathfrak{gl}(n|n) : X^{\text{st}}\mathfrak{P} + (-1)^{|X|}\mathfrak{P}X = 0\}, \tag{5.1}$$

with $\mathfrak{gl}(n|n)$ -inherited \mathbb{Z}_2 -grading. Explicitly, taking $\mathfrak{P} = \begin{pmatrix} 0 & \text{id}_n \\ \text{id}_n & 0 \end{pmatrix}$ yields

$$\mathfrak{pe}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} : A, B, C \in \mathfrak{gl}(n) : B = B^\top, C = -C^\top \right\}. \tag{5.2}$$

Above st and \top are the supertranspose and the usual transpose, respectively. We also define some related Lie superalgebras:

- *special periplectic* $\mathfrak{spe}(n) := \mathfrak{pe}(n) \cap \mathfrak{sl}(n|n)$. This is simple for $n \geq 3$.
- *conformal (special) periplectic* $\mathfrak{cpe}(n) := \mathbb{C}\text{id}_{2n} \oplus \mathfrak{pe}(n)$ and $\mathfrak{cspe}(n) := \mathbb{C}\text{id}_{2n} \oplus \mathfrak{spe}(n)$.
- $\mathfrak{spe}_{a,b}(n) := (a\tau + bz) \times \mathfrak{spe}(n)$, where $a, b \in \mathbb{C}$, $\tau = \text{diag}(\text{id}_n, -\text{id}_n)$, and $z = \text{id}_{2n}$.

Note that $\mathfrak{spe}_{a,b}(n)$ depends only on $[a : b] \in \mathbb{C}P^1$, and $\mathfrak{spe}_{1,0}(n) = \mathfrak{pe}(n)$, $\mathfrak{spe}_{0,1}(n) = \mathfrak{cspe}(n)$; we treat separately $\mathfrak{spe}_{0,0}(n) = \mathfrak{spe}(n)$. We have:

$$\dim \mathfrak{spe}_{a,b}(n) = (n^2|n^2), \quad \dim \mathfrak{spe}(n) = (n^2 - 1|n^2), \quad \dim \mathfrak{cpe}(n) = (n^2 + 1|n^2).$$

Theorem 5.2 Consider an irreducible G_0 -structure F_0 on a supermanifold M of dimension $(n|n)$, i.e., $G_0 = \exp \mathfrak{g}_0$ acts irreducibly on \mathfrak{g}_{-1} , where \mathfrak{g}_0 is one of the Lie superalgebras above. We get the following symmetry dimension bounds for $\mathfrak{s} = \text{inf}(M, F_0)$:

- (1) If $\mathfrak{g}_0 = \mathfrak{pe}(n), \mathfrak{spe}(n), \mathfrak{cspe}(n)$, or $\mathfrak{spe}_{a,b}(n)$, where $a, b \in \mathbb{C}^\times$ with $b \neq na$, then $\dim(\mathfrak{s}) \leq (n|n) + \dim(\mathfrak{g}_0)$.
- (2) If $\mathfrak{g}_0 = \mathfrak{cpe}(n)$, then $\dim(\mathfrak{s}) \leq \dim(\mathfrak{pe}(n+1)) = ((n+1)^2|(n+1)^2)$.
- (3) If $\mathfrak{g}_0 = \mathfrak{spe}_{1,n}(n)$, then $\dim(\mathfrak{s}) \leq \dim(\mathfrak{spe}(n+1)) = (n^2 + 2n|(n+1)^2)$.

Proof These results follow from applying our Theorem 1.1 to the prolongation results due to Poletaeva [32, Thm.1.2]: namely, she proved that $\text{pr}(\mathfrak{g}_{-1}, \mathfrak{g}_0) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ for (1), while $\text{pr}(\mathfrak{g}_{-1}, \mathfrak{g}_0) \cong \mathfrak{pe}(n+1)$ for (2), and $\text{pr}(\mathfrak{g}_{-1}, \mathfrak{g}_0) \cong \mathfrak{spe}(n+1)$ for (3). We remark that for (2) and (3), the prolongation height is 2, which differs from its depth being 1. See [32, Lemma 1.1] for details on this \mathbb{Z} -grading. \square

Consider now the case when \mathfrak{P} is skew-supersymmetric, i.e., $\mathfrak{P}(x, y) = -(-1)^{|x||y|}\mathfrak{P}(y, x)$ for all pure parity x, y . The same formula as in (5.1) defines the skew-periplectic Lie superalgebra $\mathfrak{pe}^{sk}(n)$ and taking $\mathfrak{P} = \begin{pmatrix} 0 & \text{id}_n \\ -\text{id}_n & 0 \end{pmatrix}$ gives

$$\mathfrak{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & C \\ B & -A^\top \end{pmatrix} : A, B, C \in \mathfrak{gl}(n) : B = B^\top, C = -C^\top \right\}. \tag{5.3}$$

We may define analogous related Lie superalgebras as above. Clearly the parity change functor $V \rightarrow \Pi(V)$ on $V = \mathbb{R}^{n|n}$ induces an isomorphism $\mathfrak{pe}(n) \cong \mathfrak{pe}^{sk}(n)$ as Lie superalgebras, but the differing representation on V is crucial as the following result shows:

Proposition 5.3 *Let $\mathfrak{g}_{-1} = V = \mathbb{R}^{n|n}$. If $\mathfrak{g}_0 \supset \mathfrak{spe}^{sk}(n)$, then $\mathfrak{g} = \text{pr}(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ has infinite odd part.*

Proof Focusing on the “ B -part” of (5.3), we see that \mathfrak{g}_0 contains a rank 1 odd element $x \otimes \omega$, where $x \in V_{\bar{1}}$ and $\omega \in V_0^*$. Considering the odd elements $\phi_k = x \otimes \omega^{k+1}$ for all $k > 0$, we inductively observe that $\phi_k \in \mathfrak{g}_k$ for all $k > 0$. □

5.1.5 Projective Superstructures

Classical projective structures are defined as equivalence classes of affine connections for which geodesics differ by a reparametrization. It is well-known that every class contains a torsion-free connection. We here omit the discussion of what a supergeodesic is since this is not uniform in the literature [15, 27] and simply follow [27] in adapting the classical interpretation of projective equivalence for the torsion-free connections: two torsion-free affine superconnections ∇ and ∇' are equivalent if and only if $\nabla - \nabla' = \text{Id} \circ \omega \in \Gamma(S^2\mathcal{T}^*M \otimes \mathcal{T}M)$ for an even 1-form $\omega \in \Omega^1(M)$. (The symmetric power is meant in the super-sense.) This is a higher order reduction of the frame bundle. Namely, using the $\mathfrak{gl}(V)$ -equivariant splitting $\mathfrak{g}_1 = S^2V^* \otimes V = V^* \oplus (S^2V^* \otimes V)_{\text{tf}} = \mathfrak{g}'_1 \oplus \mathfrak{g}''_1$ (trace and trace-free parts), the principal bundle $F_1 \rightarrow F_0 = Fr_M$ is reduced to the (Abelian) structure group \mathfrak{g}'_1 .

After this the geometric structure is prolonged. The obtained projective structure has symmetry the entire diffeomorphism group in the case of (even) line.

Proposition 5.4 *The projective structure is of finite type for $\dim V = (m|n) \neq (1|0)$.*

Proof In dimension $(0|1)$ we have $\mathfrak{g}_1 = 0$, so it is clear. Otherwise we claim that $\mathfrak{g}_i = (\mathfrak{g}'_1)^{(i-1)} = 0$ for all $i > 1$. Indeed the Spencer complex in \mathbb{Z} -degree 2 is given by

$$0 \rightarrow V^* \otimes V^* \rightarrow \Lambda^2 V^* \otimes \mathfrak{gl}(V) \rightarrow \Lambda^3 V^* \otimes V \rightarrow 0$$

and its first cohomology group vanishes, i.e., $H^{2,1}(V, V \oplus \mathfrak{gl}(V) \oplus \mathfrak{g}'_1) = 0$. Indeed, we have

$$\begin{aligned} (\delta A)(u, v, u) &= A(u, v)u + (-1)^{|u||v|}A(u, u)v \\ &\quad - (-1)^{|u||v|}A(v, u)u - (-1)^{|u||v|+|u|}A(v, u)u, \end{aligned}$$

for $u, v \in V, A \in V^* \otimes V^*$. Taking u, v independent (which is possible if $m > 1$ or $n > 1$), the condition $\delta A = 0$ tells us that $A(u, u) = 0$ for all $u \in V$. Considering the vanishing of the remaining three terms gives $A = 0$. The first cohomology groups then must vanish in higher \mathbb{Z} -gradings too and $H^{\geq 2,1}(V, V \oplus \mathfrak{gl}(V) \oplus \mathfrak{g}'_1) = 0$ is equivalent to the prolongation claim, cf. [23, 38]. \square

Consequently, the symmetry dimension of a projective structure on a supermanifold M with $\dim M \neq (1|0)$ is bounded by

$$\dim \mathfrak{s} \leq \dim V + \dim \mathfrak{gl}(V) + \dim \mathfrak{g}'_1 = (2m + n^2 + m^2 | 2n + 2mn).$$

Now we will consider examples of filtered structures in the nonholonomic case, that is $\mathcal{D} \subsetneq \mathcal{FM}$.

5.2 G(3)-Supergeometries

In [23], we studied two types of $G(3)$ -supergeometries, where $G(3)$ is the exceptional simple Lie supergroup of dimension $(17|14)$.

5.2.1 G(3)-Contact Supergeometry

Consider a contact distribution \mathcal{C} of rank $(4|4)$ on a supermanifold M of dimension $(5|4)$. The induced conformally super-symplectic structure on \mathcal{C} reduces the structure group to $\text{CSpO}(4|4)$ and it is still of infinite type. A cone structure on \mathcal{C} is given by a field of supervarieties in projectivized contact spaces. Namely for $x \in M_o$ the projective superspace $\mathbb{P}\mathcal{C}|_x$ contains a distinguished subvariety $\mathcal{V}|_x$ of dimension $(1|2)$ that is isomorphic to the unique irreducible flag manifold of the simple Lie supergroup $\text{OSp}(3|2)$, namely $\mathcal{V}|_x \cong \text{OSp}(3|2)/P_1^{\text{II}}$, where $\mathfrak{p}_1^{\text{II}}$ is the parabolic subalgebra $\begin{matrix} \circ & \text{---} & \bullet \\ \times & & \end{matrix}$. We call this subvariety the $(1|2)$ -twisted cubic, because its underlying classical manifold is a rational normal curve of degree 3, which is “deformed” in 2 odd dimensions.

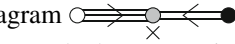
This cone field reduces the structure group to $\text{COSp}(3|2) \subset \text{CSpO}(4|4)$, and now this is of finite type: if $\mathfrak{g}_0 = \mathfrak{cosp}(3|2)$ and $\mathfrak{g} = \mathfrak{g}(3)$ is the Lie algebra of $G(3)$ then $H^{d,1}(\mathfrak{m}, \mathfrak{g}) = 0$ if $d > 0$ by [23, Theorem 3.9], so the maximal prolongation is $\mathfrak{g} = \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$ (Corollary 3.10 loc.cit.). Such a geometric structure arises on the generalized flag-supervariety $G(3)/P_1^{\text{IV}}$ with marked Dynkin diagram $\begin{matrix} \circ & \rightleftarrows & \circ & \leftarrow & \bullet \\ \times & & & & \end{matrix}$ and in [23, Theorem 4.9] we established that the maximal symmetry dimension (in the strong sense) of supergeometries $(M, \mathcal{C}, \mathcal{V})$ as above is $(17|14)$, under the assumption that the geometry is locally homogeneous. Now as a direct corollary of Theorem 1.1 we derive that the assumption of local homogeneity can be removed (we fulfill thus what is written in footnote 5 at page 54 of loc.cit.):

Theorem 5.5 *The maximal symmetry dimension of a $G(3)$ -contact supergeometry $(M, \mathcal{C}, \mathcal{V})$ is equal to $(17|14)$.*

5.2.2 Super Hilbert–Cartan Geometries

Another $G(3)$ supergeometry lives on supermanifolds of dimensions $(5|6)$ and it is given by a superdistribution with growth vector $(2|4, 1|2, 2|0)$. The symbol of a (fundamental, non-degenerate) superdistribution with such growth vector can be one of four types [23, Theorem 5.1], and just of two types if its even part is the standard symbol as for the Hilbert–Cartan equation. Moreover one of them is generic, hence rigid, and it is called SHC type symbol. More explicitly, for a basis of \mathfrak{m} (listed in the format (even|odd)) $\mathfrak{g}_{-1} = \langle e_1, e_2 | \theta'_1, \theta''_1, \theta'_2, \theta''_2 \rangle$, $\mathfrak{g}_{-2} = \langle h | \varrho_1, \varrho_2 \rangle$, $\mathfrak{g}_{-3} = \langle f_1, f_2 | \cdot \rangle$, the non-trivial commutator relations of the SHC type symbol are the following:

$$\begin{aligned} [e_1, e_2] &= h, [e_1, h] = f_1, [e_2, h] = f_2, [\theta'_1, \theta'_2] = [\theta''_1, \theta''_2] = h, \\ [e_1, \theta'_2] &= [e_2, \theta''_1] = \varrho_1, [e_1, \theta''_2] = -[e_2, \theta'_1] = \varrho_2, \\ [\theta'_1, \varrho_1] &= [\theta''_1, \varrho_2] = f_1, [\theta''_2, \varrho_1] = -[\theta'_2, \varrho_2] = f_2. \end{aligned}$$

Such a superdistribution arises on the generalized flag-supermanifold $G(3)/P_2^{IV}$ with the marked Dynkin diagram . For the grading corresponding to the parabolic P_2^{IV} the Lie superalgebra $\mathfrak{g} = \text{Lie}(G(3))$ contains \mathfrak{m} as the negative part. In [23, Theorem 3.16] we established $H^{d,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for all $d \geq 0$. Hence (Corollary 3.17 loc.cit.) \mathfrak{g} is the Tanaka-Weisfeiler prolongation of \mathfrak{m} , i.e., $\mathfrak{g} = \text{pr}(\mathfrak{m})$.

The methods of [23] allow to conclude that $(17|14)$ is the maximal symmetry dimension for locally homogeneous distributions with the SHC symbol. Using Theorem 1.1 of the present paper we derive the result in full generality without the local homogeneity assumption.

Theorem 5.6 *The maximal symmetry dimension of a superdistribution with SHC symbol is $(17|14)$.*

5.3 Super-Poincaré Structures

Let \mathbb{V} be a complex vector space with a non-degenerate symmetric bilinear form (\cdot, \cdot) and \mathbb{S} an irreducible module over the associated Clifford algebra. A supertranslation algebra is a \mathbb{Z} -graded Lie superalgebra $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$, where $\mathfrak{m}_{-2} = \mathfrak{m}_0 = \mathbb{V}$ and $\mathfrak{m}_{-1} = \mathfrak{m}_1 = \mathbb{S} \oplus \dots \oplus \mathbb{S}$ is the direct sum of an arbitrary number $N \geq 1$ of copies of \mathbb{S} , whose bracket $\Gamma : \mathfrak{m}_{-1} \otimes \mathfrak{m}_{-1} \rightarrow \mathfrak{m}_{-2}$ is of the form $(\Gamma(s, t), v) = \mathfrak{B}(v \cdot s, t)$ for $v \in \mathbb{V}, s, t \in \mathfrak{m}_{-1}$. Here \mathfrak{B} is a non-degenerate bilinear form on \mathfrak{m}_{-1} , which is admissible in the sense of [1]. We note that Γ is $\mathfrak{so}(\mathbb{V})$ -equivariant, so the semidirect sum $\mathfrak{p} = \mathfrak{m} \rtimes \mathfrak{so}(\mathbb{V})$ is a Lie superalgebra, usually referred to as Poincaré superalgebra (complex, N -extended, in dimension $\dim \mathbb{V}$).

Real supermanifolds M endowed with a strongly regular odd distribution $\mathcal{D} \subset \mathcal{TM}$ whose complexified symbol is \mathfrak{m} appear naturally in “super-space” approaches to supergravity and rigid supersymmetric field theories (see [9, 10, 12, 13, 35, 36] and references therein). The superdistribution \mathcal{D} has been called a super-Poincaré structure in [3] and the main result of that paper is the explicit description of the maximal transitive prolongation of \mathfrak{m} . Here we recall it for the reader’s convenience:

Table 1 Exceptional prolongations of super-Poincaré algebras

\mathfrak{g}	$\dim \mathfrak{g}$	Dynkin diagram	$\dim \mathbb{V}$	$\dim \mathbb{S}$	N	\mathfrak{h}_0
$\mathfrak{osp}(1 4)$	$(10 4)$		3	2	1	0
$\mathfrak{osp}(2m+1 4)$ $m \geq 1$	$(2m^2+m+10 8m+4)$		3	2	$2m+1$	$\mathfrak{so}(2m+1)$
$\mathfrak{osp}(2 4)$	$(11 8)$		3	2	2	$\mathfrak{so}(2)$
$\mathfrak{osp}(2m 4)$ $m \geq 2$	$(2m^2-m+10 8m)$		3	2	$2m$	$\mathfrak{so}(2m)$
$\mathfrak{sl}(m+1 4)$ $m \neq 3$	$(m^2+2m+16 8m+8)$		4	4	$m+1$	$\mathfrak{gl}(m+1)$
$\mathfrak{pgl}(4 4)$ $F(4)$	$(30 32)$ $(24 16)$		4	4	4	$\mathfrak{gl}(4)$
			5	4	2	$\mathfrak{sl}(2)$

Theorem 5.7 *If $\dim \mathbb{V} = 1, 2$, the prolongation \mathfrak{g} of \mathfrak{m} is infinite-dimensional. If $\dim \mathbb{V} \geq 3$, it is finite-dimensional and $\mathfrak{g}_{\leq 0} = \mathfrak{p} \oplus \mathbb{C}Z \oplus \mathfrak{h}_0$ as the vector space direct sum of the Poincaré superalgebra, the grading element Z and the algebra $\mathfrak{h}_0 = \{D \in \mathfrak{g}_0 | [D, \mathfrak{m}_{-2}] = 0\}$ of the internal symmetries of \mathfrak{m}_{-1} . If $\dim \mathbb{V} \geq 3$, then $\mathfrak{g}_p = 0$ for all $p \geq 1$ in all cases except those listed in Table 1.*

Here the simple roots of degree 1 coincide with the odd simple roots, i.e., those associated to black and gray nodes on the Dynkin diagram.

Since the complexification of the prolongation of a real symbol is the prolongation of the complexified symbol, one may combine Theorems 1.1 and 5.7 to get the bound on the dimension of the symmetry superalgebra of a super-Poincaré structure in dimension $\dim \mathbb{V} \geq 3$. For the exceptional cases with $\mathfrak{g}_1 \neq 0$, it is provided by the second column in Table 1, in all other cases it is given by

$$\dim \mathfrak{s} \leq \left(\frac{d(d+1)}{2} + 1 + \dim \mathfrak{h}_0 \mid N2^{\lfloor d/2 \rfloor} \right),$$

where $d = \dim \mathbb{V}$ and square brackets refer to the integer part. Furthermore, the subalgebra \mathfrak{h}_0 of the internal symmetries can be easily described on a case-by-case basis. It splits into the sum of its symmetric part \mathfrak{h}_0^s and skew-symmetric part \mathfrak{h}_0^a with respect to \mathfrak{B} and the condition that elements of \mathfrak{h}_0 act as derivations of \mathfrak{m} yields:

$$\mathfrak{h}_0^a = \{D \in \mathfrak{gl}(\mathfrak{m}_{-1}) \mid \mathfrak{B}(Ds, t) = -\mathfrak{B}(s, Dt), D(v \cdot s) = v \cdot Ds \ \forall v \in \mathbb{V}, s, t \in \mathfrak{m}_{-1}\},$$

$$\mathfrak{h}_0^s = \{D \in \mathfrak{gl}(\mathfrak{m}_{-1}) \mid \mathfrak{B}(Ds, t) = \mathfrak{B}(s, Dt), D(v \cdot s) = -v \cdot Ds \ \forall v \in \mathbb{V}, s, t \in \mathfrak{m}_{-1}\}.$$

It is well-known that \mathbb{S} is $\mathfrak{so}(\mathbb{V})$ -irreducible if d is odd and the direct sum of two inequivalent $\mathfrak{so}(\mathbb{V})$ -irreducible submodules if d is even. By $\mathfrak{so}(\mathbb{V})$ -equivariancy, a uniform (but not sharp) bound on $\dim \mathfrak{h}_0$ is thus given by N^2 if d is odd and $2N^2$ if d is even.

We conclude this subsection with the following direct consequence of Section 3.6. Consider, for instance, the 4- and 11-dimensional vector spaces V in Lorentzian signature. The real spinor module S is an irreducible module for the Lorentz algebra $\mathfrak{so}(V)$ and it is of Clifford real type (i.e., $S \otimes \mathbb{C} = \mathbb{S}$). It follows from Theorem 5.7 that, if we reduce the structure algebra to $\mathfrak{so}(V)$, the prolongation of the real $N = 1$ Poincaré superalgebra

$$\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 = V + S + \mathfrak{so}(V)$$

is just \mathfrak{p} . Theorem 3.17 and Remark 3.18 then imply that any super-Poincaré structure \mathcal{D} with reduced structure group $P_0 = \text{Spin}(V)$ has associated a Cartan superconnection on a P_0 -principal bundle $\pi : P \rightarrow M$. This bridges from the “super-space approach” to the so-called rheonomic approach of supergravity and supersymmetric field theories (see, e.g., the nice reviews [6, 8]). We stress that in the rheonomic approach the axioms of a Cartan superconnection follow from a Lagrangian principle on the absolute parallelism Φ , whereas our general construction affords the existence of the Cartan superconnection, from purely geometric arguments.

It would be interesting to study the normalization conditions on the Cartan superconnection in the cohomological spirit of Section 3.5.3 and compare them with those traditionally obtained in the rheonomic approach via Lagrangian principles.

5.4 Odd Ordinary Differential Equations

5.4.1 Review of Some Classical ODE

Classically, ODE are geometrically viewed as submanifolds of a jet space with the inherited structure (via pullback along the inclusion map). This leads to formulating these as manifolds M with a rank 2 distribution $C \subset TM$ (having specific symbol \mathfrak{m}) and a splitting into line fields $C = E \oplus V$:

- *2nd-order ODE* $y'' = f(x, y, y')$ (up to point transformations): Introduce local coordinates (x, y, p) on M with $C = E \oplus V = \langle \partial_x + p\partial_y + f\partial_p \rangle \oplus \langle \partial_p \rangle$. Then C has symbol $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = \langle X, e_1 \rangle \oplus \langle e_2 \rangle$ with non-trivial bracket $[X, e_1] = e_2$. (C is a contact distribution.)
- *3rd-order ODE* $y''' = g(x, y, y', y'')$ (up to contact transformations): Introduce local coordinates (x, y, p, q) on M^4 with $C = E \oplus V = \langle \partial_x + p\partial_y + q\partial_p + g\partial_q \rangle \oplus \langle \partial_q \rangle$. Then C has symbol $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} = \langle X, e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$ with non-trivial brackets $[X, e_1] = e_2, [X, e_2] = e_3$. (C is an Engel distribution.)

The splitting indicates a reduction $\mathfrak{g}_0 \hookrightarrow \text{der}_{\text{gr}}(\mathfrak{m})$. In both cases, $\dim(\mathfrak{g}_0) = 2$ with $\mathfrak{g}_0 \hookrightarrow \mathfrak{gl}(\mathfrak{g}_{-1})$ corresponding to scalings along the two distinguished directions in \mathfrak{g}_{-1} .

There are well-known \mathbb{Z} -gradings of $A_2 \cong \mathfrak{sl}_3$ and $B_2 \cong \mathfrak{so}_{2,3}$ for which the negative parts \mathfrak{g}_- are the indicated symbol algebras above and the non-negative parts $\mathfrak{p} = \mathfrak{g}_{\geq 0}$ are the respective Borel subalgebras. This implies inclusions of \mathfrak{sl}_3 and $\mathfrak{so}_{2,3}$ into the respective Tanaka prolongations $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$. One can show that these are, in fact, equalities by verifying that $H^{+1}(\mathfrak{g}_-, \mathfrak{g}) = 0$, as was done by Yamaguchi [41] using Kostant’s theorem. (Previously this was done by Tresse, Cartan and Chern using geometric methods.)

Table 2 Confirmation of $H^{+,1}(\mathfrak{g}_-, \mathfrak{g}) = 0$ for 2nd-order odd ODE

Bi-grading	Basis	Ker(δ_1)
(2,2)	$E_{31}^* \otimes E_{13}$.
(2,1)	$E_{31}^* \otimes E_{12}, E_{21}^* \otimes E_{13}$.
(2,0)	$E_{21}^* \otimes E_{12}$.
(1,2)	$E_{32}^* \otimes E_{13}, E_{31}^* \otimes E_{23}$.
(1,1)	$E_{21}^* \otimes E_{23}, E_{32}^* \otimes E_{12}, E_{31}^* \otimes Z_1, E_{31}^* \otimes Z_2$	$\delta_0(E_{13})$
(1,0)	$E_{21}^* \otimes Z_1, E_{21}^* \otimes Z_2, E_{31}^* \otimes E_{32}$	$\delta_0(E_{12})$
(0,2)	$E_{32}^* \otimes E_{23}$.
(0,1)	$E_{32}^* \otimes Z_1, E_{32}^* \otimes Z_2, E_{31}^* \otimes E_{21}$	$\delta_0(E_{23})$

5.4.2 2nd-Order Odd ODE

Consider a 2nd-order *odd* ODE $\xi'' = \mathfrak{F}(x, \xi, \xi')$, where ξ is an odd function of the even variable x , and \mathfrak{F} is an odd function. As in the classical case, the space $M = \mathbb{R}^{1|2}(x, \xi, \xi')$ is equipped with a distribution $C = E \oplus V \subset TM$, where $E = \langle \partial_x + \xi' \partial_\xi + \mathfrak{F} \partial_{\xi'} \rangle$ is *even* and $V = \langle \partial_{\xi'} \rangle$ is *odd*. The symbol is $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = (\langle X \rangle \oplus \langle \theta_1 \rangle) \oplus \langle \theta_2 \rangle$, for even X and odd θ_1, θ_2 satisfying $[X, \theta_1] = \theta_2$.

Consider $\mathfrak{g} = \mathfrak{sl}(2|1)$, i.e., supertrace-free 3×3 matrices, with \mathbb{Z}_2 -grading induced from $\mathbb{R}^{2|1}$. Write $\mathfrak{h} = \langle Z_1, Z_2 \rangle$, where $Z_1 = \text{diag}(0, -1, -1)$ and $Z_2 = \text{diag}(-1, -1, -2)$. Defining $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(\text{diag}(h_1, h_2, h_3)) = h_i$, we have $\{Z_1, Z_2\}$ being the dual basis to $\{\alpha_1 := \epsilon_1 - \epsilon_2, \alpha_2 := \epsilon_2 - \epsilon_3\}$. Letting E_{ij} be the matrix with a 1 in the (i, j) -position and 0 elsewhere, we use (Z_1, Z_2) to induce a bigrading on \mathfrak{g} , and let $Z = Z_1 + Z_2$ be the induced grading on \mathfrak{g} . In particular:

$$\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = (\mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{0,-1}) \oplus \mathfrak{g}_{-1,-1} = (\langle E_{21} \rangle \oplus \langle E_{32} \rangle) \oplus \langle E_{31} \rangle. \tag{5.4}$$

Here E_{21} is even, while E_{31}, E_{32} are odd. The only non-trivial bracket on \mathfrak{g}_- is $[E_{32}, E_{21}] = E_{31}$. We conclude that $\mathfrak{sl}(2|1)$ includes into $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$. From Table 2, we use the differentials $\delta_k : C^k(\mathfrak{g}_-, \mathfrak{g}) \rightarrow C^{k+1}(\mathfrak{g}_-, \mathfrak{g})$ to conclude that $H^{+,1}(\mathfrak{g}_-, \mathfrak{g}) = 0$, whence that $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0) \cong \mathfrak{sl}(2|1)$.

When $\mathfrak{F} = 0$, i.e., $\xi'' = 0$, we have the prolongation X of S_f (satisfying $\mathcal{L}_X E \subset E$ and $\mathcal{L}_X V \subset V$), expressed in terms of a generating superfunction f (see Appendix).

Grading	Even part		Odd part	
	f	Prolongation of S_f	f	Prolongation of S_f
+2	.	.	$x\xi\xi'$	$-x\xi\partial_x + \xi\xi'\partial_{\xi'} + (2\xi + x\xi')\xi''\partial_{\xi''}$
+1	$x\xi - x^2\xi'$	$x^2\partial_x + x\xi\partial_\xi + (\xi - x\xi')\partial_{\xi'} - 3x\xi''\partial_{\xi''}$	$\xi'\xi$	$\xi\partial_x - \xi'\xi''\partial_{\xi''}$
0	$x\xi'$	$-x\partial_x + \xi'\partial_{\xi'} + 2\xi''\partial_{\xi''}$.	.
	ξ	$\xi\partial_\xi + \xi'\partial_{\xi'} + \xi''\partial_{\xi''}$.	.
-1	ξ'	$-\partial_x$	x	$x\partial_\xi + \partial_{\xi'}$
-2	.	.	1	∂_ξ

$$\tag{5.5}$$

These symmetries are all projectable over (x, ξ) -space, i.e., they are (prolonged) point symmetries. (Equivalently, their generating functions are linear in ξ' .) This symmetry superalgebra is indeed $\mathfrak{sl}(2|1)$. In stark contrast to the classical case, 2nd-order odd ODE do not admit non-trivial deformations:

Proposition 5.8 Any 2nd-order odd ODE $\xi'' = \mathfrak{F}(x, \xi, \xi')$ is locally equivalent to the trivial equation $\xi'' = 0$ via a point transformation, and thus has symmetry dimension $(4|4)$.

Proof Since \mathfrak{F} and ξ, ξ' are odd, then any 2nd-order odd ODE must be of the form:

$$\xi'' = \mathfrak{F}_0(x)\xi + \mathfrak{F}_1(x)\xi'. \tag{5.6}$$

Let $(\tilde{x}, \tilde{\xi}) := (a(x), b(x)\xi)$, $a'(x) \neq 0 \neq b(x)$, which induces $\frac{d\tilde{\xi}}{d\tilde{x}} = \frac{d\tilde{\xi}}{dx}a' = b'\xi + b\xi'$ and $\frac{d^2\tilde{\xi}}{d\tilde{x}^2} = \frac{d^2\tilde{\xi}}{dx^2}(a')^2 + \frac{d\tilde{\xi}}{dx}a'' = b''\xi + 2b'\xi' + b\xi''$. We find that $\frac{d^2\tilde{\xi}}{d\tilde{x}^2} = \tilde{\mathfrak{F}}_0\tilde{\xi} + \tilde{\mathfrak{F}}_1\frac{d\tilde{\xi}}{d\tilde{x}}$, where

$$\tilde{\mathfrak{F}}_0 = \frac{(b''+b\mathfrak{F}_0)b-(2b'+b\mathfrak{F}_1)b'}{(a')^2b^2}, \quad \tilde{\mathfrak{F}}_1 = \frac{(2b'+b\mathfrak{F}_1)a'-ba''}{(a')^2b}.$$

This vanishes for solutions of the even 2nd-order ODE system

$$a'' = \left(2\frac{b'}{b} + \mathfrak{F}_1\right)a', \quad b'' = \frac{2}{b}b'^2 + \mathfrak{F}_1b' - \mathfrak{F}_0b,$$

and this trivializes the odd ODE (5.6). □

5.4.3 3rd-Order Odd ODE

Consider a 3rd-order odd ODE $\xi''' = \mathfrak{G}(x, \xi, \xi', \xi'')$, where ξ is an odd function of the even variable x , and \mathfrak{G} is an odd function. As in the classical case, the space $M = \mathbb{R}^{1|3}(x, \xi, \xi', \xi'')$ is equipped with a distribution $C = E \oplus V \subset TM$, where $E = \langle \partial_x + \xi'\partial_\xi + \xi''\partial_{\xi'} + \mathfrak{G}\partial_{\xi''} \rangle$ is even and $V = \langle \partial_{\xi''} \rangle$ is odd. The symbol is $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} = (\langle X \rangle \oplus \langle \theta_1 \rangle) \oplus \langle \theta_2 \rangle \oplus \langle \theta_3 \rangle$, for even X and odd $\theta_1, \theta_2, \theta_3$ satisfying $[X, \theta_1] = \theta_2, [X, \theta_2] = \theta_3$. Let us now compute the prolongation directly.

Since \mathfrak{g}_{-1} has a splitting into distinguished lines, then $\mathfrak{g}_0 = \langle T_1, T_2 \rangle \hookrightarrow \mathfrak{der}_{\mathfrak{g}_r}(\mathfrak{m})$ is even with $T_1 = \text{diag}(1, 0, 1, 2)$ and $T_2 = \text{diag}(0, 1, 1, 1)$, expressed in the $\{X, \theta_1, \theta_2, \theta_3\}$ basis. Interestingly, the height of the prolongation differs from the depth of \mathfrak{m} .

Proposition 5.9 $\dim(\mathfrak{g}_1) = (1|0), \dim(\mathfrak{g}_2) = (0|1)$, while $\dim(\mathfrak{g}_k) = (0|0)$ for all $k \geq 3$.

Proof Let $A \in \mathfrak{g}_1$ be odd, so $AX = 0$ and $A\theta_1 = a_1T_1 + a_2T_2$. We find $A = 0$ from:

$$\begin{aligned} A\theta_2 &= A[X, \theta_1] = -a_1X, & A\theta_3 &= A[X, \theta_2] = 0, & 0 &= A[\theta_1, \theta_1] = 2a_2\theta_1, \\ 0 &= A[\theta_2, \theta_2] = -2a_1\theta_3. \end{aligned}$$

Now let $A \in \mathfrak{g}_1$ be even. As a map $\mathfrak{g}_{-1} \rightarrow \mathfrak{g}_0$, we have $AX = a_1T_1 + a_2T_2$ and $A\theta_1 = 0$. Then

$$A\theta_2 = A[X, \theta_1] = a_2\theta_1, \quad A\theta_3 = A[X, \theta_2] = (a_1 + 2a_2)\theta_2, \\ 0 = A[X, \theta_3] = 3(a_1 + a_2)\theta_3,$$

so $a_2 = -a_1$. Taking $-a_2 = a_1 = 1$ yields a specific (even) generator for \mathfrak{g}_1 , which we henceforth label as A . (Since all odd-odd brackets vanish, then A is indeed a superderivation.)

Let $B \in \mathfrak{g}_2$ be even. Write $BX = bA$, $B\theta_1 = 0$. We find $B = 0$ from:

$$B\theta_2 = B[X, \theta_1] = 0, \quad B\theta_3 = B[X, \theta_2] = -b\theta_1, \quad 0 = B[X, \theta_3] = -2b\theta_2.$$

Now let $B \in \mathfrak{g}_2$ be odd. Write $BX = 0$, $B\theta_1 = bA$. We obtain:

$$B\theta_2 = B[X, \theta_1] = -b(T_1 - T_2), \quad B\theta_3 = B[X, \theta_2] = bX.$$

A direct check shows that all conditions resulting from $B[X, \theta_3] = 0 = B[\theta_i, \theta_j]$ are satisfied, so B is indeed a superderivation. Take $b = 1$ above yields a specific (odd) generator for \mathfrak{g}_2 that we henceforth label as B .

Let $C \in \mathfrak{g}_3$ be even. Write $CX = 0$, $C\theta_1 = cB$. We find $C = 0$ from:

$$C\theta_2 = C[X, \theta_1] = 0, \quad 0 = C[\theta_1, \theta_2] = c(T_2 - T_1).$$

Now let $C \in \mathfrak{g}_3$ be odd. Write $CX = cB$, $C\theta_1 = 0$. We find $C = 0$ from:

$$C\theta_2 = C[X, \theta_1] = [cB, \theta_1] = cA, \\ 0 = C[\theta_2, \theta_2] = 2[C\theta_2, \theta_2] = 2c[A, \theta_2] = -2c\theta_1.$$

□

When $\mathfrak{G} = 0$, i.e., $\xi''' = 0$, we have the symmetries $X = \mathbf{S}_f$ (satisfying $\mathcal{L}_X E \subset E$ and $\mathcal{L}_X V \subset V$), expressed in terms of a generating superfunction f (see [Appendix](#)).

	Even part		Odd part	
Grading	f	Prolongation of \mathbf{S}_f	f	Prolongation of \mathbf{S}_f
+2	.	.	$\xi\xi'$	$-\xi\partial_x + \xi'\xi''\partial_{\xi''} + 2\xi'\xi'''\partial_{\xi'''}$
+1	$x\xi - \frac{x^2}{2}\xi'$	$\frac{x^2}{2}\partial_x + x\xi\partial_{\xi} + \xi\partial_{\xi'}$ $+ (\xi' - x\xi'')\partial_{\xi''} - 2x\xi'''\partial_{\xi'''}$.	.
0	$\xi - x\xi'$ ξ	$x\partial_x + \xi\partial_{\xi} - \xi''\partial_{\xi''} - 2\xi'''\partial_{\xi'''}$ $\xi\partial_{\xi} + \xi'\partial_{\xi'} + \xi''\partial_{\xi''} + \xi'''\partial_{\xi'''}$.	.
-1	ξ'	$-\partial_x$	$\frac{x^2}{2}$	$\frac{x^2}{2}\partial_{\xi} + x\partial_{\xi'} + \partial_{\xi''}$
-2	.	.	x	$x\partial_{\xi} + \partial_{\xi'}$
-3	.	.	1	∂_{ξ}

(5.7)

These symmetries are all projectable over (x, ξ) -space, i.e., they are (prolonged) point symmetries. Abstractly, the symmetry superalgebra \mathfrak{g} has derived superalgebras

$$\mathfrak{g}^{(1)} = \langle 0 \mid \xi\xi' \rangle \times \mathfrak{g}^{(2)}, \quad \mathfrak{g}^{(2)} = \left\langle \xi', \xi - x\xi', x\xi - \frac{x^2}{2}\xi' \mid 1, x, \frac{x^2}{2} \right\rangle. \quad (5.8)$$

We note that $\mathfrak{g}^{(2)} \cong \mathfrak{sl}(2, \mathbb{R}) \ltimes S^2\mathbb{R}^2$, where the even part is $\mathfrak{sl}(2, \mathbb{R})$ and odd part is $S^2\mathbb{R}^2$ (abelian), with even-odd brackets given by the naturally induced representation. Alternatively, $\mathfrak{g}^{(2)}$ is the Euclidean Lie algebra $\mathfrak{e}(3, \mathbb{R}) := \mathfrak{so}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ regarded as a Lie superalgebra with even part $\mathfrak{so}(3, \mathbb{R})$ and odd part \mathbb{R}^3 (abelian).

Consequently Theorem 1.2 implies:

Theorem 5.10 *Any 3rd-order odd ODE $\xi''' = \mathfrak{G}(x, \xi, \xi', \xi'')$ has contact symmetry superalgebra of dimension at most (4|4) and this bound is sharp.*

In contrast to the 2nd-order odd ODE case, 3rd-order odd ODE are not in general contact-trivializable, i.e., equivalent to $\xi''' = 0$. In fact, 3rd-order odd ODE have the form

$$\xi''' = a(x)\xi + b(x)\xi' + c(x)\xi'' + d(x)\xi\xi'\xi''$$

and one can verify that the term $d(x)$ is a relative invariant. (The even part of the contact supergroup is $(x, \xi) \mapsto (\alpha(x), \beta(x)\xi)$ and the verification is straightforward; the odd part does not contribute.) Consequently, general 3rd-order odd ODE are not linearizable.

Below we exhibit two examples of 3rd-order odd ODE that are not contact-trivializable. Both have solvable symmetry superalgebras. Symmetries are given in terms of their generating superfunctions (see [Appendix](#) for the Lagrange bracket).

Odd ODE	$\xi''' = \xi''$	$\xi''' = \xi\xi'\xi''$																						
Sym dim	(2 3)	(2 2)																						
Symmetries	even part : ξ', ξ odd part : $1, x, e^x$	even part : $\xi', x\xi'$ odd part : $\xi\xi', h := 3 + x\xi\xi'$																						
Lagrange brackets	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding: 5px;"></td> <td style="padding: 5px;">$\xi' \quad \xi \quad 1 \quad x \quad e^x$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">ξ'</td> <td style="padding: 5px;">$\cdot \quad \cdot \quad \cdot \quad -1 \quad -e^x$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">ξ</td> <td style="padding: 5px;">$\cdot \quad \cdot \quad -1 \quad -x \quad -e^x$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">1</td> <td style="padding: 5px;">$\cdot \quad 1 \quad \cdot \quad \cdot \quad \cdot$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">x</td> <td style="padding: 5px;">$1 \quad x \quad \cdot \quad \cdot \quad \cdot$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">e^x</td> <td style="padding: 5px;">$e^x \quad e^x \quad \cdot \quad \cdot \quad \cdot$</td> </tr> </table>		$\xi' \quad \xi \quad 1 \quad x \quad e^x$	ξ'	$\cdot \quad \cdot \quad \cdot \quad -1 \quad -e^x$	ξ	$\cdot \quad \cdot \quad -1 \quad -x \quad -e^x$	1	$\cdot \quad 1 \quad \cdot \quad \cdot \quad \cdot$	x	$1 \quad x \quad \cdot \quad \cdot \quad \cdot$	e^x	$e^x \quad e^x \quad \cdot \quad \cdot \quad \cdot$	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding: 5px;"></td> <td style="padding: 5px;">$\xi' \quad x\xi' \quad \xi\xi' \quad h$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">ξ'</td> <td style="padding: 5px;">$\cdot \quad -\xi' \quad \cdot \quad -\xi\xi'$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">$x\xi'$</td> <td style="padding: 5px;">$\xi' \quad \cdot \quad \xi\xi' \quad \cdot$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">$\xi\xi'$</td> <td style="padding: 5px;">$\cdot \quad -\xi\xi' \quad \cdot \quad 3\xi'$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px;">h</td> <td style="padding: 5px;">$\xi\xi' \quad \cdot \quad 3\xi' \quad 6x\xi'$</td> </tr> </table>		$\xi' \quad x\xi' \quad \xi\xi' \quad h$	ξ'	$\cdot \quad -\xi' \quad \cdot \quad -\xi\xi'$	$x\xi'$	$\xi' \quad \cdot \quad \xi\xi' \quad \cdot$	$\xi\xi'$	$\cdot \quad -\xi\xi' \quad \cdot \quad 3\xi'$	h	$\xi\xi' \quad \cdot \quad 3\xi' \quad 6x\xi'$
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$\xi\xi'$	$\cdot \quad -\xi\xi' \quad \cdot \quad 3\xi'$																							
h	$\xi\xi' \quad \cdot \quad 3\xi' \quad 6x\xi'$																							

(5.9)

More explicitly, the symmetries as (prolonged) contact vector fields are:

- $\xi''' = \xi''$:

$$\begin{aligned}
 & -\partial_x, \quad \xi\partial_\xi + \xi'\partial_{\xi'} + \xi''\partial_{\xi''} + \xi''' \partial_{\xi'''} \\
 & \partial_\xi, \quad x\partial_x + \partial_{\xi'}, \quad e^x(\partial_\xi + \partial_{\xi'} + \partial_{\xi''} + \partial_{\xi'''}).
 \end{aligned}
 \tag{5.10}$$

- $\xi''' = \xi\xi'\xi''$:

$$\begin{aligned}
 & -\partial_x, \quad -x\partial_x + \xi'\partial_{\xi'} + 2\xi''\partial_{\xi''} + 3\xi''' \partial_{\xi'''} \\
 & -x\xi\partial_x + 3\partial_\xi + \xi\xi'\partial_{\xi'} + (2\xi + x\xi')\xi''\partial_{\xi''} + (3\xi'\xi'' + 3\xi\xi''' + 2x\xi'\xi''')\partial_{\xi'''} \\
 & \hspace{15em} \tag{5.11}
 \end{aligned}$$

Appendix. Generating Superfunctions and an Odd Prolongation Formula

In this section, we briefly discuss the jet spaces $J^r := J^r(\mathbb{R}^{p|0}, \mathbb{R}^{0|1})$ associated with p even variables (x^i) and one *odd* (dependent) variable ξ . Our interest is only local, so we introduce these spaces (with *Cartan distribution* $\mathcal{C}_r \subset T J^r$) from a local perspective only:

- $r = 0$: Coordinates (x^i, ξ) . No local structure.
- $r = 1$: Coordinates (x^i, ξ, ξ_i) . Local structure: $\mathcal{C}_1 = \langle \partial_{x^i} + \xi_i \partial_\xi, \partial_{\xi_i} \rangle$.
- $r = 2$: Coordinates $(x^i, \xi, \xi_i, \xi_{ij} = \xi_{ji})$. Local structure: $\mathcal{C}_2 = \langle \partial_{x^i} + \xi_i \partial_\xi + \xi_{ij} \partial_{\xi_j}, \partial_{\xi_{ij}} \rangle$.

(Here, ξ, ξ_i, ξ_{ij} , etc. are odd.) For $r \geq 3$, we proceed similarly. For notational convenience, we take J^∞ as the inverse limit and on it we introduce

$$D_{x^i} = \partial_{x^i} + \xi_i \partial_\xi + \xi_{ij} \partial_{\xi_j} + \xi_{ijk} \partial_{\xi_{jk}} + \dots \tag{A.1}$$

with corresponding truncated vector field $D_{x^i}^{(r)}$ on J^r . For example, $D_{x^i}^{(1)} = \partial_{x^i} + \xi_i \partial_\xi$, $D_{x^i}^{(2)} = \partial_{x^i} + \xi_i \partial_\xi + \xi_{ij} \partial_{\xi_j}$, etc.

A vector field \mathbf{S} on $J^r(\mathbb{R}^{p|0}, \mathbb{R}^{0|1})$ is *contact* if $\mathcal{L}_\mathbf{S} \mathcal{C}_r \subset \mathcal{C}_r$. By the Lie–Bäcklund theorem, any such vector field is projectable over J^1 , and \mathbf{S} is canonically determined from this projection via *prolongation*. On J^1 , fixing $\sigma = d\xi - (dx^i)\xi_i$ generating \mathcal{C}_1 , any contact vector field \mathbf{S} is uniquely determined by its *generating superfunction* $f = \iota_\mathbf{S} \sigma$ (which has opposite parity to \mathbf{S} since σ is odd), and conversely any local superfunction $f = f(x^i, \xi, \xi_i)$ determines a contact vector field:

Proposition A.1 *Given a superfunction $f = f(x^i, \xi, \xi_i)$ with pure parity $|f|$, its associated contact vector field has parity $|f| + 1$ and is given by the formula*

$$\mathbf{S}_f = (-1)^{|f|} (\partial_{\xi_i} f) D_{x^i}^{(1)} + f \partial_\xi + (D_{x^i}^{(1)} f) \partial_{\xi_i}. \tag{A.2}$$

We have $[\mathbf{S}_f, \mathbf{S}_g] = \mathbf{S}_{[f,g]}$ with

$$[f, g] = f(\partial_\xi g) + (-1)^{|f|} (\partial_{\xi_i} f) g + (D_{x^i}^{(1)} f) (\partial_{\xi_j} g) + (-1)^{|f|} (\partial_{\xi_j} f) (D_{x^i}^{(1)} g). \tag{A.3}$$

The prolonged vector field $\mathbf{S}_f^{(\infty)}$ on J^∞ is obtained via the prolongation formula

$$\mathbf{S}_f^{(\infty)} = \mathbf{S}_f + h_{jk} \partial_{\xi_{jk}} + h_{jkl} \partial_{\xi_{jkl}} + \dots, \tag{A.4}$$

with coefficients $h_{jk} = D_{x^j} D_{x^k} f$, $h_{jkl} = D_{x^j} D_{x^k} D_{x^l} f$, etc. (with obvious truncations on each J^r).

This result is analogous to [23, Prop.4.3]. Details are left as an exercise for the reader since it is proved similarly.

Funding Open Access funding provided by UiT The Arctic University of Norway. The research leading to these results has received funding from the Norwegian Financial Mechanism 2014-2021 (project registration number 2019/34/H/ST1/00636) and the Tromsø Research Foundation (project “Pure Mathematics in Norway”).

Declarations

Conflict of Interest The authors declare no competing interests.

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