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# Some new results and inequalities for subsequences of Nörlund logarithmic means of Walsh–Fourier series

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## Abstract

We prove that there exists a martingale  $f \in H_p$  such that the subsequence  $\{L_{2^n}f\}$  of Nörlund logarithmic means with respect to the Walsh system are not bounded from the martingale Hardy spaces  $H_p$  to the space *weak*  $-L_p$  for  $0 < p < 1$ . We also prove that for any  $f \in L_p$ ,  $p \geq 1$ ,  $L_{2^n}f$  converge to  $f$  at any Lebesgue point  $x$ . Moreover, some new related inequalities are derived.

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**Keywords:** Walsh system; Logarithmic means; Partial sums; Fejér means; Martingale Hardy space; Convergence; Divergence; Inequalities

## 1 Introduction

The terminology and notations used in this introduction can be found in Sect. 2.

It is well known that Vilenkin systems do not form bases in the space  $L_1$ . Moreover, there is a function in the Hardy space  $H_1$ , such that the partial sums of  $f$  are not bounded in the  $L_1$ -norm. Moreover, (see Tephnadze [22]) there exists a martingale  $f \in H_p$  ( $0 < p < 1$ ), such that

$$\sup_{n \in \mathbb{N}} \|S_{2^{n+1}}f\|_{\text{weak-}L_p} = \infty.$$

On the other hand, (for details see, e.g., the books [20] and [25]) the subsequence  $\{S_{2^n}\}$  of partial sums is bounded from the martingale Hardy space  $H_p$  to the space  $H_p$ , for all  $p > 0$ , that is, the following inequality holds:

$$\|S_{2^n}f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad n \in \mathbb{N}, p > 0. \quad (1)$$

It is also well known that (see [20] and [16])

$$S_{2^n}f(x) \rightarrow f(x), \quad \text{for all Lebesgue points of } f \in L_p, \text{ where } p \geq 1. \quad (2)$$

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Weisz [26] considered the norm convergence of Fejér means of Vilenkin–Fourier series and proved that the inequality

$$\|\sigma_k f\|_p \leq c_p \|f\|_{H_p}, \quad p > 1/2 \text{ and } f \in H_p, \tag{3}$$

holds. Moreover, Goginava [8] (see also [12–15, 18]) proved that the assumption  $p > 1/2$  in (3) is essential. In particular, he showed that there exists a martingale  $f \in H_{1/2}$  such that  $\sup_{n \in \mathbb{N}} \|\sigma_n f\|_{1/2} = +\infty$ . However, Weisz [26] (see also [17]) proved that for every  $f \in H_p$ , there exists an absolute constant  $c_p$ , such that the following inequality holds:

$$\|\sigma_{2^n} f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad n \in \mathbb{N}, p > 0. \tag{4}$$

Móricz and Siddiqi [11] investigated the approximation properties of some special Nörlund means of Walsh–Fourier series of  $L_p$  functions in norm. Approximation properties for general summability methods can be found in [2, 3]. Fridli, Manchanda and Siddiqi [5] improved and extended the results of Móricz and Siddiqi [11] to martingale Hardy spaces. The case when  $\{q_k = 1/k : k \in \mathbb{N}\}$  was excluded, since the methods are not applicable to Nörlund logarithmic means. In [6] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the Lebesgue space  $L_1$ . In particular, they proved that there exists a function in the space  $L_1$ , such that  $\sup_{n \in \mathbb{N}} \|L_n f\|_1 = \infty$ .

In [4] (see also [10]) it was proved that there exists a martingale  $f \in H_p$ ,  $(0 < p < 1)$  such that  $\sup_{n \in \mathbb{N}} \|L_n f\|_p = \infty$ .

In [19] (see also [24]) it was proved that there exists a martingale  $f \in H_1$  such that

$$\sup_{n \in \mathbb{N}} \|L_n f\|_1 = \infty. \tag{5}$$

However, Goginava [7] proved that

$$\|L_{2^n} f\|_1 \leq c \|f\|_1, \quad f \in L_1, n \in \mathbb{N}.$$

From this result it immediately follows that for every  $f \in H_1$ , there exists an absolute constant  $c$ , such that the inequality

$$\|L_{2^n} f\|_1 \leq c \|f\|_{H_1} \tag{6}$$

holds for all  $n \in \mathbb{N}$ . Goginava [7] also proved that for any  $f \in L_1(G)$ ,

$$L_{2^n} f(x) \rightarrow f(x), \quad \text{a.e., as } n \rightarrow \infty.$$

According to (1), (4) and (6), the following question is quite natural.

**Question 1** Is the subsequence  $\{L_{2^n}\}$  also bounded on the martingale Hardy spaces  $H_p(G)$  when  $0 < p < 1$ ?

In Theorem 2 of this paper we give a negative answer to this question. In particular, we further develop some methods considered in [1, 9] and prove that for any  $0 < p < 1$ , there exists a martingale  $f \in H_p$  such that  $\sup_{n \in \mathbb{N}} \|L_{2^n} f\|_{weak-L_p} = \infty$ . Moreover, in our Theorem 1 we generalize the result of Goginava [7] and prove that for any  $f \in L_1(G)$  and for any Lebesgue point  $x$ ,

$$L_{2^n} f(x) \rightarrow f(x), \quad \text{as } n \rightarrow \infty.$$

The main results in this paper are presented and proved in Sect. 4. Section 3 is used to present some auxiliary lemmas, where, in particular, Lemma 2 is new and of independent interest. In order not to disturb our discussions later some definitions and notations are given in Sect. 4. Finally, Sect. 5 is reserved for some open questions we hope can be a source of inspiration for further research in this interesting area.

### 2 Definitions and notations

Let  $\mathbb{N}_+$  denote the set of the positive integers,  $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ . Denote by  $Z_2$  the discrete cyclic group of order 2, that is  $Z_2 := \{0, 1\}$ , where the group operation is the modulo 2 addition and every subset is open. The Haar measure on  $Z_2$  is given so that the measure of a singleton is  $1/2$ .

Define the group  $G$  as the complete direct product of the group  $Z_2$ , with the product of the discrete topologies of  $Z_2$ s. The elements of  $G$  are represented by sequences  $x := (x_0, x_1, \dots, x_j, \dots)$ , where  $x_k = 0 \vee 1$ .

It is easy to give a base for the neighborhood of  $x \in G$ , namely:

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (n \in \mathbb{N}).$$

Denote  $I_n := I_n(0)$ ,  $\overline{I_n} := G \setminus I_n$  and  $e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G$ , for  $n \in \mathbb{N}$ . It is easy to show that  $\overline{I_M} = \bigcup_{s=0}^{M-1} I_s \setminus I_{s+1}$ .

If  $n \in \mathbb{N}$ , then every  $n$  can be uniquely expressed as  $n = \sum_{k=0}^{\infty} n_k 2^k$ , where  $n_j \in Z_2$  ( $j \in \mathbb{N}$ ) and only a finite number of  $n_j$  differ from zero. Let  $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ .

The norms (or quasinorms) of the spaces  $L_p(G)$  and  $weak-L_p(G)$ , ( $0 < p < \infty$ ) are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p d\mu, \quad \|f\|_{weak-L_p}^p := \sup_{\lambda > 0} \lambda^p \mu \quad (f > \lambda).$$

The  $k$ th Rademacher function is defined by

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

Now, define the Walsh system  $w := (w_n : n \in \mathbb{N})$  on  $G$  as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}).$$

It is well known that (see, e.g., [20])

$$w_n(x + y) = w_n(x)w_n(y). \tag{7}$$

The Walsh system is orthonormal and complete in  $L_2(G)$  (see, e.g., [20]).

If  $f \in L_1(G)$  let us define Fourier coefficients, partial sums and the Dirichlet kernel by

$$\widehat{f}(k) := \int_G f w_k d\mu \quad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \quad D_n := \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N}_+).$$

Recall that (for details see, e.g., [20]):

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{8}$$

and

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}), \quad \text{for } n = \sum_{i=0}^{\infty} n_i 2^i. \tag{9}$$

Let  $\{q_k : k \geq 0\}$  be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of  $f$  are defined by

$$t_n f := \frac{1}{l_n} \sum_{k=0}^n q_{n-k} S_k f.$$

In the special case when  $\{q_k = 1 : k \in \mathbb{N}\}$ , we obtain the Fejér means

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^n S_k f.$$

If  $q_k = 1/(k + 1)$ , then we obtain the Nörlund logarithmic means:

$$L_n f := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k f}{n - k}, \quad l_n := \sum_{k=1}^n \frac{1}{k}. \tag{10}$$

The Riesz logarithmic means are defined by

$$R_n f := \frac{1}{l_n} \sum_{k=1}^n \frac{S_k f}{k}, \quad l_n := \sum_{k=1}^n \frac{1}{k}.$$

We note that this is an inverse of the Nörlund logarithmic means.

The convolution of two functions  $f, g \in L_1(G)$  is defined by

$$(f * g)(x) := \int_G f(x + t)g(t) d\mu(t) \quad (x \in G).$$

It is well known that if  $f \in L_p(G), g \in L_1(G)$  and  $1 \leq p < \infty$ . Then,  $f * g \in L_p(G)$  and the corresponding inequality holds:

$$\|f * g\|_p \leq \|f\|_p \|g\|_1. \tag{11}$$

The representations

$$L_n f(x) = \int_G f(t) P_n(x+t) d\mu(t) \quad \text{and} \quad R_n f(x) = \int_G f(t) Y_n(x+t) d\mu(t)$$

for  $n \in \mathbb{N}$  play a central role in the following, where

$$P_n := \frac{1}{Q_n} \sum_{k=1}^n q_{n-k} D_k \quad \text{and} \quad Y_n := \frac{1}{Q_n} \sum_{k=1}^n q_k D_k$$

are called the kernels of the Nörlund logarithmic and the Reisz means, respectively. It is well known that (see, e.g., Goginava [7] and Tephnadze [23]):

$$P_{2^n}(x) = D_{2^n}(x) - \psi_{2^{n-1}}(x) Y_{2^n}(x). \tag{12}$$

Moreover, for all  $n \in \mathbb{N}$ ,

$$\|P_{2^n}\|_1 < c < \infty \quad \text{and} \quad \|Y_n\|_1 < c < \infty. \tag{13}$$

In the case  $f \in L_1(G)$  the maximal functions are given by

$$M(f)(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) d\mu(u) \right| = \sup_{n \in \mathbb{N}} 2^n \left| \int_{I_n(x)} f(u) d\mu(u) \right|.$$

It is well known (for details see, e.g., [20]) that if  $f \in L_1(G)$ , then

$$\|M(f)\|_{\text{weak-}L_1} \leq \|f\|_1.$$

According to a density argument of Calderon–Zygmund (see [20]) we obtain that if  $f \in L_1(G)$ , then

$$2^n \left| \int_{I_n(x)} f(u) d\mu(u) \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

A point  $x$  on the Walsh group is called a Lebesgue point of  $f \in L_1(G)$ , if

$$\lim_{n \rightarrow \infty} 2^n \int_{I_n(x)} f(t) d\mu(t) = f(x) \quad \text{a.e. } x \in G.$$

According to (2) we find that if  $f \in L_1(G)$ , then a.e. point is a Lebesgue point.

Let  $f := (f^{(n)}, n \in \mathbb{N})$  be a martingale with respect to  $F_n (n \in \mathbb{N})$ , which are generated by the intervals  $\{I_n(x) : x \in G\}$  (for details see, e.g., [25]).

We say that a martingale belongs to Hardy martingale spaces  $H_p(G)$ , where  $0 < p < \infty$  if  $\|f\|_{H_p} := \|f^*\|_p < \infty$ , where  $f^* := \sup_{n \in \mathbb{N}} |f^{(n)}|$ .

If  $f = (f^{(n)}, n \in \mathbb{N})$  is a martingale, then the Walsh–Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \rightarrow \infty} \int_G f^{(k)}(x) w_i(x) d\mu(x).$$

### 3 Auxiliary results

The Hardy martingale space  $H_p(G)$  has an atomic characterization (see Weisz [25, 26]).

**Lemma 1** *A martingale  $f = (f^{(n)}, n \in \mathbb{N})$  is in  $H_p$  ( $0 < p \leq 1$ ) if and only if there exist a sequence  $(a_k, k \in \mathbb{N})$  of  $p$ -atoms, which means that they satisfy the conditions*

$$\int_I a_k d\mu = 0, \quad \|a_k\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a_k) \subset I,$$

and a sequence  $(\mu_k, k \in \mathbb{N})$  of real numbers such that for every  $n \in \mathbb{N}$ :

$$\sum_{k=0}^\infty \mu_k S_{2^n} a_k = f^{(n)}, \quad \text{where } \sum_{k=0}^\infty |\mu_k|^p < \infty. \tag{14}$$

Moreover,  $\|f\|_{H_p} \sim \inf(\sum_{k=0}^\infty |\mu_k|^p)^{1/p}$ , where the infimum is taken over all decompositions of  $f$  of the form (14).

We also state and prove a new lemma of independent interest:

**Lemma 2** *Let  $n \in \mathbb{N}$  and  $x \in I_2(e_0 + e_1) \in I_0 \setminus I_1$ . Then,*

$$\left| \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} \frac{D_j}{2^{2\alpha_k+1} - j} \right| \geq \frac{1}{3}.$$

*Proof* Let  $x \in I_2(e_0 + e_1) \in I_0 \setminus I_1$ . According to (8) and (9) we obtain that

$$D_j(x) = \begin{cases} w_j, & \text{if } j \text{ is an odd number,} \\ 0, & \text{if } j \text{ is an even number,} \end{cases}$$

and

$$\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} \frac{D_j}{2^{2\alpha_k+1} - j} = \sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-1} \frac{w_{2j+1}}{2^{2\alpha_k+1} - 2j - 1} = w_1 \sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-1} \frac{w_{2j}}{2^{2\alpha_k+1} - 2j - 1}.$$

Since

$$\begin{aligned} & \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} \left| \frac{1}{2^{2\alpha_k+1} - 4j + 3} - \frac{1}{2^{2\alpha_k+1} - 4j + 1} \right| \\ &= \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} \frac{2}{(2^{2\alpha_k+1} - 4j + 3)(2^{2\alpha_k+1} - 4j + 1)} \\ &\leq \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} \frac{2}{(2^{2\alpha_k+1} - 4j)(2^{2\alpha_k+1} - 4j)} \\ &\leq \frac{1}{8} \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} \frac{1}{(2^{2\alpha_k-1} - j)(2^{2\alpha_k-1} - j)} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{1}{8} + \frac{1}{8} \sum_{k=2}^{\infty} \frac{1}{k^2} \\ &\leq \frac{1}{8} + \frac{1}{8} \sum_{k=2}^{\infty} \left( \frac{1}{k-1} - \frac{1}{k} \right) \\ &\leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}, \end{aligned}$$

if we apply  $w_{4k+2} = w_2 w_{4k} = -w_{4k}$ , for  $x \in I_2(e_0 + e_1)$ , we find that

$$\begin{aligned} \left| \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} \frac{D_j}{2^{2\alpha_k+1}-j} \right| &= \left| w_{2^{2\alpha_k+1}-2} + \frac{w_{2^{2\alpha_k+1}-4}}{3} + \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k}-1} \frac{w_{2j}}{2^{2\alpha_k+1}-2j-1} \right| \\ &= \left| \frac{w_{2^{2\alpha_k+1}-4}}{3} - w_{2^{2\alpha_k+1}-4} + \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k}-1} \frac{w_{2j}}{2^{2\alpha_k+1}-2j-1} \right| \\ &= \left| \frac{2w_{2^{2\alpha_k+1}-4}}{3} + \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k}-1} \left( \frac{w_{4j-4}}{2^{2\alpha_k+1}-4j+3} + \frac{w_{4j-2}}{2^{2\alpha_k+1}-4j+1} \right) \right| \\ &= \left| \frac{2w_{2^{2\alpha_k+1}-4}}{3} + \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k}-1} \left( \frac{w_{4j-4}}{2^{2\alpha_k+1}-4j+3} - \frac{w_{4j-4}}{2^{2\alpha_k+1}-4j+1} \right) \right| \\ &\geq \frac{2}{3} - \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k}-1} \left| \frac{1}{2^{2\alpha_k+1}-4j+3} - \frac{1}{2^{2\alpha_k+1}-4j+1} \right| \\ &\geq \frac{2}{3} - \frac{1}{4} \geq \frac{1}{3}. \end{aligned}$$

The proof is complete. □

### 4 Main results

Our first main result reads:

**Theorem 1** *Let  $p \geq 1$  and  $f \in L_p(G)$ . Then,*

$$\|L_{2^n} f - f\|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{15}$$

Moreover, for all Lebesgue points of  $f$ ,

$$\lim_{n \rightarrow \infty} L_{2^n} f(x) = f(x).$$

*Proof* Let  $n \in \mathbb{N}$ . By combining (11) and (13) we immediately obtain

$$\|L_{2^n} f\|_p \leq c_p \|f\|_p \quad \text{for all } n \in \mathbb{N},$$

which immediately implies (15).

To prove a.e. convergence we use identity (12) to obtain that

$$L_{2^n}f(x) = \int_G f(t)P_{2^n}(x+t) = \int_G f(t)D_{2^n}(x+t) d\mu(t) d\mu(t) - \int_G f(t)w_{2^{n-1}}(x+t)Y_{2^n}(x+t) d\mu(t) := I - II.$$

By applying (2) we can conclude that  $I = S_{2^n}f(x) \rightarrow f(x)$  for all Lebesgue points of  $f \in L_p$ . Moreover, by using (7) we find that

$$II = \psi_{2^{n-1}}(x) \int_G f(t)Y_{2^n}(x+t)\psi_{2^{n-1}}(t) d(t).$$

In view of (13) we see that

$$f(t)Y_{2^n}(x+t) \in L_p \quad \text{where } p \geq 1 \text{ for any } x \in G,$$

and also note that  $II$  describes the Fourier coefficients of an integrable function. Hence, according to the Riemann–Lebesgue Lemma it vanishes as  $n \rightarrow \infty$ , i.e.,  $II \rightarrow 0$  for any  $x \in G, n \rightarrow \infty$ .

The proof is complete. □

Our next main result is the following answer of Question 1.

**Theorem 2** *Let  $0 < p < 1$ . Then, there exists a martingale  $f \in H_p$  such that*

$$\sup_{n \in \mathbb{N}} \|L_{2^n}f\|_{weak-L_p} = \infty.$$

*Proof* Let  $\{\alpha_k : k \in \mathbb{N}\}$  be an increasing sequence of the positive integers such that

$$\sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty, \tag{16}$$

$$\sum_{\eta=0}^{k-1} \frac{(2^{2\alpha_\eta})^{1/p}}{\sqrt{\alpha_\eta}} < \frac{(2^{2\alpha_k})^{1/p}}{\sqrt{\alpha_k}}, \tag{17}$$

and

$$\frac{(2^{2\alpha_{k-1}})^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{2^{2\alpha_k-8}}{\alpha_k^{1/2} l_{2^{2\alpha_k+1}}}. \tag{18}$$

Let

$$f^{(n)}(x) := \sum_{\{k; 2\alpha_k < n\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{1}{\sqrt{\alpha_k}} \quad \text{and} \quad a_k = 2^{2\alpha_k(1/p-1)}(D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}).$$



From (16) and Lemma 1 we find that  $f \in H_p$ . It is easy to show that

$$\widehat{f}(j) = \begin{cases} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}, & \text{if } j \in \{2^{2\alpha_k}, \dots, 2^{2\alpha_{k+1}} - 1\}, k \in \mathbb{N}, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{2^{2\alpha_k}, \dots, 2^{2\alpha_{k+1}} - 1\}. \end{cases} \tag{19}$$

Moreover,

$$\begin{aligned} L_{2^{2\alpha_k+1}}f &= \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=1}^{2^{2\alpha_k}-1} \frac{S_j f}{2^{2\alpha_k+1}-j} + \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_{k+1}}-1} \frac{S_j f}{2^{2\alpha_k+1}-j} \\ &:= I + II. \end{aligned} \tag{20}$$

Let  $j < 2^{2\alpha_k}$ . By combining (17), (18) and (19) we can conclude that

$$\begin{aligned} |S_j f(x)| &\leq \sum_{\eta=0}^{k-1} \sum_{\nu=2^{2\alpha_\eta}}^{2^{2\alpha_{\eta+1}}-1} |\widehat{f}(\nu)| \\ &\leq \sum_{\eta=0}^{k-1} \sum_{\nu=2^{2\alpha_\eta}}^{2^{2\alpha_{\eta+1}}-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\sqrt{\alpha_\eta}} \\ &\leq \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta/p}}{\sqrt{\alpha_\eta}} \\ &\leq \frac{2^{2\alpha_{k-1}/p+1}}{\sqrt{\alpha_{k-1}}} < \frac{2^{2\alpha_k-4}}{\alpha_k^{1/2} l_{2^{2\alpha_k+1}}}. \end{aligned}$$

Hence,

$$\begin{aligned} |I| &\leq \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=1}^{2^{2\alpha_k}-1} \frac{|S_j f(x)|}{2^{2\alpha_k+1}-j} \\ &\leq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}} \sum_{j=1}^{M_{2^{2\alpha_k+1}}-1} \frac{1}{j} \\ &\leq \frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}}. \end{aligned} \tag{21}$$

Let  $2^{2\alpha_k} \leq j \leq 2^{2\alpha_{k+1}} - 1$ . We can write that

$$\begin{aligned} S_j f &= \sum_{\eta=0}^{k-1} \sum_{\nu=2^{2\alpha_\eta}}^{2^{2\alpha_{\eta+1}}-1} \widehat{f}(\nu)w_\nu + \sum_{\nu=2^{2\alpha_k}}^{j-1} \widehat{f}(\nu)w_\nu \\ &= \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\sqrt{\alpha_\eta}} (D_{2^{2\alpha_{\eta+1}}} - D_{2^{2\alpha_\eta}}) + \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} (D_j - D_{2^{2\alpha_k}}). \end{aligned}$$

It follows that

$$\begin{aligned}
 II &= \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} \frac{1}{2^{2\alpha_k+1}-j} \left( \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\sqrt{\alpha_\eta}} (D_{2^{2\alpha_\eta+1}} - D_{2^{2\alpha_\eta}}) \right) \\
 &\quad + \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} \frac{(D_j - D_{2^{2\alpha_k}})}{2^{2\alpha_k+1}-j} \\
 &:= II_1 + II_2.
 \end{aligned} \tag{22}$$

Let  $x \in I_2(e_0 + e_1) \in I_0 \setminus I_1$ . According to  $\alpha_0 \geq 1$  we obtain that  $2\alpha_k \geq 2$ , for all  $k \in \mathbb{N}$  and if we use (8) we obtain that  $D_{2^{2\alpha_k}} = 0$ ,

$$II_1 = 0 \tag{23}$$

and

$$\begin{aligned}
 II_2 &= \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k}-1} \frac{w_{2j+1}}{2^{2\alpha_k+1}-2j-1} \\
 &= \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)} w_1}{\sqrt{\alpha_k}} \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k}-1} \frac{w_{2j}}{2^{2\alpha_k+1}-2j-1}.
 \end{aligned}$$

By using Lemma 2 we can conclude that

$$|II_2| \geq \frac{1}{3} \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} \geq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-1}}{\sqrt{\alpha_k}}. \tag{24}$$

If we apply (18), (20)–(24) for  $x \in I_2(e_0 + e_1)$  and  $0 < p < 1$ , we have that

$$\begin{aligned}
 |L_{2^{2\alpha_k+1}} f(x)| &\geq II_2 - II_1 - I \\
 &\geq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-2}}{\sqrt{\alpha_k}} - \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-3}}{\sqrt{\alpha_k}} \\
 &\geq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-3}}{\sqrt{\alpha_k}} \\
 &\geq \frac{2^{2\alpha_k(1/p-1)-3}}{(\ln 2^{2\alpha_k+1} + 1)\sqrt{\alpha_k}} \\
 &\geq \frac{2^{2\alpha_k(1/p-1)-3}}{(4\alpha_k + 1)\sqrt{\alpha_k}} \geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}}.
 \end{aligned}$$

Hence, we can conclude that

$$\begin{aligned}
 &\|L_{q\alpha_k} f\|_{weak-L_p} \\
 &\geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \mu \left\{ x \in G : |L_{2^{2\alpha_k+1}} f| \geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \right\}^{1/p}
 \end{aligned}$$

$$\begin{aligned} &\geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \mu \left\{ x \in I_2(e_0 + e_1) : |L_{2^{2\alpha_k+1}}f| \geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \right\}^{1/p} \\ &\geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} (\mu(I_2(e_0 + e_1)))^{1/p} \\ &> \frac{c2^{2\alpha_k(1/p-1)}}{\alpha_k^{3/2}} \rightarrow \infty, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

The proof is complete. □

### 5 Open questions

It is known (for details see, e.g., the books [20] and [25]) that the subsequence  $\{S_{2^n}\}$  of the partial sums is bounded from the martingale Hardy space  $H_p$  to the Lebesgue space  $L_p$ , for all  $p > 0$ . On the other hand, (see Tephnadze [22]) there exists a martingale  $f \in H_p$  ( $0 < p < 1$ ), such that  $\sup_{n \in \mathbb{N}} \|S_{2^{n+1}}f\|_{weak-L_p} = \infty$ . However, Simon [21] proved that for all  $f \in H_p$ , there exists an absolute constant  $c_p$ , depending only on  $p$ , such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p < 1).$$

In [24] it was proved that for all  $f \in H_p$ , there exists an absolute constant  $c_p$ , depending only on  $p$ , such that

$$\sum_{k=1}^{\infty} \frac{\|L_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p < 1).$$

**Open Problem 1** (a) Let  $f \in H_p$ , where  $0 < p < 1$ . Does there exist an absolute constant  $c_p$ , such that the following inequality holds:

$$\sum_{k=1}^{\infty} \frac{\log^p k \|L_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p < 1)?$$

(b) For  $0 < p < 1/2$  and any nondecreasing function  $\Phi : \mathbb{N} \rightarrow [1, \infty)$  satisfying the conditions  $\lim_{n \rightarrow \infty} \Phi(n) = +\infty$ , is it possible to find a martingale  $f \in H_p$  such that

$$\sum_{n=1}^{\infty} \frac{\log^p n \|L_n f\|_p^p \Phi(n)}{n^{2-p}} = \infty?$$

**Open Problem 2** (a) Let  $f \in H_p$ , where  $0 < p \leq 1$  and

$$\omega_{H_p} \left( \frac{1}{2^n}, f \right) = o \left( \frac{\log n}{2^{n(1/p-1)} \log^{2[p]} n} \right), \quad \text{as } n \rightarrow \infty.$$

Does the following convergence result hold:

$$\|L_k f - f\|_{H_p} \rightarrow 0, \quad \text{as } k \rightarrow \infty?$$

(b) Let  $0 < p \leq 1$ . Does there exist a martingale  $f \in H_p$ , for which

$$\omega_{H_p} \left( \frac{1}{2^n}, f \right) = O \left( \frac{\log n}{2^{n(1/p-1)} \log^{2[p]} n} \right), \quad \text{as } n \rightarrow \infty$$

and  $\|L_k f - f\|_{weak-L_p} \rightarrow 0$ , as  $k \rightarrow \infty$ ?

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#### Declarations

##### Competing interests

The authors declare that they have no competing interests.

##### Authors' contributions

DB and GT proposed the idea and initiated the writing of this paper. LEP and HS followed this up with some complementary ideas. All the authors read and approved the final manuscript.

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