

On geometrical structures, associated with linear differential operators of the 1st order

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Abstract

The problem of equivalency for linear differential operators of the first order is discussed.

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1. Introduction

In this paper we continue to study invariants and structures on linear differential operators. The cases of differential operators of order $k \geq 2$, acting in line bundles, were studied in papers ([1],[2],[3]).

In this paper we study linear differential operators of the 1st order, acting in sections of a vector bundle $\pi : E(\pi) \rightarrow M$, where $\dim M = n \geq 2$, $\dim \pi = m \geq 2$.

To illustrate our approach we consider a "toy case" of the 1-st order operators, when $\dim \pi = 1$, $\dim M = 1$.

Let Δ be such an operator. Then its symbol $\sigma \in \text{End}(\pi) \otimes \Sigma_1 = \Sigma_1$ is a vector field on M .

Assume that we have a connection ∇ in the bundle π and let Q_∇ be the quantization, associated with ∇ (see, for example,[3]), i.e., in our case $Q_\nabla : \text{End}(\pi) \otimes \Sigma_1 \rightarrow \mathbf{Dif}_1(\pi, \pi)$ is the morphism splitting exact sequence $\mathbf{0} \rightarrow \text{End}(\pi) \rightarrow \mathbf{Dif}_1(\pi, \pi) \xrightarrow{\text{smbl}} \text{End}(\pi) \otimes \Sigma_1 \rightarrow \mathbf{0}$.

Then we get a decomposition $\Delta = Q_\nabla(\sigma) + \sigma_0(\nabla)$, where operator $\sigma_0(\nabla) \in \text{End}(\pi)$ depends on connection in the following way.

If $\tilde{\nabla} = \nabla + \alpha$ is another connection in the bundle π , where α is a differential 1-form on M , then $\sigma_0(\tilde{\nabla}) = \sigma_0(\nabla) + \langle \alpha, \sigma \rangle$.

Therefore, if $\sigma \neq 0$, we can choose a unique connection ∇ in such a way, that $\sigma_0(\nabla) = 0$. For this connection, we have $\Delta = Q_\nabla(\sigma)$ and equivalence of

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operators with respect to group of automorphisms of π is equivalent to joint classification of pairs (vector field, connection).

In the case, considered in this paper, we start with classification of symbols and find a class of symbols, which we call symbols in general position, such that they allow us to associate with operator a unique connection (the connection that "preserves" the symbol in the best way) and therefore transform the equivalence problem of operators to the equivalence problem of tensors and connections.

To find invariants of symbols we use the theorem of Procesi ([4],[5]), where he proved the M.Artin conjecture ([6]) on algebra of polynomial invariants of a set of linear operators. Applying the Artin's idea we construct tensor invariants of symbols (we call them Artin-Procesi invariants) and then describe the field of rational invariants of symbols. Then, we use these results and ideas of ([3]) on natural coordinates and natural atlases in order to get a local as well as a global description of regular differential operators of the first order.

The paper is organized in the following way. At first, we discuss Artin-Procesi invariants of symbols and corresponding conditions of regularity or general position. We show that the regularity conditions of symbols provide us with a series of geometrical structures on the manifold. Namely, they give us a (pseudo) Riemannian structure on M as well as a canonical frame. Secondly, we study connections in the bundle π and on the manifold, associated with symbols. On the manifold we take the Levi-Civita connection, associated with the (pseudo) Riemannian structure. The requirement on connections in π to preserve the symbol is too strong (cf. [3]) and we restrict ourselves by the class of connections (we call them minimal) that "preserve" symbols in the best way. Using the subsymbol of the operator (similar to [3]) we show that among minimal there is an unique connection associated with the operator. Existence of such connection allows us to get local and global description of operators by using natural coordinates and natural atlases.

2. Differential operators

2.1. Quantizations

The notations we've used in this paper are similar to notations that were used in paper ([3]).

Let M be an n -dimensional manifold and let $\pi : E(\pi) \rightarrow M$ be a vector bundle.

We denote by $\tau : TM \rightarrow M$ and $\tau^* : T^*M \rightarrow M$ the tangent and respectively cotangent bundles over manifold M , and by $\mathbf{1} : \mathbb{R} \times M \rightarrow M$ we denote the trivial linear bundle.

The symmetric and exterior powers of a vector bundle $\pi : E(\pi) \rightarrow M$ will be denoted by $\mathbf{S}^k(\pi)$ and $\mathbf{\Lambda}^k(\pi)$.

The module of smooth sections of bundle π we denote by $C^\infty(\pi)$, and for the cases tangent, cotangent and the trivial bundles we'll use the following notations: $\Sigma_k(M) = C^\infty(\mathbf{S}^k(\tau))$ – the module of symmetric k -vectors and $\Sigma^k(M) =$

$C^\infty(\mathbf{S}^k(\tau^*))$ – the module of symmetric k -forms, $\Omega_k(M) = C^\infty(\mathbf{\Lambda}^k(\tau))$ – the module of skew-symmetric k -vectors and $\Omega^k(M) = C^\infty(\mathbf{\Lambda}^k(\tau^*))$ – the module of exterior k -forms, $C^\infty(\mathbf{1}) = C^\infty(M)$.

Let $\mathbf{Diff}_k(\pi, \pi)$ be the module of linear differential operators of order k , acting in the sections of the bundle π . These modules are connected by exact sequences

$$\mathbf{0} \rightarrow \mathbf{Diff}_{k-1}(\pi, \pi) \rightarrow \mathbf{Diff}_k(\pi, \pi) \xrightarrow{\text{smb}l_k} \text{End}(\pi) \otimes \Sigma_k(M) \rightarrow \mathbf{0},$$

where $\text{End}(\pi) = \mathbf{Diff}_0(\pi, \pi)$ is the module of endomorphisms of $C^\infty(\pi)$, and mapping $\text{smb}l$ assign to differential operator Δ its symbol $\text{smb}l(\Delta)$.

If we consider $\text{smb}l_k(\Delta)$ as a linear operator $\text{smb}l_k(\Delta) : \Sigma^k(M) \rightarrow \text{End}(\pi)$, then

$$\text{smb}l_k(\Delta)(df^k) = \frac{1}{k!} \delta_f^k(\Delta) \in \text{End}(\pi),$$

where $f \in C^\infty(M)$, $\delta_f : \mathbf{Diff}_i(\pi, \pi) \rightarrow \mathbf{Diff}_{i-1}(\pi, \pi)$, is the commutator $\delta_f(\square) = f \circ \square - \square \circ f$, and $df^k = df \cdots df \in \Sigma^k(M)$ is the k -th symmetric power of the differential 1-form $df \in \Sigma^1(M)$.

For the case $k = 1$, which we'll consider in this paper, we respectively have:

1. Exact sequences

$$\mathbf{0} \rightarrow \text{End}(\pi) \rightarrow \mathbf{Diff}_1(\pi, \pi) \xrightarrow{\text{smb}l_1} \text{End}(\pi) \otimes \Sigma_1(M) \rightarrow \mathbf{0}, \quad (1)$$

and $\text{smb}l_1(\Delta)(df) = \delta_f(\Delta) \in \text{End}(\pi)$.

2. Any connection ∇ in the vector bundle π gives us *quantization*

$$\begin{aligned} \mathbf{Q}_\nabla & : \text{End}(\pi) \otimes \Sigma_1(M) \rightarrow \mathbf{Diff}_1(\pi, \pi), \\ \text{smb}l_1 \circ \mathbf{Q}_\nabla & = \text{id}, \end{aligned} \quad (2)$$

where

$$\mathbf{Q}_\nabla(A \otimes X) = A \circ \nabla_X, \quad (3)$$

for all $A \in \text{End}(\pi)$, $X \in \Sigma_1$.

Moreover, if $\tilde{\nabla}$ is another connection and

$$\tilde{\nabla} = \nabla + \alpha,$$

where $\alpha \in \text{End}(\pi) \otimes \Sigma^1(M)$, $\tilde{\nabla}_X = \nabla_X + \alpha(X)$, then

$$\mathbf{Q}_{\tilde{\nabla}} = \mathbf{Q}_\nabla + \langle \cdot, \alpha \rangle,$$

where $\langle \cdot, \cdot \rangle : (\text{End}(\pi) \otimes \Sigma_1(M)) \otimes (\text{End}(\pi) \otimes \Sigma^1(M)) \rightarrow \text{End}(\pi)$ is the natural pairing: $\langle A \otimes X, B \otimes \omega \rangle = \langle \omega, X \rangle A \circ B$.

In what follows, we denote by $\sigma_1(\Delta)$ and $\sigma_0^\nabla(\Delta)$ (or simply by σ_1 and σ_0) the symbol $\sigma_1(\Delta) = \text{smb}l_1(\Delta)$ and *subsymbols* $\sigma_0^\nabla(\Delta) = \Delta - \mathbf{Q}_\nabla(\Delta) \in \text{End}(\pi)$.

Then symbol σ_1 does not depend on connection but

$$\sigma_0^{\tilde{\nabla}}(\Delta) = \sigma_0^\nabla(\Delta) - \langle \sigma_1(\Delta), \alpha \rangle. \quad (4)$$

Remark 1. The quantizations introduced above are special splitting $R : \text{End}(\pi) \otimes \Sigma_1(M) \rightarrow \mathbf{Diff}_1(\pi, \pi)$ of exact sequence (1) that satisfy the following condition:

$$R(A \cdot B \otimes X) = A \cdot R(B \otimes X),$$

for all operators $A, B \in \text{End}(\pi)$ and vector fields $X \in \Sigma_1(M)$. Namely, $\nabla_X = R(\text{id} \otimes X)$.

3. Pseudogroup actions

We consider two pseudogroups: $\mathcal{G}(M)$ – the pseudogroup of local diffeomorphisms of manifold M , and $\mathbf{Aut}(\xi)$ – the pseudogroups of local automorphisms of vector bundle π over M .

There is the following sequence of pseudogroup morphisms

$$1 \rightarrow \mathbf{GL}(\pi) \rightarrow \mathbf{Aut}(\pi) \rightarrow \mathcal{G}(M) \rightarrow 1, \quad (5)$$

where $\mathbf{GL}(\pi) \subset \mathbf{Aut}(\pi)$ is the pseudogroup of automorphisms that are identity on M .

We'll consider the natural actions of these pseudogroups on sections of the bundles and operators. Namely, let $\tilde{\phi}$ be a local automorphism, $\tilde{\phi} \in \mathbf{Aut}(\pi)$, covering a local diffeomorphism $\phi \in \mathcal{G}(M)$.

Then we define action of $\tilde{\phi}$ on sections $s \in C^\infty(\xi)$ as

$$\tilde{\phi}_* : s \mapsto \tilde{\phi} \circ s \circ \phi^{-1},$$

and

$$\tilde{\phi}_* : \Delta \mapsto \tilde{\phi}_* \circ \Delta \circ \tilde{\phi}_*^{-1},$$

for differential operators.

4. Artin-Procesi invariants of symbols

Let's fix a point $a \in M$, and let $E = \pi_a$ be the fibre of the bundle π at the point a , and let $T = T_a(M)$ be the tangent space at the point, $\dim T = n \geq 2$, $\dim E = m$.

In this section we consider symbols of the 1st order differential operators at the point as tensors of the type $\text{End}(E) \otimes T$ and classify them with respect to the natural action of the group $G = \mathbf{GL}(E) \times \mathbf{GL}(T)$.

Remind, that M. Artin made the conjecture ([6]), concerning polynomial invariants of $\mathbf{GL}(E)$ -action on n -tuples (X_1, \dots, X_n) of linear operators, $X_i \in \text{End}(E)$. Namely, he stated that the algebra of polynomial invariants (we denote it by $\mathcal{A}(X_1, \dots, X_n)$) is generated by the polynomials $\text{Tr}(X_{i_1} \cdots X_{i_k})$, associated with non-commutative monomials $X_{i_1} \cdots X_{i_k}$. This conjecture was proven by C. Procesi in ([4]) with the important addition that the length of the monomials could be bounded by $2^m - 1$ as well as a description of syzygies.

We'll apply this result to our case. Namely, we denote by $\sigma_\theta \in \text{End}(E)$ the value of symbol σ at covector $\theta \in T^*$ and consider linear functions $A_{\sigma,I} : (T^*)^{\otimes k} \rightarrow \mathbb{R}$, where $I = (i_1, \dots, i_k) \in S_k$ is a permutation of k -letters, as follows

$$A_{\sigma,I}(\theta_1 \otimes \dots \otimes \theta_k) = \text{Tr}(\sigma_{\theta_{i_1}} \dots \sigma_{\theta_{i_k}}).$$

These tensors $A_{\sigma,I} \in ((T^*)^{\otimes k})^* = T^{\otimes k}$ are obviously G -invariants and we call them *Artin-Procesi tensors*.

Remark also that $A_{\sigma,I} = A_{\sigma,J}$, when permutation $J \in S_k$ is obtained from I by a cycle permutation.

The following first Artin-Procesi tensors are extremely important for us:

- In the case $k = 1$, we have a vector $\chi_\sigma \in T$, where

$$\langle \theta, \chi_\sigma \rangle = \text{Tr}(\sigma_\theta),$$

for all covectors $\theta \in T^*$.

- In the case $k = 2$, we have a symmetric 2-vector $g_\sigma \in S^2T$, where

$$g_\sigma(\theta_1, \theta_2) = \text{Tr}(\sigma_{\theta_1} \sigma_{\theta_2}).$$

- In the case $k = 3$, we have two tensors $h_i \in T^{\otimes 3}$, where

$$\begin{aligned} h_{1,\sigma}(\theta_1, \theta_2, \theta_3) &= \text{Tr}(\sigma_{\theta_1} \sigma_{\theta_2} \sigma_{\theta_3}), \\ h_{2,\sigma}(\theta_1, \theta_2, \theta_3) &= \text{Tr}(\sigma_{\theta_2} \sigma_{\theta_1} \sigma_{\theta_3}), \end{aligned}$$

and equivalently two tensors

$$\begin{aligned} h_{\sigma,s} &= \frac{1}{2}(h_{\sigma,1} + h_{\sigma,2}) \in S^2T \otimes T, \\ h_{\sigma,a} &= \frac{1}{2}(h_{\sigma,1} - h_{\sigma,2}) \in \Lambda^2T \otimes T. \end{aligned}$$

All these tensors are G -invariants.

In what follows we'll use only *symbols in general position*, or shortly *general symbols*, i.e symbols where the following regularity conditions hold.

1. Vector χ_σ is non trivial, $\chi_\sigma \neq 0$.
2. Quadratic form $g_\sigma \in S^2T$ on T^* is non degenerated.
The inverse quadratic form on T we denote by $g_\sigma^{-1} \in S^2T^*$, and the covector dual to the vector χ_σ with respect to the last quadratic form will be denoted as $\widehat{\chi}_\sigma \in T^*$, i.e. $\widehat{\chi}_\sigma \rfloor g_\sigma = \chi_\sigma$, or $\chi_\sigma \rfloor g_\sigma^{-1} = \widehat{\chi}_\sigma$.
3. Denote also by $\widetilde{h}_\sigma \in S^2T$ the following symmetric bivector

$$\widetilde{h}_\sigma(\theta_1, \theta_2) = h_{\sigma,s}(\theta_1, \theta_2, \widehat{\chi}_\sigma)$$

and let $\widehat{h}_\sigma : T^* \rightarrow T^*$ be the operator, associated with the pair quadratic forms g_σ and \widetilde{g}_σ on T^* .

We'll require that covectors

$$e_1^* = \widehat{\chi}_\sigma, e_2^* = \widehat{h}_\sigma e_1^*, \dots, e_n^* = \widehat{h}_\sigma e_{n-1}^*,$$

are linear independent.

Then covectors $e^*(\sigma) = \langle e_1^*, \dots, e_n^* \rangle$ form a coframe in T^* and vectors $e(\sigma) = \langle e_1, \dots, e_n \rangle$ of the dual basis give us the frame in T . Remark that both frames are G -invariant.

4. In the similar way each covector $\theta \in T^* \setminus 0$ defines also bivector

$$\zeta_{\theta, \sigma}(\theta_1, \theta_2) = h_{\sigma, a}(\theta_1, \theta_2, \theta) \in \Lambda^2 T.$$

Denote by $\Sigma_\sigma \subset T^*$ the variety of covectors $\theta \in T^*$ where operator σ_θ has eigenvalues of multiplicity greater the one. Then the last condition that σ is in general position requires that $\Sigma_\sigma \neq T^*$ and $\zeta_{\theta, \sigma}$ does not vanish on the complement of Σ_σ .

Definition 1. A symbol tensor $\sigma \in \text{End}(E) \otimes T$, when $m \geq 3$, is said to be in general position if the above conditions hold. In the case $m = 2$ general position requires the first two conditions and that $\langle \widehat{\chi}_\sigma, \chi_\sigma \rangle \neq 0, \Sigma_\sigma \neq T^*$.

Remark 2. In the case $m = 2$ the G -invariant frame consist of vectors $e_1 = \chi_\sigma$ and the vector e_2 orthonormal to e_1 with respect to metric g_σ^{-1} .

Let σ be a general symbol then, using the invariant frame $e(\sigma)$, we represent σ in the form

$$\sigma = \sum_{i=1}^n \sigma_i \otimes e_i,$$

where $\sigma_i \in \text{End}(E)$.

Remark that G -invariance of the frame reduce G -equivalence of symbols to $\mathbf{GL}(E)$ -equivalence of n -tuples of operators $(\sigma_1, \dots, \sigma_n)$.

Proposition 1. *Let σ be a general symbol. Then $n \leq m^2$, and $\alpha \in \text{End}(E)$ is a scalar operator if and only if commutators $[\sigma_i, \alpha] = 0$, for all $i = 1, \dots, n$.*

PROOF. Assume that operators σ_i are linear dependent, i.e. $\sum_{i=1}^n \lambda_i \sigma_i = 0$, for some $\lambda_i \in \mathbb{R}$ such that $\sum_{i=1}^n \lambda_i^2 \neq 0$. Then $\sigma_\theta = 0$, for $\theta = \sum_{i=1}^n \lambda_i e_i^* \neq 0$, and $g_\sigma(\theta, \theta') = 0$, for all $\theta' \in T^*$. Therefore, $\theta \in \ker g_\sigma$ and g_σ is degenerated.

Let's now $[\sigma_i, \alpha] = 0$, for all $i = 1, \dots, n$. The 3rd condition of generality σ states that there are covectors $\theta_1, \theta_2 \notin \Sigma_\sigma$, such that $[\sigma_{\theta_1}, \sigma_{\theta_2}] \neq 0$. Condition $[\sigma_i, \alpha] = 0$, for all i , gives $[\sigma_{\theta_1}, \alpha] = [\sigma_{\theta_2}, \alpha] = 0$. Therefore operator α has the same eigenvectors with operators σ_{θ_1} and σ_{θ_2} , that possible only when α is a scalar operator.

Corollary 2. *The stationary algebra Lie of a general symbol consist of scalar operators and codimension of G -orbit of such symbols equals to*

$$\nu = (n - 1) (m^2 - n - 1).$$

Take now Artin-Procesi tensors $A_{\sigma, I} \in T^{\otimes k}$, $k = |I|$, for general symbol σ , and write down them in the invariant frame $e(\sigma)$:

$$A_{\sigma, I} = \sum_J A_I^J(\sigma) e^J,$$

where $J = (j_1, \dots, j_k)$ and $e^J = e_{j_1} \otimes \dots \otimes e_{j_k}$.

Then coefficients $A_I^J(\sigma)$ are rational functions on $\text{End}(E) \otimes T$ and G -invariants. We call them *Artin-Procesi invariants* of the symbol.

We have the algebraic G -action on $\text{End}(E) \otimes T$. Therefore, due to Rosenlicht theorem ([7]), there are ν Artin-Procesi invariants a_1, \dots, a_ν which are algebraically independent and all rational G -invariants (in particular all Artin-Procesi invariants) are rational functions of them.

In this case we call a_1, \dots, a_ν *basic Artin-Procesi invariants*.

G -orbits $G(\sigma) \subset \{a_1 = \text{const}_1, \dots, a_\nu = \text{const}_\nu\}$ such that differentials da_1, \dots, da_ν are linear independent at the points of the orbit we call *regular* as well as the symbol σ itself.

Remark that G -orbits are connected and therefore the manifold $\{a_1 = \text{const}_1, \dots, a_\nu = \text{const}_\nu\}$ in the last case is a finite union of regular orbits.

Example 3. *The following invariants might be used as basic:*

1. For $n = 2$

$$\begin{aligned} I_i^{(1)} &= \text{Tr} \sigma_1^i, I_i^{(2)} = \text{Tr} \sigma_2^i, i = 1, \dots, m, \\ I_{i,j} &= \text{Tr} \left(\sigma_1^i \sigma_2^j \right), 1 \leq i, j \leq m - 1. \end{aligned}$$

2. For $n \geq 3$

$$\begin{aligned} I_i^{(k)} &= \text{Tr} \sigma_k^i, k = 1, \dots, n, i = 1, \dots, m, \\ I_{i,j}^{(k)} &= \text{Tr} \left(\sigma_1^i \sigma_k^j \right), 1 \leq i, j \leq m - 1, \\ J_j^{(k)} &= \text{Tr} \left(\sigma_1 \sigma_2 \sigma_k^j \right), k = 3, \dots, n, j = 1, \dots, m - 1. \end{aligned}$$

Summarizing we get the following.

Let $\sigma \in \text{End}(E) \otimes T$ be a general symbol. Then

1. Tensors

$$A_{\sigma, I} = \text{Tr} \left(\sigma_{\theta_{i_1}} \dots \sigma_{\theta_{i_k}} \right) \in T^{\otimes |I|},$$

are $\mathbf{GL}(E) \times \mathbf{GL}(T)$ -invariants.

2. The conditions that σ is in general position are conditions on tensors $A_{\sigma,I}$, with $|I| \leq 3$. Under these conditions the symbol σ determines $\mathbf{GL}(E) \times \mathbf{GL}(T)$ -invariant coframe $e(\sigma)$ in the tangent space T .
3. Coefficients $A_{\sigma,I}^J$ of the tensors $A_{\sigma,I}$ in the frame $e(\sigma)$ are $\mathbf{GL}(E) \times \mathbf{GL}(T)$ -invariants of symbols. These invariants are polynomials over the field $\mathbb{R}(\det g_\sigma^{-1})$.
4. The maximum dimensional $\mathbf{GL}(E) \times \mathbf{GL}(T)$ -orbits in $\text{End}(E) \otimes T$ have codimension $\nu = (n-1)(m^2 - n - 1)$. The $\mathbf{GL}(E) \times \mathbf{GL}(T)$ -action is algebraic and, due to the Rosenlicht theorem ([7]) the field of rational $\mathbf{GL}(E) \times \mathbf{GL}(T)$ -invariants has transcendent degree ν . We call $\mathbf{GL}(E) \times \mathbf{GL}(T)$ -orbit (and its elements) *regular* if there are *basic* invariants a_1, \dots, a_ν such that their differentials are linear independent at the points of the orbit.

5. Connections, associated with regular symbols

Let $\sigma = \sigma_1(\Delta)$ be a regular symbol and let ∇_c be the Levi-Civita connection in the tangent bundle $\tau(M)$, associated with the metric $g = g_\sigma$.

Let ∇ be a connection in the bundle π . We'll use the same notation ∇ for the associated connection in the bundle $\text{End}(\pi)$, where $\nabla_X(A) = \nabla_X \circ A - A \circ \nabla_X$, for all $X \in \Sigma_1(M)$, $A \in \text{End}(\pi)$.

Let $\nabla^c = \nabla \otimes \nabla_c$, be the connection in the bundle $\text{End}(\pi) \otimes \Sigma_1(M)$ equals to the tensor product of the connection ∇ in the bundle $\text{End}(\pi)$ and the Levi-Civita connection in the tangent bundle $\tau(M)$.

The covariant differential $d_{\nabla^c} : \text{End}(\pi) \otimes \Sigma_1(M) \rightarrow \text{End}(\pi) \otimes \Sigma_1(M) \otimes \Omega^1(M) = \text{End}(\pi) \otimes \text{End}(\tau)$, applying to the symbol σ , gives us a tensor

$$d_{\nabla^c} \sigma \in \text{End}(\pi) \otimes \text{End}(\tau).$$

Remark that for another connection $\tilde{\nabla}$ in the bundle π such that

$$\tilde{\nabla}_X - \nabla_X = \alpha(X) \in \text{End}(\pi),$$

where $\alpha \in \text{End}(\pi) \otimes \Omega^1(M)$, we have

$$d_{\tilde{\nabla}^c} \sigma - d_{\nabla^c} \sigma = [\alpha, \sigma],$$

where by $[\alpha, \sigma] \in \text{End}(\pi) \otimes \text{End}(\tau)$ we denoted the natural pairing

$$\begin{aligned} [,] : \text{End}(\pi) \otimes \Omega^1(M) \times \text{End}(\pi) \otimes \Sigma_1(M) &\rightarrow \text{End}(\pi) \otimes \text{End}(\tau) \quad , \\ [A \otimes \theta, B \otimes X] &= [A, B] \otimes (\theta \otimes X) \in \text{End}(\pi) \otimes \text{End}(\tau) \quad . \end{aligned}$$

Indeed, let $\sigma = \Sigma_i A_i \otimes X_i$, where $A_i \in \text{End}(\pi)$, and $X_i \in \Sigma_1(M)$. Then

$$\nabla_X^c(\sigma) = \Sigma_i (\nabla_X(A_i) \otimes X_i + A_i \otimes \nabla_{cX}(X_i))$$

and

$$\begin{aligned} \tilde{\nabla}_X^c(\sigma) &= \Sigma_i \left(\tilde{\nabla}_X^c(A_i) \otimes X_i + A_i \otimes \nabla_{cX}(X_i) \right) = \\ &= \nabla_X^c(\sigma) + \Sigma_i [\alpha(X), A_i] \otimes X_i = \nabla_X^c(\sigma) + [\alpha, \sigma](X). \end{aligned}$$

Proposition 4. *Operator $D(\sigma) = (\text{Tr} \otimes \text{id})(d_{\nabla^c} \sigma) \in \text{End}(\tau)$ does not depend on choice of connection ∇ .*

Let now σ be a general symbol and let $\sigma = \sum_i \sigma_i \otimes e_i$ be its decomposition in the invariant frame. Remark that $d_{\tilde{\nabla}^c} \sigma = d_{\nabla^c} \sigma$ if and only if $[\alpha, \sigma] = 0$ or if $[\alpha_i, \sigma_j] = 0$, for all i, j , where $\alpha = \sum_j \alpha_j \otimes e_j^*$. Therefore, each operator α_i is a scalar operator $\alpha_i = \lambda_i \text{id}$. In other words,

$$d_{\tilde{\nabla}^c} \sigma = d_{\nabla^c} \sigma \Leftrightarrow \tilde{\nabla} - \nabla = \text{id} \otimes \lambda,$$

where $\lambda \in \Omega^1(M)$ is a differential 1-form such that $\langle \lambda, e_i \rangle = \lambda_i$.

Denote by $W(\sigma) \subset \text{End}(\pi) \otimes \text{End}(\tau)$ the affine subbundle generated by tensors $d_{\nabla^c} \sigma$, computing for all connections ∇^c . The fibres of this bundle isomorphic to $\text{sl}(\pi) \otimes \text{End}(\tau)$, where

$$\text{sl}(\pi) = \{A \in \text{End}(\pi) \mid \text{Tr} A = 0\}.$$

Let's consider $\text{End}(\pi) \otimes \text{End}(\tau)$ as Euclidean bundle with respect to inner product $(A \otimes a, B \otimes b) = \text{Tr}(AB) \cdot \text{Tr}(ab)$, where $A, B \in \text{End}(\pi)$ and $a, b \in \text{End}(\tau)$.

A connection ∇ we call *minimal* if tensor $d_{\nabla^c} \sigma$ orthogonal to $W(\sigma)$ with respect to the above inner product.

Proposition 5. *Let σ be a symbol in general position. Then, for any connection ∇ in the bundle π , there is and unique tensor $\alpha \in \text{sl}(\pi) \otimes \Omega^1(M)$, such that*

$$(d_{\nabla^c} \sigma + [\alpha, \sigma], \text{sl}(\pi) \otimes \text{End}(\tau)) = 0. \quad (6)$$

PROOF. Let $E_{ij} = e_i \otimes e_j^*$ be the elementary operators in the invariant basis in T and let

$$d_{\nabla^c} \sigma - \text{id} \otimes D(\sigma) = \sum_{i,j} d_{ij} \otimes E_{ij},$$

for some $d_{ij} \in \text{sl}(\pi)$, and

$$\sigma = \sum_i \sigma_i \otimes e_i, \quad \alpha = \sum_i \alpha_j \otimes e_j^*,$$

where $\alpha_j \in \text{sl}(\pi)$.

Then (6) equivalent to the following linear system of $n(m^2 - 1)$ equations

$$d_{ij} - [\alpha_j, \sigma_i] = 0, \quad (7)$$

with respect to $n(m^2 - 1)$ unknowns α_j .

As we have seen above the stationary Lie algebra of the symbol consists of scalar operators. It means that the homogeneous system of (7) has the only zero solution.

The above theorem states the existence of minimal connections and, as we have seen, any two minimal connections differ on tensors of the form $\alpha = \text{id} \otimes \lambda$.

Let $R_{\nabla}(\sigma) \in \text{End}(\pi) \otimes \Omega^2(M)$ be the curvature tensor of minimal connection ∇ . Then (see, for example, [8]) we have

$$R_{\tilde{\nabla}}(\sigma) - R_{\nabla}(\sigma) = \text{id} \otimes d\lambda,$$

if $\tilde{\nabla} - \nabla = \text{id} \otimes \lambda$.

Therefore, the tensor

$$R^0(\sigma) = R_{\nabla}(\sigma) - \frac{1}{m} \text{id} \otimes \text{ch}(\nabla),$$

where $\text{ch}(\nabla) = \text{Tr}(R_{\nabla}(\sigma)) \in \Omega^2(M)$ is the first *Chern form*, does not depend on choice of minimal connection ∇ .

This tensor is also G -invariant of the symbol.

On the other hand, we have (4)

$$\sigma_0(\tilde{\nabla}) - \sigma_0(\nabla) = \sigma_{-\lambda},$$

for given operator $\Delta \in \mathbf{Diff}_1(\pi, \pi)$ and minimal connections ∇ and $\tilde{\nabla}$.

Therefore, subbundle $W_0(\Delta) \subset \text{End}(\pi)$, generated by operators $\sigma_0(\nabla)$, taking for all minimal connections ∇ , is an affine bundle isomorphic to τ^* by the injection $\lambda \in \Omega^1 \rightarrow \sigma_{\lambda} \in \text{End}(\pi)$.

We say that a minimal connection ∇ is *associated with operator* Δ if the operator $\sigma_0(\nabla)$ is orthogonal to $W_0(\Delta)$.

To find this connection we should find differential form λ such that

$$\text{Tr}((\sigma_0(\nabla) - \sigma_{\lambda})\sigma_{\theta}) = 0,$$

for all $\theta \in \Omega^1(M)$.

Remark that the last system has a unique solution because quadratic form $g_{\sigma}(\lambda, \theta) = \text{Tr}(\sigma_{\lambda}\sigma_{\theta})$ is non degenerated.

Finally we get the following result.

Theorem 6. *Let $\Delta \in \mathbf{Diff}_1(\pi, \pi)$ be a differential operator with general symbol $\sigma \in \text{End}(\pi) \otimes \Sigma_1$. Then there exists and unique associated connection ∇^{Δ} in the bundle π which is minimal and the operator $\sigma_0(\nabla^{\Delta}) \in \text{End}(\pi)$ in the decomposition*

$$\Delta = Q_{\nabla^{\Delta}}(\sigma) + \sigma_0(\nabla^{\Delta})$$

satisfies the following conditions:

$$\text{Tr}(\sigma_0(\nabla^{\Delta}) \cdot \sigma_{\theta}) = 0,$$

for all differential 1-forms $\theta \in \Omega^1(M)$.

Corollary 7. *Let $A \in \text{Aut}(\pi)$ be an automorphism, transforming operator $\Delta \in \mathbf{Diff}_1(\pi, \pi)$ to operator $\Delta' \in \mathbf{Diff}_1(\pi, \pi)$, $A_*(\Delta) = \Delta'$, and let $\Delta = Q_{\nabla^\Delta}(\sigma) + \sigma_0(\nabla^\Delta)$ and $\Delta' = Q_{\nabla^{\Delta'}}(\sigma') + \sigma_0(\nabla^{\Delta'})$ be the decompositions with respect to the associated connections.*

Then $A_(\sigma) = \sigma'$, $A_*(\nabla^\Delta) = \nabla^{\Delta'}$, and $A_*(\sigma_0(\nabla^\Delta)) = \sigma_0(\nabla^{\Delta'})$.*

PROOF. The statement follows directly from the fact that any automorphism $A \in \text{GL}(\pi)$ preserves the inner product structure in $\text{End}(\pi)$.

This corollary shows that the decomposition with respect to associated connection behave in the natural way under transformations from the pseudogroup $\text{Aut}(\pi)$. It allow us to extend Artin invariants of symbols to *Artin-Procesi invariants of operators*.

Namely, let $\Delta \in \mathbf{Diff}_1(\pi, \pi)$ be a differential operator with regular symbol at a point $a \in M$. Then the decomposition of the symbol

$$\sigma = \sum_i \sigma_i \otimes e_i$$

in the invariant frame $e(\sigma)$ and operator σ_0 gives us the Artin-Procesi invariants of operators as elements of algebra $\mathcal{A}(\sigma_0, \sigma_1)$.

Moreover, the curvature tensor

$$R_{\nabla^\Delta}(\sigma) \in \text{End}(\pi) \otimes \Omega^2(M),$$

computing for the associated connection ∇^Δ in the bundle π is an $\mathbf{Aut}(\pi)$ -invariant of the operators as well as the closed Chern differential 2-form

$$\text{ch}(\Delta) = \text{Tr}(R_{\nabla^\Delta}(\sigma)) \in \Omega^2(M),$$

which we call *Chern invariant of the operator*.

Coefficients of the form in the invariant frame

$$\text{ch}_{ij}(\Delta) = \text{ch}(\Delta)(e_i, e_j)$$

are scalar $\mathbf{Aut}(\pi)$ -invariants of the the operators.

6. Equivalence of regular differential operators

A differential operator $\Delta \in \mathbf{Diff}_1(\pi, \pi)$ is said to be *regular* at a point $a \in M$ if its symbol $\sigma \in \text{End}(\pi) \otimes \Sigma_1(M)$ is regular at the point and among Artin-Procesi invariants a_1, \dots, a_{ν_0} of the operator, defining the $\text{GL}(\pi) \times \text{GL}(T)$ -orbit of the pairs (σ_0, σ) , there are $n = \dim M$ invariants, say a_1, \dots, a_n , such that differentials of functions $a_1(\Delta), \dots, a_n(\Delta)$ are linear independent at the point $a \in M$.

In this case there is a neighborhood $a \in U \subset M$ such that $\text{GL}(\pi_b) \times \text{GL}(T_b)$ -orbits of the pairs $(\sigma_0(b), \sigma(b)) \in \text{End}(\pi_b) \oplus \text{End}(\pi_b) \otimes T_b$ are regular and are defined by values the same basic invariants a_1, \dots, a_{ν_0} , and furthermore functions $a_1(\Delta), \dots, a_n(\Delta)$ are local coordinates in U .

We call such local coordinates *natural*.

Remark 3. The regularity of symbol requires that $m^2 \geq n+1$, and the number ν_0 of basic Artin-Procesi invariants of operators equals

$$\nu_0 = \nu + m^2 > n.$$

Let's a_1, \dots, a_n be the natural coordinates then all basic Artin-Procesi invariants $a_j(\Delta)$, $1 \leq j \leq \nu_0$, are functions of $a_1(\Delta), \dots, a_n(\Delta)$ in U :

$$a_j(\Delta) = F_j(a_1(\Delta), \dots, a_n(\Delta)), \quad n+1 \leq j \leq \nu_0, \quad (8)$$

for some functions F_j .

All rational G -invariants of the pairs (σ_0, σ) are rational functions of basic Artin-Procesi invariants (a_1, \dots, a_{ν_0}) and, due to (8), their values in the neighborhood U are also functions of $(a_1(\Delta), \dots, a_n(\Delta))$ completely defined by functions F_j .

Let's $\phi_U : U \rightarrow \mathbf{D}_U \subset \mathbb{R}^n$ be the natural local chart,

$$\phi_U(b) = (a_1(\Delta)(b), \dots, a_n(\Delta)(b)),$$

and let $F_U : \mathbf{D}_U \rightarrow \mathbb{R}^{\nu_0-n}$ be the function given by (8), and

$$\text{ch}_U(\Delta) = \phi_{U*}(\text{ch}(\Delta)) \in \Omega^2(\mathbf{D}_U).$$

The data $(\phi_U, \mathbf{D}_U, F_U, \text{ch}_U(\Delta))$ we call *model of the differential operator* in neighborhood U .

Theorem 8. *Let differential operators Δ and Δ' has the same model in a simply connected open set U .*

Then there is and a unique automorphism $A_U \in \text{GL}(\pi_U)$ of the restriction bundle π on domain U such that $A_{U}(\Delta) = \Delta'$.*

PROOF. Condition that two operators have the same model means that the pairs (σ_0, σ) and (σ'_0, σ') belong to the same G -orbit at any point $a \in U$. Therefore, there are the above automorphisms A_U . Moreover, if ∇ is the connection, associated with operator Δ , then $\nabla' = A_{U*}(\nabla)$ is a minimal connection for Δ' because A_{U*} preserves the inner structure in $\text{End}(\pi) \otimes \text{End}(T)$ and transforms σ to σ' .

Let

$$\nabla^{\Delta'} = \nabla' + \text{id} \otimes \lambda$$

for some differential 1-form λ .

Then, $\text{ch}(\Delta') = \text{ch}(\nabla') + d\lambda = \text{ch}(\Delta) + d\lambda$, and $d\lambda = 0$. Therefore, $\lambda = df$ in U .

Finally, $(B_U)(\nabla) = \nabla^{\Delta'}$, where $B_U = e^{-f} A_U$, and therefore $B_{U*}(\Delta) = \Delta'$.

Corollary 9. *Let differential operators Δ and Δ' has models $(\phi_U, \mathbf{D}_U, F_U, \text{ch}_U(\Delta))$ and $(\phi_{U'}, \mathbf{D}_{U'}, F_{U'}, \text{ch}_{U'}(\Delta'))$ in open sets $U \subset M$ and $U' \subset M$ defined by the same basic invariants a_1, \dots, a_n and functions F_j .*

Let $\tilde{U} \subset U$ and $\tilde{U}' \subset U'$ be open and simply connected domains such that $\phi_U(\tilde{U}) = \phi_{U'}(\tilde{U}') \subset \mathbf{D}_U \cap \mathbf{D}_{U'}$, and where $\text{ch}_U(\Delta) = \text{ch}_{U'}(\Delta')$.

Then there is an automorphism $A_{\tilde{U}, \tilde{U}'} : \pi_{\tilde{U}} \rightarrow \pi_{\tilde{U}'}$ such that $(A_{\tilde{U}, \tilde{U}'})_ \Delta = \Delta'$.*

Let now Δ be a differential operator regular on the manifold M , i.e. regular at all points of the manifold. We say that an atlas $\{(\phi_{U^\alpha}, \mathbf{D}_{U^\alpha})\}$ given by models $\{(\phi_{U^\alpha}, \mathbf{D}_{U^\alpha}, F_{U^\alpha}, \text{ch}_{U^\alpha}(\Delta))\}$ is *natural* if the sets of basic invariants $(a_1^\alpha, \dots, a_n^\alpha)$ are different for different α .

The following result follows directly from the above theorem.

Theorem 10. *Two linear differential operators $\Delta, \Delta' \in \mathbf{Diff}_1(\pi, \pi)$ on a manifold M are $\text{Aut}(\pi)$ -equivalent if and only if a natural atlas for operator Δ is the natural atlas for Δ' , i.e. they have the same models.*

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