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Research Article

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Convergence of T means with respect to Vilenkin systems of integrable functions

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Abstract: In this paper, we derive the convergence of T means of Vilenkin–Fourier series with monotone coefficients of integrable functions in Lebesgue and Vilenkin–Lebesgue points. Moreover, we discuss the pointwise and norm convergence in L_p norms of such T means.

Keywords: Vilenkin systems, Vilenkin groups, T means, Nörlund means, a.e. convergence, Lebesgue points, Vilenkin–Lebesgue points

MSC 2010: 42C10, 42B25

1 Introduction

The definitions and notations used in this introduction can be found in the next section.

It is well known (see, e.g., the book [33]) that there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f\|_p \leq c_p \|f\|_p \quad \text{when } p > 1.$$

On the other hand, the boundedness does not hold for $p = 1$ (for details, see [7, 8, 29, 42–44]). The analogue of Carleson's theorem for the Walsh system was proved by Billard [3] for $p = 2$ and by Sjölin [36] for $1 < p < \infty$, and for the bounded Vilenkin systems by Gosselin [15]. For the Walsh–Fourier series, Schipp [31–33] gave a proof by using the methods of martingale theory. A similar proof for the Vilenkin–Fourier series can be found in [34] by Schipp and Weisz and in [47] by Weisz. In each proof, they show that the maximal operator of the partial sums is bounded on L_p, i.e. there exists an absolute constant c_p such that

$$\|S^* f\|_p \leq c_p \|f\|_p \quad \text{when } f \in L_p,\ p > 1.$$

Hence, if $f \in L_p(G_m)$, where $p > 1$, then $S_n f \to f$ a.e. on G_m. Stein [37] constructed the integrable function whose Vilenkin–Fourier (Walsh–Fourier) series diverges almost everywhere. In [33], it was proved that there exists an integrable function whose Walsh–Fourier series diverges everywhere. The a.e. convergence of subsequences of Vilenkin–Fourier series was considered in [5], where the methods of martingale Hardy spaces were used.

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If we consider the restricted maximal operator $\tilde{S}_{\#}^{*}f := \sup_{n\in\mathbb{N}}|S_{M_n}f|$, we have a weak $(1,1)$ type inequality for $f \in L_1(G_m)$. Hence, if $f \in L_1(G_m)$, then $S_{M_n}f \to f$ a.e. on G_m. Moreover, for any integrable function, it is known that an a.e. point is a Lebesgue point, and for any such point x of the integrable function f, we have that

$$S_{M_n}f(x) \to f(x) \quad \text{as } n \to \infty, \quad \text{for any Lebesgue point } x \text{ of } f \in L_1(G_m). \tag{1.1}$$

In the one-dimensional case, Yano [49] proved that

$$\|\sigma_n f - f\|_p \to 0 \quad \text{as } n \to \infty \quad (f \in L_p(G_m),\ 1 \leq p \leq \infty).$$

If we consider the maximal operator of the Féjer means

$$\sigma^* f := \sup_{n\in\mathbb{N}}|\sigma_n f|,$$

then

$$\lambda\mu\{\sigma^* f > \lambda\} \leq c\|f\|_1, \quad f \in L_1(G_m),\ \lambda > 0.$$

This result can be found in [50] by Zygmund for the trigonometric series, in [35] by Schipp and in [12, 26, 27, 38, 39, 41] for Walsh series and in [25] by Pál and Simon for bounded Vilenkin series (see also [47, 48] by Weisz). The boundedness does not hold from the Lebesgue space $L_1(G_m)$ to the space $L_1(G_m)$. The weak-$(1,1)$ type inequality implies that, for any $f \in L_1(G_m)$,

$$\sigma_n f(x) \to f(x) \quad \text{a.e.} \quad \text{as } n \to \infty.$$

Moreover, in [11] (see also [10]), it was proved that, for any integrable function, an a.e. point is the Vilenkin–Lebesgue point, and for any such point x of an integrable function f, we have

$$\sigma_n f(x) \to f(x) \quad \text{as } n \to \infty.$$

Móricz and Siddiqi [18] investigate the approximation properties of some special Nörlund means of the Walsh–Fourier series of L_p functions in norm. Similar results for the two-dimensional case can be found in [19, 20] by Nagy, [21–24] by Nagy and Tephnadze, [13, 14] by Gogolashvili and Tephnadze (see also [2, 17]). The approximation properties of general summability methods can also be found in [4, 6]. Fridli, Manchanda and Siddiqi [9] improved and extended the results of Móricz and Siddiqi [18] to martingale Hardy spaces. The a.e. convergence of Nörlund means of Vilenkin–Fourier series with monotone coefficients of $f \in L_1$ was proved in [28] (see also [30]). In [45], it was proved that the maximal operators of T means T^* defined by $T^*f := \sup_{n\in\mathbb{N}}|T_n f|$ either with non-increasing coefficients, or with a non-decreasing sequence satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O\Big(\frac{1}{n}\Big) \quad \text{as } n \to \infty, \tag{1.2}$$

are bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. Moreover, there exist a martingale and such T means for which the boundedness does not hold from the Hardy space H_p to the space L_p when $0 < p \leq 1/2$.

One of the most well-known means of T means is the Riesz summability. In [40] (see also [16]), it was proved that the maximal operator of Riesz logarithmic means

$$R^* f := \sup_{n\in\mathbb{N}}|R_n f|$$

is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$ and is not bounded from H_p to the space L_p for $0 < p \leq 1/2$. It was also proved there that the Riesz summability has better properties than Fejér means.

In this paper, we derive the convergence of T means of Vilenkin–Fourier series with monotone coefficients of integrable functions in Lebesgue and Vilenkin–Lebesgue points.

This paper is organized as follows. In order to provide the coherence of our further discussion, some definitions and notations are presented in Section 2. For the proofs of the main results, we need some auxiliary lemmas of which some are new and of independent interest. These results are presented in Section 3. The main results and some of its consequences and detailed proofs are given in Section 4.

2 Definitions and notation

Denote by $\mathbb{N}_+$ the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the Vilenkin group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j}'s (for details, see [46]). In this paper, we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$. The direct product μ of measures $\mu_k(\{j\}) := 1/m_k$ $(j \in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m) = 1$. The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \quad (x_j \in Z_{m_j}).$$

It is easy to give a basis for the neighborhoods of G_m,

$$I_0(x) := G_m,\ I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}, \quad \text{where } x \in G_m,\ n \in \mathbb{N}.$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero.

We introduce on G_m an orthonormal system which is called the Vilenkin system. First, we define the complex-valued function $r_k(x)\colon G_m \to \mathbb{C}$, which is the generalized Rademacher function, by

$$r_k(x) := \exp(2\pi i x_k/m_k) \quad (i^2 = -1,\ x \in G_m,\ k \in \mathbb{N}).$$

Next, we define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m by

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh–Paley system when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and weak-$L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \quad \|f\|_{\text{weak-}L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [46]).

Now, we introduce analogues of the usual definitions in Fourier analysis. If $f \in L_1(G_m)$, we can define Fourier coefficients, partial sums and Dirichlet kernels with respect to the Vilenkin system in the usual manner,

$$\hat{f}(n) := \int_{G_m} f\overline{\psi}_n \, d\mu, \quad S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+).$$

Recall that

$$\int_{G_m} D_n(x)\, dx = 1, \tag{2.1}$$

$$D_{M_n - j}(x) = D_{M_n}(x) - \psi_{M_n-1}(x)\overline{D}_j(x), \quad j < M_n. \tag{2.2}$$

The convolution of two functions $f, g \in L_1(G_m)$ is defined by

$$(f * g)(x) := \int_{G_m} f(x - t)g(t)\, dt \quad (x \in G_m).$$

It is easy to see that if $f \in L_p(G_m)$, $g \in L_1(G_m)$ and $1 \le p < \infty$, then $f * g \in L_p(G_m)$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1. \tag{2.3}$$

Let $\{q_k : k \ge 0\}$ be a sequence of non-negative numbers. The n-th Nörlund mean t_n for a Fourier series of f is defined by

$$t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad \text{where } Q_n := \sum_{k=0}^{n-1} q_k. \tag{2.4}$$

It is obvious that

$$t_n f(x) = \int_{G_m} f(t) F_n(x-t)\, d\mu(t), \quad \text{where } F_n := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k D_k,$$

is called the T kernel.

The next proposition can be found in [7, 28].

Proposition 2.1. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then, for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} F_{M_n}(x)\, d\mu(x) = 1,$$

$$\sup_{n \in \mathbb{N}} \int_{G_m} |F_{M_n}(x)|\, d\mu(x) \le c < \infty,$$

$$\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_{M_n}(x)|\, d\mu(x) \to 0 \quad \text{as } n \to \infty.$$

Let $\{q_k : k \ge 0\}$ be a sequence of non-negative numbers. The n-th T means T_n for a Fourier series of f are defined by

$$T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f, \quad \text{where } Q_n := \sum_{k=0}^{n-1} q_k. \tag{2.5}$$

It is obvious that

$$T_n f(x) = \int_{G_m} f(t) F_n^{-1}(x-t)\, d\mu(t), \quad \text{where } F_n^{-1} := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k D_k,$$

is called the T kernel. We always assume that $\{q_k : k \ge 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then the summability method (2.5) generated by $\{q_k : k \ge 0\}$ is regular if and only if $\lim_{n\to\infty} Q_n = \infty$.

It is easy to show that, for any real numbers $a_1, \dots, a_m, b_1, \dots, b_m$ and $a_k = A_k - A_{k-1}$, $k = n, \dots, m$, we have the so-called Abel transformation

$$\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}).$$

For $a_j = A_j - A_{j-1}$, $j = 1, \dots, n$, if we invoke the Abel transformations

$$\sum_{j=1}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_0 b_1 + \sum_{j=0}^{n-2} A_j (b_j - b_{j+1}), \tag{2.6}$$

$$\sum_{j=M_N}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_{M_N - 1} b_{M_N} + \sum_{j=M_N}^{n-2} A_j (b_j - b_{j+1}), \tag{2.7}$$

then, for $b_j = q_j$, $a_j = 1$ and $A_j = j$ for any $j = 0, 1, \dots, n$, we get the following identities:

$$Q_n = \sum_{j=0}^{n-1} q_j = q_0 + \sum_{j=1}^{n-1} q_j = q_0 + \sum_{j=1}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1), \tag{2.8}$$

$$\sum_{j=M_N}^{n-1} q_j = \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1) - (M_N - 1) q_{M_N}.$$

Moreover, if we use $D_0 = K_0 = 0$ for any $x \in G_m$ and invoke the Abel transformations (2.6) and (2.7) for $b_j = q_j$, $a_j = D_j$ and $A_j = jK_j$ for any $j = 0, 1, \dots, n-1$, then we get the identities

$$F_n^{-1} = \frac{1}{Q_n}\sum_{j=0}^{n-1} q_j D_j = \frac{1}{Q_n}\Bigg(\sum_{j=1}^{n-2}(q_j - q_{j+1})jK_j + q_{n-1}(n-1)K_{n-1}\Bigg), \tag{2.9}$$

$$\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j D_j = \frac{1}{Q_n}\Bigg(\sum_{j=M_N}^{n-2}(q_j - q_{j+1})jK_j + q_{n-1}(n-1)K_{n-1} - q_{M_N}(M_N-1)K_{M_N-1}\Bigg). \tag{2.10}$$

Analogously, if we use $S_0 f = \sigma_0 f = 0$ for any $x \in G_m$ and invoke the Abel transformations (2.6) and (2.7) for $b_j = q_j$, $a_j = S_j$ and $A_j = j\sigma_j$ for any $j = 0, 1, \dots, n-1$, then we get the identities

$$T_n f = \frac{1}{Q_n}\sum_{j=0}^{n-1} q_j S_j f = \frac{1}{Q_n}\Bigg(\sum_{j=1}^{n-2}(q_j - q_{j+1})j\sigma_j f + q_{n-1}(n-1)\sigma_{n-1} f\Bigg),$$

$$\frac{1}{Q_n}\sum_{j=M_N}^{n-1} q_j S_j f = \frac{1}{Q_n}\Bigg(\sum_{j=M_N}^{n-2}(q_j - q_{j+1})j\sigma_j f + q_{n-1}(n-1)\sigma_{n-1} f - q_{M_N}(M_N-1)\sigma_{M_N-1} f\Bigg).$$

If $q_k \equiv 1$ in (2.4) and (2.5), we respectively define the Fejér means σ_n and Fejér kernels K_n as follows:

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

It is well known that (for details, see [1])

$$n|K_n| \le c \sum_{l=0}^{|n|} M_l |K_{M_l}|, \tag{2.11}$$

and for any $n, N \in \mathbb{N}_+$,

$$\int_{G_m} K_n(x)\, d\mu(x) = 1,$$

$$\sup_{n\in\mathbb{N}} \int_{G_m} |K_n(x)|\, d\mu(x) \le c < \infty, \tag{2.12}$$

$$\sup_{n\in\mathbb{N}} \int_{G_m \setminus I_N} |K_n(x)|\, d\mu(x) \to 0 \quad \text{as } n \to \infty. \tag{2.13}$$

The well-known example of the Nörlund summability is the so-called (C, α) means (Cesàro means) for $0 < \alpha < 1$, which are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f, \quad \text{where } A_0^\alpha := 0, \quad A_n^\alpha := \frac{(\alpha+1)\cdots(\alpha+n)}{n!}.$$

We also consider the "inverse" (C, α) means, which are examples of T means,

$$U_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \quad 0 < \alpha < 1.$$

Let V_n^α denote the T mean, where $\{q_0 = 0,\ q_k = k^{\alpha-1} : k \in \mathbb{N}_+\}$, that is,

$$V_n^\alpha f := \frac{1}{Q_n}\sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \quad 0 < \alpha < 1.$$

The n-th Riesz logarithmic mean R_n and the Nörlund logarithmic mean L_n are defined by

$$R_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{k} \quad \text{and} \quad L_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{n-k}, \quad \text{where } l_n := \sum_{k=1}^{n-1}\frac{1}{k}.$$

Up to now, we have considered T means in the case where the sequence $\{q_k : k \in \mathbb{N}\}$ is bounded, but now we consider T summabilities with an unbounded sequence $\{q_k : k \in \mathbb{N}\}$. We also define the class B_n of T means with non-decreasing coefficients,

$$B_n f := \frac{1}{Q_n} \sum_{k=1}^{n-1} \log k S_k f.$$

3 Auxiliary lemmas

First, we consider the kernels of T means with non-increasing sequences.

Lemma 3.1. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers satisfying the condition*

$$\frac{q_0}{Q_n} = O\Big(\frac{1}{n}\Big) \quad \text{as } n \to \infty.$$

Then, for some constant c, we have

$$|F_n^{-1}| \leq \frac{c}{n} \Big\{ \sum_{j=0}^{|n|} M_j |K_{M_j}| \Big\}.$$

Proof. Let the sequence $\{q_k : k \in \mathbb{N}\}$ be non-increasing. Then, by using (1.2), we get that

$$\frac{1}{Q_n}\bigg(\sum_{j=1}^{n-2} |q_j - q_{j+1}| + q_{n-1}\bigg) \leq \frac{1}{Q_n}\bigg(\sum_{j=1}^{n-2} (q_j - q_{j+1}) + q_{n-1}\bigg) \leq \frac{q_0}{Q_n} \leq \frac{c}{n}.$$

Hence, if we apply (2.11) and use equalities (2.8) and (2.9), we immediately obtain

$$|F_n^{-1}| \leq \bigg(\frac{1}{Q_n}\bigg(\sum_{j=1}^{n-1} |q_j - q_{j+1}| + q_{n-1}\bigg)\bigg) \sum_{i=0}^{|n|} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{|n|} M_i |K_{M_i}|.$$

The proof is completed by just combining the estimates above. □

Lemma 3.2. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. Then, for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} F_n^{-1}(x)\, d\mu(x) = 1, \tag{3.1}$$

$$\sup_{n \in \mathbb{N}} \int_{G_m} |F_n^{-1}(x)|\, d\mu(x) < \infty, \tag{3.2}$$

$$\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_n^{-1}(x)|\, d\mu(x) \to 0 \quad \text{as } n \to \infty.$$

Proof. According to (2.1), we easily obtain the proof of (3.1). By using (2.12) combined with (2.8) and (2.9), we get that

$$\begin{aligned} &\frac{1}{Q_n}\bigg(\sum_{j=0}^{n-2} (q_j - q_{j+1}) j \int_{G_m} |K_j|\, d\mu + q_{n-1}(n-1) \int_{G_m} |K_{n-1}|\, d\mu\bigg) \\ &\qquad \leq \frac{c}{Q_n}\bigg(\sum_{j=0}^{n-2} (q_j - q_{j+1}) j + q_{n-1}(n-1)\bigg) \leq c < \infty, \end{aligned}$$

so (3.2) is also proved. By using (2.13) and inequalities (2.8) and (2.9), we can conclude that

$$\begin{aligned} \int_{G_m \setminus I_N} |F_n^{-1}|\, d\mu &\leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \int_{G_m \setminus I_N} |K_j|\, d\mu + \frac{q_{n-1}(n-1)}{Q_n} \int_{G_m \setminus I_N} |K_{n-1}| \\ &\leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \alpha_j + \frac{q_{n-1}(n-1)\alpha_{n-1}}{Q_n} = \mathrm{I} + \mathrm{II}, \end{aligned}$$

where $\alpha_n \to 0$ as $n \to \infty$. Since the sequence is non-increasing, we can conclude that

$$\mathrm{II} = \frac{q_{n-1}(n-1)\alpha_{n-1}}{Q_n} \le \alpha_{n-1} \to 0 \quad \text{as } n \to \infty.$$

Moreover, for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\alpha_n < \varepsilon$ when $n > N_0$. Furthermore,

$$\mathrm{I} = \frac{1}{Q_n}\sum_{j=0}^{n-2}(q_j - q_{j+1})j\alpha_j = \frac{1}{Q_n}\sum_{j=0}^{N_0}(q_j - q_{j+1})j\alpha_j + \frac{1}{Q_n}\sum_{j=N_0+1}^{n-2}(q_j - q_{j+1})j\alpha_j := \mathrm{I}_1 + \mathrm{I}_2.$$

The sequence $\{q_k : k \in \mathbb{N}\}$ is non-increasing, and therefore, $|q_j - q_{j+1}| < 2q_0$,

$$\mathrm{I}_1 \le \frac{2q_0N_0}{Q_n} \to 0 \quad \text{as } n \to \infty$$

and

$$\mathrm{I}_2 = \frac{1}{Q_n}\sum_{j=N_0+1}^{n-2}(q_j - q_{j+1})j\alpha_j \le \frac{\varepsilon}{Q_n}\sum_{j=N_0+1}^{n-2}(q_j - q_{j+1})j \le \frac{\varepsilon}{Q_n}\sum_{j=0}^{n-2}(q_j - q_{j+1})j < \varepsilon,$$

and we can conclude that $\mathrm{I}_2 \to 0$, so the proof is complete. □

Next, we consider the kernels of *T* means with non-decreasing sequences.

Lemma 3.3. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers satisfying the condition*

$$\frac{q_{n-1}}{Q_n} = O\Big(\frac{1}{n}\Big) \quad \text{as } n \to \infty. \tag{3.3}$$

Then, for some constant c,

$$|F_n^{-1}| \le \frac{c}{n}\bigg\{\sum_{j=0}^{|n|} M_j|K_{M_j}|\bigg\}.$$

Proof. Since the sequence $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, if we apply condition (3.3), we find that

$$\frac{1}{Q_n}\bigg(\sum_{j=1}^{n-2}|q_j - q_{j+1}| + q_{n-1}\bigg) = \frac{1}{Q_n}\bigg(\sum_{j=1}^{n-2}(q_{j+1} - q_j) + q_{n-1}\bigg) \le \frac{2q_{n-1}}{Q_n} \le \frac{c}{n}. \tag{3.4}$$

If we apply the Abel transformation (2.10) combined with (2.11) and (3.4), we get that

$$|F_n^{-1}| \le \bigg(\frac{1}{Q_n}\bigg(\sum_{j=1}^{n-1}|q_j - q_{j+1}| + q_{n-1} + q_0\bigg)\bigg)\sum_{i=0}^{|n|} M_i|K_{M_i}| \le \frac{c}{n}\sum_{i=0}^{|n|} M_i|K_{M_i}|.$$ □

Lemma 3.4. *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers satisfying condition* (3.3). *Then, for some constant c,*

$$\int_{G_m} F_n^{-1}(x)\,d\mu(x) = 1,$$
$$\sup_{n\in\mathbb{N}}\int_{G_m} |F_n^{-1}(x)|\,d\mu(x) \le c < \infty, \tag{3.5}$$
$$\sup_{n\in\mathbb{N}}\int_{G_m\setminus I_N} |F_n^{-1}(x)|\,d\mu(x) \to 0 \quad \text{as } n \to \infty.$$

Proof. If we compare the estimation of F_n in Lemma 3.2 with the estimation of F_n in Lemma 3.3, we find that they are quite the same. It follows that the proof is analogical to Lemma 3.2. So we leave out the details. □

Finally, we study some special subsequences of kernels of *T* means.

Lemma 3.5. *Let $n \in \mathbb{N}$. Then*

$$F_{M_n}^{-1}(x) = D_{M_n}(x) - \psi_{M_n-1}(x)\overline{F}_{M_n}(x).$$

Proof. By using (2.2), we get that

$$F^{-1}_{M_n}(x)=\frac{1}{Q_{M_n}}\sum_{k=0}^{M_n-1}q_kD_k(x)=\frac{1}{Q_{M_n}}\sum_{k=1}^{M_n}q_{M_n-k}D_{M_n-k}(x)$$
$$=\frac{1}{Q_{M_n}}\sum_{k=1}^{M_n1}q_{M_n-k}(D_{M_n}(x)-\psi_{M_n-1}(x)\overline{D}_k(x))=D_{M_n}(x)-\psi_{M_n-1}(x)\overline{F}_{M_n}(x).$$ □

Corollary 3.6. *Let $\{q_k : k\in\mathbb{N}\}$ be a sequence of non-decreasing numbers. Then, for some constant c,*

$$\int_{G_m}F^{-1}_{M_n}(x)\,d\mu(x)=1,$$
$$\sup_{n\in\mathbb{N}}\int_{G_m}|F^{-1}_{M_n}(x)|\,d\mu(x)\le c<\infty,$$
$$\sup_{n\in\mathbb{N}}\int_{G_m\setminus I_N}|F^{-1}_{M_n}(x)|\,d\mu(x)\to 0\quad\text{as } n\to\infty,\quad\text{for any } N\in\mathbb{N}_+.$$

Proof. The proof is a direct consequence of Proposition 2.1 and Lemma 3.5. □

4 Proof of main result

Theorem 4.1. *The following statements hold.*

(a) *Let $p\ge 1$, and let $\{q_k : k\in\mathbb{N}\}$ be a sequence of non-increasing numbers. Then $\|T_nf-f\|_p\to 0$ as $n\to\infty$, for all $f\in L_p(G_m)$. Let a function $f\in L_1(G_m)$ be continuous at a point x. Then $T_nf(x)\to f(x)$ as $n\to\infty$. Moreover, $\lim_{n\to\infty}T_nf(x)=f(x)$ for all Vilenkin–Lebesgue points of $f\in L_p(G_m)$.*

(b) *Let $p\ge 1$, and let $\{q_k : k\in\mathbb{N}\}$ be a sequence of non-decreasing numbers satisfying condition (3.3). Then $\|T_nf-f\|_p\to 0$ as $n\to\infty$, for all $f\in L_p(G_m)$. Let a function $f\in L_1(G_m)$ be continuous at a point x. Then $T_nf(x)\to f(x)$ as $n\to\infty$. Moreover, $\lim_{n\to\infty}T_nf(x)=f(x)$ for all Vilenkin–Lebesgue points of $f\in L_p(G_m)$.*

Proof. Let $\{q_k : k\in\mathbb{N}\}$ be a non-increasing sequence. Lemma 3.2 immediately implies stated norm and pointwise convergences. Suppose that x is either a point of continuity or a Vilenkin–Lebesgue point of a function $f\in L_p(G_m)$. Then $\lim_{n\to\infty}|\sigma_nf(x)-f(x)|=0$. Hence,

$$|T_nf(x)-f(x)|\le\frac{1}{Q_n}\Bigl(\sum_{j=0}^{n-2}(q_j-q_{j+1})j|\sigma_jf(x)-f(x)|+q_{n-1}(n-1)|\sigma_{n-1}f(x)-f(x)|\Bigr)$$
$$\le\frac{1}{Q_n}\sum_{j=0}^{n-2}(q_j-q_{j+1})j\alpha_j+\frac{q_{n-1}(n-1)\alpha_{n-1}}{Q_n}:=\mathrm{I}+\mathrm{II},\quad\text{where }\alpha_n\to 0\quad\text{as } n\to\infty.$$

To prove $\mathrm{I}\to 0$ as $n\to\infty$ and $\mathrm{II}\to 0$ as $n\to\infty$, we just have to make analogous steps of the proof of Lemma 3.2. It follows that part (a) is proved.

Now, we assume that the sequence is non-decreasing and satisfying condition (3.3). According to (3.5) in Lemma 3.4, we define the norm and pointwise convergence. To prove the convergence in Vilenkin–Lebesgue points, we use the estimation

$$|T_nf(x)-f(x)|\le\frac{1}{Q_n}\sum_{j=0}^{n-2}(q_{j+1}-q_j)j\alpha_j+\frac{q_{n-1}(n-1)\alpha_n}{Q_n}:=\mathrm{III}+\mathrm{IV},\quad\text{where }\alpha_n\to 0\quad\text{as } n\to\infty.$$

It is evident that

$$\mathrm{IV}\le\frac{q_{n-1}(n-1)\alpha_n}{Q_n}\le\alpha_n\to 0\quad\text{as } n\to\infty.$$

On the other hand, for any $\varepsilon>0$, there exists $N_0\in\mathbb{N}$ such that $\alpha_n<\varepsilon/2$ when $n>N_0$. We can write that

$$\frac{1}{Q_n}\sum_{j=1}^{n-2}(q_{j+1}-q_j)j\alpha_j=\frac{1}{Q_n}\sum_{j=1}^{N_0}(q_{j+1}-q_j)j\alpha_j+\frac{1}{Q_n}\sum_{j=N_0+1}^{n-2}(q_{j+1}-q_j)j\alpha_j=\mathrm{III}_1+\mathrm{III}_2.$$

Since the sequence $\{q_k\}$ is non-decreasing, we obtain that $|q_{j+1} - q_j| < 2q_{j+1} < 2q_{n-1}$. Hence,

$$\mathrm{III}_1 \leq \frac{2q_0 N_0}{Q_n} \to 0 \quad \text{as } n \to \infty$$

and

$$\begin{aligned}\mathrm{III}_2 &\leq \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{n-j-1} - q_{n-j}) j\alpha_j \leq \frac{\varepsilon(n-1)}{Q_n} \sum_{j=N_0+}^{n-2} (q_{n-j} - q_{n-j-1}) \\ &\leq \frac{\varepsilon(n-1)}{Q_n}(q_0 - q_{n-N_0}) \leq \frac{2q_0\varepsilon(n-1)}{Q_n} < \varepsilon.\end{aligned}$$

Therefore, III $\to \infty$ too so that the proof of part (b) is also complete. □

Corollary 4.2. *Let $f \in L_p$, where $p \geq 1$. Then*

$$R_n f \to f \quad \text{a.e.} \quad \text{as } n \to \infty, \qquad U_n^\alpha f \to f \quad \text{a.e.} \quad \text{as } n \to \infty,$$
$$V_n^\alpha f \to f \quad \text{a.e.} \quad \text{as } n \to \infty, \qquad B_n f \to f \quad \text{a.e.} \quad \text{as } n \to \infty.$$

Theorem 4.3. *Let $p \geq 1$, and let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers. Then $\|T_{M_n} f - f\|_p \to 0$ as $n \to \infty$, for all $f \in L_p(G_m)$. Let a function $f \in L_1(G_m)$ be continuous at a point x. Then $T_{M_n} f(x) \to f(x)$ as $n \to \infty$. Moreover, $\lim_{n\to\infty} T_{M_n} f(x) = f(x)$ for all Lebesgue points of $f \in L_p(G_m)$.*

Proof. Corollary 3.6 immediately implies the norm and pointwise convergence. To prove the a.e. convergence, we use first the identity in Lemma 3.5 to write

$$T_{M_n} f(x) = \int_{G_m} f(t) F_n^{-1}(x-t)\, d\mu(t) = \int_{G_m} f(t) D_{M_n}(x-t)\, d\mu(t) - \int_{G_m} f(t)\psi_{M_n-1}(x-t)\overline{F}_{M_n}(x-t) = \mathrm{I} - \mathrm{II}.$$

By applying (1.1), we can conclude that $\mathrm{I} = S_{M_n} f(x) \to f(x)$ for all Lebesgue points of $f \in L_p(G_m)$. By using $\psi_{M_n-1}(x-t) = \psi_{M_n-1}(x)\overline{\psi}_{M_n-1}(t)$, we can conclude that

$$\mathrm{II} = \psi_{M_n-1}(x) \int_{G_m} f(t)\overline{F}_{M_n}(x-t)\overline{\psi}_{M_n-1}(t)\, d(t).$$

By combining (2.3) and Proposition 2.1, we find that the function

$$f(t)\overline{F}_{M_n}(x-t) \in L_p, \quad \text{where } p \geq 1, \quad \text{for any } x \in G_m,$$

and II are Fourier coefficients of an integrable function. According to the Riemann–Lebesgue lemma, we get that $\mathrm{II} \to 0$ for any $x \in G_m$. □

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