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SHARP (H_p, L_p) AND $(H_p, \mathbf{weak} - L_p)$ TYPE INEQUALITIES OF WEIGHTED MAXIMAL OPERATORS OF T MEANS WITH RESPECT TO VILENKIN SYSTEMS

DAVIT BARAMIDZE

Abstract. We discuss (H_p, L_p) and $(H_p, \text{weak} - L_p)$ type inequalities of weighted maximal operators of T means with respect to the Vilenkin systems with monotone coefficients [44] and prove that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

1. Introduction

It is well-known that Vilenkin systems do not form bases in the space L_1. Moreover, there is a function in the Hardy space H_p such that the partial sums of f are not bounded in L_p-norm, for $0 < p \leq 1$. Approximation properties of Vilenkin–Fourier series with respect to one- and two-dimensional cases can be found in the works of [8, 9, 26, 31, 37, 40, 41]. In the one-dimensional case, the weak $(1,1)$-type inequality for the maximal operator of Fejér means

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|$$

can be found in [32] for Walsh series and in [23] for bounded Vilenkin series. [7] and [29] verified that σ^* is bounded from H_1 to L_1. [48] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the space L_p, for $p > 1/2$. [30] gave a counterexample, which shows that the boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ has been given by [10] (see also [33]). Moreover, [49] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the $weak - L_{1/2}$ space. In [34] and [35], the following result has been proved.

Theorem T1. *Let $0 < p \leq 1/2$. Then the weighted maximal operator of Fejér means $\widetilde{\sigma}_p^*$ defined by*

$$\widetilde{\sigma}_p^* f := \sup_{n \in \mathbb{N}_+} \frac{|\sigma_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.

Moreover, the rate of the weights $\left\{1/(n+1)^{1/p-2} \log^{2[p+1/2]}(n+1)\right\}_{n=1}^{\infty}$ in n-th Fejér mean is given exactly.

In [39] (see also [2, 15]), it was proved that the maximal operator of Riesz means

$$R^* f := \sup_{n \in \mathbb{N}} |R_n f|$$

is bounded from the Hardy space $H_{1/2}$ to the $weak - L_{1/2}$ space and is not bounded from H_p to the space L_p, for $0 < p \leq 1/2$. It was also proved that the Riesz summability has better properties than Fejér means. In particular, the following weighted maximal operators

$$\frac{\log n |R_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

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are bounded from H_p to the space L_p, for $0<p\leq 1/2$ and the rate of weights is sharp.

Similar results with respect to Walsh-Kachmarz systems were obtained in [11] for $p=1/2$ and in [36] for $0<p<1/2$. Approximation properties of Fejér means with respect to Vilenkin and Kaczmarz systems can be found in [5, 12, 25, 27, 28, 38, 43].

[18] investigates the approximation properties of some special Nörlund means of Walsh–Fourier series of L_p function in a norm. In the two-dimensional case, approximation properties of Nörlund means were considered by [19–22]. In [24] (see also [6,16,25]), it was proved that the maximal operators of Nörlund means t^* defined by

$$t^*f:=\sup_{n\in\mathbb{N}}\left|t_nf\right|,$$

either with non-decreasing coefficients, or non-increasing coefficients, satisfying the condition

$$\frac{1}{Q_n}=O\Big(\frac{1}{n}\Big),\quad \text{as}\quad n\to\infty, \tag{1}$$

are bounded from the Hardy space $H_{1/2}$ to the $weak-L_{1/2}$ space and are not bounded from the Hardy space H_p to the space L_p, when $0<p\leq 1/2$.

In [42], it was proved that the maximal operators T^* of T means defined by

$$T^*f:=\sup_{n\in\mathbb{N}}\left|T_nf\right|$$

either with non-increasing coefficients, or non-decreasing sequence satisfying the condition

$$\frac{q_{n-1}}{Q_n}=O\Big(\frac{1}{n}\Big),\quad \text{as}\quad n\to\infty, \tag{2}$$

are bounded from the Hardy space $H_{1/2}$ to the $weak-L_{1/2}$ space. Moreover, there exist a martingale and such T means for which the boundedness from the Hardy space H_p to the space L_p does not hold when $0<p\leq 1/2$.

In [44] (see also [13, 14]), it was proved that if T is either with non-increasing coefficients, or non-decreasing sequence satisfying condition (2), then the weighted maximal operator of T means $\widetilde{T}_p^*$ defined by

$$\widetilde{T}_p^*f:=\sup_{n\in\mathbb{N}_+}\frac{\left|T_nf\right|}{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)} \tag{3}$$

is bounded from the martingale Hardy space H_p to the Lebesgue space L_p.

Some general means related to T means where investigated by [3] (see also [4]).

In this paper we discuss (H_p, L_p) and $(H_p, \text{weak}-L_p)$ type inequalities of weighted maximal operators of T means with respect to the Vilenkin systems with monotone coefficients [44] and prove that the rate of the weights in (3) is the best possible in a special sense. As applications, both some the well-known and new results are pointed out.

This paper is organized as follows. Some definitions and notations are presented in Section 2. The main results with their proofs and some of consequences can be found in Section 3.

2. Definitions and Notation

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Let $m:=(m_0,m_1,\ldots)$ be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k}:=\{0,1,\ldots,m_k-1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of Z_{m_j}.

The direct product μ of the measures $\mu_k(\{j\}):=1/m_k$ $(j\in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m)=1$.

In this paper, we discuss the bounded Vilenkin groups, i.e., the case for $\sup_n m_n<\infty$.

The elements of G_m are represented by the sequences

$$x:=(x_0,x_1,\ldots,x_j,\ldots)\quad\left(x_j\in Z_{m_j}\right).$$

Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m$, the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$. It is easy to give a basis for the neighborhoods of G_m :

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\},$$

where $x \in G_m$, $n \in \mathbb{N}$.

If we define the so-called generalized number system based on m in the form

$$M_0 := 1, \ M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$, and only a finite number of n_j's differ from zero.

We introduce on G_m an orthonormal system which is called the Vilenkin system. First, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad \left(i^2 = -1, x \in G_m, \ k \in \mathbb{N}\right).$$

Next, we define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m by

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system as the Walsh–Paley system when $m \equiv 2$.

The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $weak-L_p(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \ \ \|f\|_{weak-L_p}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [45]).

Now, we introduce the analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$, we can define Fourier coefficients, partial sums and Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu, \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+).$$

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. The n-th T means T_n for a Fourier series of f are defined by

$$T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k. \tag{4}$$

We always assume that $\{q_k : k \geq 0\}$ is a sequence of non-negative numbers and $q_0 > 0$. Then the summability method (4) generated by $\{q_k : k \geq 0\}$ is regular if and only if $\lim_{n \to \infty} Q_n = \infty$.

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The n-th Nörlund mean t_n for a Fourier series of f is defined by

$$t_n f = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad \text{where} \quad Q_n := \sum_{k=0}^{n-1} q_k. \tag{5}$$

If $q_k \equiv 1$ in (4) and (5), we define respectively the Fejér means σ_n and Kernels K_n as follows:

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f \,, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k.$$

The well-known example of the Nörlund summability is the so-called (C, α) means (Cesàro means) for $0 < \alpha < 1$, which are defined by

$$\sigma_n^{\alpha} f := \frac{1}{A_n^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0, \qquad A_n^\alpha := \frac{(\alpha+1)\cdots(\alpha+n)}{n!}.$$

We also consider the "inverse" (C,α) means, which is an example of T means:

$$U_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=0}^{n-1} A_k^{\alpha-1} S_k f, \qquad 0<\alpha<1.$$

Let V_n^α denote the T mean, where $\left\{q_0=0,\ q_k=k^{\alpha-1} : k\in\mathbb{N}_+\right\}$, that is,

$$V_n^\alpha f := \frac{1}{Q_n}\sum_{k=1}^{n-1} k^{\alpha-1} S_k f, \qquad 0<\alpha<1.$$

The n-th Riesz logarithmic mean R_n and the Nörlund logarithmic mean L_n are defined by

$$R_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{k} \quad \text{and} \quad L_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{n-k},$$

respectively, where $l_n := \sum_{k=1}^{n-1} 1/k$.

If $\{q_k : k\in\mathbb{N}\}$ is a monotone and bounded sequence, then we get the class B_n of T means with the non-decreasing coefficients

$$B_n f := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_k S_k f.$$

The σ-algebra generated by the intervals $\{I_n(x) : x\in G_m\}$ will be denoted by $F_n\,(n\in\mathbb{N})$. Denote by $f=\left(f^{(n)}, n\in\mathbb{N}\right)$ a martingale with respect to $F_n\,(n\in\mathbb{N})$ (for details see, e.g., [46]). The maximal function of a martingale f is defined by $f^* := \sup_{n\in\mathbb{N}}\left|f^{(n)}\right|$. For $0<p<\infty$, the Hardy martingale spaces H_p consist of all martingales f for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f=\left(f^{(n)}, n\in\mathbb{N}\right)$ is a martingale, then the Vilenkin–Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty}\int_{G_m} f^{(k)}\overline{\psi}_i d\mu.$$

A bounded measurable function a is called a p-atom, if there exists an interval I such that

$$\int_I a\,d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \operatorname{supp}(a)\subset I.$$

We need the following auxiliary Lemmas.

Proposition 1 (see, e.g., [47])**.** *A martingale $f=\left(f^{(n)}, n\in\mathbb{N}\right)$ is in $H_p\,(0<p\le 1)$ if and only if there exist a sequence $(a_k, k\in\mathbb{N})$ of p-atoms and a sequence $(\mu_k, k\in\mathbb{N})$ of real numbers such that for every $n\in\mathbb{N}$,*

$$\sum_{k=0}^{\infty}\mu_k S_{M_n} a_k = f^{(n)}, \quad \text{a.e.}, \quad \text{where} \quad \sum_{k=0}^{\infty}|\mu_k|^p<\infty. \tag{6}$$

Moreover,

$$\|f\|_{H_p} \backsim \inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (6).

3. The Main Results and Applications

Our first main result reads as

Theorem 1. a) *Let the sequence* $\{q_k : k \geq 0\}$ *be nondecreasing, satisfying the condition*

$$\frac{q_0}{Q_{M_{2n_k}+2}} \geq \frac{c}{M_{2n_k}}, \quad \text{for some constant } c \quad \text{and} \quad n \in \mathbf{N}, \tag{7}$$

or let the sequence $\{q_k : k \geq 0\}$ *be nonincreasing, satisfying the condition*

$$\frac{q_{M_{2n_k}+1}}{Q_{M_{2n_k}+2}} \geq \frac{c}{M_{2n_k}}, \quad \text{for some constant } c \quad \text{and} \quad n \in \mathbf{N}. \tag{8}$$

Then for any increasing function $\varphi : \mathbf{N}_+ \to [1, \infty)$ *satisfying the conditions*

$$\lim_{n\to\infty} \varphi(n) = \infty$$

and

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\varphi(n+1)} = +\infty, \tag{9}$$

there exists a martingale $f \in H_{1/2}$ *such that*

$$\left\| \sup_{n\in\mathbf{N}} \frac{|T_n f|}{\varphi(n)} \right\|_{1/2} = \infty.$$

b) *Let* $0 < p < 1/2$ *and the sequence* $\{q_k : k \geq 0\}$ *be nondecreasing, or let the sequence* q_k *be nonincreasing, satisfying condition* (8)*. Then for any increasing function* $\varphi : \mathbf{N}_+ \to [1, \infty)$ *satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2}}{\varphi(n+1)} = +\infty, \tag{10}$$

there exists a martingale $f \in H_p$ *such that*

$$\left\| \sup_{n\in\mathbf{N}} \frac{|T_n f|}{\varphi(n)} \right\|_{weak-L_p} = \infty.$$

Proof. According to condition (9) in case a), we conclude that there exists an increasing sequence $\{n_k : k \in \mathbf{N}\}$ of positive integers such that

$$\lim_{k\to\infty} \frac{\log^2(M_{2n_k+1})}{\varphi(M_{2n_k+1})} = +\infty.$$

According to condition (10), we conclude that there exists an increasing sequence $\{n_k : k \in \mathbf{N}\}$ of positive integers such that (here we use the same indices n_k, but they may be different)

$$\lim_{k\to\infty} \frac{(M_{2n_k}+2)^{1/p-2}}{\varphi(M_{2n_k}+2)} = +\infty, \quad \text{for} \quad 0 < p < 1/2.$$

Let

$$f_{n_k}(x) := D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x).$$

It is evident that

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = M_{2n_k}, \ldots, M_{2n_k+1} - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{M_{2n_k}}(x), & i = M_{2n_k} + 1, \ldots, M_{2n_k+1} - 1, \\ f_{n_k}(x), & i \geq M_{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Since

$$f^*_{n_k}(x) = \sup_{n\in\mathbf{N}} |S_{M_n}(f_{n_k}; x)| = |f_{n_k}(x)|,$$

we get

$$\|f_{n_k}\|_{H_p} = \left\|f^*_{n_k}\right\|_p = \left\|D_{M_{2n_k}}\right\|_p = M^{1-1/p}_{2n_k}. \tag{12}$$

First, for case a) we consider $p = 1/2$. By using (11) and the equality (see [1])

$$D_n(x) = D_{M_{|n|}}(x) + r_{|n|}(x)\, D_{n-M_{|n|}}(x)$$

for $1 \leq s \leq n_k$, we get

$$\begin{aligned}
\frac{\left|T_{M_{2n_k}+M_{2s}} f_{n_k}\right|}{\varphi(M_{2n_k}+M_{2s})} &= \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{1}{Q_{M_{2n_k}+M_{2s}}} \left| \sum_{j=0}^{M_{2n_k}+M_{2s}-1} q_j S_j f_{n_k} \right| \\
&= \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{1}{Q_{M_{2n_k}+M_{2s}}} \left| \sum_{j=M_{2n_k}}^{M_{2n_k}+M_{2s}-1} q_j S_j f_{n_k} \right| \\
&= \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{1}{Q_{M_{2n_k}+M_{2s}}} \left| \sum_{j=M_{2n_k}}^{M_{2n_k}+M_{2s}-1} q_j \left(D_j - D_{M_{2n_k}}\right) \right| \\
&= \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{1}{Q_{M_{2n_k}+M_{2s}}} \left| \sum_{j=0}^{M_{2n_k}-1} q_{j+M_{2n_k}} \left(D_{j+M_{2n_k}} - D_{M_{2n_k}}\right) \right| \\
&= \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{1}{Q_{M_{2n_k}+M_{2s}}} \left| \sum_{j=0}^{M_{2s}-1} q_{j+M_{2m_k}} D_j \right|.
\end{aligned}$$

Let $x \in I_{2s} \setminus I_{2s+1}$. Then

$$\frac{\left|T_{M_{2n_k}+M_{2s}} f_{n_k}(x)\right|}{\varphi(M_{2n_k}+M_{2s})} = \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{1}{Q_{M_{2n_k}+M_{2s}}} \left| \sum_{j=0}^{M_{2n_k}-1} q_{j+M_{2n_k}} j \right|.$$

Let the sequence $\{q_k : k \geq 0\}$ be nondecreasing. Then according to condition (7), we find that

$$\begin{aligned}
\frac{\left|T_{M_{2n_k}+M_{2s}} f_{n_k}(x)\right|}{\varphi(M_{2n_k}+M_{2s})} &\geq \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{q_0}{Q_{M_{2n_k}+M_{2s}}} \sum_{j=0}^{M_{2s}-1} j \\
&\geq \frac{1}{\varphi(M_{2n_k+1})} \frac{q_0}{Q_{M_{2n_k}+M_{2s}}} \sum_{j=0}^{M_{2s}-1} j \geq \frac{cM^2_{2s}}{M_{2n_k}\varphi(M_{2n_k+1})}.
\end{aligned}$$

Let the sequence $\{q_k : k \geq 0\}$ be nonincreasing. Since $\varphi : \mathbf{N}_+ \to [1, \infty)$ is an increasing sequence, by using condition (8), we get

$$\frac{\left|T_{M_{2n_k}+M_{2s}} f_{n_k}(x)\right|}{\varphi(M_{2n_k}+M_{2s})} \geq \frac{1}{\varphi(M_{2n_k}+M_{2s})} \frac{q_{M_{2n_k}+M_{2s}-1}}{Q_{M_{2n_k}+M_{2s}}} \sum_{j=0}^{M_{2s}-1} j \geq \frac{cM^2_{2s}}{M_{2n_k}\varphi(M_{2n_k+1})}.$$

Hence

$$\begin{aligned}
&\int\limits_{G_m} \left(\sup_{n\in\mathbf{N}} \frac{|T_n f_{n_k}|}{\varphi(n)} \right)^{1/2} d\mu \geq \sum_{s=1}^{n_k} \int\limits_{I_{2s}\setminus I_{2s+1}} \left| \frac{T_{M_{2n_k}+M_{2s}} f_{n_k}}{\varphi(M_{2n_k}+M_{2s})} \right|^{1/2} d\mu \\
\geq & \sum_{s=1}^{n_k} \int\limits_{I_{2s}\setminus I_{2s+1}} \left(\frac{cM_{2s}^2}{M_{2n_k}\varphi(M_{2n_k+1})} \right)^{1/2} d\mu \geq \frac{c}{(M_{2n_k}\varphi(M_{2n_k+1}))^{1/2}} \sum_{s=1}^{n_k} M_{2s} |I_{2s} \setminus I_{2s+1}| \\
\geq & \frac{c}{(M_{2n_k}\varphi(M_{2n_k+1}))^{1/2}} \sum_{s=1}^{n_k} 1 \geq \frac{cn_k}{(M_{2n_k}\varphi(M_{2n_k+1}))^{1/2}}.
\end{aligned}$$

From (12), we get

$$\begin{aligned}
\frac{\left(\int\limits_{G_m} \left(\sup_{n\in\mathbf{N}} \frac{|T_n f_{n_k}|}{\varphi(n)} \right)^{1/2} d\mu \right)^2}{\|f_{n_k}\|_{H_{1/2}}} \geq & \frac{cn_k^2 M_{2n_k}}{M_{2n_k}\varphi(M_{2n_k+1})} \geq \frac{cn_k^2}{\varphi(M_{2n_k+1})} \\
\geq & \frac{c(2n_k+1)^2}{\varphi(M_{2n_k+1})} \geq \frac{c\log^2(M_{2n_k+1})}{\varphi(M_{2n_k+1})} \to \infty, \quad \text{as} \quad k \to \infty.
\end{aligned}$$

This completes the proof of part a).

Next, we consider the case $0 < p < 1/2$. In view of identities (11) of the Fourier coefficients, we find that

$$\begin{aligned}
\frac{\left| T_{M_{2n_k}+2} f_{n_k} \right|}{\varphi(M_{2n_k}+2)} = & \frac{1}{\varphi(M_{2n_k}+2)} \frac{1}{Q_{M_{2n_k}+2}} \left| \sum_{j=0}^{M_{2n_k}+1} q_j S_j f_{n_k} \right| \\
= & \frac{1}{\varphi(M_{2n_k}+2)} \frac{1}{Q_{M_{2n_k}+2}} \left| q_{M_{2n_k}+1} (D_{M_{2n_k}+1} - D_{M_{2n_k}}) \right| \\
= & \frac{1}{\varphi(M_{2n_k}+2)} \frac{1}{Q_{M_{2n_k}+2}} \left| q_{M_{2n_k}+1} \psi_{M_{2n_k}} \right| \\
= & \frac{1}{\varphi(M_{2n_k}+2)} \frac{q_{M_{2n_k}+1}}{Q_{M_{2n_k}+2}}.
\end{aligned}$$

Let the sequence $\{q_k : k \geq 0\}$ be nondecreasing. Then

$$\frac{\left| T_{M_{2n_k}+2} f(x) \right|}{\varphi(M_{2n_k}+2)} \geq \frac{1}{\varphi(M_{2n_k}+2)} \frac{q_{M_{2n_k}+1}}{q_{M_{2n_k}+1}(M_{2n_k}+2)} \geq \frac{c}{M_{2n_k}\varphi(M_{2n_k}+2)}.$$

Let the sequence $\{q_k : k \geq 0\}$ be nonincreasing. Then, according tocondition (8), we find that

$$\frac{\left| T_{M_{2n_k}+2} f(x) \right|}{\varphi(M_{2n_k}+2)} = \frac{1}{\varphi(M_{2n_k}+2)} \frac{q_{M_{2n_k}+1}}{Q_{M_{2n_k}+2}} \geq \frac{c}{M_{2n_k}\varphi(M_{2n_k}+2)}.$$

Hence

$$\mu\left\{ x \in G_m : \frac{|T_{M_{2n_k}+2} f(x)|}{\varphi(M_{2n_k}+2)} \geq \frac{c}{M_{2n_k}\varphi(M_{2n_k}+2)} \right\} = |G_m| = 1.$$

Then from (12), we get

$$\frac{\frac{c}{M_{2n_k}\varphi\left(M_{2n_k}+2\right)}\left\{\mu\left\{x\in G_m:\frac{\left|T_{M_{2n_k}+2}f_{n_k}(x)\right|}{\varphi\left(M_{2n_k}+2\right)}\geq\frac{c}{M_{2n_k}\varphi\left(M_{2n_k}+2\right)}\right\}\right\}^{1/p}}{\left\|f_{n_k}\right\|_{H_p}}$$

$$\geq\frac{c}{M_{2n_k}\varphi\left(M_{n_k}+2\right)M_{2n_k}^{1-1/p}}=\frac{cM_{2n_k}^{1/p-2}}{\varphi\left(M_{2n_k}+2\right)}$$

$$\geq\frac{c\left(M_{2n_k}+2\right)^{1/p-2}}{\varphi\left(M_{2n_k}+2\right)}\to\infty,\quad\text{as}\quad k\to\infty.$$

The proof is complete. □

As an application, we get the well-known result for the weighted maximal operator of Fejér means which was considered in [34, 35]:

Corollary 1. *Let $\varphi:\mathbf{N}_+\to[1,\infty)$ be any increasing function satisfying the conditions*

$$\lim_{n\to\infty}\varphi(n)=\infty$$

and

$$\overline{\lim_{n\to\infty}}\frac{(n+1)^{1/p-2}\log^{2[1/2+p]}(n+1)}{\varphi(n)}=+\infty. \tag{13}$$

Then

$$\frac{\left\|\sup_{n\in\mathbf{N}}\frac{|\sigma_n f|}{\varphi(n)}\right\|_{1/2}}{\|f\|_{H_{1/2}}}=\infty$$

and

$$\frac{\left\|\sup_{n\in\mathbf{N}}\frac{|\sigma_n f|}{\varphi(n)}\right\|_{weak-L_p}}{\|f\|_{H_p}}=\infty.$$

We also present some new results on T means with respect to Vilenkin systems which follow Theorem 1.

Corollary 2. *Theorem* 1 *holds true for $U_n^\alpha f$, $V_n^\alpha f$ and $B_n f$ means with respect to Vilenkin systems.*

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The University of Georgia, School of science and technology, 77a Merab Kostava Str., Tbilisi 0128, Georgia

Department of Computer Science and Computational Engineering, UiT–The Arctic University of Norway, P.O. Box 385, N-8505, Narvik, Norway

E-mail address: davit.baramidze@ug.edu.ge