

KILLING TENSORS IN KOUTRAS–MCINTOSH SPACETIMES

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ABSTRACT. The Koutras–McIntosh family of metrics include conformally flat pp-waves and the Wils metric. It appeared in a paper of 1996 by Koutras–McIntosh as an example of a pure radiation spacetime without scalar curvature invariants or infinitesimal symmetries. Here we demonstrate that these metrics have no “hidden symmetries”, by which we mean Killing tensors of low degrees. For the particular case of Wils metrics we show the nonexistence of Killing tensors up to degree 6.

The technique we use is the geometric theory of overdetermined PDEs and the Cartan prolongation-projection method. Application of those allows to prove the nonexistence of polynomial in momenta integrals for the equation of geodesics in a mathematical rigorous way. Using the same technique we can completely classify all lower degree Killing tensors and, in particular, prove that for generic pp-waves all Killing tensors of degree 3 and 4 are reducible.

1. INTRODUCTION

1.1. Formulation and motivation. Polynomial integrals of Hamiltonian ODEs were actively studied in the XIXth century classical mechanics; in particular the existence of quadratic integral for the metric of the ellipsoid allowed Jacobi in 1836 to find an explicit formula for geodesics in terms of elliptic functions.

This problem also appeared in general relativity: the famous metrics of Schwarzschild, Gödel and Kerr admit polynomial integrals allowing to describe geodesics of the corresponding spacetimes in detail. Often integrals are conserved quantities related to Killing vectors via Noether’s theorem, but sometimes there are higher degree integrals, known as Killing tensors. One of those is the Carter constant [3, 26] reducing the geodesic motion to quadratures.

There exist obstructions to the existence of polynomial integrals: according to [14] a generic metric g admits no such integrals even locally. It is thus important to realize the existence/nonexistence of Killing tensors for concrete metrics from applications, see [6, 8, 9, 13, 25].

The following is the Koutras–McIntosh family of spacetimes for $(a, b) \neq (0, 0)$:

$$(1.1) \quad g = 2(ax + b) du dw - 2aw dx du + (f(u)(ax + b)(x^2 + y^2) - a^2w^2) du^2 - dx^2 - dy^2.$$

These metrics were shown in [11] to possess neither invariants nor symmetries. The first property means that all polynomial curvature invariants, i.e., complete contractions of tensor products of the Riemann tensor and its covariant derivatives $\nabla_{i_1} \cdots \nabla_{i_s} R_{abcd}$, vanish and so cannot be used to distinguished g from the Minkowski metric.

These are so-called VSI (vanishing scalar invariants) spaces that received considerable attention in recent time [21]. They belong to a more general class of spacetimes not separated by their scalar curvature invariants [4, 5], which in dimension 4 were proven to be of degenerate Kundt type. Note that Kundt spaces can be distinguished by their Cartan [19] or differential [16] invariants, see [15] for a comparisson.

The second property above means there are no Killing vectors, or linear integrals, for (1.1). In this paper we show that it also does not possess “hidden symmetries”, by which we mean Killing tensors of low degrees.

Note that the nonexistence of Killing tensors is important in several applications. For instance, it is necessary for linearization stability of Einstein’s equations [1] and also for the inverse problem in tensor tomography [20]. Thus, even though Killing tensors do not have direct geometric interpretation (as noticed by Penrose and Walker [26], see however [2]) their existence or nonexistence carries certain dynamical implications.

1.2. Main results. Metric (1.1) is conformally flat (but nonflat for $f \neq 0$) and describes pure radiation, satisfying Einstein’s field equations of the type $R_{ab} = \phi l_a l_b$ for a null vector field l and a scalar field ϕ .

Metric (1.1) for $a = 0, b = 1$ is a pp-wave, possessing 6 Killing vectors and 1 homothety except for special cases $f(u) = c$ and $f(u) = c/u^2$, where the number of Killing vectors increases to 7 [22, 8] and the homothety persists. We will examine the existence of higher order Killing tensors (up to degree 4) and for specific cases $f(u) = cu^m$, $m = 0, 1, 2, -2$ we prove that there is only one such irreducible quadratic tensor.

Metric (1.1) for $a = 1, b = 0$ defines the Wils spacetime [27]. This metric is known to have no Killing vectors or homotheties for general functional parameter $f(u)$, so we examine it for the existence of higher order Killing tensors.

It turns out that up to order 6 no irreducible Killing tensors exist (that is with the exception of powers of the Hamiltonian and combinations with Killing vectors when they exist). These results are presented in Section 3.

For the general Koutras–McIntosh family we deduce the following statement:

Theorem 1. *For generic numerical parameters a, b and functional parameter $f(u)$ the spacetime (1.1) possesses no Killing tensors up to degree 6 except for energy and its powers H , H^2 and H^3 .*

Here and below genericity of $f(u)$ is understood in C^{k+1} topology, where k is the prolongation level determined by Algorithm 1 of §2.4, where the matrix M_k depends on the jet $j^{k+1}f$. Table 2 shows values of k for degrees $d \leq 6$.

We can be more specific on the exceptional values of the involved parameters. To find those that allow Killing vectors one may follow the general approach with metric invariants via the Cartan-Karlhede algorithm [7], however our method with counting compatibility conditions via the coefficient matrix of the prolonged PDE system gives an alternative and implies the following results.

Theorem 2. *Metrics (1.1) possess Killing vectors if and only if either $a = 0$ (then rescale $b \rightarrow 1$), so that the spacetime is a plane wave, or $b = 0$ (then rescale $a \rightarrow 1$), so that g is Wils metric with $f(u) = (c_0 + c_1u + c_2u^2)^{-2}$.*

The same approach but with much heavier computations yields the following results.

Theorem 3. *Metrics (1.1) possess Killing 2-tensors different from the Hamiltonian H in the same range of parameters as for the Killing vectors, i.e., either $a = 0$ or $b = 0$, $f(u) = (c_0 + c_1u + c_2u^2)^{-2}$.*

The proofs and further specifications will be given in Section 3. The Maple & LinBox worksheets, which demonstrate our computations, can be found in a supplement to the arXiv version of this paper.

2. GEOMETRIC THEORY OF PDES

We start with the general setup. Let (N^n, g) be a pseudo-Riemannian manifold. In this section we formalize searching for Killing tensors (or polynomial integrals of the geodesic flow on the tangent bundle but we work on the cotangent bundle using raising/lowering indices with the metric g) via a compatibility analysis of an overdetermined PDE system and discuss the prolongation-projection technique.

2.1. Hamiltonian formalism. The energy function $H = \frac{1}{2}\|p\|_g^2$ writes in local coordinates

$$H(x, p) = \frac{1}{2}g^{ij}(x)p_i p_j \quad [g^{ij}] = [g_{ij}]^{-1}.$$

It is well-known that geodesics of g are projections to the base N of trajectories of the corresponding Hamiltonian vector field $X_H = \omega^{-1}dH$ on $T^*N \cong TN$, where ω is the canonical symplectic form on the cotangent bundle.

Integrating the equations of geodesics requires conserved quantities for this Hamiltonian system. A function $I : T^*N \rightarrow \mathbb{R}$ is an *integral* (of motion) $X_H(I) = 0$ if it Poisson commutes with the Hamiltonian:

$$\{H, I\} = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial I}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial I}{\partial p_i} \right) = 0.$$

The natural action of the isometry group on the cotangent bundle T^*N is Hamiltonian and it preserves the energy H . Thus, the isometries represent infinitesimal symmetries of the geodesic flow given, by virtue of Noether's theorem, by *linear in momenta* integrals of motion. Explicitly, if $X = X^i(x)\partial_{x^i} \in \mathfrak{iso}(N, g)$ is a Killing vector field then the corresponding integral is $I(x, p) = \langle X, p \rangle = X^i(x)p_i$.

More generally, a *Killing tensor* of degree d corresponds to a *homogeneous in momenta* polynomial

$$(2.1) \quad I_d := a^{i_1 \dots i_d}(x) p_{i_1} \cdots p_{i_d},$$

which Poisson commutes with H , and is thus a polynomial integral. Since the Hamiltonian is quadratic in momenta, for any (2.1) the Poisson bracket $\{H, I_d\}$ is of degree $d+1$ in momenta. Consequently, Killing d -tensors correspond to solutions of a system of differential equations formed by vanishing of p -coefficients of the Poisson bracket, which we call the *Killing equation*,

$$(2.2) \quad \mathcal{E}_d := \{F = 0 : F \in \text{coeffs}_p(\{H, I_d\})\}.$$

This is an overdetermined system of linear first order PDEs on the coefficients $a^{i_1 \dots i_d}(x)$ of the Killing tensor. Actually, there are $\binom{n+d}{d+1}$ equations on $\binom{n+d-1}{d}$ unknown functions. Denote solutions to this system – the linear space of all Killing d -tensors – by K_d .

2.2. Jet spaces and equations. The notion of jet-space formalizes the computational device of truncated Taylor polynomials; we refer for details to [12]. If x^i are local coordinates on N then the jet-space $J^k N$ of k -jet of functions $u : N \rightarrow \mathbb{R}$ has local coordinates (x^i, u_σ) for multi-indices $\sigma = (i_1, \dots, i_n)$, $i_s \geq 0$, $|\sigma| = \sum i_s \leq k$. Similarly are defined jets of vector-valued functions, sections, etc. For a bundle $\pi : E \rightarrow N$ the jet-space of its sections is denoted by $J^k(N, E)$.

The space of k -jets of maps $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ will be simply denoted by $J^k(n, m)$. It is a bundle of rank $m \cdot \binom{n+k-1}{k}$ over n -dimensional base. Any map $u = (u^j) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ lifts to the jet-section $j^k u : \mathbb{R}^n \rightarrow J^k(n, m)$ given by $x^i \mapsto u_\sigma^j = \partial u^j(x) / \partial x^\sigma$.

Definition 4 (Geometric PDE). A partial differential equation of order k is a submanifold $\mathcal{E} \subseteq J^k(n, m)$. A *solution* of the PDE is defined to be a function $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that its k -jet $j^k u$ takes values in \mathcal{E} . A local solution is the same but defined on a domain $U \subset \mathbb{R}^n$. We denote by $\text{Sol}(\mathcal{E})$ the space of all (local) solutions of \mathcal{E} .

Elements of a k 'th order geometric PDE $\mathcal{E} \subseteq J^k(n, m)$ are solutions *up to order k* (at a point). To find the solutions of the PDE \mathcal{E} up to order $k+1$ and higher, we have to differentiate the defining equations. To encode the chain rule, we define the q 'th *total derivative* of $F : J^k \rightarrow \mathbb{R}^s$ to be a vector-function on J^{k+1} given by

$$(2.3) \quad D_q F := \frac{\partial F}{\partial x^q} + \sum_{j=1}^m \sum_{|\sigma| \leq k} \frac{\partial F}{\partial u_\sigma^j} \cdot u_{\sigma+1_q}^j.$$

(Here we use the notation $\sigma + 1_q$ for the multi-index obtained by adding 1 to the q 'th entry of σ .) Now, a point $(x^i, u_\sigma^j) \in J^{k+1}$ is said to be a solution of \mathcal{E} up to order $k+1$ if it satisfies the following system of equations:

$$\mathcal{E}^{(1)} := \left\{ F(x^i, u_\sigma^j) = 0, (D_q F)(x^i, u_\sigma^j) = 0 \ \forall q = 1, \dots, n \right\}.$$

The resulting system of equations is called the *first prolongation* of \mathcal{E} . By construction, a solution of the prolongation $\mathcal{E}^{(1)}$ is still a solution of \mathcal{E} . We inductively define the l 'th prolongation by $\mathcal{E}^{(l)} = (\mathcal{E}^{(l-1)})^{(1)} \subset J^{k+l}$. It corresponds to solutions *up to order $k+l$* (at a point).

Definition 5 (Finite Type). A PDE $\mathcal{E} \subseteq J^k(n, m)$ is called of *finite type l* if after l prolongations all the highest order derivatives of the dependent variables can be expressed algebraically in terms of the lower order derivatives. A PDE is called of *Frobenius type* if it is of finite type 0.

Given a PDE \mathcal{E} of finite type, it is readily seen that the space of formal solutions $\mathcal{E}^{(\infty)}$ is necessarily finite-dimensional. This implies that (under some regularity conditions) the solution space $\text{Sol}(\mathcal{E})$ is finite-dimensional.

The Killing PDE is represented by a first order system $\mathcal{E}_d \subset J^1(N, S^d T N)$. The following fundamental result is well-known, cf. [24] and [28].

Theorem 6 (Killing PDE is of Finite Type). *The PDE \mathcal{E}_d defining a Killing d -tensor is a first order linear PDE of finite type d with $\text{Sol}(\mathcal{E}_d) = K_d$. This equation and its prolongations possess no compatibility conditions before achieving Frobenius type.*

2.3. Prolongation-projection. Let $\mathcal{E} = \{F(x^i, u_\alpha^j) = 0\} \subseteq J^k(n, m)$ be a PDE of order k . Its solution up to order k can be extended to order $(k+l)$ if and only if it belongs to the projection of the prolongation $\pi_{k+l, k}(\mathcal{E}^{(l)}) \subseteq \mathcal{E}$. In the case of equality here, every k -jet solution can be extended to a $(k+l)$ -jet solution. In the opposite case, there is a linear combination of iterated total derivatives up to order l , $\square(F) = \sum_{|\tau| \leq l} a^\tau D_\tau F$, which has order k .

Definition 7 (Compatibility). A *compatibility condition* of \mathcal{E} is an equation defining $\pi_{k+l, k}(\mathcal{E})$ that is algebraically independent of F and that is satisfied by all formal solutions.

Associated to a PDE is the Cartan distribution. Solutions arise as integral manifolds of this distribution, cf. [12]. Therefore, in the PDE setting, Frobenius theorem implies:

Theorem 8 (Frobenius Theorem). *Solutions of a PDE $\mathcal{E} \subseteq J^k(n, m)$ of finite type l are determined uniquely by their $(k+l-1)$ -jets. If in addition \mathcal{E} has no compatibility conditions, then for every $\xi \in \mathcal{E}^{(l)}$ there exists a local solution $u \in \text{Sol}(\mathcal{E})$ satisfying $j_x^{k+l} u = \xi$.*

Note that if a PDE is of finite type, then all of its prolongations are finite type as well. There is the following more general claim, which holds also true in infinite type case, under the assumption of analyticity of the equation.

Theorem 9 (Cartan's Involution). *There exists $q \in \mathbb{N}$ such that $\mathcal{E}^{(q)}$ is compatible.*

Thus, in regular domains, there are only finitely many compatibility conditions. However to find them explicitly is generally difficult, and bringing to involution in practice is a formidable computation. We therefore substitute searching for involution by the following criterion. For the finite type l case: *If $\pi : \mathcal{E}^{(r)} \rightarrow \mathcal{E}^{(r-1)}$ is surjective for some $r > l$, then $\mathcal{E}^{(r-1)}$ is compatible.* This is especially simple for linear overdetermined PDEs: over regular domains $U \subset N$ such \mathcal{E} are vector bundles and on each step of the prolongation-projection a compatibility condition reduces its rank; once this rank is stabilized for one step, then by Theorem 8 the system is compatible, so the involution level q of Theorem 9 is achieved.

2.4. Algorithmic implementation. The above criterion allows for an effective implementation of evaluation of $\dim K_d$ for a given metric g using computer algebra systems.

The Killing PDE \mathcal{E}_d as well as its prolongations $\mathcal{E}_d^{(k)}$ are linear in $(k+1)$ -jets of the dependent variables. We convert this linear system of equations to a matrix-valued function $M_k(x)$ on the spacetime. For our class of metrics g the entries are polynomials with rational coefficients. Hence to make use of computer algebra software, we insert a *rational* point $x_0 \in N$ to obtain a matrix with *rational* coefficients (in this case computer calculations are exact!).

The first thing to do is to find the points that work nicely with Cartan's prolongation-projection method. We call a point $x_0 \in N$ *regular* if the function $x \mapsto \text{rank}(M_k(x))$ attains its maximum at x_0 for all $k \geq 0$, that is, at each step we find the maximal number of compatibility conditions. Note that a regular point is generic, i.e., the set of regular points is an open dense subset of N . A point is *singular* if it is not regular.

Algorithm 1. (Cartan's Prolongation Method for Geodesic Flow).

(**Input:** A nonnegative integer d , a regular point x_0 .)

- Step 1.) Compute the Poisson bracket $\{H, I_d\}$ of a polynomial in momenta p function I_d with the Hamiltonian H .
 - Step 2.) Collect the coefficients of $\{H, I_d\}$ with respect to the momentum variables. Define the first order linear PDE $\mathcal{E}_d := \{F = 0 : F \in \text{coeffs}_p(\{H, I_d\})\}$.
 - Step 3.) Set $k := 0$.
 - Convert the linear system of equations $\mathcal{E}_d^{(k)}$ w.r.t. the variables $\mathcal{V}_{k+1,d} := \{a_\alpha^{i_1 \dots i_d} : |\alpha| \leq k+1\}$ into a matrix $M_k(x)$ that depends on the x -coordinates.
 - Substitute x_0 to obtain a matrix $M_k := M_k(x_0)$, the k 'th *prolongation matrix*.
 - Set $\delta_k := \text{columns}(M_k) - \text{rank}(M_k)$.
- If $(k \leq d)$ or $(k > d$ and $\delta_k \neq \delta_{k-1})$, increase k by 1 and repeat Step 3.
- Step 4.) Return (δ_k, k) .

(**Output:** The dimension of the space of Killing d -tensors is $\dim K_d = \delta_k$. The integer k indicates the number of prolongations necessary to find all compatibility conditions of \mathcal{E} .)

Proposition 10. *Algorithm 1 is correct and it terminates.*

Proof. Termination is clear. We now justify correctness, i.e., that the algorithm computes the number of Killing d -tensors. Turn the prolongation matrix M_k into row reduced echelon form. The equation $\mathcal{E}_d^{(k)}$ is linear and its rank as a bundle over N , equal to $\text{columns}(M_k)$, counts the number of $(k+1)$ -jets of dependent variables. Rows of the matrix represent equations defining $\mathcal{E}_d^{(k)}$, so they consist of the original Killing PDE, their differential corollaries and compatibility conditions. Consequently, δ_k is number of free jets (coordinates on fibers of the equation $\mathcal{E}_d \rightarrow N$). In view of the Frobenius theorem, each free variable corresponds to a $(k+1)$ jet-solution of the Killing PDE.

Now consider the conditions in step 3 determining termination of the loop. The first part $(k \leq d)$ addresses whether the prolongation has achieved Frobenius type, see Theorem 6. The second part $(k > d$ and $\delta_k \neq \delta_{k-1})$ checks whether all compatibility conditions have been computed, as guaranteed by the criterion after Theorem 9. Thus every $(k+1)$ jet yields a local solution. \square

2.5. Syzygies and Irreducible Killing Tensors. The pointwise multiplication of functions gives rise to a linear map

$$(2.4) \quad K_{d_1} \otimes K_{d_2} \mapsto K_{d_1+d_2}, \quad I_{d_1} \otimes I_{d_2} \mapsto I_{d_1} \cdot I_{d_2}.$$

A *relation (syzygy)* among Killing tensors of rank d_1 and d_2 with $d_1 \neq d_2$ is an element of the kernel of the map

$$(2.5) \quad K_{d_1} \otimes K_{d_2} \rightarrow K_{d_1+d_2}.$$

If $d_1 = d_2 =: d$ a relation is given by an element in the kernel of the map $S^2K_d \rightarrow K_{2d}$.

A Killing d -tensor ($d \geq 2$) is *irreducible* if it is not a linear combination of the symmetric product of lower rank Killing tensors. The number of irreducible Killing d -tensors can be found using the number of syzygies. We demonstrate this for Killing 2-tensors. The space of irreducible Killing 2-tensors can be identified with the cokernel of map $\iota_2 : S^2K_1 \rightarrow K_2$, fitting into a short exact sequence

$$(2.6) \quad 0 \longrightarrow \text{Ker } \iota_2 \longrightarrow S^2K_1 \rightarrow K_2 \longrightarrow \text{Coker } \iota_2 \longrightarrow 0.$$

The space of irreducible Killing 3-tensors can be identified with the cokernel of the map $\iota_3 : K_1 \otimes K_2 \rightarrow K_3$, etc. The number of syzygies among Killing tensors is found as follows. (We use the notation $\text{Taylor}(a(x), x_0, k)$ for the Taylor polynomial of the function a around x_0 up to order k .)

Algorithm 2. (Finding Relations among Killing Tensors).

(**Input:** Nonnegative integers d_1, d_2 , a regular point x_0 .)

- Step 1.) For $s = 1, 2$: run algorithm 1 obtain $\dim K_{d_s}$ and the number of prolongation k_s needed to achieve compatibility.

Consider the polynomial $I_{k_s+1, d_s} := \text{Taylor}(a^{i_1 \dots i_{d_s}}, x_0, k_s + 1) \cdot p_{i_1} \dots p_{i_{d_s}}$.

- Step 2.) Consider the linear algebraic system of equations $\{\text{Taylor}(c, x_0, k_s) = 0 : c \in \text{coeffs}_p(\{H, I_{k_s+1, d_s}\})\}$ on the variables $\mathcal{V}_{k_s+1, d_s}(x_0) := \{a_\alpha^{i_1 \dots i_{d_s}}(x_0) : |\alpha| \leq k_s + 1\}$ for $s = 1, 2$. Solve these linear equations and substitute the corresponding solutions into I_{k_s+1, d_s} to obtain the truncated integrals I_{k_s+1, d_s}^j for $1 \leq j \leq \dim K_{d_s}$.

- Step 3.) Set

$$T := \sum_{l_1=1}^{\dim K_{d_1}} \sum_{l_2=1}^{\dim K_{d_2}} c_{l_1, l_2} I_{k_1+1, d_1}^{l_1} \cdot I_{k_2+1, d_2}^{l_2}.$$

Define $S := \{\text{Taylor}(c, x_0, d_1 + d_2) : c \in \text{coeffs}_p(T)\}$.

- Step 4.) Solve the linear algebraic system of equations $\{F = 0 : F \in \text{coeffs}_x(S)\}$ in terms of the coefficients c_{l_1, l_2} , and denote the resulting solution space R .
- Step 5.) Return R and $\dim R$.

(**Output:** Relations among Killing tensors of rank d_1 and d_2 ; # (indep) syzygies = $\dim R$.)

Proposition 11. *Algorithm 2 is correct and it terminates.*

Proof. For $d \geq 1$, consider the Killing PDE $\mathcal{E}_d \subseteq J^1$. A $(k+1)$ -jet $j_{x_0}^{k+1}u$ of a vector-function $u = (a^{i_1 \dots i_d}(x))$ can be identified with the Taylor polynomial $I_{k+1, d} = \text{Taylor}(a^{i_1 \dots i_d}, x_0, k+1)p_{i_1} \dots p_{i_d}$. Under this correspondence, we have that $j_{x_0}^{k+1}u \in \mathcal{E}^{(k)}$ if and only if $\{H, I_{k+1, d}\}$ vanishes up to order k at x_0 . These observations explain steps 1 and 2.

By Theorem 6 K_d is determined by d -jets in the sense that we can compute all jets of a Killing d -tensor at a point if we know its d -jet. Thus, in step 3 we must include the jets up to order $d_1 + d_2$ in order to determine uniquely the corresponding $(d_1 + d_2)$ -tensor. \square

Application. In practice we apply algorithm 2 as follows. First, using algorithm 1 we compute the dimensions of S^2K_1 and K_2 . Then we use algorithm 2 to determine the dimension of the kernel $\text{Ker } \iota_2$. Finally, the number of (lin. independent) irreducible Killing 2-tensors is given by

$$\dim \text{Coker } \iota_2 = \dim K_2 - \dim S^2K_1 + \dim \text{Ker } \iota_2.$$

This method can be readily generalized to higher order Killing tensors.

Regular and singular points. Even though the regular points are dense, it is difficult to verify (in practice) that a given point is regular. Thus, we must be careful in order to get rigorous results. For a singular point, algorithm 1 gives an upper bound on the number of Killing tensors. The number of syzygies imply lower bounds on the number of Killing tensors (indeed, the syzygies imply the number of reducible Killing tensors). Thus, whenever the algorithms suggest the existence of an irreducible Killing tensor it is important to find it explicitly. (For our metrics g it turns out to be possible to find the irreducible Killing tensors explicitly using Maple's `pdsolve`.)

2.6. Note on the computability of the algorithm. We briefly discuss the computational difficulties associated with the proposed method and how we deal with them. Dimension of the prolongation matrix M_k from algorithm 1 equals

$$\text{rows}(M_k) = \binom{n+d}{d+1} \cdot \binom{n+k}{n}, \quad \text{columns}(M_k) = \binom{n+d-1}{d} \cdot \binom{n+k+1}{n}.$$

In particular, we see that the number of rows grows faster with k than the number of columns. We highlight several elements that have made the computer implementation more efficient:

- **(LinBox).** The LinBox package [18] in Sage allows for incredibly fast rank computations of large sparse integer matrices. For example, computing the rank of the quartic prolongation matrix M_{19} for metric 2 with size $(495880) \times (371910)$ took less than an hour. In comparison, rank computations of smaller matrices (say 50000 by 40000) would take several days in Maple or not give a result at all. Thanks to LinBox, the time to compute the ranks is negligible. Generating a prolongation matrix takes by far the longest time of the steps in algorithm 1.
- **(Exploiting Sparsity.)** The prolongation matrices M_k that we encounter here are sparse (with density < 0.001). It is important that the generation of the matrix reflects this. We generate the initial matrix with all entries zeroes and then substitute the nonzero values.
- **(Combinatorial Description of Prolongations.)** For the quartic case, we used a combinatorial description of the prolongation equations. We demonstrate this for metric 2. Since I_4 is of degree 4, we have that $\{H, I_4\}$ is of degree 5 in momenta. Thus, we can write $\{H, I_4\} = \text{coeff}_\tau p^\tau$ where $p^\tau = p_1^{\tau_1} p_2^{\tau_2} p_3^{\tau_3} p_4^{\tau_4}$. Given a multi-index τ of length 5, we obtain the p^τ -coefficient in terms of the coefficients of I :

$$\begin{aligned} \text{coeff}_\tau(\{H, I_4\}) &= 2\partial_1(a^{\tau-1_1}) + 2\partial_2(a^{\tau-1_2}) - 2\partial_4(a^{\tau-1_3}) - 2\partial_3(a^{\tau-1_4}) \\ &\quad + 4x^3((x^1)^2 + (x^2)^2)\partial_4(a^{\tau-1_4}) - 2((x^1)^2 + (x^2)^2)\frac{(\tau+1_3-2\cdot 1_4)!}{(\tau-2\cdot 1_4)!} \\ &\quad - 4x^1x^3\frac{(\tau+1_1-2\cdot 1_4)!}{(\tau-2\cdot 1_4)!}a^{\tau+1_1-2\cdot 1_4} - 4x^2x^3\frac{(\tau+1_2-2\cdot 1_4)!}{(\tau-2\cdot 1_4)!}a^{\tau+1_2-2\cdot 1_4}. \end{aligned}$$

Using the Leibniz rule for multi-index notation, we can subsequently determine the general expression for the derivative $\partial^\alpha(\text{coeff}_\tau(\{H, I_4\}))$, where α is a multi-index. In this way we obtain the equations of the prolongation as a function of the multi-indices τ and α . This combinatorial description significantly reduces the time needed to generate the equations in Maple, especially as the order increases. This approach is most beneficial for Hamiltonians which are polynomials of low order in the independent x -variables. (For the Kerr metric, for example, these combinatorics would be unfeasible.)

3. COMPUTATIONS AND RESULTS

Here we discuss the results of concrete computations with the above algorithms. We begin with investigations of the special cases of pp-waves and Wils metric and then discuss the general case.

3.1. Conformally Flat pp-Waves. These are given by the following formula:

$$(3.1) \quad g = 2dx^3dx^4 + (f(x^3)((x^1)^2 + (x^2)^2))(dx^3)^2 - (dx^1)^2 - (dx^2)^2$$

Sippel and Goenner classified pp-waves in terms of their isometry groups [22]. For conformally flat pp-waves there are three classes: $f(x^3) = c$, $f(x^3) = c(x^3)^{-2}$ and the generic case with $\dim K_1 = 6$. We apply our prolongation-projection algorithm to the following four metrics (rescaling of f does not play a role for the first three metrics):

$$(i) f(x^3) = 1, \quad (ii) f(x^3) = x^3, \quad (iii) f(x^3) = (x^3)^2, \quad (iv) f(x^3) = 2(x^3)^{-2}.$$

If two subsequent values δ_k, δ_{k+1} are equal (with $k \geq d$), the sequence of δ -values stabilizes and we can read off the number of Killing d -tensors. In the table this is shown by circling this δ -value.

We see that metrics 1 and 4 have 7-dimensional isometry, which is consistent with the classification by Sippel and Goenner. Note that for the quartic case of metrics 2 and 3 we have to go all the way to the 19'th prolongation of the Killing PDE. The number of equations and variables at this stage are so large that it is unlikely that we can compute the number of Killing 5-tensors with present computational powers.

Linear	\mathcal{E}	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(3)}$	Linear	\mathcal{E}	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(3)}$
δ	10	10	7	(7)	...	δ	10	10	7	6	(6)
Quadratic	\mathcal{E}	...	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	Quadratic	\mathcal{E}	...	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$
δ	30	...	29	28	(28)	δ	30	...	24	22	(22)
Cubic	\mathcal{E}	...	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	Cubic	\mathcal{E}	...	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	$\mathcal{E}^{(13)}$
δ	65	...	87	84	(84)	δ	65	...	63	62	(62)
Quartic	\mathcal{E}	...	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	Quartic	\mathcal{E}	...	$\mathcal{E}^{(17)}$	$\mathcal{E}^{(18)}$	$\mathcal{E}^{(19)}$
δ	119	...	211	210	(210)	δ	119	...	150	148	(148)

Linear	\mathcal{E}	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(3)}$	Linear	\mathcal{E}	$\mathcal{E}^{(1)}$	$\mathcal{E}^{(2)}$	$\mathcal{E}^{(3)}$	$\mathcal{E}^{(3)}$
δ	10	10	7	6	(6)	δ	10	10	7	(7)	...
Quadratic	\mathcal{E}	...	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	Quadratic	\mathcal{E}	...	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	30	...	24	22	(22)	δ	30	...	29	28	(28)
Cubic	\mathcal{E}	...	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$	$\mathcal{E}^{(13)}$	Cubic	\mathcal{E}	...	$\mathcal{E}^{(6)}$	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$
δ	65	...	63	62	(62)	δ	65	...	87	84	(84)
Quartic	\mathcal{E}	...	$\mathcal{E}^{(17)}$	$\mathcal{E}^{(18)}$	$\mathcal{E}^{(19)}$	Quartic	\mathcal{E}	...	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	$\mathcal{E}^{(12)}$
δ	119	...	150	148	(148)	δ	119	...	211	210	(210)

TABLE 1. Left up: Metric (i) $f(x^3) = 1$;
Left down: Metric (iii) $f(x^3) = (x^3)^2$;

Right up: Metric (ii) $f(x^3) = x^3$.
Right down: Metric (iv) $f(x^3) = 2(x^3)^{-2}$.

3.2. Syzygies and irreducible Killing tensors for pp-waves. In order to find the number of irreducible Killing tensors, we have to take into account the number of syzygies among the Killing tensors. We demonstrate this for metric 2 (the other cases are similar). Consider the following short exact sequence

$$0 \longrightarrow \underbrace{\text{Ker } \iota_2}_{1 \text{ syzygy}} \longrightarrow \underbrace{S^2 K_1}_{21\text{-dim.}} \xrightarrow{\iota_2} \underbrace{K_2}_{22\text{-dim.}} \longrightarrow \underbrace{\text{Coker } \iota_2}_{2 \text{ irreducible Killing 2-tensors}} \longrightarrow 0$$

Algorithm 2 gives $\dim \text{Ker } \iota_2 = 1$, and so there are 2 irreducible Killing 2-tensors. Next, we consider

$$0 \longrightarrow \underbrace{\text{Ker } \iota_3}_{70 \text{ syzygies}} \longrightarrow \underbrace{K_1 \otimes K_2}_{132\text{-dim.}} \xrightarrow{\iota_3} \underbrace{K_3}_{62\text{-dim.}} \longrightarrow \underbrace{\text{Coker } \iota_3}_{0 \text{ irreducible Killing 3-tensors}} \longrightarrow 0$$

Algorithm 2 gives $\dim \text{Ker } \iota_3 = 70$ and so there are no irreducible Killing 3-tensors. Since there are no irreducible Killing 3-tensors, the source space of ι_4 is the second symmetric power $S^2 K_2$. (If there were irreducible Killing 3-tensors, then the source space would be $K_1 \otimes K_3 \oplus S^2 K_2$.) Thus we obtain the short exact sequence

$$0 \longrightarrow \underbrace{\text{Ker } \iota_4}_{105 \text{ syzygies}} \longrightarrow \underbrace{S^2 K_2}_{253\text{-dim.}} \xrightarrow{\iota_4} \underbrace{K_4}_{148\text{-dim.}} \longrightarrow \underbrace{\text{Coker } \iota_4}_{0 \text{ irreducible Killing 4-tensors}} \longrightarrow 0$$

There are 105 syzygies, it follows that there are no irreducible Killing 4-tensors.

For metrics (i), (ii) and (iii) we obtain that there exists one irreducible Killing 2-tensor, in addition to the Hamiltonian. Actually, we can explicitly write this Killing 2-tensor as follows:

$$(3.2) \quad J := -x^3 H + x^1 p_1 p_4 + x^2 p_2 p_4 + 2x^4 p_4^2.$$

For metric (iv) the only irreducible Killing 2-tensor is the Hamiltonian H , i.e., the Killing 2-tensor J is reducible in this case (due to the existence of an extra Killing vector).

In the general case (3.1) for $f(u) \neq c, cu^{-2}$ the Killing vectors are the following:

$$(3.3) \quad I_1 = p_1 x^2 - p_2 x^1, \quad I_2 = p_4, \quad I_{3,4} = a_{1,2}(x^3) p_1 + a'_{1,2}(x^3) x^1 p_4, \quad I_{5,6} = a_{1,2}(x^3) p_2 + a'_{1,2}(x^3) x^2 p_4,$$

where a_i ($i = 1, 2$) are fundamental solutions of the linear second order ODE $a'' + fa = 0$, i.e., solutions satisfying the initial conditions $a_1(0) = 1, a'_1(0) = 0, a_2(0) = 0, a'_2(0) = 1$. The Hamiltonian is equal to

$$(3.4) \quad H = 2p_3 p_4 - p_1^2 - p_2^2 - ((x^1)^2 + (x^2)^2) f(x^3) p_4^2$$

and the other quadratic integral J is given by (3.2) (also for general f). These results are consistent with the following theorem by Keane and Tupper [8] that was proven using the Koutras algorithm [10] (our approach is different).

Theorem 12 ([8]). *A conformally flat pp-wave with $\dim K_1 = 6$ or with $f(x^3) = c$ admits an irreducible Killing 2-tensor, independent of the (irreducible) Hamiltonian H .*

By using our computational algorithm we can also establish the nonexistence results of higher order Killing tensors for these conformally flat pp-waves.

Theorem 13. *A conformally flat pp-wave (3.1) with $f(u) = cu^m$, $m = 0, 1, 2$, or $f(u) = 2u^{-2}$, admits no irreducible Killing 3- and 4-tensors.*

Proof. The result follows straightforwardly from the above computations and a rescaling argument. \square

Corollary 14. *For a generic conformally flat pp-wave (3.1) all 3- and 4- Killing tensors are combinations of Killing vectors (3.3), the Hamiltonian H (3.4) and the Killing 2-tensor J (3.2).*

Here f is generic in C^{13} topology for Killing 3-tensors and in C^{19} topology for Killing 4-tensors, see Table 1 for $k = k_d$, however we believe that also holds in lower regularity by the approach of [14].

Proof. It follows from our computations and algebraic dependence of the matrix M_k on $j^{k+1}f$ that $\dim K_i$ ($i = 2, 3, 4$) is upper semi-continuous in this jet. Hence, for a generic $f(x^3)$ the dimension of K_2, K_3, K_4 are as indicated in the third term of the above short exact sequences. Due to full control of K_1, K_2 the second terms have dimensions as indicated. Dimension of the first term is also upper semi-continuous, so for a generic $f(x^3)$ we have at most the indicated number of syzygies. In fact, this number is realizable as follows.

In the case of Killing 2-tensor (first short exact sequence) the only syzygy is (verifying this exploits constancy of the Wronskian of a_1, a_2)

$$\mathfrak{S}_2 : I_1 I_2 + I_3 I_6 - I_4 I_5 = 0.$$

For Killing 3-tensor (second short exact sequence) the only 6 syzygies are $I_j \cdot \mathfrak{S}_2$ ($1 \leq j \leq 6$). To explain dimension 70 of the first term, note that kernel of the symmetrization operator $K_1 \otimes S^2 K_1 / K_1 \otimes \mathfrak{S}_2 \rightarrow S^3 K_1$ is 64-dimensional. Similarly one justifies the case of Killing 4-tensor (third short exact sequence).

Actually, we can also obtain the claim from the fact that the functional rank of 8 functions I_j, H, J is 7, while that of I_j is 5. Thus no syzygies can involve H, J and the only syzygy among 6 Killing vectors I_i is given by \mathfrak{S}_2 . \square

3.3. Absence of Killing Tensors for the Wils Metric. The Wils metric is given by

$$(3.5) \quad g = 2x^1 dx^3 dx^4 - 2x^4 dx^1 dx^3 + (f(x^3)x^1((x^1)^2 + (x^2)^2) - (x^4)^2)(dx^3)^2 - (dx^1)^2 - (dx^2)^2.$$

We apply our prolongation-projection algorithm to the following three cases: $f(u) = u^m$, $m = 0, 1, 2$. The results are displayed in the following table.

Linear	...	$\mathcal{E}^{(4)}$	$\mathcal{E}^{(5)}$	Linear	...	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	Linear	...	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	...	1	(1)	δ	...	0	(0)	δ	...	0	(0)
Quadratic	...	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	Quadratic	...	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$	Quadratic	...	$\mathcal{E}^{(5)}$	$\mathcal{E}^{(6)}$
δ	...	2	(2)	δ	...	1	(1)	δ	...	1	(1)
Cubic	...	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	Cubic	...	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$	Cubic	...	$\mathcal{E}^{(7)}$	$\mathcal{E}^{(8)}$
δ	...	2	(2)	δ	...	0	(0)	δ	...	0	(0)
Quartic	...	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$	Quartic	...	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$	Quartic	...	$\mathcal{E}^{(8)}$	$\mathcal{E}^{(9)}$
δ	...	3	(3)	δ	...	1	(1)	δ	...	1	(1)
Quintic	...	$\mathcal{E}^{(9)}$	$\mathcal{E}^{(10)}$	Quintic	...	$\mathcal{E}^{(9)}$	$\mathcal{E}^{(10)}$	Quintic	...	$\mathcal{E}^{(9)}$	$\mathcal{E}^{(10)}$
δ	...	3	(3)	δ	...	0	(0)	δ	...	0	(0)
Sextic	...	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	Sextic	...	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$	Sextic	...	$\mathcal{E}^{(10)}$	$\mathcal{E}^{(11)}$
δ	...	4	(4)	δ	...	1	(1)	δ	...	1	(1)

TABLE 2. Metric (i) $f(x^3) = 1$; Metric (ii) $f(x^3) = x^3$; Metric (iii) $f(x^3) = (x^3)^2$.

Theorem 15. *The Wils metric (3.5) for $f(u) = u^m$, $m = 0, 1, 2$, admits no Killing tensors up to degree 6 except for powers of the Hamiltonian.*

This statement follows directly from Table 2. It also implies that for generic values of the functional parameter f there are no lower degree Killing tensors. Now we want to be more specific on those exceptional parameters.

Theorem 16. *The Wils metric admits Killing vectors if and only if f is of the form*

$$(3.6) \quad f(x^3) = (c_0 + c_1x^3 + c_2(x^3)^2)^{-2}.$$

In this case the Killing vector is unique up to scale and is given by the formula

$$(3.7) \quad X := (c_0 + c_1x^3 + c_2(x^3)^2) \partial_{x^3} - (2c_2x^1 + c_1x^4 + 2c_2x^3x^4) \partial_{x^4}.$$

Proof. In order to simplify the calculations we evaluate at $x^1 = 1, x^2 = 2, x^4 = 4$ but leave x^3 general.

Step 1 and 2.) Using, the equations defining the PDE \mathcal{E} , we express the 1-jets $a_1^1, a_2^1, a_3^1, a_4^1, a_3^2, a_4^2, a_2^3, a_3^3, a_4^3, a_1^4, a_2^4$ in terms of the free variables $a^1, a^2, a^3, a^4, a_1^2, a_2^2, a_1^3, a_3^3, a_4^3, a_4^4$ and the function $f(x^3)$.

Step 3.) For the first prolongation $\mathcal{E}^{(1)}$, we can express all 2-jets in terms of lower order jets without making any assumptions on f .

Step 4.) Consider $\mathcal{E}^{(2)}$. If we assume that $f \neq 0$, we obtain the following compatibility conditions:

$$a_1^3 = 0, \quad a_3^3 = -\frac{a^1 f + f' a^3}{2f}, \quad a_4^4 = 0.$$

We are left with 7 free jet variables. For $\mathcal{E}^{(3)}$, we obtain the additional compatibility conditions:

$$a^1 = 0, \quad a_2^2 = a_3^4, \quad a_1^2 = \frac{2a^2 f^2 - 4a^3 f f' + 2a^3 f f'' - 3a^3 (f')^2}{6f^2}.$$

We are left with 4 free variables. The prolongation $\mathcal{E}^{(4)}$ gives three additional compatibility conditions: $a_3^4 = 0$ and two expressions for a^2 and a^4 . Only 1 free variable a^3 remains, and the next prolongation $\mathcal{E}^{(5)}$ does not give an additional compatibility condition if and only if f is a solution of the ODE

$$(3.8) \quad f''' = \frac{18f f' f'' - 15(f')^3}{4f^2}.$$

Resolving this ODE gives the required formula (3.6). Expression (3.7) follows. \square

The following theorem is proven in the same manner, but the number of steps is larger, so the proof is omitted.

Theorem 17. *The Wils metric has Killing 2-tensors if and only if it has nontrivial Killing vectors. This happens only for the functional parameter (3.6); in this case, denoting $I_1 = \langle X, p \rangle$ the linear integral corresponding to (3.7), the general quadratic integral is a linear combination $k_1 I_1^2 + k_2 H$.*

3.4. General Koutras-McIntosh metrics. Investigation of the general metric (1.1) follows the same scheme. First of all, the computation in the previous section implies that the matrix M_k of the prolonged Killing PDE for degree $d \leq 6$ tensors has minimal possible value for δ_k , i.e., 0 for odd d and 1 for even d . This implies Theorem 1.

To obtain Theorems 2 and 3 we can perform general computation with symbolic matrix for the prolongation $\mathcal{E}^{(6)}$ when $d = 1$ and $\mathcal{E}^{(7)}$ when $d = 2$. The matrix M_k has size 1260×840 for $d = 1$ and 4200×3300 for $d = 2$. To compute its rank we use the idea exploited in [17], namely successively identifying rows or columns with few non-zero terms (this means ≤ 2 for $d = 1$ and ≤ 8 for $d = 2$) and doing Gauss elimination, while storing the involved factors to check their vanishing separately. This gives the splitting $a = 0$ or $b = 0$ and the rest follows from Theorem 16. In fact, this computation also yields equation (3.8).

The exceptional functional parameters $f(u)$ in (3.6) up to transformations $u \rightarrow ku + b$ (change of coordinates: $x^1 \mapsto \lambda x^1, x^2 \mapsto \lambda x^2, x^3 \mapsto kx^3 + b, x^4 \mapsto \lambda x^4/k, g \mapsto \lambda^2 g, f \mapsto f/(\lambda k^2)$) give the following different cases

$$f(u) = 1, \quad f(u) = u^{-1}, \quad f(u) = cu^{-2}, \quad f(u) = u^{-4}, \quad f(u) = |u^2 \pm 1|^{-2}.$$

In each of these cases one can directly verify there are no irreducible Killing 3- or 4-tensors (for the middle case this was only verified for a generic parameter c), i.e. all of them are algebraic combinations of I_1 and H .

4. OUTLOOK

In this paper we obtain the nonexistence of Killing tensors of degrees d up to 6 for the Koutras-McIntosh spacetimes for generic parameters. This complements the previous result on the nonexistence of Killing vectors [11]. The problem of existence of higher order $d > 6$ Killing tensors remains open. The size of the involved matrices (163800×152880 for $d = 6$) does not allow further computational progress, and we have to stress that our success for metrics (1.1) is related to sparsity of the corresponding matrices M_k and rationality of their entries in coordinates and parameters.

The complexity of computations carried here is much higher than that in preceding works [13, 17, 25]; actually those possessed Killing vectors allowing to reduce the PDE setup to that on a 2-dimensional manifold, while our setup here is fully 4-dimensional (that is why the size of the matrix M_k of $\mathcal{E}_d^{(k)}$ grows much faster). Other works [6, 8, 9], addressing Killing 2-tensors, have in similar vein reductions to ODEs (that is, differential equations on a 1-dimensional manifold), so our work on higher degree d Killing tensors is apparently novel.

One may envision that the following approach is feasible for large d . Consider the Killing PDE \mathcal{E}_d with $\text{Sol}(\mathcal{E}_d) = K_d$. This is an overdetermined system and a compatibility analysis gives the dimension of K_d depending on certain rank invariants. Those depend on vanishing of some relative invariants. Since the construction involves only invariant algebraic operations and all absolute polynomial invariants vanish, there are only few possibilities and the answer for higher d might be the same as that for $d = 1$. This is indeed confirmed by what we have investigated.

The nonexistence of polynomial integrals of low degree raises the question whether the geodesic flow of metrics (1.1) is integrable. Depending on the class of admissible integrals the methods to approach this problem are: differential Galois theory, Painlevé test, numerical simulations. None of these have been done yet.

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