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The three missing terms in Ramanujan's septic theta function identity

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Abstract
On page 206 in his lost notebook, Ramanujan recorded the following enigmatic identity for his theta function $\varphi(q)$:

$$\varphi(e^{-7\pi\sqrt{7}}) = 7^{-3/4}\varphi(e^{-\pi\sqrt{7}})\{1 + (\quad)^{2/7} + (\quad)^{2/7} + (\quad)^{2/7}\}.$$

We give the three missing terms. In addition, we calculate the class invariant G_{343} and further special values of $\varphi(e^{-n\pi})$, for $n = 7, 21, 35$, and 49.

Keywords Theta functions · Septic theta function identities · Ramanujan's lost notebook

Mathematics Subject Classification 33C05 · 05A30 · 11F32 · 11R29

1 Introduction

Ramanujan's general theta function $f(a, b)$ is defined by [24, p. 197], [4, p. 34]

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \tag{1}$$

which provides an alternative formulation [5, p. 3] for the classical theta functions $(\theta_i(z, q))_{i=1}^4$ in [34, pp. 462–465]. The symmetry reflected in the definition of $f(a, b)$ is inherited by its representation by the Jacobi triple product identity [17, pp. 176–183], [24, p. 197], [4, p. 35], which states that

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$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1, \tag{2}$$

where

$$(a;q)_\infty := \prod_{k=0}^{\infty}\left(1-aq^k\right), \quad |q| < 1.$$

In Ramanujan's notation, the theta function $\varphi(q)$ is defined by [24, p. 197], [4, p. 36]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2\left(q^2;q^2\right)_\infty, \quad |q| < 1, \tag{3}$$

where the series and product representations are straightforward from (1) and (2), respectively. Furthermore, we set [24, p. 197], [4, p. 37]

$$\chi(q) := \left(-q;q^2\right)_\infty, \quad |q| < 1. \tag{4}$$

If $q = \exp(-\pi\sqrt{n})$, for a positive rational number n, the class invariant G_n of Ramanujan [21], [22, pp. 23–39], [7, p. 183] and Weber [33, p. 114] is defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q). \tag{5}$$

For odd n, Ramanujan's values for G_n are listed in [7, pp. 189–199], Weber's list is in [33, pp. 721–726], and motivation is in [12]. The class invariants are algebraic [15, pp. 214, 257].

A fundamental result in the theory of elliptic functions is that for a positive rational n,

$$\varphi^2(e^{-\pi\sqrt{n}}) = {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;k_n^2\right) = \frac{2}{\pi}K(k_n). \tag{6}$$

Here, ${}_2F_1$ is the ordinary or Gaussian hypergeometric function [34, pp. 24, 281], K is the complete elliptic integral of the first kind [34, pp. 499–500], [4, p. 102], [9], and k_n is a singular value or singular modulus [34, pp. 525–527], [12, 14, 19], [7, p. 183] of the elliptic integral K. The singular values are algebraic [1]. Ramanujan used the notation $\alpha_n := k_n^2$ [12], [7, p. 183]. This statement is given more generally in [24, p. 207], [4, p. 101, Entry 6]. An overview of the theory of elliptic functions can be found in [4, p. 102], [9, 12], [7, pp. 323–324].

For a positive rational n, a positive integer d, and $q = \exp(-\pi\sqrt{n})$, in the theory of modular equations, the multiplier m of degree d is defined by

$$m := \frac{\varphi^2(q)}{\varphi^2(q^d)} = \frac{\varphi^2(e^{-\pi\sqrt{n}})}{\varphi^2(e^{-d\pi\sqrt{n}})}. \tag{7}$$

The multiplier m can be defined more generally, as given in [4, p. 230], [9, 12], [7, p. 324]. In our case, m defined in (7) is algebraic [37]. An overview of the theory of modular equations can be found in [4, pp. 213–214], [9, 12], [7, p. 185].

It is classical [34, pp. 524–525], and it was also discovered by Ramanujan [24, p. 207], [4, p. 103], [23, p. 248], [9], [7, p. 325], that

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)}{\sqrt{2}\pi^{3/4}}. \tag{8}$$

For a positive rational n, Ramanujan recorded his values for $\varphi(e^{-n\pi})$ in terms of $\varphi(e^{-\pi})$, but in view of (8), $\varphi(e^{-n\pi})$ is therefore determined explicitly. At scattered places in his notebooks, Ramanujan recorded some values of $\varphi(e^{-n\pi})$ when n is a power of two, namely, for $n = 1, 2, 4, 1/2, 1/4$ [23, p. 248], [7, p. 325]; and when $n \geq 3$ is an odd integer, namely, for $n = 3, 5, 7, 9$, and 45 [23, pp. 284, 285, 297, 287, 312], [9], [7, pp. 327–337]. The values at powers of two are parts of more general results from Ramanujan's second notebook [24, p. 210], [4, pp. 122–123]. The evaluations for the odd values were established by Berndt and Chan [9], [7, pp. 327–337]. They also determined the values for $n = 13, 27$, and 63.

Selberg and Chowla [26] showed that for any singular value k_n, the elliptic integral $K(k_n)$ is expressible in terms of gamma functions. J. M. Borwein and Zucker [14, 37], [13, p. 298] evaluated $K(k_n)$, for $n = 1, \ldots, 16$. Thus, by (6), we have the value of $\varphi(e^{-\pi\sqrt{n}})$ in these cases. We give two theta function values, corresponding to k_3 and k_7 [19, 38], respectively:

$$\varphi\left(e^{-\pi\sqrt{3}}\right) = \frac{3^{1/8}\Gamma^{3/2}\left(\frac{1}{3}\right)}{2^{2/3}\pi} \tag{9}$$

and

$$\varphi\left(e^{-\pi\sqrt{7}}\right) = \frac{\left\{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)\right\}^{1/2}}{\sqrt{2}\cdot 7^{1/8}\pi} = \frac{\sqrt{2}\left\{\left(\cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{3\pi}{7}\right)\right)B\left(\frac{1}{7}, \frac{2}{7}\right)\right\}^{1/2}}{7^{3/8}\sqrt{\pi}}, \tag{10}$$

where $B(x, y) := \Gamma(x)\Gamma(y)/\Gamma(x+y)$, for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, is the beta function [34, pp. 253–256].

If we would like to calculate $\varphi(e^{-\pi\sqrt{r}})$ explicitly, for a positive integer r, then if r is square-free and the corresponding values k_r and $K(k_r)$ are known, we can use (6). If r is not square-free and the value $\varphi(e^{-\pi\sqrt{n}})$ is known, where n is the square-free part of r and $\sqrt{r} = d\sqrt{n}$, for a positive integer d, then we can calculate $\varphi(e^{-d\pi\sqrt{n}})$, with appropriate modular equations of degree d, which contains the class invariants G_n and G_{d^2n}, with known explicit values, and the multiplier m. There are other particular methods, as we see next, in Entry 1.1.

On page 206 in his lost notebook [25], Ramanujan recorded the following identities.

Entry 1.1 (p. 206) *Let*

(i) $\frac{\varphi(q^{1/7})}{\varphi(q^7)} = 1 + u + v + w$. *Then*

(ii) $p := uvw = \frac{8q^2(-q;q^2)_\infty}{(-q^7;q^{14})^7_\infty}$ *and*

(iii) $\frac{\varphi^8(q)}{\varphi^8(q^7)} - (2+5p)\frac{\varphi^4(q)}{\varphi^4(q^7)} + (1-p)^3 = 0$. *Furthermore,*

(iv) $u = \left(\frac{\alpha^2 p}{\beta}\right)^{1/7}, \quad v = \left(\frac{\beta^2 p}{\gamma}\right)^{1/7}, \quad$ *and* $\quad w = \left(\frac{\gamma^2 p}{\alpha}\right)^{1/7}$, *where* α, β, *and* γ *are the roots of the equation*

(v) $r(\xi) := \xi^3 + 2\xi^2\left(1 + 3p - \frac{\varphi^4(q)}{\varphi^4(q^7)}\right) + \xi p^2(p+4) - p^4 = 0$. *For example,*

(vi) $\varphi(e^{-7\pi\sqrt{7}}) = 7^{-3/4}\varphi(e^{-\pi\sqrt{7}})\Big\{1 + (\quad)^{2/7} + (\quad)^{2/7} + (\quad)^{2/7}\Big\}$.

We remark that (i)–(v) hold for $|q| < 1$, with $q \neq 0$ in (iv). If $q = 0$, then $u = v = w = 0$. Parts (i)–(v) were proved by Son [27], [2, pp. 180–194], [28, pp. 198–200],

Part (i) is recorded in Ramanujan's second notebook [24, p. 239], [4, p. 303] as well, in the form of

$$\varphi\left(q^{1/7}\right) - \varphi(q^7) = 2q^{1/7} f(q^5, q^9) + 2q^{4/7} f(q^3, q^{11}) + 2q^{9/7} f(q, q^{13}),$$

from where the values of u, v, and w can be determined [27], [2, p. 181], [28, p. 198] as

$$u := 2q^{1/7}\frac{f(q^5, q^9)}{\varphi(q^7)}, \qquad v := 2q^{4/7}\frac{f(q^3, q^{11})}{\varphi(q^7)}, \qquad w := 2q^{9/7}\frac{f(q, q^{13})}{\varphi(q^7)}, \tag{11}$$

since they are not clearly defined in (iv). For (i) to hold, we could give u, v, and w in any arbitrary order, but throughout the paper, we use the definitions in (11).

In (vi), Ramanujan gave an enigmatic identity, as a fragmentary example, where on the right-hand side there are three missing terms. Note that Ramanujan used the exponent $2/7$, instead of $1/7$, as he should have according to (iv). It turns out that this is correct, so it is likely that Ramanujan knew something about the structure of the terms. Ramanujan wrote $7^{3/4}$ instead of $7^{-3/4}$ on the right-hand side. We have corrected this.

Berndt [8], Son [28], and Andrews and Berndt [2, p. 181] leave the problem of the three missing terms open. They wonder why Ramanujan did not record the terms in (vi). We cannot answer this question, but Ramanujan gave us the procedure in (i)–(v) that helps us to find them. The equations in Entry 1.1(i)–(v) can be interpreted as the following:

(i) Our aim is to find the values of u, v, and w for a given $|q| < 1$.

(ii) Calculate p.

(iii) Solve the quadratic equation for $\varphi^4(q)/\varphi^4(q^7)$ and choose the *correct root*.

(v) By solving the cubic equation $r(\xi) = 0$, find α, β, and γ.

(iv) By using α, β, and γ in the *correct order*, construct u, v, and w.

Before we take these steps, we give some preliminaries in Sects. 2 and 3. Ramanujan gave us no hints on which is the correct choice for $\varphi^4(q)/\varphi^4(q^7)$ in (iii), and how to find the correct order of the roots of r in (v). Possibly, he stopped after solving (v), since the exponents $2/7$ on the right-hand side of (vi) become apparent after one finds a proper representation for the roots of r, but before their correct order is determined. This part needs most of our preparation; thus we give lemmas on the correct order of α, β, and γ in Sect. 3. Our main result is in Sect. 4, where we give the three missing terms of (vi). In Sect. 5, we give a closed-form expression for the class invariant G_{343}. In Sect. 6, we calculate the special values of $\varphi(e^{-n\pi})$, for $n = 7, 21$, and 35, and the value of $\varphi(e^{-7\pi\sqrt{3}})$. In Sect. 7, we conclude our article with the value of $\varphi(e^{-49\pi})$, given as a second example of Ramanujan's type for (i).

2 Preliminaries

We recall the transformation formula for $\varphi(e^{-\pi\sqrt{n}})$.

Lemma 2.1 *If n is a positive rational number, then*

$$\varphi(e^{-\pi/\sqrt{n}}) = n^{1/4}\varphi(e^{-\pi\sqrt{n}}).$$

Proof The transformation formula for $\varphi(q)$ states that [24, p. 199], [4, p. 43] if $a, b > 0$ with $ab = \pi$, then

$$\sqrt{a}\varphi\left(e^{-a^2}\right) = \sqrt{b}\varphi\left(e^{-b^2}\right).$$

The lemma is the special case for $a^2 = \pi/\sqrt{n}$. □

Ramanujan gave some properties of G_n. We need the following.

Lemma 2.2 *If n is a positive rational number, then $G_n = G_{1/n}$.*

Proof See Ramanujan's paper [21], [22, pp. 23–39] or Yi's thesis [35, pp. 18–19]. □

Our next lemma helps us to find p in Entry 1.1(ii). It gives a connection between p and G_n.

Lemma 2.3 *If $q = \exp(-\pi\sqrt{n})$, for a positive rational number n, then*

$$p = \frac{2\sqrt{2}G_n}{G_{49n}^7}.$$

Proof From Entry 1.1(ii), (4), and (5), we have

$$p = \frac{8q^2(-q;q^2)_\infty}{(-q^7;q^{14})_\infty^7} = \frac{8q^2\chi(q)}{\chi^7(q^7)} = \frac{2\sqrt{2}G_n}{G_{49n}^7}.$$

□

The Chebyshev polynomial U_n of the second kind [20, pp. 3–4] is defined for $|\cos\theta| \le 1$ by

$$U_n(\cos\theta) := \frac{\sin((n+1)\theta)}{\sin\theta}, \quad n = 0, 1, \dots. \tag{12}$$

The polynomial U_n satisfies the recurrence relation $U_0(x) = 1$, $U_1(x) = 2x$,

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \dots,$$

which extends the definition to all complex values x. Thus, U_n is a polynomial of degree n, with real, distinct roots, which are symmetric about zero. The first few cases are listed in [16, p. 994].

Lemma 2.4 *The roots of U_n are*

$$x_k = \cos\frac{k\pi}{n+1}, \quad k = 1, \dots, n.$$

Proof This follows directly from the definition (12). □

3 The order of the roots

In Entry 31 of Chapter 16 of Ramanujan's second notebook [24, p. 200], [4, pp. 48–49], the following general theorem is stated. Let $\mathcal{U}_k = a^{k(k+1)/2}b^{k(k-1)/2}$ and $\mathcal{V}_k = a^{k(k-1)/2}b^{k(k+1)/2}$ for each nonnegative integer k. Then, for a positive integer n,

$$f(\mathcal{U}_1, \mathcal{V}_1) = \sum_{k=0}^{n-1} \mathcal{U}_k f\left(\frac{\mathcal{U}_{n+k}}{\mathcal{U}_k}, \frac{\mathcal{V}_{n-k}}{\mathcal{U}_k}\right), \quad |ab| < 1, \quad ab \neq 0. \tag{13}$$

For a positive integer n, and $|q| < 1$, $q \neq 0$, let

$$u_k := q^{k^2/n} f\left(q^{n+2k}, q^{n-2k}\right) = \sum_{m=-\infty}^{\infty} q^{(k-mn)^2/n}, \quad k = 0, \dots, n-1, \tag{14}$$

where the series representation is obtained by (1). Note by (3) that $u_0 = \varphi(q^n)$. By setting $(a, b) = (q^{1/n}, q^{1/n})$ into (13), we obtain

$$\varphi(q^{1/n}) = \sum_{k=0}^{n-1} u_k.$$

We consider $f(a,b) = af(a^{-1}, a^2 b)$, for $|ab| < 1$ and $a \neq 0$, which is followed by Entry 18(i),(iv) in [24, p. 197], [4, pp. 34–35]. Because of this, we have $u_k = u_{n-k}$, for $k = 1, \ldots, n-1$. Since u_0 is nonzero, for each odd integer n,

$$\frac{\varphi(q^{1/n})}{u_0} = 1 + \sum_{k=1}^{(n-1)/2} \frac{2u_k}{u_0}.$$

For $n = 7$, we arrive at Entry 1.1(i), and for u, v, and w, defined in (11), we have

$$u = \frac{2u_1}{u_0}, \quad v = \frac{2u_2}{u_0}, \quad and \quad w = \frac{2u_3}{u_0}. \tag{15}$$

For finding the three missing terms, it is enough to handle u, v, and w for $0 < q < 1$, in which case these values are real. Thus in this section, we state our lemmas under this condition.

First, we would like to show that for $0 < q < 1$, the values u_k defined in (14) are in descending order, for $k = 0, \ldots, \lfloor n/2 \rfloor$, where $\lfloor n/2 \rfloor$ is the largest integer r, such that $r \leq n/2$. For this purpose, we now overview some classical results.

The third Jacobi theta function $\theta_3(z,q)$ is defined by [34, pp. 463–464]

$$\theta_3(z,q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}, \quad z \in \mathbb{C}, \quad |q| < 1, \tag{16}$$

or equivalently, $\theta_3(z \mid \tau)$ is defined by

$$\theta_3(z \mid \tau) := \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2 + 2niz}, \quad z \in \mathbb{C}, \quad \operatorname{Im} \tau > 0. \tag{17}$$

With $q = e^{\pi i \tau}$, (16) and (17) are equal, and $|q| < 1$ if and only if $\operatorname{Im} \tau > 0$. We use both notations, depending on whether we would like to indicate the dependence on q or τ. Using the identity $e^{2niz} + e^{-2niz} = 2\cos(2nz)$, we can rewrite (16) as [34, pp. 463–464]

$$\theta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz, \quad z \in \mathbb{C}, \quad |q| < 1. \tag{18}$$

A straightforward calculation shows the connection between the third Jacobi theta function and Ramanujan's general theta function. From (16) and (1), we find that [4, p. 3]

$$\theta_3(z,q) = f\left(qe^{2iz}, qe^{-2iz}\right), \quad z \in \mathbb{C}, \quad |q| < 1, \tag{19}$$

and from (17) and (1), we have

$$\theta_3(z \mid \tau) = f\left(e^{\pi i\tau+2iz}, e^{\pi i\tau-2iz}\right), \quad z \in \mathbb{C}, \quad \operatorname{Im}\tau > 0. \tag{20}$$

In the following lemma, we show a monotonicity property of the third Jacobi theta function.

Lemma 3.1 *If m is an integer and $0 < q < 1$, then*

(i) *if $z \in [m\pi, m\pi + (\pi/2)]$, then $\theta_3(z, q)$ is strictly monotonically decreasing in z, and*
(ii) *if $z \in [m\pi - (\pi/2), m\pi]$, then $\theta_3(z, q)$ is strictly monotonically increasing in z.*

Proof Since $\theta_3(z, q)$ has period π in z [34, p. 463], it is enough to show that the statement is true for $m = 0$. Because $\theta_3(z, q)$ is an even function of z [34, p. 464], it is enough to show one of the two cases.

To prove (i), we consider $\theta_3(z, q)$ for $z \in [0, \pi/2]$ and $0 < q < 1$. The zeros of $\theta_3(z, q)$ are of the form of $z = (k + (1/2))\pi + (\ell + (1/2))\pi\tau$, where $\operatorname{Im}\tau > 0$, for all integer values of k and ℓ [34, pp. 465–466], therefore it has no zeros for $z \in [0, \pi/2]$. Since the series for $\theta_3(z, q)$ in (16) is a series of analytic functions, uniformly convergent in any bounded domain of values of z [34, p. 463], $\theta_3(z, q)$ is therefore a continuous function. Furthermore, from (16) or (18), $\theta_3(z, q)$ is a real-valued function, for that $\theta_3(0, q) = \sum_{n=-\infty}^{\infty} q^{n^2} > 0$, thus by the contrapositive of the intermediate value theorem, we find that $\theta_3(z, q)$ is positive for $z \in [0, \pi/2]$.

From (19) and from the Jacobi triple product identity (2), we find that [34, p. 469]

$$\begin{aligned}\theta_3(z, q) = f\left(qe^{2iz}, qe^{-2iz}\right) &= \left(-qe^{2iz}; q^2\right)_\infty \left(-qe^{-2iz}; q^2\right)_\infty \left(q^2; q^2\right)_\infty \\ &= \prod_{n=1}^{\infty}\left(1 - q^{2n}\right)\left(1 + 2q^{2n-1}\cos 2z + q^{4n-2}\right).\end{aligned} \tag{21}$$

Since the resulting series converges uniformly [34, pp. 471, 479], we may differentiate the logarithm of (21) with respect to z. Denoting the first partial derivative of $\theta_3(z, q)$ with respect to z by $\theta_3'(z, q)$, and taking the logarithmic derivative of (21), we find that [34, p. 489]

$$\frac{\theta_3'(z, q)}{\theta_3(z, q)} = -4\sum_{n=1}^{\infty}\frac{q^{2n-1}\sin 2z}{1 + 2q^{2n-1}\cos 2z + q^{4n-2}}. \tag{22}$$

Now, note that the denominator of the summand on the right-hand side of (22) is positive, since

$$-\frac{1 + q^{4n-2}}{2q^{2n-1}} = -\frac{1}{2}\left(q^{2n-1} + \frac{1}{q^{2n-1}}\right) < -1 \le \cos 2z, \quad n = 1, 2, \ldots.$$

Thus, the sign of the sum is depending only on the sign of $\sin 2z$. Since $\theta_3(z, q) > 0$, and since $\sin 2z = 0$ for $z \in \{0, \pi/2\}$ and $\sin 2z > 0$ for $z \in (0, \pi/2)$, we find that

$\theta_3'(z, q) = 0$ for $z \in \{0, \pi/2\}$ and $\theta_3'(z, q) < 0$ for $z \in (0, \pi/2)$. Since $\theta_3(z, q)$ is continuous for $z \in [0, \pi/2]$, we conclude that $\theta_3(z, q)$ is strictly monotonically decreasing for $z \in [0, \pi/2]$. □

Now, we prove the needed monotonicity property of the values u_k defined in (14).

Lemma 3.2 *If n is a nonnegative integer and $0 < q < 1$, then u_k is positive and strictly monotonically decreasing for $k = 0, \ldots, n/2$, when n is even, and for $k = 0, \ldots, (n-1)/2$, when n is odd.*

Proof Let n and q be fixed with the given conditions. From (14), we deduce that u_k is positive. To prove the monotonicity, we rewrite u_k in terms of the third Jacobi theta function. From (14), (1), (17), and (20), we find that

$$u_k = q^{k^2/n} f\left(q^{n+2k}, q^{n-2k}\right) = q^{k^2/n} f\left(e^{\pi i\tau + 2iz_k}, e^{\pi i\tau - 2iz_k}\right) = q^{k^2/n}\theta_3(z_k \mid \tau), \tag{23}$$

where here and in the rest of the proof $k = 0, \ldots, \lfloor n/2 \rfloor$,

$$z_k = -ik\log q, \quad \text{and} \quad \tau = -\frac{in\log q}{\pi} = iC^{-1}, \quad \text{with} \quad C := \frac{\pi}{n\,|\log q|} > 0. \tag{24}$$

Since $\log q < 0$, we have $\operatorname{Im}\tau > 0$. Note that τ is independent of k, and that $e^{\pi i\tau} = e^{-\pi/C} = q^n$.

Now, we apply Jacobi's imaginary transformation formula [34, pp. 474–476], [6, pp. 140–141]

$$\theta_3(z_k \mid \tau) = (-i\tau)^{-1/2}\exp\left(\frac{z_k^2}{\pi i\tau}\right)\theta_3(z_k/\tau \mid -1/\tau), \tag{25}$$

where by (24),

$$(-i\tau)^{-1/2} = \sqrt{C} \tag{26}$$

and

$$\exp\left(\frac{z_k^2}{\pi i\tau}\right) = q^{-k^2/n}. \tag{27}$$

From (23)–(27), we find that

$$u_k = q^{k^2/n}\theta_3(z_k \mid \tau) = \sqrt{C}\cdot\theta_3(z_k/\tau \mid -1/\tau) = \sqrt{C}\cdot\theta_3(z_k/\tau, e^{-\pi i/\tau}),$$

where C is some positive value independent of k,

$$\frac{z_k}{\tau} = \frac{k\pi}{n}, \quad \text{and} \quad -\frac{1}{\tau} = iC.$$

Since $e^{-\pi i/\tau} = e^{-\pi C} \in (0, 1)$ and $(z_k/\tau) = (k\pi/n)$ is a strictly monotonically increasing sequence in $[0, \pi/2]$, applying Lemma 3.1(i) with $m = 0$, we complete the proof. □

The following lemma is a corollary of Lemma 3.2 for u, v, and w defined in (11).

Lemma 3.3 *For* $0 < q < 1$,

(i) $2 > u > v > w > 0$,
(ii) $0 < p < 8$.

Proof To prove (i), we represent u, v, and w as in (15), and we use Lemma 3.2 with $n = 7$. Part (ii) follows from (i) with $p = uvw$, as it is defined in Entry 1.1(ii). □

We need the following two statements on the values of u, v, and w defined in (11).

Lemma 3.4 *For* $|q| < 1$,

(i) $u^3v + v^3w + w^3u = 2\left(\frac{\varphi^4(q)}{\varphi^4(q^7)} - 3p - 1\right)$ *and*
(ii) $u^7 + v^7 + w^7 = \frac{\varphi^8(q)}{\varphi^8(q^7)} - 7(p-2)\frac{\varphi^4(q)}{\varphi^4(q^7)} + 7p^2 - 49p - 15.$

Proof See Son's article [27] or the Andrews–Berndt book [2, pp. 185–186, 194].

Next, we give the correct root of Entry 1.1(iii) for $\varphi^4(q)/\varphi^4(q^7)$, when $0 < q < 1$.

Lemma 3.5 *For* $0 < q < 1$,

$$\frac{\varphi^4(q)}{\varphi^4(q^7)} = 1 + \frac{5p}{2} + \frac{1}{2}\sqrt{(2+5p)^2 - 4(1-p)^3}.$$

Proof From Lemma 3.3(i), we know that $u, v, w > 0$, thus by Lemma 3.4(i) we have

$$\frac{\varphi^4(q)}{\varphi^4(q^7)} \geq 1 + 3p.$$

By solving Entry 1.1(iii) for $\varphi^4(q)/\varphi^4(q^7)$, we find that only the given solution satisfies this. □

The cubic polynomial r defined in Entry 1.1(v) has the following property.

Lemma 3.6 *For* $0 < q < 1$, *r has three distinct positive roots.*

Proof We recall that if a cubic polynomial with real coefficients has a positive discriminant, then it has three distinct real roots. For $|q| < 1$, depending on the value of $\varphi^4(q)/\varphi^4(q^7)$ from Entry 1.1(iii), r has one of the following two possible discriminants:

$$\Delta_{\pm} = p^5\left(p^3 + 104p^2 + 608p + 512 \pm (8p^{3/2} + 160\sqrt{p})\sqrt{4p^2 + 13p + 32}\right).$$

From Lemma 3.5, we know that for $0 < q < 1$, the discriminant of r is Δ_-. It follows by elementary algebra that

$$\Delta_+\Delta_- = p^{10}(p-8)^6.$$

It is clear that for $p > 0$, we have $\Delta_+ > 0$, and for $p > 0$ and $p \neq 8$, we have $p^{10}(p-8)^6 > 0$. Since by Lemma 3.3(ii) if $0 < q < 1$, then $0 < p < 8$; thus we find that $\Delta_- > 0$.

From Lemma 3.3 we know that if $0 < q < 1$, then $p > 0$ and $u, v, w > 0$. Thus, from the construction in Entry 1.1(iv) it follows that the roots of r are positive. □

The next lemma helps to choose the correct order of the roots of r in the case of $0 < q < 1$.

Lemma 3.7 *For $0 < q < 1$, suppose that the roots of r are given in order (α, β, γ) such that they satisfy the following two conditions:*

(i) $\frac{\alpha^2 p}{\beta} + \frac{\beta^2 p}{\gamma} + \frac{\gamma^2 p}{\alpha} = \frac{\varphi^8(q)}{\varphi^8(q^7)} - 7(p-2)\frac{\varphi^4(q)}{\varphi^4(q^7)} + 7p^2 - 49p - 15$ *and*

(ii) $\frac{\alpha^2}{\beta} > \frac{\beta^2}{\gamma} > \frac{\gamma^2}{\alpha}$.

Then

$$u = \left(\frac{\alpha^2 p}{\beta}\right)^{1/7}, \quad v = \left(\frac{\beta^2 p}{\gamma}\right)^{1/7}, \quad \textit{and} \quad w = \left(\frac{\gamma^2 p}{\alpha}\right)^{1/7}.$$

Condition (i) guarantees the correct order of α, β, and γ in Entry 1.1(iv), so that Entry 1.1(i) holds. Condition (ii) provides the correct order of u, v, and w, so that (11) holds.

Proof First, for each possible order of α, β, and γ, consider the set of possible values of u, v, and w given in Entry 1.1(iv). For (α, β, γ), (β, γ, α), (γ, α, β) and for (γ, β, α), (β, α, γ), (α, γ, β), we have

$$\left\{\left(\frac{\alpha^2 p}{\beta}\right)^{1/7}, \left(\frac{\beta^2 p}{\gamma}\right)^{1/7}, \left(\frac{\gamma^2 p}{\alpha}\right)^{1/7}\right\} \quad \text{and}$$
$$\left\{\left(\frac{\gamma^2 p}{\beta}\right)^{1/7}, \left(\frac{\beta^2 p}{\alpha}\right)^{1/7}, \left(\frac{\alpha^2 p}{\gamma}\right)^{1/7}\right\},$$

respectively. Thus, it is enough to consider the order (α, β, γ) and its reverse (γ, β, α). By Lemma 3.4(ii), we know that for at least one of these two sets it is true that the sum of their seventh powers fulfills the condition in (i). We show that exactly one of the two sets fulfills it. Suppose that

$$\frac{\alpha^2 p}{\beta} + \frac{\beta^2 p}{\gamma} + \frac{\gamma^2 p}{\alpha} = \frac{\gamma^2 p}{\beta} + \frac{\beta^2 p}{\alpha} + \frac{\alpha^2 p}{\gamma}.$$

After rearrangement, we find that

$$\frac{p(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta+\gamma)}{\alpha\beta\gamma}=0,$$

but this is contradiction, since p is positive by Lemma 3.3(ii) and α, β, and γ are distinct, positive numbers by Lemma 3.6.

Since p is positive by Lemma 3.3(ii), the condition (ii) guarantees that $u>v>w$, which holds by Lemma 3.3(i). □

Lastly, we need the following trigonometric identity.

Lemma 3.8 *We have*

$$\left(\frac{\cos\frac{\pi}{7}}{2\cos^2\frac{2\pi}{7}}\right)^2+\left(\frac{\cos\frac{2\pi}{7}}{2\cos^2\frac{3\pi}{7}}\right)^2+\left(\frac{\cos\frac{3\pi}{7}}{2\cos^2\frac{\pi}{7}}\right)^2=41.$$

Proof Let $\theta:=\pi/7$, and let

$$\begin{aligned}a&:=\cos\theta=-\cos 6\theta=-\cos 8\theta,\\ b&:=\cos 2\theta=-\cos 5\theta=\cos 12\theta,\\ c&:=\cos 3\theta=-\cos 4\theta=-\cos 10\theta=-\cos 18\theta.\end{aligned}$$

By using power-reduction [16, p. 32] and product-to-sum [16, p. 29] identities, we derive that

$$\begin{aligned}&\left(\frac{a}{2b^2}\right)^2+\left(\frac{b}{2c^2}\right)^2+\left(\frac{c}{2a^2}\right)^2=\frac{a^6c^4+b^6a^4+c^6b^4}{4(abc)^4}\\ &=\frac{1}{1024(abc)^4}\{(10+15b-6c-a)(3-4a+b)\\ &\quad+(10-15c-6a+b)(3+4b-c)\\ &\quad+(10-15a+6b-c)(3-4c-a)\}\\ &=\frac{1}{1024(abc)^4}\{90+116(b-c-a)+91(ca-ab-bc)+19(a^2+b^2+c^2)\}\\ &=\frac{1}{1024(abc)^4}\Big\{90+116(b-c-a)+91\Big(\tfrac{1}{2}(b-c)-\tfrac{1}{2}(a+c)-\tfrac{1}{2}(a-b)\Big)\\ &\quad+19\Big(\tfrac{1}{2}(1+b)+\tfrac{1}{2}(1-c)+\tfrac{1}{2}(1-a)\Big)\Big\}\\ &=\frac{433(b-c-a)+237}{2048(abc)^4}.\end{aligned}$$

Since we know [3] that $b-c-a=-1/2$ and $abc=1/8$, the proof is complete. □

4 The three missing terms

We complete Ramanujan's enigmatic septic theta function identity Entry 1.1(vi).

Theorem 4.1 *We have*

$$\varphi\left(e^{-7\pi\sqrt{7}}\right) = 7^{-3/4}\varphi\left(e^{-\pi\sqrt{7}}\right) \times\left\{1+\left(\frac{\cos\frac{\pi}{7}}{2\cos^2\frac{2\pi}{7}}\right)^{2/7}+\left(\frac{\cos\frac{2\pi}{7}}{2\cos^2\frac{3\pi}{7}}\right)^{2/7}+\left(\frac{\cos\frac{3\pi}{7}}{2\cos^2\frac{\pi}{7}}\right)^{2/7}\right\}.$$

Proof We use the results in Entry 1.1(i)–(v) with $q = \exp(-\pi/\sqrt{7})$. First, by using Lemma 2.1, we rewrite Entry 1.1(i) as

$$\varphi\left(e^{-7\pi\sqrt{7}}\right) = 7^{-3/4}\varphi\left(e^{-\pi\sqrt{7}}\right)\{1+u+v+w\}.$$

With G_7 given in [18, 30, 31], [7, p. 189], by Lemma 2.2, we find that $G_{1/7} = G_7 = 2^{1/4}$. By Lemma 2.3 with $n = 1/7$, for Entry 1.1(ii) we have

$$p = \frac{2\sqrt{2}G_{1/7}}{G_7^7} = \frac{2\sqrt{2}\cdot 2^{1/4}}{2^{7/4}} = 1.$$

Next, we solve the equation in Entry 1.1(iii). By Lemma 3.5, or in this special case by Lemma 2.1, we have

$$\frac{\varphi^4(q)}{\varphi^4(q^7)} = \frac{\varphi^4\left(e^{-\pi/\sqrt{7}}\right)}{\varphi^4\left(e^{-\pi\sqrt{7}}\right)} = 7.$$

Now, we have all the coefficients of r given in Entry 1.1(v). We have to determine the zeros of

$$r(\xi) = \xi^3 - 6\xi^2 + 5\xi - 1.$$

With an appropriate polynomial transformation, we relate r to U_6 defined in (12) or given in [16, p. 994]. Note that for $\xi \neq 0$,

$$-(2\xi)^6 r\left(\frac{1}{(2\xi)^2}\right) = 64\xi^6 - 80\xi^4 + 24\xi^2 - 1 = U_6(\xi),$$

and $r(0) = U_6(0) = -1$. From Lemma 2.4, we know that U_6 has the roots $\xi_k = \cos(k\pi/7)$, for $k = 1, \ldots, 6$, for which $\xi_k \neq 0$ and $|\xi_k| = |\xi_{7-k}|$. Thus, we find that

$$r\left(\frac{1}{(2\cos\frac{k\pi}{7})^2}\right) = 0, \quad k = 1, 2, 3.$$

Lastly, we determine the appropriate order of the roots α, β, and γ in Entry 1.1(iv). The choice

$$(\alpha, \beta, \gamma) = \left(\frac{1}{(2\cos\frac{3\pi}{7})^2}, \frac{1}{(2\cos\frac{2\pi}{7})^2}, \frac{1}{(2\cos\frac{\pi}{7})^2} \right)$$

is correct, since the condition Lemma 3.7(i) holds by Lemma 3.8, and Lemma 3.7(ii) is satisfied by the inequalities $0 < \cos(3\pi/7) < \cos(2\pi/7) < \cos(\pi/7) < 1$. We arrive at

$$u = \left(\frac{\cos\frac{2\pi}{7}}{2\cos^2\frac{3\pi}{7}} \right)^{2/7}, \quad v = \left(\frac{\cos\frac{\pi}{7}}{2\cos^2\frac{2\pi}{7}} \right)^{2/7}, \quad \text{and} \quad w = \left(\frac{\cos\frac{3\pi}{7}}{2\cos^2\frac{\pi}{7}} \right)^{2/7},$$

which completes the proof. □

By combining the value (10) and Theorem 4.1, we find the evaluation of $\varphi(e^{-7\pi\sqrt{7}})$, i.e.,

$$\begin{aligned} \varphi\left(e^{-7\pi\sqrt{7}}\right) &= \frac{\left\{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)\right\}^{1/2}}{\sqrt{2}\cdot 7^{7/8}\pi} \\ &\quad \times \left\{ 1 + \left(\frac{\cos\frac{\pi}{7}}{2\cos^2\frac{2\pi}{7}} \right)^{2/7} + \left(\frac{\cos\frac{2\pi}{7}}{2\cos^2\frac{3\pi}{7}} \right)^{2/7} + \left(\frac{\cos\frac{3\pi}{7}}{2\cos^2\frac{\pi}{7}} \right)^{2/7} \right\}. \end{aligned}$$

5 The class invariant G_{343} in closed-form

Berndt [8], Son [27], and Andrews and Berndt [2, p. 181] proposed the explicit value of the class invariant G_{343} as an open problem. Actually, Watson showed in [30, 31] that $G_{343} = 2^{1/4}x$, where $x^7 - 7x^6 - 7x^5 - 7x^4 - 1 = 0$. This can be proved by using a modular equation of degree 7, given in Entry 19(ix) of Chapter 19 of Ramanujan's second notebook [24, p. 240], [4, p. 315], [11, Lemma 3.5], [36, Theorem 2.4]. Watson [31] solved this septic polynomial as well. Thus, we have

$$G_{343} = 2^{1/4}7\{b_1 + b_2 + b_3 + c_1 + c_2 + c_3\}^{-1},$$

where

$$\begin{aligned} b_1 &= \left(b_1'^4\, b_2'^2\, b_3'\right)^{1/7}, & c_1 &= \left(c_1'^4\, c_2'^2\, c_3'\right)^{1/7}, \\ b_2 &= \left(b_2'^4\, b_3'^2\, b_1'\right)^{1/7}, & c_2 &= \left(c_2'^4\, c_3'^2\, c_1'\right)^{1/7}, \\ b_3 &= \left(b_3'^4\, b_1'^2\, b_2'\right)^{1/7}, & c_3 &= \left(c_3'^4\, c_1'^2\, c_2'\right)^{1/7}, \end{aligned}$$

and

$$b_r' = -\tfrac{1}{3}(3\sigma_r + 5\tau_r), \quad c_r' = -\tfrac{1}{3}(7 + 4\sigma_r + 2\tau_r), \quad r = 1, 2, 3,$$
$$\tau_1 = \sigma_3 - \sigma_2, \quad \tau_2 = \sigma_1 - \sigma_3, \quad \tau_3 = \sigma_3 - \sigma_1,$$
$$\sigma_r = \frac{1}{2} + 3\cos\frac{2^r\pi}{7}, \quad r = 1, 2, 3.$$

In this section, we give a closed-form expression for G_{343}, based on our previous results.

Theorem 5.1 *We have*

$$G_{343} = 2^{1/4} p^{-1/7},$$

where

$$p = 1 + \frac{10m^2}{s^{1/3}} - \frac{s^{1/3}}{6}, \tag{28}$$

and

$$s = 12m^2\left(9(7 - m^2) + \sqrt{3}\sqrt{27(m^4 + 49) + 122m^2}\right),$$
$$m = 7^{3/2}\left\{1 + \left(\frac{\cos\frac{\pi}{7}}{2\cos^2\frac{2\pi}{7}}\right)^{2/7} + \left(\frac{\cos\frac{2\pi}{7}}{2\cos^2\frac{3\pi}{7}}\right)^{2/7} + \left(\frac{\cos\frac{3\pi}{7}}{2\cos^2\frac{\pi}{7}}\right)^{2/7}\right\}^{-2}.$$

Proof By taking $q = \exp(-\pi\sqrt{7})$, the expression for m is obtained by Theorem 4.1 as

$$m = \frac{\varphi^2(q)}{\varphi^2(q^7)} = \frac{\varphi^2\left(e^{-\pi\sqrt{7}}\right)}{\varphi^2\left(e^{-7\pi\sqrt{7}}\right)}.$$

For Entry 1.1(ii), by Lemma 2.3 with $n = 7$ and with $G_7 = 2^{1/4}$ [18, 30, 31], [7, p. 189], we find that

$$p = \frac{2\sqrt{2}G_7}{G_{343}^7} = \frac{2\sqrt{2}\cdot 2^{1/4}}{G_{343}^7},$$

from which we have $G_{343} = 2^{1/4}p^{-1/7}$. On the other hand, for Entry 1.1(iii), we have

$$m^4 - (2 + 5p)m^2 + (1 - p)^3 = 0.$$

After rearrangement, the following cubic polynomial in p can be deduced:

$$p^3 - 3p^2 + (3 + 5m^2)p - (m^2 - 1)^2 = 0.$$

We solve this equation, and choose the only real root, which is given by (28). □

6 Examples for Entry 1.1(iii)

The Borwein brothers [13, p. 145] observed first [12] that class invariants can be used to calculate certain values of $\varphi(e^{-n\pi})$. By using ideas from Berndt's proof [4, p. 347] of Entry 1(iii) of Chapter 20 of Ramanujan's second notebook [24, p. 241], [4, p. 345], one can deduce that [7, p. 330, (4.5)], [9, (3.10)]

$$\frac{\varphi(e^{-3\pi\sqrt{n}})}{\varphi(e^{-\pi\sqrt{n}})} = \frac{1}{\sqrt{3}}\left(1 + \frac{2\sqrt{2}G_{9n}^3}{G_n^9}\right)^{1/4}. \tag{29}$$

Similarly, as in [7, p. 339, (8.11)] and [7, p. 334, (5.7)], we have

$$\frac{\varphi(e^{-5\pi\sqrt{n}})}{\varphi(e^{-\pi\sqrt{n}})} = \frac{1}{\sqrt{5}}\left(1 + \frac{2G_{25n}}{G_n^5}\right)^{1/2} \tag{30}$$

and

$$\frac{\varphi(e^{-9\pi\sqrt{n}})}{\varphi(e^{-\pi\sqrt{n}})} = \frac{1}{3}\left(1 + \frac{\sqrt{2}G_{9n}}{G_n^3}\right). \tag{31}$$

There are two groups of values for $\varphi(q)$, which can by deduced from Entry 1.1. The first one is from Entry 1.1(iii), and the second is from Entry 1.1(i). Now, in the spirit of Entry 1.1(iii), we give a result for $\varphi(e^{-7\pi\sqrt{n}})/\varphi(e^{-\pi\sqrt{n}})$, which is similar to those in (29)–(31), and then we calculate the values of $\varphi(e^{-7\pi})$, $\varphi(e^{-7\pi\sqrt{3}})$, $\varphi(e^{-21\pi})$, and $\varphi(e^{-35\pi})$.

Lemma 6.1 *If n is a positive rational number, then*

$$\frac{\varphi\left(e^{-7\pi\sqrt{n}}\right)}{\varphi\left(e^{-\pi\sqrt{n}}\right)} = \frac{1}{\sqrt{7}}\left(1 + \frac{5\sqrt{2}G_{49n}}{G_n^7} + \frac{1}{2}\sqrt{\left(2 + \frac{10\sqrt{2}G_{49n}}{G_n^7}\right)^2 - 4\left(1 - \frac{2\sqrt{2}G_{49n}}{G_n^7}\right)^3}\right)^{1/4}.$$

For

$$p = \frac{2\sqrt{2}G_{49n}}{G_n^7}, \tag{32}$$

we define

$$m(p) := \left(1 + \frac{5p}{2} + \frac{1}{2}\sqrt{(2+5p)^2 - 4(1-p)^3}\right)^{1/2}. \tag{33}$$

Using $m(p)$, we can state Lemma 6.1 as

$$\frac{\varphi\left(e^{-7\pi\sqrt{n}}\right)}{\varphi\left(e^{-\pi\sqrt{n}}\right)} = \frac{m^{1/2}(p)}{\sqrt{7}}. \tag{34}$$

Note that $m(p)$ is a multiplier of degree 7, defined in (7), with the substitution $n \mapsto (49n)^{-1}$.

Proof We apply Entry 1.1(ii),(iii). For $q = \exp(-\pi/\sqrt{49n})$, by Lemmas 2.3 and 2.2, we find that $p = 2\sqrt{2}G_{49n}/G_n^7$. By using Lemma 3.5 with Lemma 2.1, we complete the proof. □

The next result is from Ramanujan's first notebook [23, p. 297], [7, p. 328], and proved first by Berndt and Chan [9], [7, pp. 336–337]. Our proof uses Entry 1.1(iii), but all of these proofs depend on some of the modular equations given in Entry 19 of Chapter 19 in Ramanujan's second notebook [24, p. 240], [4, pp. 314–324]. The value of $\varphi(e^{-7\pi})$, in terms of $\varphi(e^{-\pi})$ given in (8), is stated as follows.

Theorem 6.2 *We have*

$$\frac{\varphi^2\left(e^{-7\pi}\right)}{\varphi^2\left(e^{-\pi}\right)} = \frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{14}(28)^{1/8}.$$

Proof We apply Lemma 6.1 with $n = 1$. From [7, p. 189], $G_1 = 1$, and from [21], [22, p. 26], [7, p. 191],

$$G_{49} = \frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}. \tag{35}$$

Using $4+\sqrt{7} = (\sqrt{7}+1)^2/2$, from (32), we find that

$$p = \frac{2\sqrt{2}G_{49}}{G_1^7} = \sqrt{2}\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right) = \sqrt{7}+\sqrt{2}\cdot 7^{1/4}+1. \tag{36}$$

We make two observations. Let a be a real number, and let

$$p_a := \frac{(a+1)^2+1}{2} = \frac{1}{2}(a^2+2a+2) = \frac{a^2}{2}+a+1. \tag{37}$$

Then, straightforward algebra shows that

$$(2+5p_a)^2 - 4(1-p_a)^3 = \frac{1}{4}(2a^3+3a^2+10a+14)^2 + \frac{a^2}{2}(a^4-28), \tag{38}$$

and for $a \neq 0$,

$$\left(\sqrt{13+\frac{a^2}{2}}+\sqrt{7+\frac{3a^2}{2}}\right)^2 = 2a^2+\sqrt{\left(8a+\frac{28}{a}\right)^2+\frac{(3a^2+28)(a^4-28)}{a^2}}+20. \tag{39}$$

Now, set $a := (28)^{1/4} = \sqrt{2}\cdot 7^{1/4}$. Comparing (36) with (37), we see that $p = p_a$. Furthermore, note in (38) and (39) that the terms with a factor of a^4-28 vanish. Thus, from (33)–(39), we find that

$$\begin{aligned}\frac{\varphi^2(e^{-7\pi})}{\varphi^2(e^{-\pi})} = \frac{m(p)}{7} &= \frac{1}{7}\left(1+\frac{5p}{2}+\frac{1}{2}\sqrt{(2+5p)^2-4(1-p)^3}\right)^{1/2} \\ &= \frac{1}{7}\left(1+\frac{5}{4}(a^2+2a+2)+\frac{1}{4}(2a^3+3a^2+10a+14)\right)^{1/2} \\ &= \frac{1}{7}\left(\frac{a}{4}\left(2a^2+8a+\frac{28}{a}+20\right)\right)^{1/2} \\ &= \frac{\sqrt{a}}{14}\left(\sqrt{13+\frac{a^2}{2}}+\sqrt{7+\frac{3a^2}{2}}\right) \\ &= \frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{14}(28)^{1/8}. \end{aligned} \tag{40}$$

□

The next theorems appear to be new.

Theorem 6.3 *We have*

$$\frac{\varphi^2\left(e^{-7\pi\sqrt{3}}\right)}{\varphi^2\left(e^{-\pi\sqrt{3}}\right)} = \frac{1}{42\sqrt{3}}\left(\left(\sqrt{21}+3\right)(28)^{1/3}+8\sqrt{3}(28)^{1/6}+4\sqrt{21}+6\right).$$

Proof We apply Lemma 6.1 with $n = 3$. From [31], [7, p. 189], $G_3 = 2^{1/12}$, and from [21], [22, p. 28], [32], [7, p. 194],

$$G_{147} = 2^{1/12}\left(\frac{1}{2}+\frac{1}{\sqrt{3}}\left\{\sqrt{\frac{7}{4}}-(28)^{1/6}\right\}\right)^{-1}.$$

Thus, from (32), we find that

$$p = \frac{2\sqrt{2}G_{147}}{G_3^7} = 2\left(\frac{1}{2}+\frac{1}{\sqrt{3}}\left\{\sqrt{\frac{7}{4}}-(28)^{1/6}\right\}\right)^{-1}. \tag{41}$$

The next observation follows by elementary algebra. Let a be a real number, and let

$$p_a := \frac{1}{54}(a^4 + 3a^3 + 12a^2 + 18a + 90) \tag{42}$$

and

$$m_a := \frac{1}{6\sqrt{3}}\left(\frac{a}{18}(a^2+6)^2 + a(a+6) + 6\right). \tag{43}$$

Then

$$(2+5p_a)^2 - 4(1-p_a)^3 = 4\left(m_a^2 - 1 - \frac{5p_a}{2}\right)^2 - \frac{(a^6-756)P}{306110016}, \tag{44}$$

where

$$\begin{aligned} P \ &= a^{14} + 48a^{12} + 72a^{11} + 1440a^{10} + 3024a^9 + 27108a^8 + 68040a^7 + 375840a^6 \\ &+843696a^5 + 3005424a^4 + 5762016a^3 \\ &+13576896a^2 + 15536448a + 5878656. \end{aligned}$$

Now, set $a := (756)^{1/6} = 2^{1/3} \cdot \sqrt{3} \cdot 7^{1/6}$. A straightforward calculation shows that

$$(a^4 + 3a^3 + 12a^2 + 18a + 90)\left(\frac{1}{2} + \frac{1}{\sqrt{3}}\left\{\sqrt{\frac{7}{4}} - (28)^{1/6}\right\}\right) = 108.$$

Thus, by comparing (41) with (42), we see that $p = p_a$. Furthermore, note in (44) that the term with a factor of $a^6 - 756$ vanishes. Thus, from (33), (34), and (41)–(44), with some simplification, we find that

$$\begin{aligned} \frac{\varphi^2\left(e^{-7\pi\sqrt{3}}\right)}{\varphi^2\left(e^{-\pi\sqrt{3}}\right)} = \frac{m(p)}{7} = \frac{m_a}{7} &= \frac{1}{42\sqrt{3}}\left(\frac{a}{18}(a^2+6)^2 + a(a+6) + 6\right) \\ &= \frac{1}{42\sqrt{3}}\left(\left(\sqrt{21}+3\right)(28)^{1/3} + 8\sqrt{3}(28)^{1/6} + 4\sqrt{21} + 6\right). \end{aligned}$$

□

By combining the value (9) and Theorem 6.3, we find the evaluation of $\varphi(e^{-7\pi\sqrt{3}})$, i.e.,

$$\varphi\left(e^{-7\pi\sqrt{3}}\right) = \frac{\Gamma^{3/2}\left(\frac{1}{3}\right)}{2^{7/6}3^{5/8}\sqrt{7}\pi}\left(\left(\sqrt{21}+3\right)(28)^{1/3} + 8\sqrt{3}(28)^{1/6} + 4\sqrt{21} + 6\right)^{1/2}.$$

The next values are derived in terms of $\varphi(e^{-\pi})$ given in (8).

Theorem 6.4 *We have*

$$\frac{\varphi\left(e^{-21\pi}\right)}{\varphi\left(e^{-\pi}\right)}=\left(\frac{m(p)}{7(6\sqrt{3}-9)^{1/2}}\right)^{1/2},$$

where

$$p=\sqrt{2}(2-\sqrt{3})\sqrt{\sqrt{3}+\sqrt{7}}\sqrt{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}\times\sqrt{\frac{\sqrt{3+\sqrt{7}}+(6\sqrt{7})^{1/4}}{\sqrt{3+\sqrt{7}}-(6\sqrt{7})^{1/4}}} \tag{45}$$

and $m(p)$ is given in (33).

Proof We apply Lemma 6.1 with $n=9$. From [21], [22, p. 24], [7, p. 189],

$$G_9=\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3},$$

and from [21], [22, p. 29], [10], [7, p. 197],

$$G_{441}=\sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}(2+\sqrt{3})^{1/6}\sqrt{\frac{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}{2}}\times\sqrt{\frac{\sqrt{3+\sqrt{7}}+(6\sqrt{7})^{1/4}}{\sqrt{3+\sqrt{7}}-(6\sqrt{7})^{1/4}}}. \tag{46}$$

An elementary calculation shows that

$$\frac{2^{5/3}(2+\sqrt{3})^{1/6}}{(1+\sqrt{3})^{7/3}}=\sqrt{2}(2-\sqrt{3}).$$

From (32), $p=2\sqrt{2}G_{441}/G_9^7$. Using the values for G_9 and G_{441} given above, we deduce (45).

From [23, p. 284], [9], [7, pp. 327, 329–331],

$$\frac{\varphi(e^{-3\pi})}{\varphi(e^{-\pi})}=\frac{1}{(6\sqrt{3}-9)^{1/4}}. \tag{47}$$

Combining Lemma 6.1 and (47) completes the proof. □

Another expression for $\varphi(e^{-21\pi})$ can be obtained by using (29) with $n=49$ and with the values G_{441} from (46), and G_{49} from (35). Combining the result with the value of $\varphi(e^{-7\pi})$ from Theorem 6.2, after some simplification we find that

$$\begin{aligned}\frac{\varphi\left(e^{-21\pi}\right)}{\varphi\left(e^{-\pi}\right)} &= \left(\frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{42}\right)^{1/2}(28)^{1/16}\\ &\times\left\{1+\frac{1}{4}\sqrt{2}\sqrt{2+\sqrt{3}}\left((\sqrt{3}+\sqrt{7})\left(2+\sqrt{7}+\sqrt{7+4\sqrt{7}}\right)\right.\right.\\ &\left.\left.\times\left(22+8\sqrt{7}-\frac{1}{2}(19+7\sqrt{7})\sqrt{2\sqrt{7}}\right)\frac{\sqrt{3+\sqrt{7}}+(6\sqrt{7})^{1/4}}{\sqrt{3+\sqrt{7}}-(6\sqrt{7})^{1/4}}\right)^{3/2}\right\}^{1/4}.\end{aligned}$$

Theorem 6.5 *We have*

$$\frac{\varphi\left(e^{-35\pi}\right)}{\varphi\left(e^{-\pi}\right)} = \left(\frac{m(p)}{35(\sqrt{5}-2)}\right)^{1/2},$$

where

$$\begin{aligned}p &= \frac{1}{4}(9-4\sqrt{5})\sqrt{\sqrt{14}+\sqrt{10}}\left(7^{1/4}+\sqrt{4+\sqrt{7}}\right)^{3/2}\\ &\times\left(\sqrt{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}+\sqrt{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}\right)\end{aligned} \tag{48}$$

and $m(p)$ is given in (33).

Proof We apply Lemma 6.1 with $n = 25$. From [21], [22, p. 26], [7, p. 190],

$$G_{25} = \frac{1+\sqrt{5}}{2},$$

and from [21], [22, p. 30], [29], [7, p. 199],

$$\begin{aligned}G_{1225} &= \frac{1+\sqrt{5}}{2}(6+\sqrt{35})^{1/4}\left(\frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}\right)^{3/2}\\ &\times\left(\sqrt{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right.\\ &\left.+\sqrt{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right).\end{aligned} \tag{49}$$

A simple calculation shows that

$$\frac{1}{G_{25}^6} = \left(\frac{2}{1+\sqrt{5}}\right)^6 = 9 - 4\sqrt{5}.$$

We use

$$\left(6+\sqrt{35}\right)^{1/4} = \sqrt{\frac{\sqrt{14}+\sqrt{10}}{2}},$$

as it is given by Watson [29], which also can be verified directly. From (32), $p = 2\sqrt{2}G_{1225}/G_{25}^7$. Using the values for G_{25} and G_{1225} given above, we deduce (48).

From [23, p. 285], [9], [7, pp. 327–329],

$$\frac{\varphi(e^{-5\pi})}{\varphi(e^{-\pi})} = \frac{1}{(5\sqrt{5}-10)^{1/2}}. \tag{50}$$

Combining Lemma 6.1 and (50) completes the proof. □

Another expression for $\varphi(e^{-35\pi})$ can be obtained by using (30) with $n = 49$ and with the values G_{1225} from (49), and G_{49} from (35). Combining the result with the value of $\varphi(e^{-7\pi})$ from Theorem 6.2, after some simplification we find that

$$\begin{aligned}\frac{\varphi(e^{-35\pi})}{\varphi(e^{-\pi})} &= \left(\frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{70}\right)^{1/2}(28)^{1/16} \\ &\times\left\{1+\frac{1}{4}(1+\sqrt{5})\sqrt{\sqrt{7}+\sqrt{5}}\left(16466+6223\sqrt{7}-\frac{7}{2}(2045+773\sqrt{7})\sqrt{2\sqrt{7}}\right)^{1/4}\right. \\ &\left.\times\left(\sqrt{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}+\sqrt{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}\right)\right\}^{1/2}.\end{aligned}$$

7 Examples for Entry 1.1(i)

Now, we review our proof for Theorem 4.1, which is in the form of Entry 1.1(i). Then, we give the value of $\varphi(e^{-49\pi})$, as a second illustration of Entry 1.1(i). We use the results from the proof of Theorem 6.2.

Let $a \in \{0, (28)^{1/4}\}$. After (37), let

$$p_a := \frac{(a+1)^2+1}{2} = \frac{1}{2}(a^2+2a+2). \tag{51}$$

Since the second term on the right-hand side of (38) vanishes for $a \in \{0, (28)^{1/4}\}$, after (40), let

$$m_a := \left(\frac{1}{2}\left(a^3+4a^2+10a+14\right)\right)^{1/2}. \tag{52}$$

Furthermore, let

$$\begin{aligned} r_a(\xi) &:= \xi^3 + 2\xi^2\left(1 + 3p_a - m_a^2\right) + \xi p_a^2(p_a + 4) - p_a^4 \\ &= \xi^3 - (a^3 + a^2 + 4a + 6)\xi^2 + \frac{1}{8}(a^2 + 2a + 2)^2(a^2 + 2a + 10)\xi \\ &\quad - \frac{1}{16}(a^2 + 2a + 2)^4. \end{aligned} \tag{53}$$

Comments on the proof of Theorem 4.1 Set $a := 0$. Using the notations of the proof of Theorem 4.1, from (51), (52), and (53), we find that $p = p_a = 1$, $\varphi^4(q)/\varphi^4(q^7) = m_a^2 = 7$, and

$$r(\xi) = r_a(\xi) = \xi^3 - 6\xi^2 + 5\xi - 1.$$

By Lemma 2.4, we know that $\cos(\pi/7)$ is a root of U_6. Thus, by using the power-reduction formula [16, p. 32]

$$\cos^2\left(\frac{\pi}{7}\right) = \frac{1}{2}\left(\cos\left(\frac{2\pi}{7}\right) + 1\right),$$

we can factor r over $\mathbb{Q}(\cos(\pi/7))$ as

$$r(\xi) = (\xi - \alpha)(\xi - \beta)(\xi - \gamma), \tag{54}$$

where

$$\begin{aligned} \alpha &= 2 + 2\cos\left(\frac{\pi}{7}\right) + 2\cos\left(\frac{2\pi}{7}\right), \\ \beta &= 3 - 4\cos\left(\frac{\pi}{7}\right) + 2\cos\left(\frac{2\pi}{7}\right), \\ \gamma &= 1 + 2\cos\left(\frac{\pi}{7}\right) - 4\cos\left(\frac{2\pi}{7}\right). \end{aligned}$$

In the same manner as in the proof of Lemma 3.8, we deduce

$$(\alpha, \beta, \gamma) = \left(\frac{1}{(2\cos\frac{3\pi}{7})^2}, \frac{1}{(2\cos\frac{2\pi}{7})^2}, \frac{1}{(2\cos\frac{\pi}{7})^2}\right),$$

where the order of the roots is determined by Lemmas 3.7 and 3.8. We construct u, v, and w, and the proof is complete. □

After these preliminaries, we derive the value of $\varphi(e^{-49\pi})$. The corresponding polynomial r, and its roots are more complicated than in Ramanujan's example in Entry 1.1(vi). We used *Mathematica* for polynomial factorization and numerical evaluations.

Theorem 7.1 *We have*

$$\frac{\varphi(e^{-49\pi})}{\varphi(e^{-\pi})} = \frac{1}{7}(1 + u + v + w), \tag{55}$$

where

$$u = \left(\frac{\alpha^2 p}{\beta}\right)^{1/7}, \quad v = \left(\frac{\beta^2 p}{\gamma}\right)^{1/7}, \quad w = \left(\frac{\gamma^2 p}{\alpha}\right)^{1/7},$$

and

$$\begin{aligned}
p &= \sqrt{7} + \sqrt{2} \cdot 7^{1/4} + 1, \\
\alpha &= \frac{1}{\sqrt{7}} \left\{ \frac{2}{3}\big(\sqrt{7} + 2\big)\big(5 + 3\sqrt{2} \cdot 7^{1/4} - \sqrt{7}\big) \right. \\
&\quad + 2\big(\sqrt{7} - 1\big)\big(1 - \sqrt{2} \cdot 7^{1/4} + \sqrt{7}\big) \cos\left(\frac{\pi}{7}\right) \\
&\quad \left. + \frac{2}{3}\big(\sqrt{7} - 1\big)\big(3\sqrt{2} \cdot 7^{1/4} - \sqrt{7} - 1\big) \cos\left(\frac{2\pi}{7}\right) \right\}, \\
\beta &= \frac{1}{\sqrt{7}} \left\{ \frac{1}{9}\big(\sqrt{7} + 5\big)\big(13 + 9\sqrt{2} \cdot 7^{1/4} + \sqrt{7}\big) \right. \\
&\quad \left. - 8 \cos\left(\frac{\pi}{7}\right) + 2\big(\sqrt{7} - 1\big)\big(1 - \sqrt{2} \cdot 7^{1/4} + \sqrt{7}\big) \cos\left(\frac{2\pi}{7}\right) \right\}, \\
\gamma &= \frac{1}{\sqrt{7}} \left\{ \frac{1}{3}\big(\sqrt{7} + 5\big)\big(1 + 3\sqrt{2} \cdot 7^{1/4} + \sqrt{7}\big) \right. \\
&\quad \left. + \frac{2}{3}\big(\sqrt{7} - 1\big)\big(3\sqrt{2} \cdot 7^{1/4} - \sqrt{7} - 1\big) \cos\left(\frac{\pi}{7}\right) - 8 \cos\left(\frac{2\pi}{7}\right) \right\}.
\end{aligned}$$

Proof We use the results in Entry 1.1(i)–(v) with $q = \exp(-\pi/7)$. First, by using Lemma 2.1, we rewrite Entry 1.1(i) as in (55).

Then, set $a := (28)^{1/4} = \sqrt{2} \cdot 7^{1/4}$, and consider p_a, m_a, and r_a from (51), (52), and (53). By Lemma 2.2, and by Lemma 2.3 with $n = 1/49$, for Entry 1.1(ii), we have $p = 2\sqrt{2}G_{49}/G_1^7$. Thus, comparing (36) with (51), we see that $p = p_a = (a^2/2) + a + 1$. For Lemma 1.1(iii), by using Lemmas 3.5 and 2.1, we arrive at

$$\frac{\varphi^4(q)}{\varphi^4(q^7)} = \frac{\varphi^4\left(e^{-\pi/7}\right)}{\varphi^4\left(e^{-\pi}\right)} = 49\frac{\varphi^4\left(e^{-7\pi}\right)}{\varphi^4\left(e^{-\pi}\right)} = m_a^2 = \frac{1}{2}(a^3 + 4a^2 + 10a + 14),$$

where the last equation is obtained by the comparison of (40) and (52). Thus, by comparing Entry 1.1(v) with (53), we find that $r(\xi) = r_a(\xi)$, where we remind readers that

$$\begin{aligned} r_a(\xi) &= \xi^3 - (a^3 + a^2 + 4a + 6)\xi^2 \\ &\quad + \frac{1}{8}(a^2 + 2a + 2)^2(a^2 + 2a + 10)\xi - \frac{1}{16}(a^2 + 2a + 2)^4. \end{aligned}$$

Now, because of Lemma 2.4, following (54), we factor r_a over $\mathbb{Q}(\cos(\pi/7), a)$ as

$$r_a(\xi) = (\xi - \alpha)(\xi - \beta)(\xi - \gamma),$$

where

$$\alpha = \frac{2}{a^2}\left\{a^3 + a^2 + 4a + 2 + (2a - a^3 + 12)\cos\left(\frac{\pi}{7}\right) + (a^3 - 2a - 4)\cos\left(\frac{2\pi}{7}\right)\right\},$$

$$\beta = \frac{2}{a^2}\left\{\frac{a^3}{2} + a^2 + 5a + 8 - 8\cos\left(\frac{\pi}{7}\right) + (2a - a^3 + 12)\cos\left(\frac{2\pi}{7}\right)\right\},$$

$$\gamma = \frac{2}{a^2}\left\{\frac{a^3}{2} + a^2 + 5a + 4 + (a^3 - 2a - 4)\cos\left(\frac{\pi}{7}\right) - 8\cos\left(\frac{2\pi}{7}\right)\right\}.$$

Numerical evaluations exclude all possible orders of α, β, and γ, which do not meet the conditions in Lemma 3.7(i),(ii). Thus, we find that (α, β, γ) is the correct order of the roots, and by Entry 1.1(iv), the proof is complete.

From Theorem 7.1, similarly as we have seen in Theorem 5.1, the value of G_{2401} can be determined. As from Theorem 6.2 in Theorem 7.1, by using Theorems 6.3, 6.4, and 6.5, analogous results can be obtained for $\varphi(e^{-49\pi\sqrt{3}})$, $\varphi(e^{-147\pi})$, and $\varphi(e^{-245\pi})$, respectively. These values can be expressed by using the solutions of the corresponding cubic polynomials, but their structure seems much more complicated.

Based on a remark at the end of Section 12 of Chapter 20 of Ramanujan's second notebook [24, p. 247], [4, p. 400], we believe that the septic identity in Entry 1.1 is a special case of a much more general result. We will continue our investigation in this direction with the description of the analogous cubic and quintic identities.

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References

1. Abel, N.H.: Recherches sur les fonctions elliptiques. J. Reine Angew. Math. **3**, 160–190 (1828)
2. Andrews, G.E., Berndt, B.C.: Ramanujan's Lost Notebook. Part II. Springer, New York (2009)
3. Bankoff, L., Garfunkel, J.: The heptagonal triangle. Math. Magn. **46**, 7–19 (1973)
4. Berndt, B.C.: Ramanujan's Notebooks. Part III. Springer, New York (1991)
5. Berndt, B.C.: Ramanujan's theory of theta-function. In: Ram Murty, M. (ed.) Theta Functions, From the Classical to the Modern, Centre de Recherches Mathématiques Proceedings and Lecture Notes, vol. 1, pp. 1–63. American Mathematical Society, Providence (1993)
6. Berndt, B.C.: Ramanujan's Notebooks. Part IV. Springer-Verlag, New York (1994)
7. Berndt, B.C.: Ramanujan's Notebooks. Part V. Springer, New York (1998)
8. Berndt, B.C.: The remaining 40% of Ramanujan's Lost Notebook. In Number Theory and its Applications, Surikaisekikenkyuusho Kokyuuroku, No. 1060, pp. 111–118, RIMS Kyoto University, Kyoto, (1998)
9. Berndt, B.C., Chan, H.H.: Ramanujan's explicit values for the classical theta function. Mathematika **42**(2), 278–294 (1995)
10. Berndt, B.C., Chan, H.H., Zhang, L.-C.: Ramanujan's class invariants and cubic continued fraction. Acta Arith. **73**(1), 67–85 (1995)
11. Berndt, B.C., Chan, H.H., Zhang, L.-C.: Ramanujan's class invariants, Kronecker's limit formula, and modular equations. Trans. Am. Math. Soc. **349**(6), 2125–2173 (1997)
12. Berndt, B.C., Chan, H.H., Zhang, L.C.: Ramanujan's class invariants with applications to the values of q-continued fractions and theta functions. In: Ismail, M.E.H., Masson, D.R., Rahman, M. (eds.) Special Functions, q-Series and Related Topics, Fields Institute Communications Series, vol. 14, pp. 37–53. American Mathematical Society, Providence (1997)
13. Borwein, J.M., Borwein, P.B.: Pi and the AGM. Wiley, New York (1987)
14. Borwein, J.M., Zucker, I.J.: Fast evaluation of the gamma function for small rational fractions using complete elliptic integrals of the first kind. IMA J. Numer. Anal. **12**(4), 519–526 (1992)
15. Cox, D.A.: Primes of the Form $x^2 + ny^2$. Wiley, New York (1989)
16. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Academic Press, New York (2007)
17. Jacobi, C.G.J.: Fundamenta nova theoriae functionum ellipticarum. Sumptibus fratrum Borntræger, Regiomonti (1829)
18. Joubert, P.: Sur la théorie des fonctions elliptiques et son application à la théorie des nombres. Comptes rendus **50**, 907–912 (1860)
19. Joyce, G.S., Zucker, I.J.: Special values of the hypergeometric series. Math. Proc. Camb. Philos. Soc. **109**(2), 257–261 (1991)
20. Mason, J.C., Handscomb, D.C.: Chebyshev Polynomials. CRC Press, Boca Raton (2003)
21. Ramanujan, S.: Modular equations and approximations to π. Quart. J. Math. **45**, 350–372 (1914)
22. Ramanujan, S.: Collected Papers of Srinivasa Ramanujan. Cambridge Univ. Press, Cambridge (1927)
23. Ramanujan, S.: Notebooks of Srinivasa Ramanujan, vol. I. Tata Institute of Fundamental Research, Bombay (1957)
24. Ramanujan, S.: Notebooks of Srinivasa Ramanujan, vol. II. Tata Institute of Fundamental Research, Bombay (1957)
25. Ramanujan, S.: The Lost Notebook and Other Unpublished Papers. Narosa, New Delhi (1988)
26. Selberg, A., Chowla, S.: On Epstein's zeta function. J. Reine Angew. Math. **227**, 86–110 (1967)
27. Son, S.H.: Septic theta function identities in Ramanujan's Lost Notebook. Acta Arith. **98**(4), 361–374 (2001)
28. Son, S.H.: Ramanujan's symmetric theta functions in his lost notebook. In: D. Dominici, D., Maier, R.S. (eds.) Special Functions and Orthogonal Polynomials, American Mathematical Society, Providence, RI. Contemp. Math. **471**, 187–202. (2008)
29. Watson, G.N.: Some singular moduli (II). Quart. J. Math. **3**(1), 189–212 (1932)
30. Watson, G.N.: Singular moduli (4). Acta Arith. **1**(2), 284–323 (1935)
31. Watson, G.N.: Singular moduli (3). Proc. Lond. Math. Soc. **40**(1), 83–142 (1936)
32. Watson, G.N.: Singular moduli (5). Proc. Lond. Math. Soc. **42**(1), 377–397 (1937)
33. Weber, H.: Lehrbuch der Algebra, Dritter Band, 2nd edn. Druck und Verlag von Friedrich Vieweg und Sohn, Braunschweig (1908)

34. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis, 4th edn. Cambridge University Press, Cambridge (1950)
35. Yi., J.: The Construction and Applications of Modular Equations. PhD thesis, University of Illinois at Urbana–Champaign, Urbana, Illinois (2001)
36. Zhang, L.-C.: Ramanujan's class invariants, Kronecker's limit formula and modular equations (III). Acta Arith. **82**(4), 379–392 (1997)
37. Zucker, I.J.: The evaluation in terms of Γ-functions of the periods of elliptic curves admitting complex multiplication. Math. Proc. Camb. Philos. Soc. **82**(1), 111–118 (1977)
38. Zucker, I.J., Joyce, G.S.: Special values of the hypergeometric series II. Math. Proc. Camb. Philos. Soc. **131**(2), 309–319 (2001)