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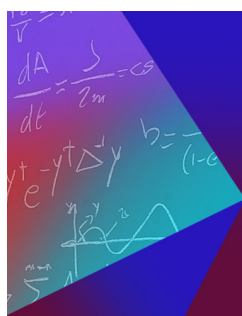
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A frame based approach to computing symmetries with non-trivial isotropy groups

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ABSTRACT

A frame approach to determining the most general solution admitting a desired symmetry group has previously been examined in Riemannian and teleparallel geometries with some success. In teleparallel geometries, one must determine the general form of the frame and spin connection to generate a general solution admitting the desired symmetry group. Current approaches often rely on the use of the proper frame, where the spin connection is zero. However, this leads to particular theoretical and practical problems. In this paper, we introduce an entirely general approach to determining the most general Riemann–Cartan geometries that admit a given symmetry group and apply these results to teleparallel geometries. To illustrate the approach, we determine the most general geometries, with the minimal number of arbitrary functions, for particular choices of symmetry groups with dimension one, three, six, and seven. In addition, we rigorously show how the teleparallel analog of the Robertson–Walker, de Sitter, and Einstein static spacetimes can be determined.

I. INTRODUCTION

There is a wealth of alternative theories of gravity, including analogues of general relativity (GR), where the torsion tensor and non-metricity tensors are employed in addition to the curvature tensor. In any theory where the torsion tensor or non-metricity tensors are non-zero, the Levi–Civita connection is now a secondary object constructed from the contorsion tensor and the connection that gives rise to the torsion and non-metricity tensors. It is often easier to formulate such theories with their metric tensor and the non-Levi–Civita connection as independent quantities. Furthermore, in some theories, such as teleparallel gravity, it is necessary to work with a frame basis instead of a metric tensor. In such theories, the role of symmetry is no longer as clearly defined as in pseudo-Riemannian geometry, where the Levi–Civita connection relates solutions to the Killing equations as symmetries, and so we can no longer formulate a symmetry as an isometry of the metric. Instead, for any theory constructed from a metric tensor and a non-Levi–Civita connection, the symmetry must satisfy the following conditions:

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 0 \quad \text{and} \quad \mathcal{L}_{\mathbf{X}} \omega^a{}_b = 0, \tag{1}$$

where the vector field $\mathbf{X}$ is the infinitesimal generator of the symmetry, g_{ab} is the metric tensor, $\omega^a{}_b$ is the connection one-form, and $\mathcal{L}$ is the Lie derivative. We note that this definition applies only if the isotropy group of the associated geometry is trivial.

The development of a frame based approach to determining the symmetries of a spacetime has been explored in four-dimensional (4D) GR.[1–3] However, the primary obstacle to these approaches is the potential existence of a non-trivial linear isotropy group, which is a Lie subgroup of Lorentz frame transformations that leave the associated tensors of the geometry invariant. If a given spacetime has a non-trivial

linear isotropy group, determining the group of symmetries requires solving a set of inhomogeneous differential equations with unknown functions, $\lambda^a{}_b$,[4]

$$\mathcal{L}_{\mathbf{X}}\mathbf{h}^a = \lambda^a{}_b\mathbf{h}^b \text{ and } \mathcal{L}_{\mathbf{X}}\boldsymbol{\omega}^a{}_b = \omega^a{}_{bc}\lambda^c{}_d\mathbf{h}^d, \tag{2}$$

where $\mathbf{h}^a$ denotes the orthonormal coframe basis and $\lambda^a{}_b$ is an element of the Lie algebra of Lorentz transformations. We note that the second condition imposes that the vector field $\mathbf{X}$ is an affine collineation $\mathcal{L}_{\mathbf{X}}\omega^a{}_{bc} = 0$.

If we have a representation of the symmetry group as a set of vector fields, then solving for the coframe basis and connection is complicated by the unknown functions in $\lambda^a{}_b$. This problem can be avoided by prolonging to a higher dimensional Lorentzian manifold where the parameters of the basis of the linear isotropy group are now coordinates. While this approach is mathematically well-defined, it is unsatisfactory from a physical perspective where all calculations should be performed on the original manifold.

In this paper, we introduce a new approach to determining the symmetries of any geometry based on a frame and an independent connection, which admits the torsion tensor and the curvature tensor as geometric objects. In such theories, the connection will be an independent object. For each example, we will also specialize to teleparallel gravity by setting the curvature tensor to zero. We will call any geometry where the non-metricity and curvature tensors vanish a *teleparallel geometry*. The approach introduced here relies on the existence of a particular class of invariantly defined frames known as symmetry frames, which is amenable to solving the differential equations arising from (2). In this paper, we have assumed that the connection is metric compatible in order to discuss Riemann–Cartan geometries. However, the approach outlined in this paper can readily be extended to geometries where the non-metricity tensor is non-zero, i.e., where the connections are no longer metric-compatible.

As an alternative to Riemannian geometries, which are typically characterized by the curvature tensor of a Levi–Civita connection calculated from the metric, teleparallel geometries are characterized by the torsion tensor, torsion being a function of the coframe, $\mathbf{h}^a$, derivatives of the coframe, and a zero curvature and metric compatible spin connection. In teleparallel geometries, the spacetime indexed representation of the connection, sometimes called the Weitzenböck connection, can be expressed via the derivatives of the frame added to the spin connection,

$$\Gamma^\rho{}_{\nu\mu} = h_c{}^\rho\left(\partial_\mu h^c{}_\nu + \omega^c{}_{b\mu}h^b_\nu\right).$$

Assuming an orthonormal frame, the spin connection $\omega^a{}_{bc}$ is defined in terms of an arbitrary Lorentz transformation $\Lambda^a{}_b$ through the equation

$$\omega^a{}_{bc} = \Lambda^a{}_d\mathbf{h}_c((\Lambda^{-1})^d{}_b), \tag{3}$$

where $\mathbf{h}_c((\Lambda^{-1})^d{}_b)$ denotes a frame vector acting as a vector field derivation on the components of $(\Lambda^{-1})^d{}_b$.

Teleparallel geometries provide an alternative framework in which to build a theory of gravity. A variety of teleparallel gravitational theories based on a Lagrangian can be constructed using various scalars built from the torsion and functions thereof. One subclass of teleparallel gravitational theories is dynamically equivalent to GR and is appropriately called the Teleparallel Equivalent to General Relativity (TEGR).[5] A particularly interesting generalization of the TEGR is $F(T)$ teleparallel gravity.[6–8] In the *covariant* approach to $F(T)$ teleparallel gravity,[9] the teleparallel geometry is defined in a gauge invariant manner as a geometry with zero curvature, having a spin connection that vanishes in a very special class of frames where all inertial effects are absent, and non-zero in all other frames.[5,9,10] Therefore, the resulting teleparallel gravity theory has Lorentz covariant field equations and is, therefore, locally Lorentz invariant.[11]

In teleparallel geometries, assuming an orthonormal frame, it is always possible to transform to a frame where the components of the spin connection are zero. Such a frame–connection pair is called a *proper frame* and is sometimes desirable when analyzing the $F(T)$ field equations. However, a proper frame is not invariantly defined since it is defined in terms of the connection, which is not a tensorial quantity. There have been attempts to use the proper frame to determine the symmetries of a teleparallel geometry and, for a given group of symmetries, determine the largest class of teleparallel geometries that admit this group.[12–14]

There are two significant problems with using the proper frame to determine symmetries. These problems are related to the fact that the proper frame is not invariantly defined. The first problem is that in the process of solving the equations in (2), arbitrary functions of integration will be introduced and it is not possible to determine when such a function is essential to the geometry or if it can be removed by an appropriate coordinate transformation. The second problem is that one cannot determine when two given solutions to (2) are inequivalent and this can lead to an incomplete list of all geometries that admit a given symmetry group. Our approach differs by choosing a particular class of symmetry frames, which we will define later, which are invariantly defined up to the linear isotropy group and admit a non-zero spin connection. Then, by restricting to a particular coordinate system and imposing a chosen vielbein form, all of the remaining frame parameters can be fixed, giving rise to an invariant frame (for this choice of coordinate system). With this class of frames, all inequivalent geometries admitting a given symmetry group can be determined along with the minimal number of functions in the vielbein and spin connection to specify them.

We note as well that the proper frame approach cannot predict the existence or non-existence of permitted symmetry groups without extensive calculation. As an example of this, in Ref. 15, it was proven that the only maximally symmetric teleparallel geometry was Minkowski space and that the (anti-)de Sitter geometries admit, at most, a seven-dimensional symmetry group. This was later verified using the proper frame approach by exhaustively proving that the resulting differential equations from (2) have no solution.[16] In addition, the proper frame approach is applicable only to teleparallel geometries and cannot be used for Riemann–Cartan geometries.

The outline of this paper is as follows: in the remainder of the Introduction, we briefly specify the notation used in this paper. In Sec. II, we introduce the class of symmetry frames and define affine frame symmetries for any Riemann–Cartan geometry. In Sec. III, we employ symmetry frames to determine the equations required to determine the most general geometries that admit a given symmetry group and we specialize this to teleparallel geometries. In Sec. IV, we provide several examples of this approach by considering the one-dimensional symmetry group, spherical symmetry, Robertson–Walker (RW) symmetry, and two seven-dimensional symmetry groups, which are special subcases of the RW geometries. In each case, we specialize to teleparallel geometries. In Sec. V, we summarize our results and discuss further applications. Note that when discussing teleparallel geometries, the associated proper frames will not be discussed.

A. Notation

We will denote the coordinate indices by $\mu, \nu, \ldots$ and the tangent space indices by $a, b, \ldots$. Unless otherwise indicated, the spacetime coordinates will be x^μ. The frame fields are denoted as $\mathbf{h}_a$, and the dual coframe one-forms are $\mathbf{h}^a$. The vielbein components are $h_a{}^\mu$ or $h^a{}_\mu$. The spacetime metric will be denoted as $g_{\mu\nu}$, while the Minkowski tangent space metric is η_{ab}.

To denote a local Lorentz transformation leaving η_{ab} unchanged, we write $\Lambda_a{}^b(x^\mu)$. The spin connection one-form $\boldsymbol{\omega}^a{}_b$ is designated by $\boldsymbol{\omega}^a{}_b = \omega^a{}_{bc}\mathbf{h}^c$. The curvature and torsion tensors will, respectively, be denoted as $R^a{}_{bcd}$ and $T^a{}_{bc}$.

Covariant derivatives with respect to a metric-compatible connection will be denoted using a semi-colon, $T_{abc;e}$. When considering the integral curves of a vector field $\mathbf{X}$, with parameter τ, given as $\phi_\tau : M \to M$, we will write the corresponding Lorentz frame transformation as $\phi_\tau^* \mathbf{h}^a = \Lambda_{(\tau)}{}^a{}_b \mathbf{h}^b$.

II. AFFINE FRAME SYMMETRIES WITH ISOTROPY

The starting point for the determination of symmetries of a Riemann–Cartan geometry, M, is the Cartan–Karlhede (CK) algorithm.[15,17] In this algorithm, a frame is chosen and the parameters of the Lorentz frame transformations are fixed by normalizing the components of the curvature tensor, torsion tensor, and their covariant derivatives in an invariant manner. This algorithm provides a wealth of invariants, and these can be used to uniquely characterize a geometry locally and determine the dimension of the symmetry group.

Two useful sequences of discrete invariants are particularly important: the dimension of the linear isotropy group, $\dim H_p$, at each iteration, p, and the number of functionally independent components of the curvature tensor, torsion tensor, and their covariant derivatives up to order p, denoted as t_p. The linear isotropy group, H_p, is a subgroup of the Lorentz frame transformations that leave the curvature tensor, the torsion tensor, and their covariant derivatives up to pth order invariant. If H_p is the trivial group, we say that the frame is now an invariant frame. The algorithm ends when $t_{p-1} = t_p$ and $\dim H_{p-1} = \dim H_p$, which indicates that all necessary and sufficient geometric information for the geometry has been produced.

At the conclusion of the algorithm, say at the qth iteration, the linear isotropy group H_q encodes the freedom in specifying an invariantly defined (co)frame. That is, the CK algorithm produces an invariantly defined (co)frame, $\mathbf{h}^a$, up to the linear isotropy H_q such that for any diffeomorphism, $\phi : M \to M$, the invariantly defined (co)frame satisfies

$$\phi^* \mathbf{h}^a = \Lambda^a{}_b \mathbf{h}^b, \tag{4}$$

where $\Lambda^a{}_b$ belongs to H_q and arises from the coordinate form of ϕ, i.e., $y^\mu = \phi^\mu(x^\nu)$.[4] If $\phi = \phi_\tau$ is generated by exponentiating the vector field locally representing an infinitesimal generator of a symmetry, $\mathbf{X}$, with some parameter τ, then we can calculate the effect of the Lie derivative of $\mathbf{X}$ on the frame as

$$\mathcal{L}_{\mathbf{X}} \mathbf{h}^a = \lambda^a{}_b \mathbf{h}^b, \tag{5}$$

where $\lambda^a{}_b$ is the Lie algebra generator for $\Lambda_{(\tau)}{}^a{}_b$. By using an invariantly defined frame, up to the linear isotropy group, we have reduced the number of unknown functions in $\lambda^a{}_b$ and restricted their functional dependence to only those coordinates that are affected by ϕ_τ. To distinguish between those coordinates that are affected or unaffected by the symmetry generators, we will denote these as $x^{\alpha'}$ and $x^{\tilde{\alpha}}$, respectively.

This is one part of the recipe for determining or fixing a symmetry in a Riemann–Cartan geometry. Such manifolds are equipped with a metric-compatible connection one-form $\boldsymbol{\omega}^a{}_b = \omega^a{}_{bc}\mathbf{h}^c$, where $\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}$. If $\mathbf{X}$ is an infinitesimal generator of a symmetry for the geometry, then it must also be an affine collineation,[18]

$$\mathcal{L}_{\mathbf{X}} \omega^a{}_{bc} = 0, \tag{6}$$

or equivalently, by using the Leibniz rule for the Lie derivative, this condition can be restated for the connection one-form $\boldsymbol{\omega}^a{}_b = \omega^a{}_{bc}\mathbf{h}^c$,

$$\mathcal{L}_{\mathbf{X}} \boldsymbol{\omega}^a{}_b = \omega^a{}_{bc}\mathcal{L}_{\mathbf{X}}\mathbf{h}^c = \omega^a{}_{bc}\lambda^c{}_d\mathbf{h}^d. \tag{7}$$

We note that this condition can be rewritten as Eq. (40) in Ref. 13 using the identity in Eq. (18). From conditions (5) and (6), it follows that the Lie derivative of the curvature tensor, the torsion tensor, and their covariant derivatives with respect to $\mathbf{X}$ will vanish. We will call any

vector field satisfying Eqs. (5) and (6) an *affine frame symmetry*. We note that if $\lambda^a_{\ b} = 0$, then we recover the original definition of an affine frame symmetry in Ref. 15, where the linear isotropy group is trivial and the frame resulting from the CK algorithm is an invariant frame.

While the above definition is necessary and sufficient to determine a symmetry of any Riemann–Cartan geometry, in practice, it is difficult to implement as the parameters of the Lorentz transformations must be determined in an invariant manner for each of the possible canonical forms of the curvature tensor and torsion tensor. We will expand this definition to a larger class of frames that are acted on by the Lie algebra generators of the isotropy group under Lie differentiation with the affine frame symmetry generators.

Definition II.1. Consider a Riemann–Cartan geometry, $(\mathbf{h}^a, \boldsymbol{\omega}^a_{\ b})$, admitting an affine frame symmetry, $\mathbf{X}$. The class of **symmetry frames**, $\mathbf{h}^a$, satisfies the following condition:

$$\mathcal{L}_{\mathbf{X}}\mathbf{h}^a = f_X^{\hat{\imath}}\lambda_{\hat{\imath}}^{\ a}_{\ b}\mathbf{h}^b, \tag{8}$$

where $\lambda_{\hat{\imath}}^{\ a}_{\ b}$ are the basis elements of the Lie algebra of the isotropy group (so that $\lambda_{\hat{\imath}ab} = -\lambda_{\hat{\imath}ba}$) and $\hat{\imath}$ ranges from 1 to the dimension of the isotropy group. The components of $f_X^{\hat{\imath}}$ are functions dependent on the coordinates affected by the symmetry generators $x^{\alpha'}$ and have been introduced to permit the possibility of an arbitrary infinitesimal generator of an element of the isotropy group. This is a general definition and is not (strictly speaking) an invariantly defined frame until the components $f_X^{\hat{\imath}}$ are fixed in some coordinate independent way.

Assuming that Eqs. (6) and (8) are satisfied so that $\mathbf{X}$ generates an affine frame symmetry, we find that for $g_{\mu\nu} = \eta_{ab}\mathbf{h}^a_{\ \mu}\mathbf{h}^b_{\ \nu}$,

$$\mathcal{L}_{\mathbf{X}}(\eta_{ab}\mathbf{h}^a\mathbf{h}^b) = f_X^{\hat{\imath}}[\lambda_{\hat{\imath}bc}\mathbf{h}^b\mathbf{h}^c + \lambda_{\hat{\imath}cb}\mathbf{h}^b\mathbf{h}^c] = 0, \tag{9}$$

which ensures that $\mathbf{X}$ is a Killing vector field as well.

The class of symmetry frames is guaranteed to exist since they are constructed by taking the class of frames generated from the CK algorithm, which are invariantly defined up to the linear isotropy group, and acting upon them by the group of Lorentz transformations that leave the representation of the isotropy group unchanged (which we have denoted as $\overline{Iso}$). This can be determined using the chosen Lie algebra basis for the Lorentz group and picking those Lie algebra generators whose commutator with the Lie algebra elements of the isotropy group lies in the isotropy group. To be precise, we can determine the Lie algebra generators of $\overline{Iso}$, $\tilde{\lambda}$, by solving the n equations, $[\lambda_{\hat{\imath}}, \tilde{\lambda}] \in \text{span}\{\lambda_{\hat{1}}, \ldots, \lambda_{\hat{n}}\}$, and then exponentiating the linearly independent parts of $\tilde{\lambda}$ to produce $\overline{Iso}$. For example, in the case of spherical symmetry, there are three generators of the isotropy group, $\mathbf{X}_I$, $I = 1, 2, 3$, and the group $\overline{Iso}$ consists of all spatial rotations.

Since the components of $f_X^{\hat{\imath}}$ are associated with the Lie derivative of a symmetry generating vector field, they are tensor quantities that depend on the frame. Under a change of frame, $\tilde{\mathbf{h}}^a = \tilde{\Lambda}^a_{\ b}\mathbf{h}^b$, the components transform as

$$\mathbf{X}_I(\tilde{\Lambda}^a_{\ b})[\tilde{\Lambda}^{-1}]^b_{\ c} + \tilde{\Lambda}^a_{\ b}f_I^{\hat{\imath}}\lambda_{\hat{\imath}}^{\ b}_{\ d}[\tilde{\Lambda}^{-1}]^d_{\ c} = \tilde{f}_I^{\hat{\imath}}\lambda_{\hat{\imath}}^{\ a}_{\ c}. \tag{10}$$

We can employ them to construct invariantly defined frames up to some minimal subgroup of $\overline{Iso}$ or invariant frames if this subgroup is trivial. To be precise, if we employ the frame freedom in $\overline{Iso}$ to fix the form of the $f_X^{\hat{\imath}}$ as much as possible so that some elements are constant or zero, the symmetry frame is then an *invariantly defined symmetry frame up to the isotropy group* $\overline{H}_q$, where $\overline{H}_q$ is some subgroup of $\overline{Iso}$ that leaves the chosen form of $f_X^{\hat{\imath}}$ unchanged. Again, using the spherically symmetric case as an example, a choice of $f_{X_I}^{\hat{\imath}} \equiv f_I^{\hat{\imath}}$, $I = 1, 2, 3$, for each symmetry generator can be made so that this subgroup, $\overline{H}_q$, consists of rotations in the $\mathbf{h}^3$–$\mathbf{h}^4$ plane where their Lorentz parameters are constant. In Subsection IV, we will outline the procedure to fix the components of $f_X^{\hat{\imath}}$ and determine $\overline{H}_q$ for the axially symmetric, spherically symmetric, and spatially homogeneous and isotropic symmetric geometries, respectively.

We note that once a class of symmetry frames is invariantly defined up to the group $\overline{H}_q$, any additional fixing of the remaining frame parameters should be maximally fixed, up to the linear isotropy group of the CK algorithm, H_q, in an invariant manner. This can be done using the curvature tensor, the torsion tensor, or their respective covariant derivatives, as in the CK algorithm. While this can be done readily for a given geometry, or even particular subclasses of geometries, it is difficult to implement for a general class of geometries. However, since the solution to Eq. (8) yields a particular frame ansatz, we can sidestep the problem of fixing the frame invariantly by restricting the set of permitted diffeomorphisms of the geometry to the "shape preserving" transformations, those that preserve the function form of the metric or coframe[19] but alter the expressions for the metric or vielbein functions. That is, we can make a choice of vielbein, which provides a way to further fix the frame fully and construct an invariant frame, relative to the given coordinate system. This is sufficient for finding inequivalent solutions but leads to difficulties if they are to be subclassified.

III. DETERMINING THE MOST GENERAL GEOMETRY FOR A GIVEN ISOMETRY GROUP

Let us assume that we have a Lie group of affine frame symmetries of dimension N, with a non-trivial isotropy group with dimension n ($n < N$). We choose coordinates so that the symmetry group is represented as a set of vector fields, $\mathbf{X}_I$, where $I, J, K \in [1, N]$, with the corresponding Lie algebra,

$$[\mathbf{X}_I, \mathbf{X}_J] = C^K_{IJ}\mathbf{X}_K, \tag{11}$$

where C^K_{IJ} are structure constants of the Lie algebra.

Then, we can determine the most general Riemann–Cartan geometries admitting this Lie group as affine symmetries by solving the following equations for an orthonormal symmetry coframe,[20] $\mathbf{h}^a$, and the associated metric, $g_{\mu\nu} = \eta_{ab}h^a{}_\mu h^b{}_\nu$, with a metric-compatible connection, $\omega^a{}_{bc}$,

$$\mathcal{L}_{\mathbf{X}_I}\mathbf{h}^a = f_I{}^{\hat{i}}\lambda^a_{\hat{i}\ b}\mathbf{h}^b, \tag{12}$$

$$\mathcal{L}_{\mathbf{X}_I}\omega^a{}_{bc} = 0, \tag{13}$$

where $\hat{i}, \hat{j}, \hat{k} \in [1, n]$ and the components of $f_I{}^{\hat{i}}$ are functions of the coordinates, $x^{\alpha'}$. The form of $f_I{}^{\hat{i}}$ can be specified by choosing some components to be equal to others, to be constant or to be zero. This defines a class of symmetry frames invariantly up to $\overline{H}_q$. For the moment, we will consider an arbitrary symmetry frame. In addition, due to the properties of the Lie derivative, we have an additional relationship coming from (11),

$$[\mathcal{L}_{\mathbf{X}_I}, \mathcal{L}_{\mathbf{X}_J}]\mathbf{h}^a = \mathcal{L}_{[\mathbf{X}_I,\mathbf{X}_J]}\mathbf{h}^a = C^K_{IJ}\mathcal{L}_{\mathbf{X}_K}\mathbf{h}^a. \tag{14}$$

We will focus for the moment on Eqs. (12) and (14). These can be expanded to give the following equations:

$$X_I{}^\nu\partial_\nu h^a{}_\mu + \partial_\mu X_I{}^\nu h^a{}_\nu = f_I{}^{\hat{i}}\lambda^a_{\hat{i}\ b}h^b{}_\mu \tag{15}$$

and

$$2\mathbf{X}_{[I}(f^{\hat{k}}_{J]}) - f_I{}^{\hat{i}}f_J{}^{\hat{j}}C^{\hat{k}}_{\hat{i}\hat{j}} = C^K_{IJ}f^{\hat{k}}_K, \tag{16}$$

where the associated Lie algebra of the isotropy group is given as

$$[\lambda_{\hat{i}}, \lambda_{\hat{j}}] = \lambda^a_{\hat{i}\ b}\lambda^b_{\hat{j}\ c} - \lambda^a_{\hat{j}\ b}\lambda^b_{\hat{i}\ c} = C^{\hat{k}}_{\hat{i}\hat{j}}\lambda^a_{\hat{k}\ c}.$$

These equations determine the **most general** frame basis that admits a given symmetry group with a non-trivial isotropy group. To determine the connection, we must solve Eq. (13) relative to this frame. Using the coordinate basis expression for the Lie derivative of the connection in Ref. 21, the corresponding frame basis expression is

$$\mathcal{L}_{\mathbf{X}_I}(\omega^a{}_{bc})\mathbf{h}_a = \mathbf{R}(\mathbf{X}, \mathbf{h}_c)\mathbf{h}_b - \mathbf{T}(\mathbf{X}, \nabla_c\mathbf{h}_b) + \nabla_c\mathbf{T}(\mathbf{X}, \mathbf{h}_b) + \nabla_c\nabla_b\mathbf{X} - \nabla_{\nabla_c\mathbf{h}_b}\mathbf{X}, \tag{17}$$

where $\mathbf{R}(\mathbf{h}_d, \mathbf{h}_c)\mathbf{h}_b = R^a{}_{bcd}\mathbf{h}_a$ and $\mathbf{T}(\mathbf{h}_c, \mathbf{h}_b) = T^a{}_{bc}\mathbf{h}_a$, respectively, denote the curvature tensor and torsion tensor of the connection $\omega^a{}_{bc}$ and $\nabla_a = \nabla_{\mathbf{h}_a}$.

We can use Eqs. (12) and (13) to find the following set of equations for $\omega^a{}_{bc}$:

$$\mathcal{L}_{\mathbf{X}_I}(\omega^a{}_{bc}) = X_I{}^d\mathbf{h}_d(\omega^a{}_{bc}) + \omega^d{}_{bc}f_I{}^{\hat{i}}\lambda^a_{\hat{i}\ d} - \omega^a{}_{dc}f_I{}^{\hat{i}}\lambda^d_{\hat{i}\ b} - \omega^a{}_{bd}f_I{}^{\hat{i}}\lambda^d_{\hat{i}\ c} - \mathbf{h}_c(f_I{}^{\hat{i}})\lambda^a_{\hat{i}\ b} = 0, \tag{18}$$

where $\mathbf{h}_c(f(x^\alpha))$ denotes the frame derivative of a function $f(x^\alpha)$ with respect to $\mathbf{h}_c$.

Theorem III.1. *The most general Riemann–Cartan geometry admitting a given group of symmetries, $\mathbf{X}_I, I, J, K \in \{1, \ldots, N\}$, with a non-trivial isotropy group of dimension n can be determined by solving for the unknowns $h^a{}_\mu$, $f_I{}^{\hat{i}}$ (with $\hat{i}, \hat{j}, \hat{k} \in \{1, \ldots, n\}$), and $\omega^a{}_{bc}$ from the following equations:*

$$X_I{}^\nu\partial_\nu h^a{}_\mu + \partial_\mu X_I{}^\nu h^a{}_\nu = f_I{}^{\hat{i}}\lambda^a_{\hat{i}\ b}h^b{}_\mu,$$

$$2\mathbf{X}_{[I}(f^{\hat{k}}_{J]}) - f_I{}^{\hat{i}}f_J{}^{\hat{j}}C^{\hat{k}}_{\hat{i}\hat{j}} = C^K_{IJ}f^{\hat{k}}_K, \tag{19}$$

$$X_I{}^d\mathbf{h}_d(\omega^a{}_{bc}) + \omega^d{}_{bc}f_I{}^{\hat{i}}\lambda^a_{\hat{i}\ d} - \omega^a{}_{dc}f_I{}^{\hat{i}}\lambda^d_{\hat{i}\ b} - \omega^a{}_{bd}f_I{}^{\hat{i}}\lambda^d_{\hat{i}\ c} - \mathbf{h}_c(f_I{}^{\hat{i}})\lambda^a_{\hat{i}\ b} = 0,$$

where $\{\lambda^a_{\hat{i}\ b}\}^n_{\hat{i}=1}$ is a basis of the Lie algebra of the isotropy group, $[\lambda_{\hat{i}}, \lambda_{\hat{j}}] = C^{\hat{k}}_{\hat{i}\hat{j}}\lambda_{\hat{k}}$, $[\mathbf{X}_I, \mathbf{X}_J] = C^K_{IJ}\mathbf{X}_K$.

We remark that while the above equations apply to any symmetry frame, solving these equations can be more easily attained by using an invariantly defined symmetry frame up to $\overline{H}_q$, where the components of $f_I{}^{\hat{i}}$ have been invariantly specified. As discussed in Subsection II, this entails using the group $\overline{Iso}$ to fix the form of the matrix $f_I{}^{\hat{i}}$ in an invariant manner, for example by setting some components to be constant, to be zero, or equal to other components.

To specialize this result to a teleparallel geometry, in addition to the differential equation coming from Eq. (6), the connection must be flat and so the curvature tensor expressed in terms of the spin connection and its derivatives must vanish. This condition is equivalent to (3) for some choice of Lorentz transformation $\Lambda^a{}_b$. In summary,

Theorem III.2. *The most general teleparallel geometry that admits a given group of symmetries, $\mathbf{X}_I, I, J, K \in \{1, \ldots, N\}$, with a non-trivial isotropy group of dimension n can be determined by solving for the unknowns $h^a{}_\mu$, $f_I{}^{\hat{i}}$ (with $\hat{i}, \hat{j}, \hat{k} \in \{1, \ldots, n\}$), and $\omega^a{}_{bc}$ from Eq. (19) along with an additional condition,*

$$R^a{}_{bcd} = h_c^\mu \partial_\mu \omega^a{}_{bd} - h_d^\nu \partial_\nu \omega^a{}_{bc} + \omega^a{}_{fc}\omega^f{}_{bd} - \omega^a{}_{fd}\omega^f{}_{bc} = 0. \tag{20}$$

IV. EXAMPLES

A. One-dimensional symmetry groups

If the symmetry group is one-dimensional, one may always choose a neighborhood and rectify the infinitesimal generator, $\mathbf{X}$, so that it is of the form $\mathbf{X} = \partial_\chi$ with coordinates $x^\mu = (x^{\tilde{\alpha}}, \chi)$ with $\tilde{\alpha} = 1, 2, 3$. We note that the coordinate χ can be spacelike, timelike, or null, depending on the nature of the affine frame symmetry. To distinguish the behavior of the resulting affine-frame symmetry, we must examine the norm of the vector field $\mathbf{X}$ and its effect on the symmetry frame,

$$\mathcal{L}_{\mathbf{X}}\mathbf{h}^a = f(\chi)\lambda^a{}_b \mathbf{h}^b, \tag{21}$$

where $f(\chi)$ is an arbitrary function to be determined. If f is identically zero, the resulting affine frame symmetry is translational, while if f is non-trivial, the affine frame symmetry is an affine frame isotropy. By computing the squared norm of $\lambda^a{}_b$, we can use global Lorentz transformations to simplify the form of this anti-symmetric matrix into four classes: a boost, a null rotation, a loxodromic transformation, or a spatial rotation. Following Ref. 22, we will use two-forms to describe the matrices $\lambda_{ab} = \eta_{ac}\lambda^c{}_b$ so that these transformations are described relative to an orthonormal frame, respectively, as

$$\lambda_{ab} \in \{\mathbf{h}^1 \wedge \mathbf{h}^4, \ \mathbf{h}^1 \wedge \mathbf{h}^2 + \mathbf{h}^2 \wedge \mathbf{h}^4, \mathbf{h}^1 \wedge \mathbf{h}^4 + a_0\mathbf{h}^2 \wedge \mathbf{h}^4, \mathbf{h}^2 \wedge \mathbf{h}^3\}, \tag{22}$$

where a_0 is a real-valued constant acting as a fixed parameter in the loxodromic transformation. In this general setting, determining $\overline{Iso}$ and $\overline{H}_q$ for each of the possible Lie algebra generators in this representation is a tedious but straightforward calculation. We will employ axisymmetry in order to illustrate this approach in Sec. IV A 1.

Without loss of generality, we can use a Lorentz transformation to eliminate the arbitrary function f using $\tilde{\Lambda}^a{}_b = [e^{\tau\lambda}]^a{}_b$ and requiring that the parameter τ is a function of χ determined by Eq. (10). Explicitly, we will solve the following differential equation:

$$X(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c + \tilde{\Lambda}^a{}_b f\lambda^b{}_e[\tilde{\Lambda}^{-1}]^e{}_c = 0. \tag{23}$$

Since $\tilde{\Lambda}^a{}_b$ has the same Lie algebra generator as $\lambda^a{}_b$, $\lambda^a{}_b$ in the second term is invariant under this transformation and the matrix equation reduces to a simple differential equation,

$$\tau_{,\chi} + f = 0.$$

Thus, by integrating τ, we can always remove f from the Lie derivative of $\mathbf{h}^a$ and, instead, involve it in the connection, according to the transformation rule for the connection one-form,

$$\tilde{\boldsymbol{\omega}}^a{}_b = \Lambda^a{}_c \boldsymbol{\omega}^c{}_d (\Lambda^{-1})^d{}_b + \Lambda^a{}_c (d\Lambda^{-1})^c{}_b. \tag{24}$$

The new equation for the symmetry frame is now

$$\mathcal{L}_{\mathbf{X}}\tilde{\mathbf{h}}^a = 0, \tag{25}$$

which implies that the vielbein matrix components are independent of χ, i.e., $\mathbf{h}^a = h^a{}_\alpha(x^{\tilde{\alpha}})dx^\alpha$. This frame is another symmetry frame; however, it is now invariantly defined up to $\overline{H}_q$ since $f_I{}^{\hat{i}} = 0$. Imposing the affine frame symmetry conditions on the new connection gives an additional condition on the function f.

In this coordinate system, with the appropriate symmetry frame determined, we can impose the second condition for an affine frame symmetry,

$$\mathcal{L}_{\mathbf{X}}\omega^a{}_{bc} = \mathbf{X}(\omega^a{}_{bc}) = 0. \tag{26}$$

This is satisfied if and only if the connection coefficients, $\omega^a{}_{bc}$, are independent of χ and so any choice of $\omega^a{}_{bc} = W^a{}_{bc}(x^{\tilde{\alpha}})$ is sufficient to define the connection. Using the transformation rule for the connection one-form with the Lorentz transformation $\tilde{\Lambda}^a{}_b$ with the parameter τ such that $\partial_\chi\tau = -f$, Lie differentiation with respect to χ of this new connection one-form must be zero and this implies that $\partial_\chi^2\tau = 0$. Since $\partial_\chi\tau = -f$, this function is independent of χ and, hence, is a constant, ϵ. With the function $f(\chi)$ specified, we have the following result.

Proposition IV.1. The most general Riemann–Cartan geometry admitting a one-dimensional symmetry $\mathbf{X} = \partial_\chi$ is given by a coframe and a connection of the form

$$\mathbf{h}^a = h^a{}_\alpha(x^{\tilde{\alpha}})dx^\alpha, \quad \omega^a{}_b = W^a{}_{bc}(x^{\tilde{\alpha}})\mathbf{h}^c - \mathbf{h}_c(\epsilon\chi)\lambda^a{}_b\mathbf{h}^c, \tag{27}$$

where $\mathbf{h}_c(\epsilon\chi)$ denotes the derivation of the linear function $\epsilon\chi$ by the frame element $\mathbf{h}_c$ and $\epsilon = 0, 1$.

The magnitude of the infinitesimal generator $|\mathbf{X}|$ determines whether the affine frame symmetry is timelike, null, or spacelike at each point, while the Lie algebra element $\lambda^a{}_b$ indicates whether the symmetry is translational or it admits a fixed point and, hence, is an isotropy.

We remark here that the second term in the connection one-form in Eq. (27) can always be absorbed into the first term. However, where our intention is to construct the most general class of Riemann–Cartan geometries that admit one affine frame symmetry, we have left this second term explicit, in order to emphasize the role of $\lambda^a{}_b$ in choosing the affine frame symmetry.

We also note that the frame in Eq. (27) is invariantly defined (up to $\overline{H}_q$), as this example is very general, and we cannot explicitly fix the frame parameters further but point out that it is always possible to construct an invariantly defined frame when the symmetry group is one dimensional.

In the case of teleparallel geometries, there exists a local Lorentz transformation, $\hat{\Lambda}^a{}_b$, that yields the spin connection through the equation

$$\omega^a{}_b = \hat{\Lambda}^a{}_c(d\hat{\Lambda}^{-1})^c{}_b. \tag{28}$$

In this case, it is easiest to formulate the most general frame and connection admitting a single affine frame symmetry by switching to a proper coframe first, i.e.,

$$\hat{\mathbf{h}}^a = (\hat{\Lambda}^{-1})^a{}_b\mathbf{h}^b, \quad \omega^a{}_b = \mathbf{0}.$$

Relative to the proper frame, the symmetry generator acts as

$$\mathcal{L}_{\mathbf{X}}\hat{\mathbf{h}}^a = \lambda^a{}_b\hat{\mathbf{h}}^a,$$

where $\lambda^a{}_b$ is some arbitrary element of the Lie algebra of the Lorentz group.

In previous papers, it has been shown that $\lambda^a{}_b$ must have constant components.[12] Due to this, it is tempting to use the proper frame as the preferred frame to determine symmetries. However, the determination of proper frames is not invariantly defined using tensor quantities, and so it is difficult to determine when an arbitrary function in a given solution is essential, or only a coordinate artifact. On the contrary, in our approach, the arbitrary functions in the vielbein describe the minimal number of functions required to be a teleparallel geometry admitting one affine frame symmetry.

If $\lambda^a{}_b$ vanishes, then the symmetry generator is translational. Otherwise, this constant matrix can be specified using global Lorentz transformations depending on the squared norm of $\lambda^a{}_b$. If it is negative, positive, or zero, the matrix $\lambda^a{}_b$ can be chosen as a simple generator of a boost, rotation, or null rotation, respectively.

Once this invariant is determined, the above process can be repeated for a teleparallel geometry. Using the fact that we can always pick the proper frame in Eq. (21), we can employ global Lorentz transformations to bring $\lambda^a{}_b$ into one of the canonical representations for a boost, rotation, or null rotation. Then, by applying the Lorentz transformation associated with $\lambda^a{}_b$ and solving a simple differential equation, it is possible to shift this term to the spin connection so that the Lie derivative of the frame basis with respect to $\mathbf{X}$ is zero. The role of $\lambda^a{}_b$ is even more apparent in the coordinate basis as only the $d\chi$ term appears in the connection one-form,

$$-\mathbf{h}_c(\epsilon\chi)\lambda^a{}_b\mathbf{h}^c = -\epsilon\lambda^a{}_b d\chi. \tag{29}$$

Summarizing these observations, we have the following result.

Proposition IV.2. The most general teleparallel geometry admitting a one-dimensional symmetry $\mathbf{X} = \partial_\chi$ is given by a frame and a connection of the form

$$\mathbf{h}^a = h^a{}_\alpha(x^{\tilde{\alpha}})dx^\alpha, \quad \omega^a{}_b = -\epsilon\lambda^a{}_b d\chi, \tag{30}$$

where $\mathbf{h}_c(\epsilon\chi)$ denotes the derivation of the linear function $\epsilon\chi$ by the frame element $\mathbf{h}_c$ and $\epsilon = 0, 1$.

The magnitude of the infinitesimal generator $|\mathbf{X}|$ determines whether the affine frame symmetry is timelike, null, or spacelike at each point, while the Lie algebra element $\lambda^a{}_b$ indicates whether the symmetry is translational or it admits a fixed point and, hence, is an isotropy.

As in the Riemann–Cartan case, the frame is invariantly defined up to $\overline{H}_q$. In each example of a one dimensional symmetry group, an invariantly defined symmetry frame can be found.

The frame and spin connection given here are the most general as the majority of the spin connection has been absorbed into the frame that consists of fully arbitrary components dependent on $x^{\tilde\alpha}$. The remaining non-zero part of the spin connection can only be removed using a Lorentz transformation with χ-dependent parameters, and the resulting symmetry frame is no longer invariantly defined.

1. An application: Axisymmetric geometries

As an example that will be relevant in the remaining examples, we will consider the class of geometries that admit rotational isotropy. These have been discussed previously for teleparallel solutions, either by using the axially symmetric field equations[23] or by using the proper frame approach.[13]

In Ref. 13, two solutions were found to admit axial symmetry. One choice relied on a non-trivial homomorphism from $SO(2)$ into $SO(1, 3)$, while the second arose from the trivial homomorphism from $SO(2)$ into $SO(1, 3)$. The solution corresponding to the trivial homomorphism is not axisymmetric, in the sense of admitting a symmetry in the isotropy group, but, instead, admits a translational affine frame symmetry, which can be seen by noting that the proper frame is already a symmetry frame with $\lambda^a{}_b = 0$.

Our approach recovers both cases as solutions where, relative to the appropriate symmetry frame, the connection one-form in Eq. (29) is either non-trivial or identically zero, respectively. This provides an explanation why the second class of frames corresponding to the trivial homomorphism in Ref. 13 does not have a proper limit to spherically symmetric geometries. In order for this to occur, the affine-symmetry generator ∂_ϕ must lie in the isotropy group in the context of spherical symmetry, but the vanishing of the connection one-form, relative to the invariantly defined symmetry frame, implies that ∂_ϕ does not belong to the isotropy subgroup and, hence, cannot be expanded into the spherically symmetric group.

For these axisymmetric geometries, we will work in the coordinate system $x^\mu = (t, r, \theta, \phi)$ and treat the vector field $\mathbf{X} = \partial_\phi$ as the generator of an axisymmetric symmetry. The Lie algebra generator is then

$$\lambda^a{}_b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{31}$$

which corresponds to the representation $\lambda = \mathbf{h}^3 \wedge \mathbf{h}^4$.

With this choice, we can determine the frame transformations that leave this form invariant by solving for $\tilde\lambda$ from $[\lambda, \tilde\lambda] = \lambda$ and exponentiating $\tilde\lambda$ to determine the elements of the Lorentz frame transformation group. $\overline{Iso}$ consists of rotations in the $\mathbf{h}^3$–$\mathbf{h}^4$ plane and boosts in the $\mathbf{h}^1$–$\mathbf{h}^2$ subspace. As in the general case of a Riemann–Cartan geometry, the function $f(\phi)$ can be set to zero, which fixes the frame invariantly. To determine $\overline{H}_q$, we solve Eq. (10) with $f_I^{\hat i} = \tilde f_I^{\hat i} = 0$ for $\tilde\Lambda^a{}_b$ in $\overline{Iso}$. This implies that

$$\mathbf{X}_I(\tilde\Lambda^a{}_b) = 0 = \partial_\phi(\tilde\Lambda^a{}_b) = 0,$$

and so $\overline{H}_q$ consists of elements of $\overline{Iso}$ but with constant parameters.

The remaining frame freedom can be fixed either by using the curvature and torsion tensor or by employing shape-preserving transformations to simplify the vielbein. Due to the large degree of generality in the vielbein, we will omit this step, as further frame fixing will not reduce the number of arbitrary functions. However, we note that it is always possible to construct an invariantly defined frame using either approach for a given solution.

Relative to a generic coframe, where the vielbein are arbitrary but independent of ϕ,

$$\mathbf{h}^a = h^a{}_\beta(x^{\tilde\alpha})dx^\beta, \tag{32}$$

with $x^{\tilde\alpha} = (t, r, \theta)$. To ensure the traditional axisymmetric geometry, the norm of $\mathbf{X}$ must be spatial and this gives an inequality for the vielbein,

$$g_{\alpha\beta}X^\alpha X^\beta = \eta_{ab}h^a{}_\alpha h^b{}_\beta X^\alpha X^\beta = h^a{}_\phi h_{a\phi} > 0, \tag{33}$$

in a neighborhood of the axis, and zero only at points on the axis. The connection one-form for a Riemann–Cartan geometry then takes the form

$$\boldsymbol{\omega}^a{}_b = W^a{}_{bc}(x^{\tilde\alpha})\mathbf{h}^c - \mathbf{h}_c(\epsilon\phi)\lambda^a{}_b\mathbf{h}^c, \tag{34}$$

where $\epsilon = 0, 1$.

In the case of a teleparallel geometry, we can employ the same argument leading to Proposition IV.2 to describe an axisymmetric teleparallel geometry in terms of the following coframe and connection one-form:

$$\mathbf{h}^a = h^a{}_\beta(x^{\tilde\alpha})dx^\beta \text{ and } \boldsymbol{\omega}^a{}_b = -\mathbf{h}_c(\epsilon\phi)\lambda^a{}_b\mathbf{h}^c = -\epsilon\lambda^a{}_b d\phi, \tag{35}$$

where $\lambda^a{}_b$ is given in Eq. (31) and the vielbein must satisfy Eq. (33).

Within the class of axisymmetric geometries defined in Riemann–Cartan geometries with non-zero torsion, the stationary axisymmetric solutions are particularly important from a physical perspective as these will contain solutions similar to the Kerr metric. To the authors' knowledge, these geometries have not been investigated using symmetry methods.[23] Relative to this coordinate system where the symmetry generator takes the form $\mathbf{X} = \partial_\phi$, stationary axisymmetric geometries can be generated by the inclusion of a second affine frame symmetry $\mathbf{Y}$ with $Y^a Y_a < 0$ locally, which commutes with $\mathbf{X}$; i.e., $[\mathbf{X}, \mathbf{Y}] = 0$. A new coordinate system can be chosen so that $\mathbf{X} = \partial_\phi$ and $\mathbf{Y} = \partial_t$. The timelike condition on $\mathbf{Y}$ imposes an additional condition on the vielbein.

In a similar manner to the stationary axisymmetric geometries, a rigorous investigation of the static axisymmetric geometries in Riemann–Cartan geometries with non-zero torsion has not been performed. However, within the current framework, it is not difficult to formulate these geometries. Static axisymmetric geometries arise by requiring that $\mathbf{Y}$ is hypersurface orthogonal and, hence, satisfies

$$\mathbf{Y} \wedge d\mathbf{Y} = 0. \tag{36}$$

The larger symmetry group introduces an additional function, $f_I{}^{\hat i}$, into Eq. (12), and this can be determined using Eq. (14). The analysis of these equations is not difficult and leads to the condition that the vielbein and the components of the connection one-form must be independent of the coordinates t and ϕ.

B. A three-dimensional symmetry group: Spherical symmetry

Working in coordinates $x^\mu = (t, r, \theta, \phi)$, the affine frame symmetry generators associated with spherical symmetry are

$$\mathbf{X}_z = \partial_\phi, \quad \mathbf{X}_y = -\cos\phi\,\partial_\theta + \frac{\sin\phi}{\tan\theta}\partial_\phi, \quad \mathbf{X}_x = \sin\phi\,\partial_\theta + \frac{\cos\phi}{\tan\theta}\partial_\phi. \tag{37}$$

Writing $\{\mathbf{X}_I\}_{I=1}^3 = \{\mathbf{X}_x, \mathbf{X}_y, \mathbf{X}_z\}$, the non-zero commutator constants $C^I{}_{JK}$ are

$$C^3{}_{12} = -1, \quad C^2{}_{13} = 1, \quad C^1{}_{23} = -1. \tag{38}$$

The basis for the isotropy group is chosen to be

$$\lambda_{\hat 1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \lambda_{\hat 2} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \lambda_{\hat 3} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{39}$$

with the corresponding commutator constants

$$C^{\hat 3}{}_{\hat 1 \hat 2} = -1, \quad C^{\hat 2}{}_{\hat 1 \hat 3} = 1, \quad C^{\hat 1}{}_{\hat 2 \hat 3} = -1. \tag{40}$$

To determine the group $\overline{Iso}$ for this choice of isotropy generators, we again consider an arbitrary element of the Lie algebra of Lorentz frame transformations, $\tilde\lambda$, and impose that $[\lambda_{\hat i}, \tilde\lambda] \in \text{span}\{\lambda_{\hat 1}, \lambda_{\hat 2}, \lambda_{\hat 3}\}$. The result of this calculation implies that $\tilde\lambda \in \text{span}\{\lambda_{\hat 1}, \lambda_{\hat 2}, \lambda_{\hat 3}\}$ and so exponentiating each of the generators $\lambda_{\hat i}$, $\hat i = 1, 2, 3$, shows that $\overline{Iso}$ consists of all spatial rotations.

We will exploit the freedom of choice in the components of $f_I{}^{\hat i}(x^{\alpha'})$ using the isotropy group, where the isotropy group affects a change to these components through Eq. (10). Since $\mathbf{X}_3 = \mathbf{X}_z$ is a generator of a spatial rotation, we will choose our frame so that $\mathbf{X}_3$ acts as a rotation on the basis elements $\mathbf{h}^3$ and $\mathbf{h}^4$, i.e.,

$$f_3{}^{\hat i}\lambda_{\hat i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_3{}^{\hat 1} \\ 0 & 0 & -f_3{}^{\hat 1} & 0 \end{bmatrix}.$$

Then, by applying a rotation about $\mathbf{h}^3$ and $\mathbf{h}^4$, the remaining component, $f_3{}^{\hat{1}}$, can be set to zero using Eq. (10),

$$\mathbf{X}_3(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c + \tilde{\Lambda}^a{}_b f_3{}^{\hat{i}}\lambda^b_{\hat{i}\ e}[\tilde{\Lambda}^{-1}]^e{}_c = 0 \tag{41}$$

Choosing $f_3{}^{\hat{i}}$ in this way, we can consider the combinations $\mathbf{X}_1$ with $\mathbf{X}_3$ and $\mathbf{X}_2$ with $\mathbf{X}_3$ in Eq. (14). These combinations yield the following non-trivial equations:

$$\begin{aligned}
\partial_\phi f_1{}^{\hat{1}} + f_2{}^{\hat{1}} = 0, \quad \partial_\phi f_1{}^{\hat{2}} + f_2{}^{\hat{2}} = 0, \quad \partial_\phi f_1{}^{\hat{3}} + f_2{}^{\hat{3}} = 0, \\
\partial_\phi f_2{}^{\hat{1}} - f_1{}^{\hat{1}} = 0, \quad \partial_\phi f_2{}^{\hat{2}} - f_1{}^{\hat{2}} = 0, \quad \partial_\phi f_2{}^{\hat{3}} - f_1{}^{\hat{3}} = 0,
\end{aligned} \tag{42}$$

which leads to the following solutions:

$$\begin{aligned}
f_2{}^{\hat{i}} &= f_2{}^{\hat{i}}(\theta) \cos \phi + g_2{}^{\hat{i}}(\theta) \sin \phi, \\
f_1{}^{\hat{i}} &= \partial_\phi f_2{}^{\hat{i}}.
\end{aligned} \tag{43}$$

If we use Lorentz transformations that are dependent on θ alone, i.e., $\partial_\phi \tilde{\Lambda}^a{}_b = 0$, then we can, without loss of generality, apply a rotation to set the coefficients of $\cos \phi$ in $f_2{}^{\hat{i}}$ to zero using Eq. (10),

$$\mathbf{X}_2(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c + \tilde{\Lambda}^a{}_b f_2{}^{\hat{i}}\lambda^b_{\hat{i}\ e}[\tilde{\Lambda}^{-1}]^e{}_c = -f_2{}^{\hat{i}}\lambda^a_{\hat{i}\ c}\cos\phi. \tag{44}$$

Doing so, Eq. (14) with $\mathbf{X}_2$ and $\mathbf{X}_1$ gives

$$\partial_\theta g_2{}^{\hat{i}} + \tan(\theta)g_2{}^{\hat{i}} = 0, \tag{45}$$

which has a simple solution

$$g_2{}^{\hat{i}} = \frac{C_2{}^{\hat{i}}}{\sin(\theta)}. \tag{46}$$

This fully determines $f_1{}^{\hat{i}}$ and $f_2{}^{\hat{i}}$, as the first is the ϕ-derivative of the other. Applying a rotation with constant parameters, we can set $f_2{}^{\hat{2}} = f_2{}^{\hat{3}} = 0$; without loss of generality, we will choose the first component to be non-zero and set the arbitrary constant to one; and the solutions for the remaining components are

$$f_2{}^{\hat{1}} = \frac{\sin \phi}{\sin \theta}, \quad f_1{}^{\hat{1}} = \frac{\cos \phi}{\sin \theta}. \tag{47}$$

This gives the following matrix representation of the coefficients:

$$f_I{}^{\hat{i}} = \begin{bmatrix} \dfrac{\cos(\phi)}{\sin(\theta)} & 0 & 0 \\[2mm] \dfrac{\sin(\phi)}{\sin(\theta)} & 0 & 0 \\[2mm] 0 & 0 & 0 \end{bmatrix}. \tag{48}$$

By making this choice, we have effectively fixed the frame up to $\overline{H}_q$. To determine this subgroup of $\overline{Iso}$, we will determine the form of $\tilde{\Lambda}$ that satisfy Eq. (10) with $f_I{}^{\hat{i}} = \tilde{f}_I{}^{\hat{i}}$ taking the form of Eq. (48). We have the following three matrix equations:

$$\begin{aligned}
\mathbf{X}_1(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c + f_1{}^{\hat{1}}[\tilde{\Lambda}^a{}_b \lambda^b_{\hat{1}\ d}[\tilde{\Lambda}^{-1}]^d{}_c - \lambda^a_{\hat{1}\ c}] &= 0, \\
\mathbf{X}_2(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c + f_2{}^{\hat{1}}[\tilde{\Lambda}^a{}_b \lambda^b_{\hat{1}\ d}[\tilde{\Lambda}^{-1}]^d{}_c - \lambda^a_{\hat{1}\ c}] &= 0, \\
\mathbf{X}_3(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c &= 0.
\end{aligned} \tag{49}$$

The first equation implies that the parameters of $\tilde{\Lambda}^a{}_b$ must be independent of ϕ. Furthermore, using the chain rule, the first term can be rewritten as $\mathbf{X}_I(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c = \mathbf{X}_I(g^{\hat{i}})\lambda^a_{\hat{i}\ c}$, where $g^{\hat{i}}$ are the parameters of $\tilde{\Lambda}^a{}_b$. Finally, we can write

$$\tilde{\Lambda}^a{}_b \lambda^b_{\hat{1}\ d}[\tilde{\Lambda}^{-1}]^d{}_c = \alpha^{\hat{j}}\lambda^a_{\hat{j}\ c},$$

where $\alpha^{\hat{j}}$ are expressions in terms of the parameters $g^{\hat{i}}$. The remaining equations can be summarized in a single equation,

$$\sin\phi\,\partial_\theta g^{\hat{i}}\lambda_{\hat{i}\ c}^a - \frac{\cos\phi}{\sin\theta}[\alpha^{\hat{j}}\lambda_{\hat{j}\ c}^a - \lambda_{\hat{1}\ c}^a] = 0. \tag{50}$$

The only solution, up to rescaling, of this differential equation is $g^{\hat{1}} = 1$ and $g^{\hat{2}} = g^{\hat{3}} = 0$. This implies that the subgroup $\overline{H}_q$ consists of rotations in the $\mathbf{h}^3$–$\mathbf{h}^4$ plane with constant parameters.

Relative to the representation for the isotropy group in Eq. (39), we can solve the first equation in (19),

$$X_I^{\ \nu}\partial_\nu h^a_{\ \mu} + \partial_\mu X_I^{\ \nu}h^a_{\ \nu} = f_I^{\ \hat{i}}\lambda_{\hat{i}\ b}^a h^b_{\ \mu}, \tag{51}$$

to determine the most general frame admitting this symmetry group,

$$h^a_{\ \mu} = \begin{bmatrix} A_1(t,r) & A_2(t,r) & 0 & 0 \\ A_3(t,r) & A_4(t,r) & 0 & 0 \\ 0 & 0 & A_5(t,r) & -A_6(t,r)\sin(\theta) \\ 0 & 0 & A_6(t,r) & A_5(t,r)\sin(\theta) \end{bmatrix}. \tag{52}$$

We can choose a new coordinate system where the top block is diagonal and apply a spatial rotation to diagonalize the second block. The effect of these choices will be absorbed into the connection, which can be determined relative to this frame. The resulting vielbein is

$$h^a_{\ \mu} = \begin{bmatrix} A_1(t,r) & 0 & 0 & 0 \\ 0 & A_2(t,r) & 0 & 0 \\ 0 & 0 & A_3(t,r) & 0 \\ 0 & 0 & 0 & A_3(t,r)\sin(\theta) \end{bmatrix}. \tag{53}$$

As the form of $f_I^{\ \hat{i}}$ in Eq. (48) has been defined invariantly, the frame in Eq. (52) is invariantly defined up to a linear isotropy subgroup $\overline{H}_q$ consisting of rotations in the $\mathbf{h}^3$–$\mathbf{h}^4$ plane with constant parameters. Relative to this coordinate system, the linear isotropy group can be further restricted by imposing a particular form on the vielbein as in Eq. (53). With this choice, the frame in Eq. (53) is an invariant symmetry frame, since the linear isotropy group is trivial.

With the chosen coframe, we can take the third equation in (19) and solve for the most general connection. This is essentially a large set of differential equations for the connection coefficients and can be solved in a straightforward, albeit tedious manner.

Proposition IV.3. The most general Riemann–Cartan geometry admitting spherical symmetry is given by the vielbein (53) and the metric compatible connection with components,

$$\begin{aligned} &\omega_{341} = W_1(t,r), \quad \omega_{342} = W_2(t,r), \\ &\omega_{233} = \omega_{244} = W_3(t,r), \quad \omega_{234} = -\omega_{243} = W_4(t,r), \\ &\omega_{121} = W_5(t,r), \quad \omega_{122} = W_6(t,r), \\ &\omega_{133} = \omega_{144} = W_7(t,r), \quad \omega_{134} = -\omega_{143} = W_8(t,r), \\ &\omega_{344} = -\frac{\cos(\theta)}{A_3\sin(\theta)}. \end{aligned} \tag{54}$$

To determine the most general connection for a teleparallel geometry, we next impose the flatness condition in (20). The resulting equations are large, but determining the solution is straightforward albeit tedious.

Proposition IV.4. Any spherically symmetric teleparallel geometry is defined by the three arbitrary functions in the vielbein (53) along with the following spin connection components:

$$\begin{aligned} W_1 &= -\frac{\partial_t\chi}{A_1}, \quad W_2 = -\frac{\partial_r\chi}{A_2}, \\ W_3 &= \frac{\cosh(\psi)\cos(\chi)}{A_3}, \quad W_4 = \frac{\cosh(\psi)\sin(\chi)}{A_3}, \\ W_5 &= -\frac{\partial_t\psi}{A_1}, \quad W_6 = -\frac{\partial_r\psi}{A_2}, \\ W_7 &= \frac{\sinh(\psi)\cos(\chi)}{A_3}, \quad W_8 = \frac{\sinh(\psi)\sin(\chi)}{A_3}, \end{aligned} \tag{55}$$

where χ and ψ are arbitrary functions of the coordinates t and r.

Any choice of the arbitrary functions ψ and χ picks out a unique teleparallel geometry, as any change in the form of the spin connection that could affect the form of ψ or χ leads to a change in the form of the vielbein. For a given pair of functions, the invariantly defined frame up to the linear isotropy group H_q arising from the CK algorithm could be computed to further provide sub-classification. We note that there are only five arbitrary functions required to specify a geometry: A_1, A_2, A_3, ψ, and χ. This characterization of spherically symmetric geometries in terms of ψ and χ has not been presented before in the literature and provides a formulation of spherically symmetric teleparallel geometries using the smallest number of arbitrary functions. With this, we can determine new families of geometries by imposing conditions on the arbitrary functions where these conditions are related to the inertial effects of the geometries.

While these geometries have been investigated in the teleparallel context before using the proper frame approach,[12,13,24] a thorough investigation in the context of Riemann–Cartan geometries with non-vanishing torsion has not been performed. We note that subclasses of these geometries have been studied earlier in teleparallel gravity[25] using the Killing equations for an arbitrary spherically symmetric metric. However, such an approach does not consider the sensitivity of the frame basis to Lorentz transformations and does not pick out the most general classes of spherically symmetric teleparallel geometries as the parameters of the Lorentz transformations do not appear at all in their formulation.

In the context of teleparallel gravity, our approach determines the minimal number of arbitrary functions, at most five, in the vielbein and spin connection to define the geometries. It also readily distinguishes four inequivalent branches of spherically symmetric teleparallel geometries, depending on whether χ or ψ vanishes, as the frame in (53) has been fully fixed by restricting the vielbein to a particular form. In comparison, the proper frame approach requires at most six arbitrary functions and the inequivalence of two spherically symmetric teleparallel geometries will require the full machinery of the CK algorithm.[12,13]

We will conclude this section with a brief discussion of stationary spherically symmetric geometries. To the authors' knowledge, very little is known about the stationary spherically symmetric Riemann–Cartan geometries with non-vanishing torsion. Stationary or static spherically symmetric teleparallel geometries have been discussed in applications.[24–26] In these papers, the stationarity or static condition is not rigorously derived but, instead, arises from restricting the arbitrary functions to be dependent on the radial coordinate alone. Although such a choice intuitively makes sense, it does not necessarily constitute the most general class of stationary or static spherically symmetric geometries since it does not consider the inequivalent subclasses of spherically symmetric metrics.

In a similar manner to the axisymmetric geometries, stationary spherically symmetric solutions can be produced by including an additional affine frame symmetry $\mathbf{Y}$ with $Y^a Y_a < 0$, locally. This vector field must commute with the original frame symmetries, $[\mathbf{X}_I, \mathbf{Y}] = 0$ for $I \in \{1, \ldots, 3\}$. Therefore, a new coordinate system can be chosen so that $\mathbf{Y} = \partial_t$, while preserving the vielbein form in (52) instead of the more specialized vielbein in (53). The timelike condition on $\mathbf{Y}$ imposes an additional condition on the vielbein functions $A_i, i \in \{1, \ldots, 6\}$, in Eq. (52). Static spherically symmetric geometries are produced by requiring that $\mathbf{Y}$ is hypersurface orthogonal,

$$\mathbf{Y} \wedge d\mathbf{Y} = 0. \tag{56}$$

In the case of static solutions, the vielbein form (53) is applicable. The expansion of the symmetry group introduces three additional functions into (12), and this can be determined using (14). As the commutator constants are zero, the analysis of these equations is straightforward and implies that the vielbein and the components of the connection one-form must be dependent on the coordinate r alone. As no other conditions are imposed, this proves that the static spherically symmetric solutions can be found by requiring that the functions $A_i, i = 1, 2, 3$, ψ, and χ are functions of the r coordinate.

We emphasize that we have not attempted to address the anti-symmetric field equations for $F(T)$ theories. While this has been considered previously using the proper frame approach[12] for particular choices of the arbitrary functions, a general analysis of the field equations has not been carried out.[16] However, the phenomenology of spherically symmetric static and stationary solutions in $F(T)$ theories has been explored in Refs. 27–30. In our approach, the minimal number of arbitrary functions for each inequivalent subclass can potentially allow for a general analysis of the field equations in $F(T)$ theories.

C. Robertson–Walker geometries

Working in the coordinate system (t, r, θ, ϕ), the affine frame symmetry generators associated with the RW metric are

$$
\begin{aligned}
\mathbf{X}_1 &= \zeta \, \sin\theta \, \cos\phi \partial_r + \frac{\zeta}{r} \cos\theta \, \cos\phi \partial_\theta - \frac{\zeta \, \sin\phi}{r \, \sin\theta} \partial_\phi, \\
\mathbf{X}_2 &= \zeta \, \sin\theta \, \sin\phi \partial_r + \frac{\zeta}{r} \cos\theta \, \sin\phi \partial_\theta + \frac{\zeta \, \cos\phi}{r \, \sin\theta} \partial_\phi, \\
\mathbf{X}_3 &= \zeta \, \cos\theta \partial_r - \frac{\zeta}{r} \sin\theta \partial_\theta, \\
\mathbf{X}_z &= \partial_\phi, \quad \mathbf{X}_y = -\cos\phi \partial_\theta + \frac{\sin\phi}{\tan\theta} \partial_\phi, \quad \mathbf{X}_x = \sin\phi \partial_\theta + \frac{\cos\phi}{\tan\theta} \partial_\phi,
\end{aligned}
\tag{57}
$$

where $\zeta = \sqrt{1 - kr^2}$ and $k \in \mathbb{R}$. Writing $\{\mathbf{X}_I\}_{I=1}^6 = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_x, \mathbf{X}_y, \mathbf{X}_z\}$, the non-zero commutator constants $C^I{}_{JK}$ are

$$
\begin{aligned}
C^6{}_{12} &= -k, & C^5{}_{13} &= -k, & C^3{}_{15} &= 1, \\
C^2{}_{16} &= 1, & C^4{}_{23} &= k, & C^3{}_{24} &= -1, \\
C^1{}_{26} &= -1, & C^2{}_{34} &= 1, & C^1{}_{35} &= -1, \\
C^6{}_{45} &= -1, & C^5{}_{46} &= 1, & C^4{}_{56} &= -1.
\end{aligned}
\tag{58}
$$

The basis for the isotropy group is of the form

$$
\lambda_{\hat{1}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad
\lambda_{\hat{2}} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
\lambda_{\hat{3}} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
\tag{59}
$$

with the corresponding commutator constants

$$
C^{\hat{3}}{}_{\hat{1}\hat{2}} = -1, \quad C^{\hat{2}}{}_{\hat{1}\hat{3}} = 1, \quad C^{\hat{1}}{}_{\hat{2}\hat{3}} = -1.
\tag{60}
$$

The form of $\overline{Iso}$ can be determined in the same manner as in case of spherical symmetry since the isotropy group is unchanged. Thus, $\overline{Iso}$ consists of spatial rotations.

With these quantities, we can substitute them into the first two equations in (19), first by solving the second equation to determine the form of the functions in $f_I{}^{\hat{i}}$,

$$
2\mathbf{X}_{[I}(f^{\hat{k}}_{J]}) - f_I{}^{\hat{i}} f_J{}^{\hat{j}} C^{\hat{k}}{}_{\hat{i}\hat{j}} = C^K{}_{IJ} f_K{}^{\hat{k}}.
\tag{61}
$$

We will exploit the freedom of choice in the components of $f_I{}^{\hat{i}}$ using the isotropy group to transform the components with Eq. (10). Since $\mathbf{X}_z$ is a generator of a spatial rotation, we will choose our frame representation so that it acts as a rotation on the basis elements $\mathbf{h}^3$ and $\mathbf{h}^4$, i.e.,

$$
f_6{}^{\hat{i}} \lambda_{\hat{i}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_6{}^{\hat{1}} \\ 0 & 0 & -f_6{}^{\hat{1}} & 0 \end{bmatrix}.
$$

Then, by applying a rotation about $\mathbf{h}^3$ and $\mathbf{h}^4$, the remaining component, $f_6{}^{\hat{1}}$, can be set to zero using

$$
\mathbf{X}_6(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c + \tilde{\Lambda}^a{}_b f_6{}^{\hat{i}} \lambda^b_{\hat{i}\,e}[\tilde{\Lambda}^{-1}]^e{}_c = 0.
\tag{62}
$$

From the second equation in (19), using $\mathbf{X}_6$ paired with $\mathbf{X}_{\bar{I}}, \bar{I} \in [1,5]$, we have the following differential equations:

$$
\begin{aligned}
\partial_\phi f_1{}^{\hat{1}} + f_2{}^{\hat{1}} &= 0, & \partial_\phi f_1{}^{\hat{2}} + f_2{}^{\hat{2}} &= 0, & \partial_\phi f_1{}^{\hat{3}} + f_3{}^{\hat{1}} &= 0, \\
\partial_\phi f_2{}^{\hat{1}} + f_1{}^{\hat{1}} &= 0, & \partial_\phi f_2{}^{\hat{2}} + f_1{}^{\hat{2}} &= 0, & \partial_\phi f_2{}^{\hat{3}} + f_1{}^{\hat{3}} &= 0, \\
\partial_\phi f_3{}^{\hat{1}} &= 0, & \partial_\phi f_3{}^{\hat{2}} &= 0, & \partial_\phi f_3{}^{\hat{3}} &= 0, \\
\partial_\phi f_4{}^{\hat{1}} + f_5{}^{\hat{1}} &= 0, & \partial_\phi f_4{}^{\hat{2}} + f_5{}^{\hat{2}} &= 0, & \partial_\phi f_4{}^{\hat{3}} + f_5{}^{\hat{3}} &= 0, \\
\partial_\phi f_5{}^{\hat{1}} + f_4{}^{\hat{1}} &= 0, & \partial_\phi f_5{}^{\hat{2}} + f_4{}^{\hat{2}} &= 0, & \partial_\phi f_5{}^{\hat{3}} + f_4{}^{\hat{3}} &= 0,
\end{aligned}
\tag{63}
$$

leading to the following solutions:

$$f_5{}^{\hat{i}} = f_5{}^{\hat{i}}(r,\theta)\cos\phi + g_5{}^{\hat{i}}(r,\theta)\sin\phi,$$
$$f_4{}^{\hat{i}} = \partial_\phi f_5{}^{\hat{i}},$$
$$f_3{}^{\hat{i}} = f_3{}^{\hat{i}}(r,\theta),\tag{64}$$
$$f_2{}^{\hat{i}} = f_2{}^{\hat{i}}(r,\theta)\cos\phi + g_2{}^{\hat{i}}(r,\theta)\sin\phi,$$
$$f_1{}^{\hat{i}} = \partial_\phi f_2{}^{\hat{i}}.$$

If Lorentz transformations are restricted to transformations that are dependent on r and θ, we can apply a rotation to set the coefficients of $\cos\phi$ in $f_5{}^{\hat{i}}$ to zero using Eq. (10), which takes the form

$$\mathbf{X}_5(\tilde{\Lambda}^a{}_b)[\tilde{\Lambda}^{-1}]^b{}_c + \tilde{\Lambda}^a{}_b f_5{}^{\hat{i}}\lambda^b_{\hat{i}\ e}[\tilde{\Lambda}^{-1}]^e{}_c = -f_5{}^{\hat{i}}\lambda^a{}_{\hat{i}\ c}\cos\phi.\tag{65}$$

Doing so, the second equation in (19) with $\mathbf{X}_5$ and $\mathbf{X}_4$ gives

$$\partial_\theta g_5{}^{\hat{i}} + \cos(\theta)g_5{}^{\hat{i}} = 0,\tag{66}$$

with the resulting solution

$$g_5{}^{\hat{i}} = \frac{G_5{}^{\hat{i}}(r)}{\sin(\theta)}.\tag{67}$$

This fully determines $f_5{}^{\hat{i}}$ and $f_4{}^{\hat{i}}$ up to functions of r, as the second is the ϕ-derivative of the first. Applying a rotation dependent on only the coordinates r, we can fix some of the components in $f_5{}^{\hat{i}}$ to be zero. Without loss of generality, we will choose the first component to be non-zero,

$$f_5{}^{\hat{1}} = \frac{\sin\phi}{\sin\theta},$$
$$f_4{}^{\hat{1}} = \frac{\cos\phi}{\sin\theta}.\tag{68}$$

The remaining equations in (61) can now be solved by looking at the coefficients of $\cos\phi$ and $\sin\phi$. With these choices, $f_I{}^{\hat{i}}$ takes the form

$$f_I{}^{\hat{i}} = \begin{bmatrix} -\dfrac{\sqrt{1-kr^2}\sin(\phi)\cos(\theta)}{r\sin(\theta)} & -\dfrac{\cos(\theta)\cos(\phi)}{r} & \dfrac{\sin(\phi)}{r} \\[3mm] \dfrac{\sqrt{1-kr^2}\cos(\phi)\cos(\theta)}{r\sin(\theta)} & -\dfrac{\cos(\theta)\sin(\phi)}{r} & -\dfrac{\cos(\phi)}{r} \\[3mm] 0 & \dfrac{\sin(\theta)}{r} & 0 \\[3mm] \dfrac{\cos(\phi)}{\sin(\theta)} & 0 & 0 \\[3mm] \dfrac{\sin(\phi)}{\sin(\theta)} & 0 & 0 \\[3mm] 0 & 0 & 0 \end{bmatrix}.\tag{69}$$

The form of $f_I{}^{\hat{i}}$ determines the parameters of Lorentz transformations in $\overline{Iso}$ up to $\overline{H}_q$.

To determine $\overline{H}_q$, we employ Eq. (10) with $f_I{}^{\hat{i}} = \tilde{f}_I{}^{\hat{i}}$ defined as in Eq. (69). The analysis proceeds in the same manner as the spherically symmetric case, except that the parameters of $\tilde{\Lambda}^a{}_b$ now depend on r, θ, and ϕ. Considering the Lie algebra generator of $\tilde{\Lambda}^a{}_b$ as $\tilde{\lambda}^a{}_c = g^{\hat{i}}\lambda^a{}_{\hat{i}\ b}$ and then using $\mathbf{X}_4, \mathbf{X}_5$, and $\mathbf{X}_6$, Eq. (10) implies that $g^{\hat{1}} = g(r)$ and $g^{\hat{2}} = g^{\hat{3}} = 0$. Finally, Eq. (10) with $\mathbf{X}_3$ gives

$$\zeta\cos\theta\partial_r f\lambda^a_{\hat{1}\ c} + \frac{\sin\theta}{r}[(\cos f)\lambda^a_{\hat{2}\ c} + (\sin f)\lambda^a_{\hat{3}\ c} - \lambda^a_{\hat{2}\ c}] = 0.\tag{70}$$

The only solution to this equation is $f = 0$, implying that $\tilde{\Lambda}^a{}_b$ is the identity Lorentz transformation. Thus, the subgroup $\overline{H}_q$ is the trivial subgroup.

Using the representation of the isotropy group in (59) and setting the components of $f_I{}^{\hat{i}}$ into the first equation in (19) yields the following general solution for the vielbein:

$$h^a{}_\mu = \begin{bmatrix} A_1(t) & 0 & 0 & 0 \\ 0 & \dfrac{A_2(t)}{\sqrt{1-kr^2}} & 0 & 0 \\ 0 & 0 & A_2(t)r & 0 \\ 0 & 0 & 0 & A_2(t)r\sin(\theta) \end{bmatrix}. \tag{71}$$

Imposing the vielbein form in Eq. (71), since $\overline{H}_q$ is the trivial group, this frame is now an invariant frame. In general, a coordinate transformation can be made to set $A_1 = 1$ and $A_2 = a(t)$. Doing so, we will work with the coframe associated with

$$h^a{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{a(t)}{\sqrt{1-kr^2}} & 0 & 0 \\ 0 & 0 & a(t)r & 0 \\ 0 & 0 & 0 & a(t)r\sin(\theta) \end{bmatrix}. \tag{72}$$

Using the coframe associated with this matrix, we can readily solve the resulting equations coming from Eq. (6) for each of the affine frame symmetries. The resulting differential equations can be solved readily by hand or by using the frame and connection ansatz from the spherically symmetric case.

Proposition IV.5. The most general Riemann–Cartan geometry that admits a RW symmetry group consists of the vielbein (72) along with the metric-compatible connection with components,

$$\begin{aligned} \omega_{122} &= \omega_{133} = \omega_{144} = W_1(t), \\ \omega_{234} &= -\omega_{243} = \omega_{342} = W_2(t), \\ \omega_{233} &= \omega_{244} = -\frac{\sqrt{1-kr^2}}{a(t)r}, \\ \omega_{344} &= -\frac{\cos(\theta)}{a(t)r\sin(\theta)}. \end{aligned} \tag{73}$$

The class of connections that describe a teleparallel geometry can be determined by solving the flatness condition in Eq. (20). This gives the following differential constraints on W_1 and W_2:

$$\begin{aligned} & a\partial_t W_1 + W_1\partial_t a = 0, \\ & W_1 W_2 = 0, \\ & a\partial_t W_2 + W_2\partial_t a = 0, \\ & W_2^2 a^2 - W_1^2 a^2 - k. \end{aligned} \tag{74}$$

These equations have four distinct solutions, leading to the following proposition.

Proposition IV.6. The most general teleparallel geometry admitting the RW symmetry group is specified by the arbitrary function in the vielbein (72) along with the choice of a metric-compatible connection (73) with the functions W_1 and W_2 given as follows:

1. $W_1 = 0$ *and* $W_2 = -\frac{\sqrt{k}}{a}$.
2. $W_1 = 0$ *and* $W_2 = \frac{\sqrt{k}}{a}$.
3. $W_1 = -\frac{\sqrt{-k}}{a}$ *and* $W_2 = 0$.
4. $W_1 = \frac{\sqrt{-k}}{a}$ *and* $W_2 = 0$.

Each choice of W_1 and W_2 yields an inequivalent teleparallel geometry.

Imposing the reasonable condition that the torsion tensors must be real-valued, this naturally distinguishes two cases for $k > 0$ and two cases for $k < 0$. When k vanishes, these four inequivalent cases coincide. Without loss of generality, the coordinate r can be rescaled to set

$k = -1, 0$, or 1. Note that the antisymmetric field equations are automatically satisfied in all cases. In Ref. 31, the proper frames for these teleparallel geometries are given and the $F(T)$ field equations are analyzed for each inequivalent case.

Alternatively, the connection coefficients for this case can be determined using the connection coefficients of the spherically symmetric case in Eq. (55) by setting

$$k < 0 : \chi = 0, \text{ or } \pi, \ \psi = \cosh^{-1}(-\cos(\chi)\sqrt{1 - kr^2}),$$
$$k \geq 0 : \chi = \cos^{-1}(-\sqrt{1 - kr^2}), \psi = 0. \tag{75}$$

This choice of representation of the arbitrary functions in the connection coefficients shows that the sign difference in the case of $k > 0$ is purely a choice of representation and, hence, these subcases are equivalent. We note that the sign difference for the $k < 0$ case does, indeed, give two inequivalent subcases.

While the RW Riemann–Cartan geometries with non-vanishing torsion have not been explored at all, the teleparallel RW geometries have previously been discussed. Using a metric ansatz and the Killing equations, a very limited subclass was determined in Ref. 25 and the flat case has been explored by many authors.[32] The first attempts at using a frame approach to determine TRW geometries with $k \neq 0$ were in Refs. 6 and 7. However, the solutions required a complex orthonormal frame that was undesirable from a physical standpoint.

The general class was determined using the proper frame approach in Ref. 14. We note that the field equations studied in Ref. 31 agree with the field equations found in Ref. 14. However, there are two significant problems with the approach in Ref. 14: certain branches of solutions require complex orthonormal proper frames[12] and the proper frames contain an additional arbitrary function of t, which is purely a coordinate artifact. Despite the appearance of complex orthonormal proper frames, these have been used to study the RW models with non-zero k in Refs. 33 and 34.

Many of the applications in the literature of these geometries have concentrated on the analysis of the flat ($k = 0$) TRW cosmological models.[16,32] In addition, particular forms for $F(T)$ have been considered. For example, employing reconstruction methods, a functional form for the solution, in terms of the scale factor, is assumed and the corresponding $F(T)$ theory is then constructed. However, geometries with non-zero k have recently been studied in bounce models[35] and in inflation where the use of a complex orthonormal frame potentially restricts to the $k = 1$ case.[36] Finally, the TRW solutions have been used to study perturbations in non-flat cosmology.[37] More generally, the TRW solutions with non-zero k have been employed in $f(T, B)$ theories, where B is a boundary term.[38]

D. de Sitter and Einstein static geometries

In GR and Lorentzian geometry, in general, the de Sitter solution is a special case of the RW solutions, where the six-dimensional group is expanded to a ten-dimensional group, namely $SO(1, 4)$. It has been proven that the only maximally symmetric geometry that admits a non-vanishing torsion tensor is Minkowski space and that the largest symmetry group permitted in such a geometry is at most seven.[15,16] In addition, it follows from the CK algorithm that if a geometry admits a seven-dimensional symmetry group, then it must have constant Cartan invariants relative to some class of invariantly defined frames.[15]

To determine the possible special cases of the TRW geometries that admit a seven-dimensional symmetry group, we will first consider the general case of the Riemann–Cartan geometries and require that their components are constant. The components of the curvature tensor and the torsion tensor are, respectively,

$$R_{1212} = R_{1313} = R_{1414} = \frac{W_{2,t}a + a_{,t}W_2}{a},$$
$$R_{1234} = R_{1324} = R_{1423} = 2W_1 W_2,$$
$$R_{2314} = R_{2413} = R_{3412} = \frac{W_{1,t}a + a_{,t}W_1}{a},$$
$$R_{2323} = R_{2424} = R_{3434} = \frac{W_1^2 a^2 - W_2^2 a^2 - k}{a^2}, \tag{76}$$

and

$$T_{212} = T_{313} = T_{414} = \frac{W_2 a + a_{,t}}{a},$$
$$T_{234} = T_{324} = T_{432} = 2W_1, \tag{77}$$

requiring that these components are constant give two distinct cases.

Proposition IV.7. The subclass of RW Riemann–Cartan geometries admitting a seventh symmetry are given by the vielbein (72) and the connection (73) where the functions W_1, W_2, and $a(t)$ take the form

$$Case \ 1 : W_1 = C_1, W_2 = C_2, \quad k = 0, \ and \ a(t) = C_3 e^{C_4 t},$$
$$Case \ 2 : W_1 = C_1, W_2 = C_2, \ k \ arbitrary, \ and \ a(t) = C_3. \tag{78}$$

Requiring that the curvature tensor vanishes leads to strong conditions on W_1 and W_2. In the remainder of this subsection, we will only discuss the teleparallel geometries admitting a seven-dimensional symmetry group, G_7.

Proposition IV.8. *The subclass of RW teleparallel geometries admitting a seventh symmetry are given by the vielbein (72) and the connection (73) with*

1. $W_1 = 0$ *and* $W_2 = -\frac{\sqrt{k}}{a}$,
2. $W_1 = 0$ *and* $W_2 = \frac{\sqrt{k}}{a}$,
3. $W_1 = -\frac{\sqrt{-k}}{a}$ *and* $W_2 = 0$, *and*
4. $W_1 = \frac{\sqrt{-k}}{a}$ *and* $W_2 = 0$,

where either

$$Case\ 1: a = C_3 e^{C_4 t} \ and \ k = 0, \ or$$
$$Case\ 2: a = C_3 \ and \ k \ is \ \pm 1. \tag{79}$$

Using (8) and (6), we can determine the form of the seventh symmetry as

$$Case\ 1: \mathbf{X}_7 = \frac{1}{C_3 C_4}\partial_t + r\partial_r,$$
$$Case\ 2: \mathbf{X}_7 = \partial_t. \tag{80}$$

In the first case, the resulting Lie algebra of $\{\mathbf{X}_I\}_{I=1}^7 = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_x, \mathbf{X}_y, \mathbf{X}_z, \mathbf{X}_7\}$ is given by the following non-zero Lie brackets:

$$[\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_3, \quad [\mathbf{X}_1, \mathbf{X}_6] = \mathbf{X}_2, \quad [\mathbf{X}_1, \mathbf{X}_7] = \mathbf{X}_1,$$
$$[\mathbf{X}_2, \mathbf{X}_4] = -\mathbf{X}_3, \quad [\mathbf{X}_2, \mathbf{X}_6] = -\mathbf{X}_1, \quad [\mathbf{X}_2, \mathbf{X}_7] = \mathbf{X}_2,$$
$$[\mathbf{X}_3, \mathbf{X}_4] = \mathbf{X}_2, \quad [\mathbf{X}_3, \mathbf{X}_5] = -\mathbf{X}_1, \quad [\mathbf{X}_3, \mathbf{X}_7] = \mathbf{X}_3,$$
$$[\mathbf{X}_4, \mathbf{X}_5] = -\mathbf{X}_6, \quad [\mathbf{X}_4, \mathbf{X}_6] = \mathbf{X}_5, \quad [\mathbf{X}_5, \mathbf{X}_6] = -\mathbf{X}_4.$$

By inspection, this is a subalgebra of the Lie algebra for the group of metric (Killing) symmetries of de Sitter spacetime. We, therefore, propose the following definition.

Definition IV.9. The teleparallel de Sitter geometry (TdS) is a teleparallel geometry with the G_7 Lie group of affine symmetries, $\mathcal{R} \rtimes E(3)$, which is a subgroup of $O(1, 4)$.

We note that in this geometry, the covariant derivative of the torsion tensor is zero.

In the second case, $a(t) = C_3$ is a constant and $k = \pm 1$. These geometries correspond to the direct product $\mathbb{R} \times M_3$, where M_3 is a locally homogeneous and isotropic Riemannian manifold. This is reflected in the Lie algebra structure of the affine frame symmetries $\{\mathbf{X}_I\}_{I=1}^7 = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_x, \mathbf{X}_y, \mathbf{X}_z, \mathbf{X}_7\}$, where $\mathbf{X}_7 = \partial_t$ satisfies $[\mathbf{X}_i, \mathbf{X}_7] = 0$, $i \in [1, 6]$, and the remaining non-zero Lie brackets are

$$[\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_3, \quad [\mathbf{X}_1, \mathbf{X}_6] = \mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_4] = -\mathbf{X}_3,$$
$$[\mathbf{X}_2, \mathbf{X}_6] = -\mathbf{X}_1, \quad [\mathbf{X}_3, \mathbf{X}_4] = \mathbf{X}_2, \quad [\mathbf{X}_3, \mathbf{X}_5] = -\mathbf{X}_1,$$
$$[\mathbf{X}_4, \mathbf{X}_5] = -\mathbf{X}_6, \quad [\mathbf{X}_4, \mathbf{X}_6] = \mathbf{X}_5, \quad [\mathbf{X}_5, \mathbf{X}_6] = -\mathbf{X}_4.$$

Since $\mathbf{X}_7 = \partial_t$ is the additional affine frame symmetry, this geometry is necessarily static due to the form of the frame basis. This geometry can be considered as the analog of the Einstein static geometry in GR, which we call the Teleparallel Einstein Static (TES) geometry. We, therefore, propose the following definition.

Definition IV.10. The Teleparallel Einstein Static (TES) geometry is a teleparallel geometry with the G_7 Lie group of affine symmetries, $\mathcal{R} \times E(3)$, which is a subgroup of $O(1, 4)$.

By working with the RW geometries, which are a subclass of the spherically symmetric geometries, admitting six affine frame symmetries, we have determined two subclasses of Riemann–Cartan geometries that admit two distinct seven-dimensional groups of affine frame symmetries, respectively. In the case of teleparallel geometries, we have explicitly determined the analog of the de Sitter (TdS) and Einstein static (TES) geometries. We note that the proper frames and the $F(T)$ field equations for the TdS and TES geometries were presented in Ref. 39. These are only two examples of geometries admitting a seven-dimensional group of affine frame symmetries, and we anticipate a much larger set of geometries admitting a six-dimensional group of affine frame symmetries with different six-dimensional groups. In the case of geometries admitting a seven-dimensional group of affine frame symmetries, these will necessarily belong to the class of locally homogeneous Riemann–Cartan geometries.[40]

V. DISCUSSION

We have introduced an invariant definition for symmetries in frame based theories of gravity. This is a generalization of the Killing equations in pseudo-Riemannian geometry to the more general class of Riemann–Cartan geometries. As an illustration of our approach, we have provided examples from teleparallel geometries where the connection is chosen so that the torsion tensor and its covariant derivatives are the only non-vanishing geometric quantities. This definition relies on a geometrically preferred class of frames, called *symmetry frames*, which are a generalization of the class of invariantly defined frames, or invariant frames used in the Cartan–Karlhede (CK) algorithm. Relative to the symmetry frames, the equations for an affine frame symmetry (2) are put into a simple form given in Eqs. (12) and (13), which can be solved readily for a coframe $\mathbf{h}^a$ connection one-form $\boldsymbol{\omega}^a{}_b$ and Lie algebra generator of the isotropy group, $\lambda^a{}_b$. In general, an invariantly defined frame can, then, be constructed by fixing the form of $f_I{}^i$ in Eq. (12) in a coordinate independent manner. Furthermore, an invariant frame, where all parameters of the Lorentz transformations are fixed, can be found by restricting the form of the vielbein. This permits the inequivalent subclasses of geometries to be readily distinguished.

In addition to this definition, we have introduced an approach to determining the most general Riemann–Cartan geometries that admit a specified symmetry group. Once a particular coordinate system and representation of the symmetry group as vector fields are given, the equations for an affine frame symmetry, (12) and (13), along with the affine frame symmetry commutator condition (14) can be solved for the coframe $\mathbf{h}^a$ and connection one-form $\boldsymbol{\omega}^a{}_b$. In order to restrict to teleparallel geometries, additional conditions must be imposed on the connection one-form, namely the vanishing of the curvature tensor. We show that for the connection one-form that satisfies (13), imposing the flatness condition can be achieved by solving a system of differential equations.

The geometrical approach in the proper frame approach[12–14] is similar to that given here. The definitions are not identical, and we are careful only to discuss affine symmetries and avoid any complex quantities. Since we have made no assumptions *a priori*, our results are general. A significant difference between the approaches is that, in general, the invariantly defined symmetry frame formulation requires a non-vanishing spin connection. While this may complicate the presentation of the geometries that admit a given symmetry group, the inclusion of the spin connection permits a complete classification of solutions of Eqs. (12) and (13).

While we have determined two distinct classes of Riemann–Cartan geometries admitting two particular seven-dimensional groups of symmetries, it is known that there are other Riemann–Cartan geometries that admit a group of seven affine frame symmetries. Within the framework of the symmetry frame approach, it is possible to explicitly find all possible Riemann–Cartan geometries admitting a seven dimensional symmetry group, and in future work, we will present an exhaustive list of these. We will also investigate the class of geometries that permit a six-dimensional group of symmetries in order to determine the teleparallel analog of the locally homogeneous pp-waves. The class of all Riemann–Cartan geometries admitting six-dimensional groups of affine frame symmetries is more difficult to determine as these can permit two-dimensional linear isotropy groups in addition to three-dimensional linear isotropy groups.

We note that the approach discussed in this paper can be extended to connections that are not metric compatible and, hence, admit a non-vanishing non-metricity tensor $Q_{abc} = \nabla_{\mathbf{h}_c} g_{ab}$ and more general groups of frame transformations such as $SO(p,q)$. The authors believe that this can provide a helpful tool for exploring more exotic geometries or more general pseudo-Riemannian geometries with indefinite signature and give insights into the alignment classification.[41]

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

D. D. McNutt: Conceptualization (lead); Formal analysis (equal); Project administration (lead); Writing – original draft (equal); Writing – review & editing (equal). **A. A. Coley**: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal). **R. J. van den Hoogen**: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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