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$(H_p - L_p)$-Type inequalities for subsequences of Nörlund means of Walsh–Fourier series

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Abstract

We investigate the subsequence $\{t_{2^n}f\}$ of Nörlund means with respect to the Walsh system generated by nonincreasing and convex sequences. In particular, we prove that a large class of such summability methods are not bounded from the martingale Hardy spaces H_p to the space *weak*–L_p for $0 < p < 1/(1+\alpha)$, where $0 < \alpha < 1$. Moreover, some new related inequalities are derived. As applications, some well-known and new results are pointed out for well-known summability methods, especially for Nörlund logarithmic means and Cesàro means.

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Keywords: Walsh system; Nörlund means; Cesàro means; Nörlund logarithmic means; Martingale Hardy space; Convergence; Divergence; Inequalities

1 Introduction

The terminology and notations used in this introduction can be found in Sect. 2.

The fact that the Walsh system is the group of characters of a compact abelian group connects Walsh analysis with abstract harmonic analysis was discovered independently by Fine [7] and Vilenkin [28]. For general references to the Haar measure and harmonic analysis on groups see Pontryagin [22], Rudin [23], and Hewitt and Ross [14]. In particular, Fine investigated the group G, which is a direct product of the additive groups $Z_2 =: \{0,1\}$ and introduced the Walsh system $\{w_j\}_{j=0}^{\infty}$.

It is well known (for details see, e.g., the books [21, 24], and [29]) that Walsh systems do not form bases in the space L_1. Moreover, there exists a martingale $f \in H_p$ $(0 < p \leq 1)$, such that $\sup_{n\in\mathbb{N}} \|S_{2^n+1}f\|_p = \infty$. On the other hand, by the definition of Hardy spaces, the subsequence $\{S_{2^n}\}$ of partial sums is bounded from the space H_p to the space H_p, for all $p > 0$.

Weisz [30] proved that the Fejér means of Vilenkin–Fourier series are bounded from the martingale Hardy space H_p to the space H_p, for $p > 1/2$. Goginava [11] (see also [19]) proved that there exists a martingale $f \in H_{1/2}$ such that

$$\sup_{n\in\mathbb{N}} \|\sigma_n f\|_{1/2} = +\infty.$$

However, Weisz [30] (see also [18]) proved that for every $f \in H_p$, there exists an absolute constant c_p, such that the following inequality holds:

$$\|\sigma_{2^n} f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad n \in \mathbb{N}, p > 0. \tag{1}$$

Móricz and Siddiqi [17] investigated the approximation properties of some special Nörlund means of Walsh–Fourier series of L_p functions in norm. Approximation properties for general summability methods can be found in [3, 4]. Fridli, Manchanda, and Siddiqi [8] improved and extended the results of Móricz and Siddiqi [17] to martingale Hardy spaces. The case when $\{q_k = 1/k : k \in \mathbb{N}\}$ was excluded, since the methods are not applicable to Nörlund logarithmic means. In [9] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the Lebesgue space L_1. In particular, they proved that there exists a function f in the space L_1, such that $\sup_{n\in\mathbb{N}} \|L_n f\|_1 = \infty$. In [1] it was proved that there exists a martingale $f \in H_p$, $(0 < p < 1)$ such that

$$\sup_{n\in\mathbb{N}} \|L_{2^n} f\|_p = \infty.$$

A counterexample for $p = 1$ was proved in [20]. However, Goginava [10] proved that for every $f \in H_1$, there exists an absolute constant c, such that the following inequality holds:

$$\|L_{2^n} f\|_1 \leq c \|f\|_{H_1}, \quad n \in \mathbb{N}. \tag{2}$$

The convergence of subsequences of Nörlund logarithmic means of Walsh–Fourier series in martingale Hardy spaces was investigated by Goginava [13] and Memić [16].

In [19] it was proved that for any nondecreasing sequence $(q_k, k \in \mathbb{N})$ satisfying the conditions

$$\frac{1}{Q_n} = O\left(\frac{1}{n^\alpha}\right), \quad \text{where } Q_n = \sum_{k=0}^{n-1} q_k \tag{3}$$

and

$$q_n - q_{n+1} = O\left(\frac{1}{n^{2-\alpha}}\right), \quad \text{as } n \to \infty, \tag{4}$$

then, for every $f \in H_p$, where $p > 1/(1+\alpha)$, there exists an absolute constant c_p, depending only on p, such that the following inequality holds:

$$\|t_n f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad n \in \mathbb{N}. \tag{5}$$

Boundedness does not hold from H_p to *weak*–L_p, for $0 < p < 1/(1+\alpha)$. As a consequence, (for details see [31]) we obtain that the Cesàro means σ_n^α is bounded from H_p to L_p, for $p > 1/(1+\alpha)$, but they are not bounded from H_p to *weak*–L_p, for $0 < p < 1/(1+\alpha)$. In the endpoint case $p = 1/(1+\alpha)$, Weisz and Simon [26] (see also [25]) proved that the maximal operator $\sigma^{\alpha,*}$ of Cesàro means defined by

$$\sigma^{\alpha,*} f := \sup_{n\in\mathbb{N}} \left|\sigma_n^\alpha f\right|$$

is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $weak\text{–}L_{1/(1+\alpha)}$. Goginava [12] gave a counterexample, which shows that boundedness does not hold for $0 < p \leq 1/(1+\alpha)$.

In this paper we develop some methods considered in [1, 2, 15] (see also the new book [21]) and prove that for any $0 < p < 1$, there exists a martingale $f \in H_p$ such that

$$\sup_{n\in\mathbb{N}} \|t_{2^n} f\|_{weak-L_p} = \infty.$$

Moreover, we prove that a class of subsequence $\{t_{2^n} f\}$ of Nörlund means with respect to the Walsh system generated by nonincreasing and convex sequences are not bounded from the martingale Hardy spaces H_p to the space $weak\text{–}L_p$ for $0 < p < 1/(1+\alpha)$, where $0 < \alpha < 1$. Moreover, some new related inequalities are derived. As applications, some well-known and new results are pointed out for well-known summability methods, especially for Nörlund logarithmic means and Cesàro means.

The main results in this paper are presented and proved in Sect. 4. Section 3 is used to present some auxiliary results, where, in particular, Lemma 2 is new and of independent interest. In order not to disturb our discussions later some definitions and notations are given in Sect. 2.

2 Definitions and notations

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is 1/2.

Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2s.

The elements of G are represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots), \quad \text{where } x_k = 0 \vee 1.$$

It is easy to give a base for the neighborhood of $x \in G$ namely:

$$I_0(x) := G, \qquad I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \backslash I_n$ and

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G, \quad \text{for } n \in \mathbb{N}.$$

If $n \in \mathbb{N}$, then every n can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j 2^j$, where $n_j \in Z_2$ $(j \in \mathbb{N})$ and only a finite number of n_js differ from zero. Let

$$|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}.$$

The norms (or quasinorms) of the spaces $L_p(G)$ and $weak\text{–}L_p(G)$, $(0 < p < \infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu \quad \text{and} \quad \|f\|_{weak-L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda).$$

The kth Rademacher function is defined by

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}).$$

It is well known that (see, e.g., [24]) the Walsh system is orthonormal and complete in $L_2(G)$. Moreover, for any $n \in \mathbb{N}$,

$$w_n(x + y) = w_n(x)w_n(y). \tag{6}$$

If $f \in L_1(G)$ we define the Fourier coefficients, partial sums, and Dirichlet kernel by

$$\widehat{f}(k) := \int_G f w_k \, d\mu \quad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \qquad D_n := \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N}_+).$$

Recall that (for details see, e.g., [24]):

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n \end{cases} \tag{7}$$

and

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}), \quad \text{for } n = \sum_{i=0}^{\infty} n_i 2^i. \tag{8}$$

Let $\{q_k, k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad \text{where } Q_n := \sum_{k=0}^{n-1} q_k.$$

In this paper we consider convex $\{q_k, k \geq 0\}$ sequences, that is

$$q_{n-1} + q_{n+1} - 2q_n \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

If the function $\psi(x)$ is any real-valued and convex function (for example $\psi(x) = x^{\alpha-1}$, $0 \leq \alpha \leq 1$), then the sequence $\{\psi(n), n \in \mathbb{N}\}$ is convex.

Since $q_{n-2} - q_{n-1} \geq q_{n-1} - q_n \geq q_n - q_{n+1} \geq q_{n+1} - q_{n+2}$ we find that

$$q_{n-2} + q_{n+2} \geq q_{n-1} + q_{n+1}$$

and we also obtain that

$$q_{n-2}+q_{n+2}-2q_n\geq 0,\quad \text{for all } n\in\mathbb{N}. \tag{9}$$

In the special case when $\{q_k=1, k\in\mathbb{N}\}$, we have the Fejér means

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f.$$

Moreover, if $q_k = 1/(k+1)$, then we obtain the Nörlund logarithmic means:

$$L_n f := \frac{1}{l_n}\sum_{k=1}^{n}\frac{S_k f}{n+1-k},\quad \text{where } l_n := \sum_{k=1}^{n}\frac{1}{k}. \tag{10}$$

The Cesàro means σ_n^α (sometimes also denoted (C,α)) is also a well-known example of Nörlund means defined by

$$\sigma_n^\alpha f =: \frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0,\qquad A_n^\alpha := \frac{(\alpha+1)\dots(\alpha+n)}{n!},\quad \alpha\neq -1,-2,\dots.$$

It is well known that

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1},\qquad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}\quad \text{and}\quad A_n^\alpha \sim n^\alpha. \tag{11}$$

We also define U_n^α means as

$$U_n^\alpha f := \frac{1}{Q_n}\sum_{k=1}^{n}(n+1-k)^{(\alpha-1)} S_k f,\quad \text{where } Q_n := \sum_{k=1}^{n} k^{\alpha-1}.$$

Let us also define V_n^α means as

$$V_n f := \frac{1}{Q_n}\sum_{k=1}^{n}\ln(n+1-k) S_k f,\quad \text{where } Q_n := \sum_{k=1}^{n}\frac{1}{\ln(k+1)}.$$

The σ-algebra generated by the intervals $\{I_n(x): x\in G\}$ will be denoted by $\digamma_n$ $(n\in\mathbb{N})$. Denote by $f := (f^{(n)}, n\in\mathbb{N})$ the martingale with respect to $\digamma_n$ $(n\in\mathbb{N})$ (for details see, e.g., [29]).

We say that this martingale belongs to the Hardy martingale spaces $H_p(G)$, where $0 < p < \infty$, if

$$\|f\|_{H_p} := \|f^*\|_p < \infty,\quad \text{with } f^* := \sup_{n\in\mathbb{N}}|f^{(n)}|.$$

When $f \in L_1(G)$, the maximal functions are also given by

$$M(f)(x) := \sup_{n\in\mathbb{N}} \left(\frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right| \right).$$

If $f \in L_1(G)$, then it is easy to show that the sequence $F = (S_{2^n} f : n \in \mathbb{N})$ is a martingale and $F^* = M(f)$.

If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Walsh–Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_G f^{(k)}(x) w_i(x)\, d\mu(x).$$

A bounded measurable function a is p-atom, if there exists an interval I, such that

$$\operatorname{supp}(a) \subset I, \qquad \int_I a\, d\mu = 0 \quad \text{and} \quad \|a\|_\infty \le \mu(I)^{-1/p}.$$

3 Auxiliary results

The Hardy martingale space $H_p(G)$ has an atomic characterization (see Weisz [29, 30]):

Lemma 1 *A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in H_p $(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that for every $n \in \mathbb{N}$:*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)}, \quad \textit{where} \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{12}$$

Moreover, the following two-sided inequality holds

$$\|f\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (12).

We also state and prove the following new lemma of independent interest:

Lemma 2 *Let $k \in \mathbb{N}$, $\{q_k : k \in \mathbb{N}\}$ be any convex and nonincreasing sequence and $x \in I_2(e_0 + e_1) \in I_0 \backslash I_1$. Then, for any $\{\alpha_k\}$, the following inequality holds*:

$$\left| \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} q_{2^{2\alpha_k+1}-j} D_j \right| \ge q_1 - \frac{3}{2} q_3.$$

Proof Let $x \in I_2(e_0 + e_1) \in I_0 \backslash I_1$. According to (7) and (8) we obtain that

$$D_j(x) = \begin{cases} -w_j, & \text{if } j \text{ is an odd number,} \\ 0, & \text{if } j \text{ is an even number} \end{cases}$$

and

$$\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} q_{2^{2\alpha_k+1}-j} D_j = -\sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-1} q_{2^{2\alpha_k+1}-2j-1} w_{2j+1} = -w_1 \sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-1} q_{2^{2\alpha_k+1}-2j-1} w_{2j}.$$

By using (9) we find that

$$\begin{aligned}
&\sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} |q_{2^{2\alpha_k+1}-4j+3} - q_{2^{2\alpha_k+1}-4j+1}| \\
&\quad= \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} (q_{2^{2\alpha_k+1}-4j+1} - q_{2^{2\alpha_k+1}-4j+3}) \\
&\quad= (q_{2^{2\alpha_k}-3} - q_{2^{2\alpha_k}-1}) + (q_{2^{2\alpha_k}-7} - q_{2^{2\alpha_k}-5}) + \cdots + (q_5 - q_7) \\
&\quad\le \frac{1}{2}(q_{2^{2\alpha_k}-3} - q_{2^{2\alpha_k}-1}) + \frac{1}{2}(q_{2^{2\alpha_k}-5} - q_{2^{2\alpha_k}-3}) \\
&\qquad+ \frac{1}{2}(q_{2^{2\alpha_k}-7} - q_{2^{2\alpha_k}-5}) + \frac{1}{2}(q_{2^{2\alpha_k}-9} - q_{2^{2\alpha_k}-7}) \\
&\qquad+ \ldots + \frac{1}{2}(q_5 - q_7) + \frac{1}{2}(q_3 - q_5) \le \frac{1}{2}q_3 - \frac{1}{2}q_{2^{2\alpha_k}-1}.
\end{aligned}$$

Hence, if we apply

$$w_{4k+2} = w_2 w_{4k} = -w_{4k}, \quad \text{for } x \in I_2(e_0 + e_1),$$

we find that

$$\begin{aligned}
&\left|\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} q_{2^{2\alpha_k+1}-j} D_j\right| \\
&\quad= \left| q_1 w_{2^{2\alpha_k+1}-2} + q_3 w_{2^{2\alpha_k+1}-4} + \sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-3} q_{2^{2\alpha_k+1}-2j-1} w_{2j} \right| \\
&\quad= \left| (q_3 - q_1) 2 w_{2^{2\alpha_k+1}-4} + \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} (q_{2^{2\alpha_k+1}-4j+3} w_{4j-4} - q_{2^{2\alpha_k+1}-4j+1} w_{4j-4}) \right| \\
&\quad\ge q_1 - q_3 - \sum_{j=2^{2\alpha_k-2}+1}^{2^{2\alpha_k-1}-1} |q_{2^{2\alpha_k+1}-4j+3} - q_{2^{2\alpha_k+1}-4j+1}| \\
&\quad\ge q_1 - q_3 - \frac{1}{2}(q_3 - q_{2^{2\alpha_k}-1}) \ge q_1 - \frac{3}{2}q_3.
\end{aligned}$$

The proof is complete. □

4 The main result

In previous sections we have discussed a number of inequalities and sometimes their sharpness. Our main result is the following new sharpness result:

Theorem 1 *Let* $0 \leq \alpha \leq 1$, β *be any nonnegative real number and* t_n *be Nörlund means with a convex and nonincreasing sequence* $\{q_k : k \in \mathbb{N}\}$ *satisfying the condition*

$$\frac{q_1 - (3/2)q_3}{Q_n} \geq \frac{C}{n^\alpha \ln^\beta n}, \tag{13}$$

for some positive constant C. *Then, for any* $0 < p < 1/(1+\alpha)$ *there exists a martingale* $f \in H_p$ *such that*

$$\sup_{n\in\mathbb{N}} \|t_{2^n} f\|_{weak-L_p} = \infty.$$

Proof Let $0 < p < 1/(1+\alpha)$. Under condition (13) there exists a sequence $\{n_k : k \in \mathbb{N}\}$ such that

$$\frac{2^{2n_k(1/p-1)}}{n_k Q_{2^{2n_k+1}}} \geq \frac{2^{2n_k(1/p-1-\alpha)}}{n_k^{\beta+1}} \to \infty, \quad \text{as } k \to \infty.$$

Let $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty, \tag{14}$$

$$\sum_{\eta=0}^{k-1} \frac{(2^{2\alpha_\eta})^{1/p}}{\sqrt{\alpha_\eta}} < \frac{(2^{2\alpha_k})^{1/p}}{\sqrt{\alpha_k}} \tag{15}$$

and

$$\frac{(2^{2\alpha_{k-1}})^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{q_1 - (3/2)q_3}{Q_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-3}}{\alpha_k}. \tag{16}$$

Let

$$f^{(n)} := \sum_{\{k;2\alpha_k<n\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{1}{\sqrt{\alpha_k}} \quad \text{and} \quad a_k = 2^{2\alpha_k(1/p-1)}(D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}).$$

From (14) and Lemma 1 we find that $f \in H_p$.

It is easy to prove that

$$\widehat{f}(j) = \begin{cases} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}, & \text{if } j \in \{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1}-1\}, k \in \mathbb{N}, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1}-1\}. \end{cases} \tag{17}$$

Moreover,

$$\begin{aligned} &t_{2^{2\alpha_k+1}}f \\ &= \frac{1}{Q_{2^{2\alpha_k+1}}}\sum_{j=1}^{2^{2\alpha_k}-1} q_{2^{2\alpha_k+1}-j}S_jf + \frac{1}{Q_{2^{2\alpha_k+1}}}\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} q_{2^{2\alpha_k+1}-j}S_jf \\ &:= I + II. \end{aligned} \tag{18}$$

Let $j < 2^{2\alpha_k}$. By combining (15), (16), and (17) we can conclude that

$$\begin{aligned} |S_jf| &\le \sum_{\eta=0}^{k-1}\sum_{\nu=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \left|\widehat{f}(\nu)\right| \\ &\le \sum_{\eta=0}^{k-1}\sum_{\nu=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \frac{2^{2\alpha_\eta(1/p-1)}}{\sqrt{\alpha_\eta}} \le \sum_{\eta=0}^{k-1}\frac{2^{2\alpha_\eta/p}}{\sqrt{\alpha_\eta}} \le \frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}}. \end{aligned}$$

Hence,

$$\begin{aligned} |I| &\le \frac{1}{Q_{2^{2\alpha_k+1}}}\sum_{j=1}^{2^{2\alpha_k}-1} q_{2^{2\alpha_k+1}-j}|S_jf| \\ &\le \frac{1}{Q_{2^{2\alpha_k+1}}}\frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}}\sum_{j=0}^{2^{2\alpha_k+1}-1} q_j \le \frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}}. \end{aligned} \tag{19}$$

Let $2^{2\alpha_k} \le j \le 2^{2\alpha_k+1}$. Since

$$\begin{aligned} S_jf &= \sum_{\eta=0}^{k-1}\sum_{\nu=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1}\widehat{f}(\nu)w_\nu + \sum_{\nu=2^{2\alpha_k}}^{j-1}\widehat{f}(\nu)w_\nu \\ &= \sum_{\eta=0}^{k-1}\frac{2^{2\alpha_\eta(1/p-1)}}{\sqrt{\alpha_\eta}}\left(D_{2^{2\alpha_\eta+1}} - D_{2^{2\alpha_\eta}}\right) + \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}\left(D_j - D_{2^{2\alpha_k}}\right), \end{aligned}$$

for II we can conclude that

$$\begin{aligned} II &= \frac{1}{Q_{2^{2\alpha_k+1}}}\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} q_{2^{2\alpha_k+1}-j}\left(\sum_{\eta=0}^{k-1}\frac{2^{2\alpha_\eta(1/p-1)}}{\sqrt{\alpha_\eta}}\left(D_{2^{2\alpha_\eta+1}} - D_{2^{2\alpha_\eta}}\right)\right) \\ &+ \frac{1}{Q_{2^{2\alpha_k+1}}}\frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} q_{2^{2\alpha_k+1}-j}\left(D_j - D_{2^{2\alpha_k}}\right). \end{aligned} \tag{20}$$

Let $x \in I_2(e_0 + e_1) \in I_0\backslash I_1$. According to the fact that $\alpha_0 \ge 1$ we obtain that $2\alpha_k \ge 2$, for all $k \in \mathbb{N}$ and if we use (7) we obtain that $D_{2^{2\alpha_k}} = 0$ and if we use Lemma 2 we can also conclude that

$$|II| = \frac{1}{Q_{2^{2\alpha_k+1}}}\frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} q_{2^{2\alpha_k+1}-j}D_j \tag{21}$$

$$\geq \frac{q_1-(3/2)q_3}{Q_{2^{2\alpha_k+1}}}\frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}.$$

By combining (16), and (18)–(21) for $x \in I_2(e_0+e_1)$ we have that

$$\begin{aligned}
\left|t_{2^{2\alpha_k+1}}f(x)\right| &\geq |II|-|I| \\
&\geq \frac{q_1-(3/2)q_3}{Q_{2^{2\alpha_k+1}}}\frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}-\frac{q_1-(3/2)q_3}{Q_{2^{2\alpha_k+1}}}\frac{2^{2\alpha_k(1/p-1)-3}}{\alpha_k} \\
&\geq \frac{q_1-(3/2)q_3}{Q_{2^{2\alpha_k+1}}}\frac{2^{2\alpha_k(1/p-1)-3}}{\sqrt{\alpha_k}} \geq \frac{C2^{2\alpha_k(1/p-1-\alpha)-3}}{(\ln 2^{2\alpha_k+1}+1)^{\beta}\sqrt{\alpha_k}} \\
&\geq \frac{C2^{2\alpha_k(1/p-1-\alpha)-3}}{\alpha_k^{\beta+1}}.
\end{aligned}$$

Hence, we can conclude that

$$\begin{aligned}
&\|t_{2^{2\alpha_k+1}}f\|_{weak-L_p} \\
&\geq \frac{C2^{2\alpha_k(1/p-1-\alpha)-3}}{\alpha_k^{\beta+1}}\mu\left\{x\in G: |t_{2^{2\alpha_k+1}}f| \geq \frac{C2^{2\alpha_k(1/p-1)-3}}{\alpha_k^{\beta+1}}\right\}^{1/p} \\
&\geq \frac{C2^{2\alpha_k(1/p-1-\alpha)-3}}{\alpha_k^{\beta+1}}\mu\left\{x\in I_2(e_0+e_1): |t_{2^{2\alpha_k+1}}f| \geq \frac{C2^{2\alpha_k(1/p-1)-3}}{\alpha_k^{\beta+1}}\right\}^{1/p} \\
&\geq \frac{C2^{2\alpha_k(1/p-1-\alpha)-3}}{\alpha_k^{\beta+1}}\left(\mu\left(I_2(e_0+e_1)\right)\right)^{1/p} \\
&> \frac{c2^{2\alpha_k(1/p-1-\alpha)}}{\alpha_k^{\beta+1}} \to \infty, \quad \text{as } k\to\infty.
\end{aligned}$$

The proof is complete. □

In an actual case we obtain a result for Nörlund logarithmic means $\{L_n\}$ proved in [1]:

Corollary 1 *Let* $0<p<1$. *Then, there exists a martingale* $f \in H_p$ *such that*

$$\sup_{n\in\mathbb{N}}\|L_{2^n}f\|_{weak-L_p}=\infty.$$

Proof It is easy to show that

$$q_1-(3/2)q_3=\frac{1}{2}-\frac{3}{2}\cdot\frac{1}{4}=\frac{1}{8}>0,$$

and condition (13) holds true for $\alpha=\beta=0$. □

We also obtain a similar new result for the V_n means:

Corollary 2 *Let* $0<p<1$. *Then, there exists a martingale* $f \in H_p$ *such that*

$$\sup_{n\in\mathbb{N}}\|V_{2^n}f\|_{weak-L_p}=\infty.$$

Proof It is easy to show that

$$q_1-(3/2)q_3=\frac{1}{\ln 2}-\frac{3}{2}\cdot\frac{1}{\ln 4}=\log_2^e-(3/2)\frac{\log_2^e}{\log_2^4}=\log_2^e\left(1-\frac{3}{4}\right)>0,$$

and condition (13) holds true for $\alpha=\beta=0$. □

We also obtain a corresponding new result for the Cesàro means $\sigma_{2^n}^{\alpha}$.

Corollary 3 *Let* $0<p<1/(1+\alpha)$, *for some* $0<\alpha\leq 0.56$. *Then, there exists a martingale* $f\in H_p$ *such that*

$$\sup_{n\in\mathbb{N}}\left\|\sigma_{2^n}^{\alpha}f\right\|_{weak-L_p}=\infty.$$

Proof By a routine calculation we find that

$$q_1-(3/2)q_3=\alpha-\frac{\alpha(\alpha+1)(\alpha+2)}{4}=\alpha\cdot\frac{2-3\alpha-\alpha^2}{4}.$$

It is easy to show that when $0<\alpha<0.56$ this expression is positive. Hence, condition (13) holds true for $\beta=0$ and $0<\alpha<1$. □

Corollary 4 *Let* $0<p<1/(1+\alpha)$, *for some* $0<\alpha\leq 0.41$. *Then, there exists a martingale* $f\in H_p$ *such that*

$$\sup_{n\in\mathbb{N}}\left\|U_{2^n}^{\alpha}f\right\|_{weak-L_p}=\infty.$$

Proof By a straightforward calculation, we find that

$$q_1-(3/2)q_3=2^{\alpha-1}-(3/2)4^{\alpha-1}=2^{\alpha-1}\left(1-3/2^{2-\alpha}\right).$$

It is easy to show that when $0<\alpha<0.41$ this expression is positive. Hence, condition (13) holds true for $\beta=0$ and $0<\alpha<1$. □

5 Open questions and final remarks

Remark 1 This article can be regarded as a complement to the new book [21]. In this book a number of open problems are also raised. Also, this new investigation implies some corresponding open questions.

Open Problem 1 Let $0<p<1/(1+\alpha)$, for some $0.56<\alpha<1$. Does there exist a martingale $f\in H_p$ such that

$$\sup_{n\in\mathbb{N}}\left\|\sigma_{2^n}^{\alpha}f\right\|_{weak-L_p}=\infty?$$

Open Problem 2 Let $0<p<1/(1+\alpha)$, for some $0.41<\alpha<1$. Does there exist a martingale $f\in H_p$ such that

$$\sup_{n\in\mathbb{N}}\left\|U_{2^n}^{\alpha}f\right\|_{weak-L_p}=\infty?$$

We also can investigate similar problems for more general summability methods:

Open Problem 3 Let $0 < p < 1/(1+\alpha)$, for some $0.56 < \alpha < 1$ and t_n be Nörlund means of Walsh–Fourier series with nonincreasing and convex sequence $\{q_k : k \in \mathbb{N}\}$, satisfying the condition (13).

Does there exist a martingale $f \in H_{1/(1+\alpha)}$ $(0 < p < 1)$, such that

$$\sup_{n\in\mathbb{N}} \|t_{2^n} f\|_{H_{1/(1+\alpha)}} = \infty?$$

Open Problem 4 Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$. Does there exist an absolute constant C_α, such that the following inequality holds

$$\left\| \sigma_{2^n}^\alpha f \right\|_{1/(1+\alpha)} \le C_\alpha \|f\|_{H_{1/(1+\alpha)}}?$$

Open Problem 5 Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$. Does there exist an absolute constant C_α, such that the following inequality holds

$$\left\| U_{2^n}^\alpha f \right\|_{1/(1+\alpha)} \le C_\alpha \|f\|_{H_{1/(1+\alpha)}}?$$

Open Problem 6 Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$ and t_n are Nörlund means of Walsh–Fourier series with a nonincreasing and convex sequence $\{q_k : k \in \mathbb{N}\}$, satisfying the condition (13). Does there exist an absolute constant C_α, such that the following inequality holds

$$\left\| t_{2^n}^\alpha f \right\|_{1/(1+\alpha)} \le C_\alpha \|f\|_{H_{1/(1+\alpha)}}?$$

Remark 2 There is an important relation between Walsh–Fourier series and wavelet theory, see, e.g., [21] and the papers [5] and [6]. This is of special interest also for applications as described in the recent PhD thesis of K. Tangrand [27].

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Author contributions
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