SOME NEW WEAK-$(H_p - L_p)$ TYPE INEQUALITIES FOR WEIGHTED MAXIMAL OPERATORS OF FEJÉR MEANS OF WALSH-FOURIER SERIES

DAVIT BARAMIDZE AND GEORGE TEPHNADZE

Abstract. In this paper we introduce some new weighted maximal operators of the Fejér means of the Walsh-Fourier series. We prove that for some "optimal" weights these new operators indeed are bounded from the martingale Hardy space $H_p(G)$ to the space weak $- L_p(G)$, for $0 < p < 1/2$. Moreover, we also prove sharpness of this result. As a consequence we obtain some new and well-known results.

2020 Mathematics Subject Classification. 42C10, 42B30.
Key words and phrases: Walsh system, Fejér means, martingale Hardy space, maximal operators, weighted maximal operators.

1. Introduction

All symbols used in this introduction can be found in Section 2.

In the one-dimensional case, the weak (1,1)-type inequality for the maximal operator σ^* of Fejér means σ_n with respect to the Walsh system

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$$

can be found in Schipp [19] and Pál, Simon [14] (see also [4], [13] and [16]). Fujii [7] and Simon [21] proved that σ^* is bounded from H_1 to L_1. Weisz [29] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [20] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [9]. Moreover, in [10] (see also [23]) he proved that there exists a martingale $F \in H_p$ $(0 < p \leq 1/2)$, such that

$$\sup_{n\in\mathbb{N}} \|\sigma_n F\|_p = \infty.$$

Weisz [29, 32] proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$.

For $0 < p < 1/2$ in [25] it was investigated the weighted maximal operator

$$\widetilde{\sigma}^{*,p} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{(n+1)^{1/p-2}} \tag{1}$$

The research was supported by Shota Rustaveli National Science Foundation grant no. FR-21-2844.

was investigated and it was proved that the following estimate holds:

$$\left\|\widetilde{\sigma}^{*}F\right\|_{p}\leq c_{p}\left\|F\right\|_{H_{p}}$$

and

$$\left\|\widetilde{\sigma}^{*}F\right\|_{weak-L_{p}}\leq c_{p}\left\|F\right\|_{H_{p}}. \tag{2}$$

Moreover, it was proved that the rate of sequence $\{(n+1)^{1/p-2}\}$, given in denominator of (1) can not be improved. In the case $p=1/2$ analogical results for the maximal operator

$$\widetilde{\sigma}^{*}F:=\sup_{n\in\mathbb{N}}\frac{|\sigma_{n}F|}{\log^{2}(n+1)}$$

was proved in [11] for Walsh system and [24] for Vilenkin systems.

In the study of convergence of subsequences of Fejér means and their restricted maximal operators on the martingale Hardy spaces $H_p(G)$ for $0<p\leq 1/2$, the central role is played by the fact that any natural number $n\in\mathbb{N}$ can be uniquely expression as $n=\sum_{k=0}^{\infty}n_j2^j,\ \ n_j\in Z_2\ (j\in\mathbb{N})$, where only a finite numbers of n_j differ from zero and their important characters $[n],\ |n|,\ \rho(n)$ and $V(n)$ are defined by

$$[n]:=\min\{j\in\mathbb{N},n_j\neq 0\},\quad |n|:=\max\{j\in\mathbb{N},n_j\neq 0\},\quad \rho(n)=|n|-[n],$$

$$V(n):=n_0+\sum_{k=1}^{\infty}|n_k-n_{k-1}|,\quad \text{for all}\quad n\in\mathbb{N}.$$

Weisz [31] (see also [30]) also proved that for any $F\in H_p(G)$ $(p>0)$, the maximal operator $\sup_{n\in\mathbb{N}}|\sigma_{2^n}F|$ is bounded from the Hardy space H_p to the Lebesgue space L_p. Persson and Tephnadze [15] (see also [4]) generalized this result and proved that if $0<p\leq 1/2$ and $\{n_k:k\geq 0\}$ be a sequence of positive numbers, such that

$$\sup_{k\in\mathbb{N}}\rho(n_k)\leq c<\infty, \tag{3}$$

then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla}F:=\sup_{k\in\mathbb{N}}|\sigma_{n_k}F| \tag{4}$$

is bounded from the Hardy space $H_p(G)$ to the space $L_p(G)$. Moreover, if $0<p<1/2$ and $\{n_k:k\geq 0\}$ be a sequence of positive numbers, such that

$$\sup_{k\in\mathbb{N}}\rho(n_k)=\infty,$$

then there exists a martingale $F\in H_p$ such that

$$\sup_{k\in\mathbb{N}}\|\sigma_{n_k}F\|_p=\infty.$$

From these fact it follows that if $0<p<1/2$, $F\in H_p$ and $\{n_k:k\geq 0\}$ is any sequence of positive numbers, then the maximal operator defined by (4) is bounded from the Hardy space H_p to the Lebesgue space L_p if and only if the condition (3) is fulfilled.

For $0<p<1/2$ in [28] it was proved that if $F\in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\|\sigma_n F\|_{H_p} \le c_p 2^{\rho(n)(1/p-2)} \|F\|_{H_p},$$

using this it follows that

$$\left\| \frac{\sigma_n F}{2^{\rho(n)(1/p-2)}} \right\|_p \le c_p \|F\|_{H_p}$$

and

$$\left\| \frac{\sigma_n F}{2^{\rho(n)(1/p-2)}} \right\|_{weak-L_p} \le c_p \|F\|_{H_p}. \tag{5}$$

Moreover, if $\{\Phi_n\}$ be any nondecreasing sequence, such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) = \infty, \quad \overline{\lim_{k\to\infty}} \frac{2^{\rho(n_k)(1/p-2)}}{\Phi_{n_k}} = \infty,$$

then there exists a martingale $F\in H_p$ $(0<p<1/2)$, such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{\sigma_{n_k} F}{\Phi_{n_k}} \right\|_{\text{weak}-L_p} = \infty.$$

In [28] it was proved that if $F\in H_{1/2}$, then there exists an absolute constant c, such that

$$\|\sigma_n F\|_{H_{1/2}} \le c V^2(n) \|F\|_{H_{1/2}}.$$

Moreover, the rate of sequence $V^2(n)$ can not be improved.

The $(H_{1/2}-L_{1/2})$-type inequalities for the the restricted and weighted maximal operators of Walsh-Fejér means were studied in [2] and [3]. Analogical problems for partial sums of Walsh-Fourier series for $0<p<1$ were proved in [5] and [6] (see also [26, 27]).

In this paper we generalize estimates (2) and (5). In particular, we prove that the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{2^{\rho(n)(1/p-2)}} \tag{6}$$

of Fejér means of Walsh-Fourier series is bounded from the Hardy space $H_p(G)$ to the space weak $-L_p(G)$. Moreover, we prove that the rate of the sequence $\{2^{\rho(n)(1/p-2)}\}$ in (6) is sharp. We also prove that maximal operator defined by (6) is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. As a consequence we obtain some new and well-know results.

This paper is organized as follows: In order not to disturb our discussions later on some preliminaries are presented in Section 2. The main result and some of its consequences can be found in Section 3. The detailed proof of the main result is given in Section 4. Some open questions and final remarks are given in Section 5.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is $1/2$.

Define the group G as the complete direct product of infinite copies of the group Z_2, with the product of the discrete topologies of Z_2 and product of the measures on Z_2 (it will be denoted by μ). The elements of G are represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, where $x_k = 0 \vee 1$.

It is easy to give a base for the neighborhood of $x \in G$

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y_0 = x_0, ..., y_{n-1} = x_{n-1}\} \ (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \setminus I_n$ and $e_n := (0, ..., 0, x_n = 1, 0, ...) \in G$, for $n \in \mathbb{N}$. Then it is easy to show that

$$\overline{I_M} = \bigcup_{i=0}^{M-1} I_i \backslash I_{i+1} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l) \right) \bigcup \left(\bigcup_{k=0}^{M-1} I_M(e_k) \right), \tag{7}$$

where

$$I_N^{k,l} =: \begin{cases} I_N(0, \ldots, 0, x_k \neq 0, 0, \ldots, 0, x_l \neq 0, x_{l+1}, \ldots, x_{N-1}, \ldots), \\ \text{for} \qquad k < l < N, \\ I_N(0, \ldots, 0, x_k \neq 0, x_{k+1} = 0, \ldots, x_{N-1} = 0, x_N, \ldots), \\ \text{for} \qquad l = N. \end{cases}$$

If $n \in \mathbb{N}$, then every n can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j 2^j$, where $n_j \in Z_2 \ (j \in \mathbb{N})$ and only a finite numbers of n_j differ from zero.

Every $n \in \mathbb{N}$ can be also represented as $n = \sum_{i=1}^{r} 2^{n^i}, n^1 > n^2 > ... n^r \geq 0$. For such representation of $n \in \mathbb{N}$, let denote numbers

$$n^{(i)} = 2^{n^{i+1}} + ... + 2^{n^r}, i = 1, ..., r.$$

The norms (or quasi-norms) of the spaces $L_p(G)$ and $weak{-}L_p(G)$, $(0 < p < \infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu, \qquad \|f\|_{\text{weak}-L_p(G)}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty,$$

The k-th Rademacher function is defined by

$$r_k(x) := (-1)^{x_k} \qquad (x \in G, \ k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [18]).

If $f \in L_1(G)$, we can define the Fourier coefficients, partial sums of Fourier series, Fejér means, Dirichlet and Fejér kernels in the usual manner:

$$\begin{aligned} \widehat{f}(n) &:= \int_G f w_n d\mu, \ (n \in \mathbb{N}), \\ S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \ \ (n \in \mathbb{N}_+, S_0 f := 0), \\ \sigma_n f &:= \frac{1}{n} \sum_{k=1}^{n} S_k f, \\ D_n &:= \sum_{k=0}^{n-1} w_k, \\ K_n &:= \frac{1}{n} \sum_{k=1}^{n} D_k, \quad (n \in \mathbb{N}_+). \end{aligned}$$

Recall that (see [8], [12] and [18]) for any $t, n \in \mathbb{N}$,

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n \\ 0 & \text{if } x \notin I_n. \end{cases} \tag{8}$$

and

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, \text{ if } \ x \in I_n(e_t), \ n > t, \ x \in I_t \backslash I_{t+1}, \\ (2^n + 1)/2, \text{ if } \ x \in I_n, \\ 0, \text{ otherwise.} \end{cases} \tag{9}$$

Let $n = \sum_{i=1}^{r} 2^{n^i}, \quad n^1 > n^2 > ... > n^r \geq 0$. Then (see [12] and [18])

$$nK_n = \sum_{A=1}^{r} \left(\prod_{j=1}^{A-1} w_{2^{n^j}} \right) \left(2^{n^A} K_{2^{n^A}} + n^{(A)} D_{2^{n^A}} \right). \tag{10}$$

The next two lemmas can be found in [17] (see also [15]):

Lemma 1. *Let* $n \geq 2^M$ *and* $x \in I_M^{k,l}, \ k = 0, ..., M-1, \ l = k+1, ..., M.$ *Then*

$$\int_{I_M} |K_n(x+t)| \, d\mu(t) \leq c 2^{k+l-2M}.$$

Lemma 2. *Let* $n \in \mathbb{N}_+$, $[n] \neq |n|$ *and* $x \in I_{[n]+1}\left(e_{[n]-1} + e_{[n]}\right)$. *Then*

$$|nK_n(x)| = \left| \left(n - 2^{|n|}\right) K_{n-2^{|n|}}(x) \right| \geq \frac{2^{2[n]}}{4}.$$

The σ-algebra, generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n $(n \in \mathbb{N})$. Denote by $F = (F_n, n \in \mathbb{N})$ a martingale with respect to ζ_n $(n \in \mathbb{N})$ (for details see e.g. [30]).

The maximal function F^* of a martingale F is defined by

$$F^* := \sup_{n \in \mathbb{N}} |F_n|.$$

In the case $f\in L_1(G)$, the maximal function f^* is given by

$$f^*(x):=\sup_{n\in\mathbb{N}}\frac{1}{\mu(I_n(x))}\left|\int\limits_{I_n(x)} f(u)\,d\mu(u)\right|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p(G)$ consists of all martingales for which (for details see e.g. [17], [22] and [30])

$$\|F\|_{H_p}:=\|F^*\|_p<\infty.$$

It is easy to check that for every martingale $F=(F_n,n\in\mathbb{N})$ and every $k\in\mathbb{N}$ the limit

$$\widehat{F}(k):=\lim_{n\to\infty}\int_G F_n(x)\,w_k(x)\,d\mu(x)$$

exists and it is called the k-th Walsh-Fourier coefficients of F.

If $F:=(S_{2^n}f:n\in\mathbb{N})$ is a regular martingale, generated by $f\in L_1(G)$, then $\widehat{F}(k)=\widehat{f}(k)$, $k\in\mathbb{N}$.

A bounded measurable function a is called p-atom, if there exists a dyadic interval I, such that

$$\int_I a\,d\mu=0,\quad \|a\|_\infty\leq\mu(I)^{-1/p},\quad \operatorname{supp}(a)\subset I.$$

The dyadic Hardy martingale spaces H_p for $0<p\leq 1$ have an atomic characterization. Namely, the following theorem holds (see [17], [30], [31]):

Lemma 3. *A martingale $F=(F_n,\ n\in\mathbb{N})$ belongs to $H_p\,(0<p\leq 1)$ if and only if there exists a sequence $(a_k,k\in\mathbb{N})$ of p-atoms and a sequence $(\mu_k,k\in\mathbb{N})$ of real numbers, such that for every $n\in\mathbb{N}$*

$$\sum_{k=0}^{\infty}\mu_k S_{2^n}a_k=F_n,\qquad \sum_{k=0}^{\infty}|\mu_k|^p<\infty. \tag{11}$$

Moreover, $\|F\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of F of the form (11).

From this result follows the following important lemma proved by Weisz [30]:

Lemma 4. *Suppose that an operator T is σ-sublinear and*

$$\sup_{\rho>0}\rho^p\mu\left\{x\in\overline{I}:|Ta(x)|>\rho\right\}\leq C_p<\infty,$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|TF\|_{weak-L_p}\leq c_p\|F\|_{H_p}.$$

3. The Main Result and its Consequences

Theorem 1. *a) Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by (6), is bounded from the Hardy space H_p to the space weak $-L_p$.*

b) Let $\varphi : \mathbb{N} \to [1, \infty)$ be a nondecreasing function, satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{2^{\rho(n)(1/p-2)}}{\varphi(n)} = \infty.$$

Then, there exist a sequence $\{f_{n_k},\ k \in \mathbb{N}_+\}$ of p-atoms and sequence $\{q_{n_k},\ k \in \mathbb{N}_+\}$ of real numbers satisfying the condition $|q_{n_k}| = n_k$, such that

$$\sup_{k\in\mathbb{N}} \frac{\left\| \frac{\sigma_{q_{n_k}} f_{n_k}}{\varphi\left(q_{n_k}\right)} \right\|_{weak-L_p}}{\left\| f_{n_k} \right\|_{H_p}} = \infty.$$

We also prove that the following theorem holds:

Theorem 2. *Let $0 < p < 1/2$. There exists a sequence $\{f_k,\ k \in \mathbb{N}_+\}$ of p-atoms, such that*

$$\sup_{k\in\mathbb{N}} \frac{\left\| \widetilde{\sigma}^{*,\nabla} f_k \right\|_p}{\left\| f_k \right\|_{H_p}} = \infty.$$

From Theorem 1 immediately follows the mentioned result of Weisz [31] (see also [30]):

Corollary 1. *Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the maximal operator*

$$\sup_{n\in\mathbb{N}} \left| \sigma_{2^n} F \right|$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space weak $-L_p(G)$.

We also obtain results of Persson and Tephnadze [15] (see also [4]):

Corollary 2. *Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the maximal operator, defined by (4) is bounded from the Hardy space $H_p(G)$ to the space weak $-L_p(G)$ if and only if condition (3) is fulfilled.*

Corollary 3. *a) Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the weighted maximal operator*

$$\sup_{n\in\mathbb{N}} \frac{\left| \sigma_{2^n+2^{n/2}} F \right|}{2^{\frac{n}{2}(1/p-2)}}$$

is bounded from the martingale Hardy space $H_p(G)$ to the space weak$-L_p(G)$.

b) Let $\varphi : \mathbb{N} \to [1, \infty)$ be a nondecreasing function, satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{2^{\frac{n}{2}(1/p-2)}}{\varphi(n)} = \infty.$$

Then, there exists a p-atom a such that

$$\sup_{n\in\mathbb{N}}\frac{\left\|\frac{\sigma_{2^n+2^{n/2}}a}{\varphi\left(2^n+2^{n/2}\right)}\right\|_{weak-L_p}}{\|a\|_{H_p}}=\infty.$$

Corollary 4. *a) Let $0<p<1/2$ and $f\in H_p(G)$. Then the weighted maximal operator*

$$\sup_{n\in\mathbb{N}}\frac{|\sigma_{2^n+1}F|}{2^{n(1/p-2)}}$$

is bounded from the Hardy space H_p to the space weak $-\,L_p$.

b) Let $\varphi:\mathbb{N}\to[1,\infty)$ be a nondecreasing function, satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{2^{n(1/p-2)}}{\varphi(n)}=\infty.$$

Then, there exists a p-atom a such that

$$\sup_{n\in\mathbb{N}}\frac{\left\|\frac{\sigma_{2^n+1}a}{\varphi(2^n+1)}\right\|_{weak-L_p}}{\|a\|_{H_p}}=\infty.$$

Theorem 1 immediately follows result given in [25]:

Corollary 5. *a) Let $0<p<1/2$ and $f\in H_p(G)$. Then the weighted maximal operator $\widetilde{\sigma}^*$, defined by*

$$\widetilde{\sigma}^*F:=\sup_{n\in\mathbb{N}}\frac{|\sigma_nF|}{(n+1)^{1/p-2}}$$

is bounded from the martingale Hardy space $H_p(G)$ to the space weak$-L_p(G)$.

b) Let $\{\varphi_n\}$ be any nondecreasing sequence satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{(n+1)^{1/p-2}}{\varphi_n}=+\infty.$$

Then there exists a martingale $f\in H_p$, such that

$$\sup_{n\in\mathbb{N}}\left\|\frac{\sigma_nf}{\varphi_n}\right\|_p=\infty.$$

4. Proof of the Theorems

Proof. Since σ_n is bounded from L_∞ to L_∞, by Lemma 4, the proof of Theorem 1 will be complete, if we show that

$$t\mu\left\{x\in\overline{I_M}:\widetilde{\sigma}^{*,\nabla}a(x)\geq t^{1/p}\right\}\leq c<\infty,\qquad t\geq 0 \tag{12}$$

for every p-atom a. We may assume that a be an arbitrary p-atom, with support I, $\mu(I)=2^{-M}$ and $I=I_M$. It is easy to see that

$$\sigma_na(x)=0,\quad\text{when}\quad n<2^M.$$

Therefore, we can suppose that $n \geq 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, we obtain that

$$\begin{aligned} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}} &\leq \frac{1}{2^{\rho(n)(1/p-2)}} \|a\|_\infty \int_{I_M} |K_n(x+t)| \, d\mu(t) \\ &\leq \frac{1}{2^{\rho(n)(1/p-2)}} 2^{M/p} \int_{I_M} |K_n(x+t)| \, d\mu(t) . \end{aligned}$$

Let $x \in I_{l+1}(e_k + e_l)$, $0 \leq k, l \leq [n] \leq M$ or $0 \leq k, l \leq M < [n]$. Then, it is easy to see that $x + t \in I_{l+1}(e_k + e_l)$ for $t \in I_M$ and if we combine (8) and (9) with (10) we get that

$$K_n(x+t) = 0, \quad \text{for} \quad t \in I_M$$

and

$$\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}} = 0. \tag{13}$$

Let $x \in I_{l+1}(e_k + e_l)$, $[n] \leq k, l \leq M$ or $k \leq [n] \leq l \leq M$. By using Lemma 1 we can conclude that

$$\begin{aligned} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}} &\leq c_p 2^{M/p} \frac{2^{k+l-2M}}{2^{\rho(n)(1/p-2)}} \\ &\leq c_p \frac{2^{[n](1/p-2)+k+l+M(1/p-2)}}{2^{|n|(1/p-2)}} \\ &\leq c_p 2^{[n](1/p-2)+k+l} \\ &\leq c_p 2^{k+l(1/p-1)}. \end{aligned} \tag{14}$$

By applying (13) and (14) for any $x \in I_{l+1}(e_k + e_l)$, $1 \leq k < l \leq M$ we find that

$$\widetilde{\sigma}^{*,\nabla} a(x) = \sup_{n \in \mathbb{N}} \left(\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}} \right) \leq c_p 2^{k+l(1/p-1)}.$$

It immediately follows that for such $k < l \leq M$ we have the following estimate

$$\widetilde{\sigma}^{*,\nabla} a(x) \leq C_p 2^{M/p} \quad \text{for} \quad x \in I_M^{k,l}$$

and also that

$$\mu\left\{x \in I_N^{k,l} : \widetilde{\sigma}^{*,\nabla} a(x) > C_p 2^{s/p}\right\} = 0, \quad s = M+1, M+2, \ldots \tag{15}$$

Suppose that

$$2^{k+l(1/p-1)} > 2^{s/p} \quad \text{for some} \quad s \leq M \tag{16}$$

It is evident that inequality (16) does not hold when $k < l \leq s$. On the other hand, inequality (16) holds for all $l > k \geq s$, that is,

$$2^{k+l(1/p-1)} > 2^{s/p}, \quad \text{where} \quad l > k \geq s. \tag{17}$$

If $l > s > k$, from (16) we can conclude that

$$\begin{aligned} k + l(1/p-1) &> s/p \\ l &> (s/p - k)/(1/p-1) \end{aligned}$$

and

$$2^{k+l(1/p-1)} > 2^{s/p}, \quad \text{where} \quad s > k, \quad l > (s/p-k)/(1/p-1). \tag{18}$$

By combining (7), (17) and (18) we get that

$$\begin{aligned}
&\left\{x \in \overline{I_M} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\} \\
\subset &\left(\bigcup_{k=s}^{M-1} \bigcup_{l=k+1}^{M} \left\{x \in I_M^{k,l} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\}\right) \\
&\bigcup \left(\bigcup_{k=0}^{s} \bigcup_{l>(s/p-k)(1/p-1)}^{M} \left\{x \in I_M^{k,l} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mu\left\{x \in \overline{I_M} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\} \\
&\leq \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} \mu\left(I_M^{k,l}\right) + \sum_{k=0}^{s} \sum_{l>(s/p-k)/(1/p-1)}^{M} \mu\left(I_M^{k,l}\right) \\
&\leq \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} \frac{1}{2^l} + \sum_{k=0}^{s} \sum_{l>(s/p-k)/(1/p-1)}^{M} \frac{1}{2^l} \\
&\leq \sum_{k=s}^{M-1} \frac{1}{2^k} + \sum_{k=0}^{s} \frac{1}{2^{(s/p-k)/(1/p-1)-1}} \leq \frac{c_p}{2^s}.
\end{aligned} \tag{19}$$

In view of (15) and (19) we can conclude that

$$2^s \mu\left\{x \in \overline{I_M} : \widetilde{\sigma}^{*,\nabla} a(x) \geq C_p 2^{s/p}\right\} < c_p < \infty,$$

which shows (12) as well as part a).

Let $q_{n_k} \in \mathbb{N}$ be sequence such that $|q_{n_k}| = n_k,\ [q_{n_k}] = s_k$ and

$$\lim_{k\to\infty} \frac{2^{\rho(q_{n_k})(1/p-2)}}{\varphi(q_{n_k})} = \infty \tag{20}$$

Set

$$f_{n_k}(x) = D_{2^{n_k+1}}(x) - D_{2^{n_k}}(x), \qquad n_k \geq 3.$$

It is evident

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, \text{ if } i = 2^{n_k}, ..., 2^{n_k+1}-1, \\ 0, \text{otherwise.} \end{cases}$$

Then we can write that

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{2^{n_k}}(x), \text{ if } i = 2^{n_k}, ..., 2^{n_k+1}-1, \\ f_{n_k}(x), \text{ if } i \geq 2^{n_k+1}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{21}$$

Since

$$D_{j+2^{n_k}}(x)-D_{2^{n_k}}(x)=w_{2^{n_k}}D_j(x),\qquad j=1,2,..,2^{n_k}, \tag{22}$$

from (8) we get

$$\begin{aligned}\|f_{n_k}\|_{H_p} &= \left\|\sup_{n\in\mathbb{N}} S_{2^n} f_{n_k}\right\|_p = \|D_{2^{n_k+1}}-D_{2^{n_k}}\|_p \\ &= \|D_{2^{n_k}}\|_p \leq 2^{n_k(1-1/p)}.\end{aligned} \tag{23}$$

By applying (21) we can conclude that

$$\begin{aligned}\left|\sigma_{q_{n_k}} f_{n_k}(x)\right| &= \frac{1}{q_{n_k}}\left|\sum_{j=0}^{q_{n_k}-1} S_j f_{n_k}(x)\right| = \frac{1}{q_{n_k}}\left|\sum_{j=2^{n_k}}^{q_{n_k}-1} S_j f_{n_k}(x)\right| \\ &= \frac{1}{q_{n_k}}\left|\sum_{j=2^{n_k}}^{q_{n_k}-1}\left(D_j(x)-D_{2^{n_k}}(x)\right)\right| \\ &= \frac{1}{q_{n_k}}\left|\sum_{j=0}^{q_{n_k}-2^{n_k}-1}\left(D_{j+2^{n_k}}(x)-D_{2^{n_k}}(x)\right)\right|.\end{aligned}$$

By using (22) we find that

$$\begin{aligned}\left|\sigma_{q_{n_k}} f_{n_k}(x)\right| &= \frac{1}{q_{n_k}}\left|\sum_{j=0}^{q_{n_k}-2^{n_k}-1} D_j(x)\right| \\ &= \frac{q_{n_k}-2^{n_k}-1}{q_{n_k}}\left|K_{q_{n_k}-2^{n_k}-1}(x)\right|.\end{aligned} \tag{24}$$

Let $x\in I_{[q_{n_k}]+1}\left(e_{[q_{n_k}]-1}+e_{[q_{n_k}]}\right)$. By using Lemma 2 we obtain that

$$\left|\sigma_{q_{n_k}} f_{n_k}(x)\right| \geq \frac{c2^{2s_k}}{2^{n_k}}$$

and

$$\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi(q_{n_k})} \geq \frac{c2^{2s_k}}{2^{n_k}\varphi(q_{n_k})}.$$

Hence, we can conclude that

$$\begin{aligned}&\mu\left\{x\in G: \frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi(q_{n_k})} \geq \frac{c2^{2[q_{n_k}]}}{2^{n_k}\varphi(q_{n_k})}\right\} \\ &\geq \mu\left(I_{[q_{n_k}]+1}(e_{[q_{n_k}]-1}+e_{[q_{n_k}]})\right) > c/2^{[q_{n_k}]}.\end{aligned} \tag{25}$$

By combining (20), (23) and (25) we get that

$$\frac{\frac{c2^{2[q_{n_k}]}}{2^{n_k}\varphi(q_{n_k})}\left(\mu\left\{x\in G:\frac{\left|\sigma_{q_{n_k}}f_{n_k}(x)\right|}{\varphi(q_{n_k})}\geq\frac{c2^{2[q_{n_k}]}}{2^{n_k}\varphi(q_{n_k})}\right\}\right)^{1/p}}{\left\|f_{n_k}(x)\right\|_{H_p}}$$

$$\geq \frac{c_p2^{2[q_{n_k}]}}{2^{n_k}\varphi\left(q_{n_k}\right)2^{n_k(1-1/p)}}\frac{1}{2^{[q_{n_k}]/p}}$$

$$= \frac{c_p2^{n_k(1/p-2)}}{2^{[q_{n_k}](1/p-2)}\varphi\left(q_{n_k}\right)}=\frac{c_p2^{\rho\left(q_{n_k}\right)(1/p-2)}}{\varphi\left(q_{n_k}\right)}\to\infty,\quad \text{as}\quad k\to\infty.$$

The proof is complete. □

Proof. Let f_{n_k} be the p-atom from part b) of Theorem 1. If we replace q_{n_k} by $q^s_{n_k}=2^{n_k}+2^s$ (we note that $\left|q^s_{n_k}\right|=n_k,\ \left[q^s_{n_k}\right]=s$) from (24) we find that

$$\left|\sigma_{q^s_{n_k}}f_{n_k}(x)\right|\geq\frac{c2^{2s}}{2^{n_k}},\quad \text{for}\quad x\in I_{s+1}\left(e_{s-1}+e_s\right)$$

and

$$\frac{\left|\sigma_{q^s_{n_k}}f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q^s_{n_k}\right)}}\geq\frac{c_p2^{s/p}}{2^{n_k(1/p-1)}},\quad \text{for}\quad x\in I_{s+1}\left(e_{s-1}+e_s\right).$$

Hence,

$$\int_G\left(\sup_{k\in\mathbb{N}}\frac{\left|\sigma_{q^s_{n_k}}f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q^s_{n_k}\right)}}\right)^p d\mu(x) \tag{26}$$

$$\geq \sum_{s=1}^{n_k-1}\int_{I_{s+1}(e_{s-1}+e_s)}\left(\frac{\left|\sigma_{q_{n^s_k}}f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q^s_{n_k}\right)}}\right)^p d\mu(x)$$

$$\geq c_p\sum_{s=1}^{n_k-1}\frac{1}{2^s}\frac{2^s}{2^{n_k(1-p)}}\geq\frac{C_pn_k}{2^{n_k(1-p)}}.$$

Finally, by combining (23) and (26) we find that

$$\frac{\left(\int_G\left(\sup_{k\in\mathbb{N}}\sup_{0<s<n_k}\frac{\left|\sigma_{q^s_{n_k}}f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q^s_{n_k}\right)}}\right)^p d\mu(x)\right)^{1/p}}{\left\|f_{n_k}\right\|_{H_p}}$$

$$\geq \frac{\left(\frac{C_pn_k}{2^{n_k(1-p)}}\right)^{1/p}}{2^{n_k(1/p-1)}}\geq c_p{n_k}^{1/p}\to\infty,\quad \text{as}\quad k\to\infty.$$

The proof is complete. □

5. Open questions and Final Remarks

Remark 1. *This article can be regarded as a complement of the new book* [17]*. In this book also a number of open problems are raised. Also this new investigation implies some corresponding open questions.*

From Theorem 2 we can conclude the following result:

Theorem 3. *a) Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by* (6)*, is not bounded from the Hardy space H_p to the Lebesgue space L_p.*

Open Problem 1. Let us introduce some new weighted maximal operator of the Fejér means of the Walsh-Fourier series with some "optimal" weights such that this new operator is bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$, for $0 < p < 1/2$.

To study boundedness of restricted maximal operators from the martingale Hardy spaces $H_p(G)$ to the Lebesgue space $L_p(G)$, where $0 < p \leq 1/2$, for any natural number satisfying the condition

$$2^s \leq n_{s_1} \leq n_{s_2} \leq ... \leq n_{s_r} < 2^{s+1}, \quad s \in \mathbb{N},$$

we define numbers

$$s_- := \min\{[n_{s_j}]\}, \quad s_+ := \max\{[n_{s_j}]\} = s, \quad \rho_s(n_{s_j}) := s_+ - s_-. \tag{27}$$

Conjecture 1. *Let $0 < p < 1/2$, $f \in H_p(G)$ and $\{n_k : k \geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ be numbers such that*

$$2^s \leq n_{s_1} \leq n_{s_2} \leq ... \leq n_{s_r} \leq 2^{s+1}, \ s \in \mathbb{N}.$$

a) Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla}F := \sup_{s\in\mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_n F|}{2^{\rho_s(n_{s_i})(1/p-2)}},$$

where $\rho_s(n_{s_i})$ are defined by (27)*, is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

b) Then for any nonnegative and nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ satisfying the condition

$$\sup_{s\in\mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{2^{\rho_s(n_{s_i})(1/p-2)}}{\varphi(n_{s_i})} = \infty, \tag{28}$$

there exists p-atoms f_s, such that

$$\frac{\left\| \sup_{s\in\mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} f_s|}{\varphi(n_{s_i})} \right\|_p}{\|f_s\|_{H_p}} \to \infty, \quad as \quad s \to \infty.$$

Acknowledgments

We thank both reviewers for the good suggestions which have improved the final version of this paper.

References

[1] D. Baramidze, N. Nadirashvili, L.-E. Persson, G. Tephnadze, Some weak-type inequalities and almost everywhere convergence of Vilenkin-Nörlund means, *J. Inequal. Appl.*, (2023), paper no. 66, 17 pp.

[2] D. Baramidze, L-E. Persson and G. Tephnadze, Some now restricted maximal operators of Fejér means of Walsh-Fourier series, *Banach J. Math. Anal.*, **17**, (2023), Paper No. 75, 20 pp.

[3] D. Baramidze, I. Blahota, G. Tephnadze and R. Toledo, Martingale Hardy spaces and some new weighted maximal operator of Fejér means of Walsh-Fourier series, *J. Geom. Anal.*, **34**, (2024), Paper No. 3, 17 pp.

[4] I. Blahota, K. Nagy, L. E. Persson, G. Tephnadze, A sharp boundedness result concerning maximal operators of Vilenkin-Fourier series on martingale Hardy spaces, *Georgian Math. J.*, **26**, 3 (2019), 351-360.

[5] D. Baramidze, L.-E. Persson, H. Singh and G. Tephnadze, Some new weak (H_p-L_p) type inequality for weighted maximal operators of partial sums of Walsh-Fourier series, *Mediterr. J. Math.*, **20**, 5 (2023) 284.

[6] D. Baramidze, L.-E. Persson and G. Tephnadze, Some new (H_p-L_p) type inequalities for weighted maximal operators of partial sums of Walsh-Fourier series, *Positivity*, **27**, 3 (2023) 38.

[7] N. J. Fujii, A maximal inequality for H_1 functions on the generalized Walsh-Paley group, *Proc. Amer. Math. Soc.*, **77** (1979), 111-116.

[8] G. Gát, Investigations of certain operators with respect to the Vilenkin system, *Acta Math. Hung.*, **61** (1993), 131-149.

[9] U. Goginava, Maximal operators of Fejér means of double Walsh-Fourier series, *Acta Math. Hungar.*, **115** (2007), 333–340.

[10] U. Goginava, The martingale Hardy type inequality for Marcinkiewicz-Fejér means of two-dimensional conjugate Walsh-Fourier series, *Acta Math. Sinica*, **27** (2011), 1949-1958.

[11] U. Goginava, Maximal operators of Fejér–Walsh means, *Acta Sci. Math. (Szeged)* **74**, 3-4 (2008), 615-624.

[12] B. Golubov, A. Efimov, V. Skvortsov, *Walsh series and transformations*, Dordrecht, Boston, London, 1991. Kluwer Acad. publ, 1991.

[13] N. Nadirashvili, L.-E. Persson, G. Tephnadze, F. Weisz, Vilenkin-Lebesgue points and almost everywhere convergence of Vilenkin-Fejér means and applications, *Mediterr. J. Math.*, **19**, 5, (2022) 239.

[14] J. Pál, P. Simon, On a generalization of the concept of derivate, *Acta Math. Hung.*, **29** (1977), 155-164.

[15] L. E. Persson, G. Tephnadze, A sharp boundedness result concerning some maximal operators of Vilenkin-Fejér means, *Mediterr. J. Math.*, **13**, 4 (2016), 1841-1853.

[16] L. E. Persson, G. Tephnadze, P. Wall, On the maximal operators of Vilenkin-Nörlund means, *J. Fourier Anal. Appl.*, **21**, 1 (2015), 76-94.

[17] L. E. Persson, G. Tephnadze, F. Weisz, *Martingale Hardy Spaces and Summability of one-dimensional Vilenkin-Fourier Series*, book manuscript, Birkhäuser/Springer, 2022.

[18] F. Schipp, W. Wade, P. Simon, J. Pál, *Walsh series, An Introduction to Dyadic Harmonic Analysis*, Adam-Hilger, Ltd. Bristol, 1990.

[19] F. Schipp, Certain rearrangements of series in the Walsh series, *Mat. Zametki*, **18** (1975), 193-201.

[20] P. Simon, Cesáro summability with respect to two-parameter Walsh systems, *Monatsh. Math.*, **131**, 4 (2000), 321–334.

[21] P. Simon, Investigations with respect to the Vilenkin system, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **2** (1985), 87-101.

[22] P. Simon, A note on the Sunouchi operator with respect to the Vilenkin system, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **43** (2000), 101-116.

[23] G. Tephnadze, Fejér means of Vilenkin-Fourier series, *Stud. sci. math. Hung.*, **49** (2012), 79-90.

[24] G. Tephnadze, On the maximal operator of Vilenkin-Fejér means, *Turk. J. Math.*, **37** (2013), 308-318.

[25] G. Tephnadze, On the maximal operators of Vilenkin-Fejér means on Hardy spaces, *Math. Inequal. Appl.*, **16**, 2 (2013), 301-312.

[26] G. Tephnadze, On the partial sums of Vilenkin-Fourier series, *J. Contemp. Math. Anal.*, **49**, 1 (2014), 23-32.

[27] G. Tephnadze, Strong convergence theorems of Walsh-Fejér means, *Acta Math. Hungar.*, **142**, 1 (2014), 244–259.

[28] G. Tephnadze, On the convergence of Fejér means of Walsh-Fourier series in the space H_p, *J. Contemp. Math. Anal.*, **51**, 2 (2016), 90-102.

[29] F. Weisz, Cesáro summability of one- and two-dimensional Walsh-Fourier series, *Anal. Math.*, **22** (1996), 229-242.

[30] F. Weisz, *Martingale Hardy spaces and their applications in Fourier Analysis*, Springer, Berlin-Heideiberg-New York, 1994.

[31] F. Weisz, *Summability of multi-dimensional Fourier series and Hardy space*, Kluwer Academic, Dordrecht, 2002.

[32] F. Weisz, Weak type inequalities for the Walsh and bounded Ciesielski systems, *Anal. Math.*, **30**, 2 (2004), 147-160.

D. Baramidze, The University of Georgia, School of science and technology, 77a Merab Kostava St, Tbilisi 0128, Georgia and Department of Computer Science and Computational Engineering, UiT - The Arctic University of Norway, P.O. Box 385, N-8505, Narvik, Norway.

Email address: davit.baramidze@ug.edu.ge

G. Tephnadze, The University of Georgia, School of Science and Technology, 77a Merab Kostava St, Tbilisi, 0128, Georgia.

Email address: giorgitephnadze@gmail.com g.tephnadze@ug.edu.ge