UiT The Arctic University of Norway

Department of Mathematics and Statistics

On elementary particles as representations of the Poincaré group

Pau Martínez Marín

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Abstract

This thesis is concerned with the definition of elementary particles as irreducible projective unitary representations of the Poincaré group. During the contents of this work, we will introduce the relevant prerequisites and results. Concerning differential geometry, we will discuss smooth manifolds, Lie groups and Lie algebras. About quantum mechanics, we will introduce Hilbert spaces and the basic structures of quantum mechanics, together with Wigner's theorem on symmetries. With respect to special relativity, we will present the Minkowski spacetime as an affine space and derive its group of automorphisms, the Poincaré group. We will finally talk about representations of Lie groups and define an elementary particle to be an irreducible projective representation of the Poincaré group.

Introduction

The interest in breaking down the matter of the physical world around us into its most elementary constituents has always been a central part of humanity's scientific study of nature. This work is occupied with the current understanding of mathematical physics about what those elementary constituents are. The goal of the work is to get to the definition of an elementary particle and to do so in a natural way. They are going to be defined as irreducible projective unitary representations of the Poincaré group and, if the goal of this work is accomplished, this definition will be a natural one. During the course of the work, we will introduce the objects that will allow us to reach that definition and the results upon which it is built. Some of the main results that we will need are Wigner's theorem on the symmetries of quantum systems and Bargmann's theorem about projective unitary representations, both of them central pieces of the mathematical study of physics.

Most of the results of this work are not original. Wigner's theorem dates from 1939 and the later contributions to the theory date from the central years of the last century. All of the material in this work is, however, original in its presentation, in the sense that it attempts to present those topics while keeping a strictly mathematical presentation. Discussion about the natural world is excluded from the presentation by such choices as defining a quantum system to *be* a complex Hilbert space, and thus making it clear that we are just concerned about the mathematical properties of our model. A similar situation happens in our discussion of Minkowski spacetime and references on it. The section about references in spacetime, in particular, presents the subject with a detail that is often omitted in special relativity textbooks. Overall, the present work manages to stay within the strict boundaries of the mathematical discourse by giving to certain mathematical objects the name of the physical reality that they are modeling.

It is also important to note the peculiarity in the approach that we have taken since the culminating point of the work is a definition. Usually, definitions are the starting point upon which one builds a richer theory by deriving results from them. It would have been a perfectly valid approach too if instead of spending all of our efforts into trying to define what an elementary particle is in a natural way, we would just have taken the definition as given and started building upon it. In such an approach, we would possibly have managed to get to Wigner's classification of elementary particles, which would have been an arguably stronger punchline for the work. However, since the background motivation for the work was always to *understand* how mathematics are used to model nature as understood by physics, motivating the definitions was of higher priority and the present slower and longer approach was preferred.

Lastly, this work serves as a basis and toolbox that allows the reader (and the author as well) to move on to the next topics in the study of the Standard Model of particle physics, such as gauge theory and the quantum theory of fields.

Structure of the work

- Chapter 1: We introduce the basics of differential geometry that will be needed during the rest of the work, including a concise introduction to Lie groups and Lie algebras.

- Chapter 2: After a short introduction to Hilbert spaces, the notions of quantum mechanics that we will work with are introduced as mathematical objects. We discuss Wigner's theorem on symmetries, a central result of the theory.

- Chapter 3: We will introduce the Minkowski spacetime as an affine space with a signature (1,3) bilinear form. The Poincaré group will be defined to be the group of bijections that preserve the structure on the Minkowski spacetime. A short discussion on reference frames in spacetime will follow.

- Chapter 4: We discuss unitary and projective unitary representations of Lie groups and present Bargmann's theorem as providing a link between them. Finally, we define an elementary particle to be a projective unitary representation of the Poincaré group, or equivalently, a unitary representation of its universal cover.

A word on the present work

My starting motivation for this work was to study and provide an account of J. Baez and J. Huerta's article [BH10] *"The Algebra of Grand Unified Theories"*. In the introductory part of this article, the authors present the algebraic structure underlying the Standard Model of particle physics and how the finite-dimensional representation theory of Lie groups comes into the picture. Lie groups were meant to model the symmetries of the fundamental interactions of physics (electromagnetic, weak nuclear and strong nuclear interactions, with gravity being left out) and the classification of their representations closely reproduced the classification of particles in the Standard Model. However, the reader who compares the outline of that article with the outline of the present work will quickly realize that they have nothing to do with one another. The reason for that is that after being engaged in studying the aforementioned article, it was quickly obvious that the amount of prerequisites and dedication needed to begin to understand the Standard Model was out of the scope of a master's thesis. Instead of that, my main goal for the work became to try and get as far as it was possible within the limits of the course.

With this motivation in mind, and after trying to break down the study of the Standard Model into smaller problems, some questions came up quite naturally. One of those questions was about the Standard Model and how it uses Lie groups as groups of symmetries, and it was just to ask: symmetries of what? This question led me naturally to the subject of elementary particles that will occupy the whole of this work. However, this has, by far, not been the only topic that I have been studying about the Standard Model. Some of my efforts have been directed toward studying gauge theory, which uses principal fiber bundles to model field theories in physics. In gauge theory, the mathematical objects of principal curvature and principal connection turn out to play a key role in physics by representing the fields and potentials themselves. Also, some basic ideas about quantum field theory were required to try to put all these things together into a basic understanding of how the Standard Model works. Also, the advanced topics that would start right at the end of this work (the representation theory of the Poincaré group through the theory of induced representations of G. Mackey) have used lots of effort in the late part of the work.

However, all those subjects (except for the one that will occupy us in this work: elementary particles) had to be left out for not having gathered enough understanding to write about them. This leads me to the last topic that I want to write about before starting with the proper contents of this work: the difficulty of doing physics from within the boundaries of mathematics. During the course of this work, I have constantly struggled with various difficulties that arise when trying to think about physics from within the domain of mathematics. In the first place, it is not obvious at all that such a task (doing physics mathematically) can be accomplished, as can be illustrated by the impossibility of the converse idea (that of doing pure mathematics from within the domain of physics). If the task of doing physics within the domain of mathematics can be carried out (as I think has been the case for this present work), it will be at the expense of both effort and amount of progress. The absolute precision that is required for a subject to be able to be treated within mathematics makes progress tough and greatly restricts how far one can get in a limited amount of time.

That said, the mathematical study of physics produces beautiful results (such as the ones presented in this work). That together with the fact that such fundamental notions in physics as that of an elementary particle can be satisfactorily defined as mathematical objects is a temptation that is hard to resist. The Standard Model itself is a great example of how studying physics from the mathematics side can produce beautiful descriptions of nature, where reality seems to be playing with Lie groups and their representations. I hope that by the end of this work, the reader will feel some of this curiosity to know more about the things that ensue when mathematics tries to describe nature.

Aknowledgements

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1

Review of differential geometry

In this chapter, we are going to review fundamental notions of differential geometry that we will need during the rest of the work. They can be found in most textbooks on the subject and references will be provided for more extensive and pedagogic treatments of the subject. The goal of the chapter is to provide a brief review to settle the notation and to make sure there is some degree of self-containment in the present work.

In the first section, we define the notion of a smooth manifold and introduce some of the structure that comes with it, such as smooth maps, tangent vectors and differentials. In the second section, we introduce the slightly more advanced idea of a fiber bundle on a smooth manifold. This allows us to define tangent bundles, sections and also the metric tensor. Lastly, in the third section, we will briefly present some topics in Lie theory that are going to be necessary for the coming chapters.

1.1 Smooth manifolds and tangent vectors

One of the main purposes of differential geometry is to provide tools for doing calculus on spaces that locally look like $\mathbb{R}^n$ Euclidean spaces but are not necessarily the same. The study of differential geometry allows us to bring such familiar notions as those of derivatives, integrals, curves, surfaces, distances, etc. into a larger class of spaces than just Euclidean spaces. Those more general spaces are called smooth manifolds, and they are the central topic of the section. Some standard references on the subject are [Lee13], [dC92] and [O'N83].

From now on, by a **smooth function** we mean a real function $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, with U open in the Euclidean topology, so that all partial derivatives of f of all orders exist and are continuous. Thus,

Definition 1.1.1. *A* ***smooth n-dimensional manifold*** *is a tuple $(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\}_{\alpha\in I})$, where:*

(i) $\mathcal{M}$ is a set.

(ii) $U_\alpha \subseteq \mathbb{R}^n$ are open subsets for all $\alpha \in I$ (with the euclidean topology).

(iii) For all $\alpha \in I$, $\phi_\alpha : U_\alpha \longrightarrow \mathcal{M}$ is an injective map.

(iv) If there are $\alpha, \beta \in I$ so that $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_b^{-1} \circ \phi_\alpha : U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^n$ is a smooth map.

We say that $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ is an **atlas** for $\mathcal{M}$ and that (U_α, ϕ_α) is a **local chart** or a local coordinate system.

A smooth manifold possesses a natural topology induced by the charts. We define a subset $W \subseteq \mathcal{M}$ to be open if and only if $\forall \alpha \in I$ such that $\phi_\alpha(U_\alpha) \cap W \neq \emptyset$, then $\phi_\alpha^{-1}(\phi_\alpha(U_\alpha) \cap W) \subseteq \mathbb{R}^n$ is an open set in the euclidean sense. This gives rise to a topology on $\mathcal{M}$ since $\mathcal{M}$ and $\emptyset$ are open and also:

(i) If $\{W_i\}_{i\in I}$ are all open sets of $\mathcal{M}$, then $W = \bigcup_i W_i$ is also an open set, since if $U_\alpha \cap W \neq \emptyset$ then:

$$\phi_\alpha^{-1}(U_\alpha \cap W) = \phi_\alpha^{-1}\Big(\bigcup_i (W_i \cap U_\alpha)\Big) = \bigcup_i \phi_\alpha^{-1}(W_i \cap U_\alpha) \tag{1.1}$$

which is a union of open sets in $\mathbb{R}^n$ and is therefore open.

(ii) If $\{W_i\}_{i=1}^n$ is a finite collection of open sets of $\mathcal{M}$, then $W = \bigcap_{i=1}^n W_i$ is also an open set, since if α is such that $U_\alpha \cap W \neq \emptyset$ then:

$$\phi_\alpha^{-1}(W \cap U_\alpha) = \phi_\alpha^{-1}(\bigcap_{i=1}^{n}(W_i \cap U_\alpha)) = \bigcap_{i=1}^{n} \phi_\alpha^{-1}(W_i \cap U_\alpha) \tag{1.2}$$

which is also open in the Euclidean topology.

Notice that with this induced topology the charts are continuous maps and the sets $\phi_\alpha(U_\alpha)$ are open sets. This topology is an example of a final topology induced by the collection of topological spaces $\{U_\alpha\}_\alpha$ and the maps $\{\phi_\alpha\}_\alpha$.

Example 1.1.2. A trivial example of a smooth n-manifold is just $\mathbb{R}^n$ with the atlas consisting of only one chart, $(\mathbb{R}^n, \mathrm{id})$, where $\mathrm{id} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map.

Another less trivial example of a manifold is the sphere $\mathcal{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, together with the atlas $\{(\mathbb{R}^n, \phi_N), (\mathbb{R}^n, \phi_S)\}$ formed by two stereographic projections of the sphere through the north and through the south pole, for example. The north pole projection $\phi_N : \mathbb{R}^n \longrightarrow \mathcal{S}^n \setminus \{(0, \dots, 0, 1)\} \subseteq \mathbb{R}^{n+1}$ can be given as the inverse of the map

$$\psi_N(y_1, \dots, y_{n+1}) = (\frac{y_1}{1 - y_{n+1}}, \dots, \frac{y_n}{1 - y_{n+1}}) \tag{1.3}$$

As we said above, one of the main purposes of differential geometry is to be able to bring the familiar notions of real analysis into a more general kind of domains than just subsets of $\mathbb{R}^n$, that is, to smooth manifolds. To do that, we have to define some basic notions such as what will we mean by "smooth functions" on the manifold or between different manifolds.

Definition 1.1.3. *Let $(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I})$ be a smooth n-manifold and let $W \subseteq \mathcal{M}$ be an open set. We will say that a function $f : W \longrightarrow \mathbb{R}^m$ is smooth at a point $p \in W$ if for every chart (U_α, ϕ_α) so that $p \in U_\alpha$, then the function $f \circ \phi_\alpha^{-1} : U_\alpha \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is smooth in the usual sense. We will say that f is smooth on W if it is smooth at all points of W.*

A particular instance of a smooth function that we will use are smooth real functions. We will denote $C_p^\infty(\mathcal{M})$ the set of smooth functions from some open set $W \subseteq \mathcal{M}$ containing p to the real numbers, $f : W \longrightarrow \mathbb{R}$. In the same vein, we can define what we mean by a smooth function between smooth manifolds.

Definition 1.1.4. *Let $(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A})$ be a smooth m-manifold and $(\mathcal{N}, \{(V_\beta, \psi_\beta)\}_{\beta \in B})$ be smooth n-manifold. A function $f : W \subseteq \mathcal{M} \longrightarrow \mathcal{N}$ is* *said to be smooth at* $p \in W$ *if for every chart* (U_α, ϕ_α) *containing* p *and for every chart* (V_β, ψ_β) *containing* $f(p)$*, the composition* $\psi_\beta^{-1} \circ f \circ \phi_\alpha : W \cap U_\alpha \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *is smooth in the usual sense.*

Remark 1.1.5. It is noteworthy that all the above definitions could be rewritten in the following manner. We say, for example, that $f : \mathcal{M} \longrightarrow \mathbb{R}$ is smooth at $p \in \mathcal{M}$ when for every chart (U_α, ϕ_α) where $p \in U_\alpha$ we have that $f \circ \phi_\alpha : U_\alpha \longrightarrow \mathbb{R}$ is smooth at $\phi_\alpha^{-1}(p)$ in the usual sense. But it is enough for us to just ask $f \circ \phi_\alpha$ to be differentiable in the usual sense for *one* chart, and then the definition of a smooth manifold will give the differentiability of $f \circ \phi_\beta$ for any other chart where $p \in U_\beta$.

After introducing this basic structure and definitions, we must now introduce how are we going to do calculus in those smooth manifolds that we have just defined. How are we going to locally describe functions in this manifold? The key to this question is the introduction of the tangent spaces to the manifold. This definition is motivated by the notion of tangent space to a surface embedded in $\mathbb{R}^n$, where one can intuitively define them. Since giving a proper introduction to differential geometry is not the goal of this section, and since there are multiple ways to motivate the definition of the tangent space in the more general setting of a smooth manifold (where we cannot use any of the properties of the space where the manifold is embedded because we are not assuming that such an embedding exists), we will simply give the definition below and briefly motivate it afterward. For a complete discussion of the different ways in which one can think about tangent vectors, see [Spi79].

Definition 1.1.6. *Let* $(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\})$ *be a smooth manifold and* $p \in \mathcal{M}$*. Denote as before* $C_p^\infty(\mathcal{M}) = \{f : W \longrightarrow \mathbb{R} : W \subseteq \mathcal{M} \text{ open and } p \in W,\ f \text{ smooth}\}$*. This set is a real vector space. A* ***derivation*** v *of* $\mathcal{M}$ *at the point* p *is a linear map* $v : C_p^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ *that satisfies the following condition: given* $f, g \in C_p^\infty(\mathcal{M})$*, then*

$$v(fg) = f(p)v(g) + g(p)v(f) \tag{1.4}$$

We will denote as $T_p\mathcal{M}$, the tangent space at p, the set of all such derivations and call its elements **tangent vectors** at p. Given $v, w \in T_p\mathcal{M}$, λ, $\mu \in \mathbb{R}$ and $f \in C_p^\infty(\mathcal{M})$, define the derivation

$$(\lambda v + \mu w)(f) = \lambda v(f) + \mu w(f). \tag{1.5}$$

That turns $T_p\mathcal{M}$ into a real vector space.

Remark 1.1.7. Above, it is clear that fg is just the product of the two functions, which is also smooth.

The following example of tangent vectors will help motivate the previous definition, which is a bit obscure if seen for the first time.

Example 1.1.8. Let $(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\})$ be a smooth n-manifold. Given a chart (U_α, ϕ_α), we will define:

(i) The functions defined, for $i = 1, \dots, n$, $\phi_\alpha^i : \phi_\alpha(U_\alpha) \longrightarrow \mathbb{R}$, so that $\phi_\alpha^{-1}(p) = (\phi_\alpha^1(p), \dots, \phi_\alpha^n(p)) \in \mathbb{R}^n$. That is, the ϕ_α^i are the coordinates of the chart.

(ii) For $p \in \phi_\alpha(U_\alpha)$, for $i = 1, \dots, n$, the derivations $(\partial_i \phi_\alpha)_p : C_p^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ defined by

$$(\partial_i \phi_\alpha)_p(f) = \frac{d}{dt} f \circ \phi_\alpha(\phi_\alpha^1(p), \dots, \phi_\alpha^i(p)+t, \dots, \phi_\alpha^n(p)), \qquad \forall f \in C_p^\infty(\mathcal{M}) \tag{1.6}$$

The functions from i) are smooth, and the maps from ii) are derivations. Then, one can see (rather non-trivially) that for a given $v \in T_p\mathcal{M}$, with $p \in \phi_\alpha(U_\alpha)$, we can write:

$$v(f) = \sum_i v(\phi_\alpha^i) \cdot (\partial_i \phi_\alpha)_p(f), \qquad \forall f \in C_p^\infty(\mathcal{M}) \tag{1.7}$$

and see from here that $v = \sum_i v(\phi_\alpha^i)(\partial_i \phi_\alpha)_p$. Since this decomposition can be done uniquely, we conclude that $\{(\partial_1 \phi_\alpha)_p, \dots, (\partial_n \phi_\alpha)_p\}$ is a basis for $T_p\mathcal{M}$.

The next concept that we will introduce is that of the differential of a map.

Definition 1.1.9. *Let $(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A})$ be a smooth m-manifold and $(\mathcal{N}, \{(V_\beta, \psi_\beta)\}_{\beta \in B})$ be smooth n-manifold. Let $\varphi : W \subseteq \mathcal{M} \longrightarrow \mathcal{N}$ be a smooth function and $p \in W$. Then, the differential of φ at the point p, denoted by $d_p\varphi$, is a linear map between the tangent spaces $d_p\varphi : T_p\mathcal{M} \longrightarrow T_{\varphi(p)}\mathcal{N}$, defined as follows. For $v \in T_p\mathcal{M}$, then $d_p\varphi(v) : C_{\varphi(p)}^\infty(\mathcal{N}) \longrightarrow \mathbb{R}$ is the derivation given by:*

$$d_p\varphi(v)(f) = v(f \circ \varphi), \qquad \forall f \in C_{\varphi(p)}^\infty(\mathcal{N}) \tag{1.8}$$

This definition works since $f \circ \varphi : W \subseteq \mathcal{M} \longrightarrow \mathbb{R}$.

1.2 Fiber bundles

In this section we are going to set up the basic structures that we will need to talk about vector fields later on, mainly to discuss Lie algebras. This structure is that of a fiber bundle.

Definition 1.2.1. *Let $\mathcal{E}$, $\mathcal{M}$, $\mathcal{F}$ be smooth manifolds and let $\pi : \mathcal{E} \longrightarrow \mathcal{M}$ be a surjective smooth function. A* ***smooth fiber bundle*** *is then the tuple $(\mathcal{E}, \mathcal{M}, \mathcal{F}, \pi, \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A})$, where:*

(i) $\bigcup_\alpha U_\alpha = \mathcal{M}$ *is an open cover of* $\mathcal{M}$

(ii) For any $\alpha \in A$, $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathcal{F}$ is a diffeomorphism and the restriction to $\{x\} \subseteq U_\alpha$ is a diffeomorphism $\psi_{\alpha|x} : \pi^{-1}(\{x\}) \longrightarrow \{x\} \times \mathcal{F}$.

In this context, $\mathcal{E}$ is called the fiber bundle manifold, $\mathcal{M}$ is called the base manifold, $\mathcal{F}$ is the typical fiber, π is the bundle projection map and $\{(U_\alpha, \psi_\alpha)\}$ are the local trivializations. We will often just denote by $\mathcal{E}_x = \pi^{-1}(x)$ the fiber at x.

Example 1.2.2. As an example of a fiber bundle, we can construct a trivial bundle. Let $\mathcal{M}$ and $\mathcal{F}$ be smooth manifolds and define $\mathcal{E} = \mathcal{M} \times \mathcal{F}$, with the product smooth structure. Then, define $\pi : \mathcal{E} \longrightarrow \mathcal{M}$ to be just the projection of the first component, $\pi(x, u) = x$, $\forall (x, u) \in \mathcal{M} \times \mathcal{F} = \mathcal{E}$. One can consider the local trivialization given by (U, ψ), where $U = \mathcal{M}$ is just the whole base manifold and $\psi : \pi^{-1}(U) \longrightarrow U \times \mathcal{F}$ is just the identity map, since $\pi^{-1}(U) = \mathcal{E}$ by definition.

Definition 1.2.3. *Let $(\mathcal{E}, \mathcal{M}, \mathcal{F}, \pi)$ be a smooth fiber bundle. A* ***section*** *of the fiber bundle is a smooth map $\sigma : U \subseteq \mathcal{M} \longrightarrow \mathcal{E}$ so that $\pi \circ \sigma = \mathrm{id}$. We will denote the set of sections $\sigma : U \subseteq \mathcal{M} \longrightarrow E$ as $\Gamma(U, E)$ or just as $\Gamma(E)$ when the base manifold or domain is understood.*

That is, $\forall x \in U$, $\sigma(x) \in \mathcal{E}_x$, the image of the section belongs to the fiber at that point. The next natural notion that we usually want to define after introducing a new object is that of homomorphisms between those objects.

Definition 1.2.4. *Let $(\mathcal{E}, \mathcal{M}, \mathcal{F}, \pi)$ and $(\mathcal{E}', \mathcal{M}, \mathcal{F}', \pi')$ be smooth bundles with the same base manifold $\mathcal{M}$ (we will omit the local trivializations from now on). A bundle homomorphism is then a smooth map $f : \mathcal{E} \longrightarrow \mathcal{E}'$ so that $\pi' \circ f = \pi$ (i.e, it maps the fiber $\mathcal{E}_x = \pi^{-1}(x)$ to the fiber $\mathcal{E}'_x = \pi'^{-1}(x)$). If f is a diffeomorphism of smooth manifolds, then we will say that those fiber bundles are isomorphic.*

We will leave it here, for now, concerning bundle homomorphisms. It is important to note, however, that they play a key role in modern physics in that they encode the idea of a gauge transformation. This topic, though, is out of the scope of the work.

Let us now introduce the different kinds of fiber bundle structures that we can define based on the previous definition and that we will use from now on. We will consider fiber bundles where the typical fiber is, apart from a smooth manifold, also a vector space.

Definition 1.2.5. *A* ***vector bundle*** *is a smooth fiber bundle* $(\mathcal{E}, \mathcal{M}, \mathcal{V}, \pi, \{(U_\alpha, \psi_\alpha)\}_{\alpha\in A})$ *so that the typical fiber* $\mathcal{V}$ *is also a (real or complex) vector space. Then, the restrictions induced by the local trivializations* $\psi_{\alpha|x} : \pi^{-1}(x) \longrightarrow \mathcal{V}$ *are also vector space isomorphisms.*

Example 1.2.6. Let $(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\}_\alpha)$ be a smooth n-manifold. We will construct a vector bundle with $\mathcal{M}$ as its base manifold by considering the tangent spaces $T_p\mathcal{M}$ at each point. We will define the vector bundle $\mathcal{E}$ with projection $\pi : \mathcal{E} \longrightarrow \mathcal{M}$ as:

$$\mathcal{E} = \bigcup_{x\in\mathcal{M}} \{x\} \times T_x\mathcal{M}, \qquad \pi(x, v) = x, \forall x \in \mathcal{M}, \forall v \in T_x\mathcal{M}$$

We can see that the typical fiber is isomorphic to $\mathbb{R}^n$. We will denote $T\mathcal{M} = \mathcal{E}$ for the tangent bundle from now on. Then, we have the vector bundle given by $(T\mathcal{M}, \mathcal{M}, \mathbb{R}^n, \pi)$. For the local trivializations, we can see that the charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$ of $\mathcal{M}$ induce a set of trivializations $\{(U_\alpha, \psi_\alpha)\}_\alpha$ where:

$$\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n, \qquad \psi_\alpha(x, v) = (x, (v(\phi_\alpha^1(v)), \ldots, \phi_\alpha^n(v)))$$

Where ϕ_α^i are the coordinate functions of the chart.

Note also that from the tangent bundle constructed above, one can also consider the cotangent bundle (denoted as $T^*\mathcal{M}$) by simply considering the cotangent spaces (the dual vector spaces to the tangent spaces), denoted as $T_x^*\mathcal{M}$.

Example 1.2.7. Let $\mathcal{M}$ be a smooth n-manifold. For a given point $p \in \mathcal{M}$, we define the cotangent space to be the dual space of the tangent space. We denote it by $T_p^*\mathcal{M}$. We define the cotangent bundle to be the vector bundle $(T^*\mathcal{M}, \mathcal{M}, \pi)$, where

$$T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} \{p\} \times T_p^*\mathcal{M} \tag{1.9}$$

and where π is just the usual projection onto the base manifold.

Given a local chart (U, ϕ) for the manifold $\mathcal{M}$ and $p \in U$, we can consider the induced basis of the tangent space $T_p\mathcal{M}$, $\{(\partial_i\phi)_p\}_{i=1}^n$. We can denote the basis of the cotangent space that is dual to this tangent basis as $\{(d\phi^i)_p\}_{i=1}^n$, that is, so that $(d\phi^i)_p((\partial_j\phi)_p) = \delta_{ij}$.

From those two, one can form arbitrary tensor bundles as follows:

Example 1.2.8. Given a smooth n-manifold $\mathcal{M}$, we can consider the (r,s)-tensor bundle, denoted by $T^{r,s}\mathcal{M}$, by just considering the tensor product of tangent and cotangent spaces as follows:

$$T^{r,s}\mathcal{M} = \bigcup_{x\in\mathcal{M}} \{x\} \times \left(\underbrace{T_x\mathcal{M} \otimes \cdots \otimes T_x\mathcal{M}}_{r} \otimes \underbrace{T_x^*\mathcal{M} \otimes \cdots \otimes T_x^*\mathcal{M}}_{s} \right) \tag{1.10}$$

This is also a vector bundle over $\mathcal{M}$. It has $\bigotimes_1^{r+s}\mathbb{R}^n$ as its typical fiber. The projection and the trivializations can be constructed from the charts as in the previous example.

As a commentary, sections of those tensor bundles over a manifold that represents a spacetime are used to represent physical fields in general relativity and classical field theories. We now briefly turn to the topic of semi-Riemannian geometry. We have delayed the introduction of metrics into our treatment of differential geometry until after we introduced the notion of a fiber bundle so that we can use those notions when thinking about vector and tensor fields.

Definition 1.2.9. *Let $\mathcal{M}$ be a smooth n-manifold. A **tensor field** of rank (r,s) is a section of the vector bundle $(T^{r,s}\mathcal{M}, \mathcal{M}, \pi)$.*

Therefore, we can think of a tensor field $T \in \Gamma(T^{r,s}\mathcal{M})$ as a function associating to every point, an element of its tensor tangent space $T(p) \in T_p^{r,s}\mathcal{M}$.

Definition 1.2.10. *Given a local chart (U,ϕ) and the induced tangent basis at every $p \in \phi(U)$, $\{(\partial_i\phi)_p\}_{i=1}^n$, we define the components of a tensor field $T \in \Gamma(T^{r,s}\mathcal{M})$ to be the functions*

$$T^{i_1,\dots,i_r}_{j_1,\dots,j_s} : \mathcal{M} \longrightarrow \mathbb{R} \tag{1.11}$$

given by

$$T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(p) = T\left((d\phi^{i_1})_p, \dots, (d\phi^{i_r})_p, (\partial_{j_1}\phi)_p, \dots, (\partial_{j_s}\phi)_p\right) \tag{1.12}$$

The components of the tensor field are smooth functions since sections are smooth maps. Then, we can see that we can write

$$T(p) = \sum_{i_1}\cdots\sum_{i_r}\sum_{j_1}\cdots\sum_{j_s} T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(p)(\partial_{i_1}\phi)_p\otimes\dots\otimes(\partial_{i_r}\phi)_p\otimes(d\phi^{j_1})_p\otimes\dots\otimes(d\phi^{j_s})_p \tag{1.13}$$

Now, we will introduce the notion of a metric.

Definition 1.2.11. *Let $\mathcal{M}$ be a smooth n-manifold. A metric g on $\mathcal{M}$ is a tensor field $g \in \Gamma(T^{0,2}\mathcal{M})$ so that $g(p) \in T_p^{0,2}\mathcal{M}$ is symmetric and non-degenerate at every point $p \in \mathcal{M}$. A manifold together with a metric, $(\mathcal{M}, g)$, is called a **semi-Riemannian manifold**.*

Thus, at a given point, the metric tensor is just a symmetric non-degenerate bilinear form on the tangent space. We will give more details about symmetric non-degenerate bilinear forms further on.

1.3 Lie groups and Lie algebras

After this review of the fundamental objects of differential geometry, we can give a brief introduction to the subject of Lie theory. The objects introduced in this section will be used extensively in the last chapters of the work. The key importance of Lie theory in mathematical physics is that they are used to model the idea of a smooth symmetry group. From that approach, a Lie group is the object that allows us to talk about symmetries (the idea at the core of a *group*) in a smooth way (the idea behind calculus) by treating the group of symmetries geometrically.

In this section, we will start by introducing the definitions of a Lie group and a Lie algebra. We will give some examples and results about their mutual connection. Lastly, we will talk about the notion of a universal cover of a Lie group.

Definition 1.3.1. *A **Lie group** is a (finite-dimensional) smooth manifold G that is also a group $(G, *)$ so that the group structure is smooth. That is, the group operations $* : G \times G \longrightarrow G$ and $^{-1} : G \longrightarrow G$ are smooth maps.*

Remark 1.3.2. A Lie group, since it is, in particular, a group, can have any property related to groups: it can be abelian, solvable, etc. Since it is a smooth manifold, it can also have any property related to manifolds: it has a dimension, a topology, it can be compact, connected, etc.

Perhaps the simplest examples of Lie groups are matrix Lie groups.

Definition 1.3.3. *A **matrix Lie group** is a subgroup of $\mathrm{GL}_n(\mathbb{C})$, for some $n \in \mathbb{N}$ which is also closed with respect to the topology induced by the smooth structure of $\mathrm{GL}_n(\mathbb{C})$. The smooth structure of $\mathrm{GL}_n(\mathbb{C})$ is given by the chart $\varphi : \mathbb{R}^{2n^2} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ that associates each of the n^2 pairs of real numbers (a_i, b_i) to the complex entry of the matrix $a_i + ib_i$. Note here that $\mathrm{GL}_n(\mathbb{C})$ is a Lie group (and a matrix*

Lie group too).

Given a n-dimensional vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, its group of automorphisms, denoted by $\mathrm{Aut}(V)$, is isomorphic to $\mathrm{GL}_n(\mathbb{K})$. Similarly, its (additive) group of endomorphisms (linear maps not necessarily isomorphisms), denoted by $\mathrm{End}(V)$, is isomorphic to $\mathrm{M}_n(\mathbb{K})$.

Example 1.3.4. Remember that if $f : \mathrm{GL}_n(\mathbb{K}) \longrightarrow \mathrm{GL}_m(\mathbb{K})$ is a continuous function ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$), then the preimage of a closed set is going to be again a closed set. That is a way in which we can see whether some subgroups of the general linear group are in fact matrix Lie groups.

(i) We can see that the set of matrices with determinant one, the $\mathrm{SL}(n, \mathbb{K}) = \{A \in \mathrm{GL}_n(\mathbb{K}) : \det(A) = 1\}$, named **special linear group**, is a matrix Lie group. Indeed, one can check that it is a group under matrix multiplication, and further, the map

$$\begin{aligned} \det : \mathrm{GL}(\mathbb{R}) &\longrightarrow \mathbb{K} \\ A &\mapsto \det(A) \end{aligned} \tag{1.14}$$

is continuous and $\det^{-1}(1) = \mathrm{SL}(n, \mathbb{K})$.

(ii) Similarly, the set of orthogonal matrices $\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^{-1} = A^T\}$ is a matrix Lie group too, named the **orthogonal group**, since one can check that products and inverses of orthogonal matrices are still orthogonal and if one considers the map

$$f : \mathrm{GL}(\mathbb{R}) \longrightarrow \mathrm{GL}(\mathbb{R}) \tag{1.15}$$

$$A \mapsto A^T A \tag{1.16}$$

then f is continuous and $\mathrm{O}(n) = f^{-1}(Id)$. We can also consider the subgroup $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$, which is also a matrix Lie group and is named the **special orthogonal group**.

(iii) A last example, which generalizes the previous one and we will use further on, is the set $O(n, m) = \{A \in \mathrm{GL}_{n+m}(\mathbb{R}) : A^T I_{n,m} A = I_{n,m}\}$, where $I_{n,m} = \mathrm{diag}(+1, \dots, +1, -1, \dots, -1) \in \mathrm{GL}_{n+m}(\mathbb{R})$ with n positive entries and m negative entries. This is also a matrix Lie group, as can be seen by using the map $f(A) = A^T I_{n,m} A$ and seeing that $\mathrm{O}(n, m) = f^{-1}(I_{n,m})$. Similarly, $\mathrm{SO}(n, m) = \mathrm{O}(n, m) \cap \mathrm{SL}(n + m, \mathbb{R})$ is also a matrix Lie group. They are called respectively the **(n,m) pseudo-orthogonal group** and the **special (n,m) pseudo-orthogonal group**.

Just as with any smooth manifold, we can consider the tangent spaces $T_g G$ at a point $g \in G$ and the tangent and cotangent bundles TG and T^*G (to be properly introduced later on). We can consider vector fields too, that is, elements of $\chi(G)$. There is a very important kind of vector fields, the ones that are compatible with the group structure of G. Basically, for every $g \in G$ we have a canonically defined diffeomorphism in G which we call the left translation map:

Definition 1.3.5. *Let G be a Lie group. Given $g \in G$, we have the left g-translation map $l_g : G \longrightarrow G$ given by $l_g(h) = gh$. This is a smooth map (since group multiplication is smooth) and we can also consider its differential $d_h l_g : T_h G \longrightarrow T_{gh} G$*

The subset of the vector fields in G that will be compatible with the group structure is precisely the ones that commute with the action of this left-translation map.

Definition 1.3.6. *Given a Lie group G and a smooth vector field $X \in \chi(G)$, we will say that X is a* ***left-invariant vector field*** *if for every $g, h \in G$ we have that $d_g l_h(X_g) = X_{hg}$. That is, if the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "l_h"] \arrow[d, "X"] & G \arrow[d, "X"] \\ T_g G \arrow[r, "d_g l_h"] & T_{hg} G \end{tikzcd} \tag{1.17}$$

This set of left-invariant vector fields inherits a fundamental structure of the vector fields, that of being a Lie algebra.

Definition 1.3.7. *A* ***Lie algebra*** *is a vector space V over the real or complex numbers field together with a binary operation $[\,,\,] : V \times V \longrightarrow V$, called a* ***Lie bracket****, that satisfies the following properties:*

i) It is bilinear.

ii) It is antisymmetric.

iii) It satisfies the Jacobi identity, $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$.

Then, a vector space with such an operation is called a Lie algebra $(V, [\,,\,])$.

The vector fields over a smooth manifold naturally possess a Lie algebra structure in the following manner. Remember that given a smooth manifold $\mathcal{M}$, a vector field $X \in \chi(M)$ is at every point $p \in \mathcal{M}$ a map $X_p : C_p^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ with certain properties that make them analogous to tangent vectors to curves in the manifold. In particular, vector fields can be added and multiplied by real numbers, so they form a vector space.

Proposition 1.3.8. *Let $\mathcal{M}$ be a smooth manifold and consider the vector space of smooth vector fields $\chi(\mathcal{M})$. Then, given a vector field $X \in \chi(\mathcal{M})$ and a smooth function $f \in C^\infty(\mathcal{M}, \mathbb{R})$, we can define the function:*

$$X \bullet f \in C^\infty(\mathcal{M}, \mathbb{R}) : \qquad (X \bullet f)(p) = X_p(f) \in \mathbb{R}, \ \forall p \in \mathcal{M} \tag{1.18}$$

Therefore, we can define the Lie bracket $[\ ,\ ] : \ \chi(\mathcal{M}) \times \chi(\mathcal{M}) \longrightarrow \chi(\mathcal{M})$ *as:*

$$[X, Y]_p(f) = X(Y \bullet f) - Y(X \bullet f), \qquad \forall X, Y \in \chi(\mathcal{M}), \ \forall f \in C^\infty(M, \mathbb{R}) \tag{1.19}$$

This operation satisfies the conditions above and therefore turns $(\chi(\mathcal{M}), [\ ,\ ])$ *into a Lie algebra.*

Then, we can now see that while for a Lie group G we still have this Lie bracket structure in the space of its vector fields $\chi(G)$, we also have the set of left-invariant vector fields defined above (those fields that commute with the group operation) and that, further, it has the crucial property that this Lie bracket turns it into a Lie subalgebra of $\chi(G)$. This Lie subalgebra will be the Lie algebra of G, denoted as $\mathrm{Lie}(G)$. The reason why it is $\mathrm{Lie}(G)$ that we regard as the important Lie algebra for our purposes instead of the bigger algebra $\chi(G)$ is that, as we will see, $\mathrm{Lie}(G)$ turns out to be isomorphic as a Lie algebra to T_eG, the tangent space at the identity element of G. That means that $\mathrm{Lie}(G)$ encodes in some sense the local smooth structure of G. We will now make those statements precise.

Proposition 1.3.9. *Given a Lie group G, the left-invariant vector fields on G are a Lie subalgebra of $\chi(G)$.*

Proof. It is immediate to check that left-invariant vector fields form a vector space, since if $X, Y \in \chi(G)$ are left-invariant and $\lambda, \mu \in \mathbb{R}$, then by the linearity of the differential,

$$d_g l_h(Z_g) = d_g l_h(\lambda X_g + \mu Y_g) = \lambda d_g l_h(X_g) + \mu d_g l_h(Y_g) = \lambda X_{hg} + \mu Y_{hg} = Z_{hg}$$

We only need to check that if X, Y are left-invariant, then $[X, Y]$ is also left-invariant. But this is also true, since by expanding the expression $d_g l_h([X, Y]_g)(f)$ for given $X, Y \in \chi(G)$ left-invariant, $g, h \in G$ and $f \in C^\infty(G, \mathbb{R})$, we get:

$$\begin{aligned} &d_g l_h([X, Y]_g)(f) = [X, Y]_g(f \circ l_h) = X(Y \bullet (f \circ l_h)) + Y(X \bullet (f \circ l_h)) = \\ &= d_g l_h(X_g)(Y \bullet f) + d_g l_h(Y_g)(X \bullet f) = X_{hg}(Y \bullet f) + Y_{hg}(X \bullet f) = [X, Y]_{hg}(f) \end{aligned}$$

And so, $\mathrm{Lie}(G)$ is indeed a Lie algebra. $\square$

Now we can finally see that this Lie algebra of invariant vector fields is indeed isomorphic to the tangent space at the identity. We can see that for every $g \in G$, its tangent space T_gG is also a Lie algebra by just inheriting the Lie bracket from $\chi(G)$. That is, if $X, Y \in T_gG$, we can define the tangent vector $[X, Y] \in T_gG$ as acting on a smooth function $f \in C_g^\infty(G)$ as

$$[X, Y](f) = X(Y \bullet f) - Y(X \bullet f). \tag{1.20}$$

Proposition 1.3.10. *Let G be a Lie group and Lie(G) its Lie algebra, that is, the Lie algebra of left-invariant vector fields. Then, considering T_eG as a Lie algebra with the inherited Lie bracket from $\chi(G)$, we have a Lie algebra isomorphism Lie$(G) \cong T_eG$.*

Proof. For this, we only need to consider the left-translation map again. Let the map $\phi : T_eG \longrightarrow \chi(G)$ be defined as

$$\phi(X_e)_g = d_e l_g(X_e) \in T_gG \tag{1.21}$$

Then, $\phi(X_e)$ is a vector field in G. It is linear by the properties of the differential. We can see that $\phi(X_e)$ is left-invariant for every $X_e \in T_eG$, since by using the definitions we have:

$$d_g l_h(\phi(X_e)_g) = d_g l_h(d_e l_g(X_e)) = d_e(l_h \circ l_g)(X_e) = d_e l_{hg}(X_e) = \phi(X_e)_{hg} \tag{1.22}$$

So that $\mathrm{Im}(\phi) \subseteq \mathrm{Lie}(G)$. But further, we can see that indeed $\mathrm{Im}(\phi) = \mathrm{Lie}(G)$ by seeing that the map $\phi^{-1} : \mathrm{Lie}(G) \longrightarrow T_eG$ defined as

$$\phi^{-1}(X) = X_e \tag{1.23}$$

is indeed an inverse of ϕ, since trivially we have that $\phi^{-1} \circ \phi(X_e) = X_e$ and because $X \in \mathrm{Lie}(G)$ we have that

$$(\phi \circ \phi^{-1}(X))_g = \phi(X_e)_g = d_e l_g(X_e) = X_{ge} = X_g \tag{1.24}$$

Finally, it is obvious to see that $[\phi(X_e), \phi(Y_e)] = \phi([X_e, Y_e])$, where the bracket on the left-hand side refers to the bracket in $\mathrm{Lie}(G)$ and the one on the right refers to the bracket in T_eG. But since one bracket is induced by the other, it is easy to see that it is preserved under the map ϕ.

Therefore, $\phi : T_eG \longrightarrow \mathrm{Lie}(G)$ is a Lie algebra isomorphism. □

Example 1.3.11. Using any of the equivalent definitions of the Lie algebra of a Lie group that we have seen (as the subalgebra of invariant vector fields or as the tangent space at the identity), we can see that the Lie algebra of the general linear group $\mathrm{GL}_n(\mathbb{K})$ is just $M_n(\mathbb{K})$, the set of all n by n matrices.

Lie group homomorphisms induce Lie algebra homomorphisms between their respective Lie algebras.

Proposition 1.3.12. *Let G and H be Lie groups with Lie algebras* $\mathrm{Lie}(G)$ *and* $\mathrm{Lie}(H)$. *Let $\phi : G \longrightarrow H$ be a Lie group homomorphism. Then, the differential at the identity is a Lie algebra homomorphism:*

$$d_e\phi : T_eG \cong \mathrm{Lie}(G) \longrightarrow T_eH \cong \mathrm{Lie}(H) \tag{1.25}$$

There is a very natural way to pass from the Lie algebra of a Lie group to the Lie group itself and it is given by the exponential map:

Theorem 1.3.13. *Given a Lie group G and a left-invariant vector field $X \in$* $\mathrm{Lie}(G)$, *there is a unique one-dimensional subgroup $G_X \subseteq G$ satisfying the property that $\forall g \in G_X$, $X_g \in T_gG_X \subseteq T_gG$.*

A proof for the above theorem can be found on [Lee13] (Chapter 20, Theorem 20.1).

Remark 1.3.14. This result is also usually stated as saying that given $X \in \mathrm{Lie}(G)$ there exists a unique smooth group homomorphism $\phi : \mathbb{R} \longrightarrow G$ so that $\frac{d}{ds}\phi(s)|_t = X_{\phi(t)}$. That is, ϕ is an integral curve of X.

We will call this map the exponential map:

Definition 1.3.15. *Let G be a Lie group with Lie algebra* $\mathrm{Lie}(G)$. *We define the exponential map* $\exp : \mathrm{Lie}(G) \longrightarrow G$ *as*

$$\exp(X) = \phi_X(1) \tag{1.26}$$

Where $\phi_X : \mathbb{R} \longrightarrow G$ is the integral curve of the vector field X defined above.

Now, we can see that the Lie algebra homomorphism induced by a Lie group homomorphism preserves the exponential map:

Proposition 1.3.16. *Let, as above, G, H be Lie groups and* $\mathrm{Lie}(G), \mathrm{Lie}(H)$ *their Lie algebras. Let $\varphi : G \longrightarrow H$ be a Lie group homomorphism and $d_e\varphi :$* $\mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$ *the induced Lie algebra homomorphism. Then, if* $\exp_G, \exp_H$ *are the corresponding exponential maps, we have that $\forall X \in$* $\mathrm{Lie}(G)$,

$$\varphi \circ \exp_G(X) = \exp_H \circ\, d_e\varphi(X) \tag{1.27}$$

That is, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\exp_G"] & H \arrow[d, "\exp_H"] \\ \mathrm{Lie}(G) \arrow[r, "d_e\varphi"] & \mathrm{Lie}(H) \end{tikzcd} \tag{1.28}$$

Without getting in too much detail, for matrix Lie groups the exponential map has a particularly simple expression and allows us to think of the Lie algebra of a matrix Lie group in a much simpler way.

Definition 1.3.17. *Let $A \in M_n(\mathbb{C})$ be a square complex matrix. We define the exponential of the matrix e^A as the series:*

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \in M_n(\mathbb{C}) \tag{1.29}$$

The above series converges for all matrices (in, for example, the operator norm). With this definition, the Lie algebra of a matrix Lie group can be given as follows:

Example 1.3.18. Let $G \subseteq GL_n(\mathbb{C})$ be a matrix Lie group. Then, its Lie algebra $\mathrm{Lie}(G)$ is given by

$$\mathrm{Lie}(G) = \{X \in \mathrm{M}_n(\mathbb{C}) : e^{tX} \in G, \forall t \in \mathbb{R}\}. \tag{1.30}$$

Also, the exponential map $\exp : \mathrm{Lie}(G) \longrightarrow G$ is given by $\exp(X) = e^X$ (cf. [Hal03] section 3.3).

Example 1.3.19. The Lie algebra of $\mathrm{GL}_n(\mathbb{C})$, denoted $\mathfrak{gl}_n(\mathbb{C})$, is just $\mathrm{M}_n(\mathbb{C})$, since e^X is an invertible matrix for all X. Indeed, e^{-X} is the inverse of e^X. Given a finite-dimensional vector space, the Lie algebra of its group of automorphisms $\mathrm{Aut}(V)$ is just its group of endomorphisms:

$$\mathrm{Lie}(\mathrm{Aut}(V)) = \mathrm{End}(V), \tag{1.31}$$

Example 1.3.20. Lie algebras for the matrix Lie groups in Example 1.3.4:

(i) The Lie algebra of the special linear groups, denoted $\mathfrak{sl}(n, \mathbb{K})$, is the set of traceless matrices, as can be seen from the condition that $\det(e^{tX}) = 1$ for all t and by properties of the trace and the matrix exponential. Thus,

$$\mathfrak{sl}(n, \mathbb{K}) = \{X \in \mathrm{M}_n(\mathbb{K}) : \mathrm{Tr}(X) = 0\} \tag{1.32}$$

(ii) For the Lie groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, one can see by using properties of the matrix exponential that their Lie algebras, denoted $\mathfrak{o}(n)$ and $\mathfrak{so}(n)$ respectively, are given by

$$\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in \mathrm{M}_n(\mathbb{R}) : X^{tr} + X = 0\} \tag{1.33}$$
$$\tag{1.34}$$

(iii) Lastly, for the pseudo-orthogonal group $\mathrm{O}(n,m)$ and special pseudo-orthogonal group $\mathrm{SO}(n,m)$, their Lie algebras are given by:

$$\mathfrak{o}(n,m) = \mathfrak{so}(n,m) = \{X \in \mathrm{M}_{n+m}(\mathbb{R}) : I_{n,m}X^{tr}I_{n,m} + X = 0\}, \tag{1.35}$$

where $I_{n,m} = \mathrm{diag}(+1,\dots,+1,-1,\dots,-1) \in \mathrm{GL}_{n+m}(\mathbb{R})$ as in Example 1.3.4.

Finally, we will just mention the correspondence between Lie algebras and Lie groups, which is turned into a bijection if one considers only simply connected Lie groups.

Theorem 1.3.21. (Lie) *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. Then, there exists a connected Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$. *Further, there exists a unique connected and simply connected Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$, *in the sense that if another such group exists, they will be isomorphic.*

A proof for this result can be found in [Lee13] (Chapter 20, Theorem 20.21). This correspondence between Lie algebras and simply connected Lie groups is known as the Lie correspondence. Thus by this last theorem, there exists a bijective correspondence between finite-dimensional Lie algebras and simply connected Lie groups. Also, given any connected Lie group, we can always find a connected and simply connected Lie group that is, in a way, an extension of it. We call these extensions universal covers:

Definition 1.3.22. *Let G be a connected Lie group. A* ***universal cover*** *of G is a pair (H,φ) where H is a connected and simply connected Lie group and $\varphi : H \longrightarrow G$ is a Lie group homomorphism so that the induced Lie algebra homomorphism $d_e\varphi$:* $\mathrm{Lie}(H) \longrightarrow \mathrm{Lie}(G)$ *is a Lie algebra isomorphism. That is, H and G have isomorphic Lie algebras.*

It is not difficult to see that if a Lie group has a universal cover, it is unique (up to isomorphism). It can also be seen that a universal cover exists for every connected Lie group.

Theorem 1.3.23. *Let G be a connected Lie group. Then, there exists a unique universal cover of G which we will denote $(\tilde{G},\varphi)$, where φ is the covering map.*

A proof can be found in [Lee13] (chapter 7, theorem 7.7).

Example 1.3.24. Consider the group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of unit norm complex numbers. It is a Lie group under complex multiplication. It can be seen that its Lie algebra is isomorphic to the real numbers $\mathrm{Lie}(\mathbb{T}) = \mathbb{R}$. This can be seen either by computation using 1.3.18 or by seeing that the tangent space at every point is the real line. $\mathbb{T}$ is connected but not simply connected. The universal cover of $\mathbb{T}$ is the Lie group $\mathbb{R}$, with covering map $\varphi : \mathbb{R} \longrightarrow \mathbb{T}$ given by $\varphi(x) = e^{i2\pi x}$. We can see that $\mathrm{Ker}(\varphi) = \mathbb{Z}$ and that by the isomorphism theorem of groups, $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$.

Example 1.3.25. It can be seen in a much more involved way that the universal cover of the pseudo-orthogonal groups $\mathrm{O}(1,3)$ and $\mathrm{SO}(1,3)$ is the special linear group $\mathrm{SL}(2,\mathbb{C})$. We will sketch a proof of this result after introducing the required notions in the following chapters.

Example 1.3.26. Let G be an arbitrary Lie group. We define the identity component of G, and denote it by G_0, as the set of elements of the group that are path-connected to the identity element of the group. Then, G_0 is connected, and it is a Lie subgroup of G. Further, $T_eG_0 \cong T_eG$, so that their Lie algebras are also isomorphic.

2

Symmetries of quantum systems

We are moving towards a mathematical description of elementary particles. The study of elementary particles in physics belongs to the field of quantum mechanics. The theory of quantum mechanics provides a generalization of classical mechanics by being able to describe a larger class of systems that were not accessible with only the usual machinery of the Hamiltonian and the Euler-Lagrange equations.

In classical mechanics, a system is understood to have a fixed number of degrees of freedom and the state of a system is completely determined by specifying the values for all of those degrees of freedom and their derivatives at that point. One can think of a system of n particles moving freely in three-dimensional space as an example of a system. Then, each particle has three degrees of freedom (one for each dimension), making up for a total of $3n + 3n$ values that need to be specified for completely determining the state of a system (three for their positions and three for their velocities). All information about the system is contained in those $6n$ numbers and the state of the system is a point in $\mathbb{R}^{6n}$.

There is a series of phenomena, however, that doesn't fit in such a model. Briefly, the framework of classical mechanics assumes the continuity of those degrees of freedom and functions of those degrees of freedom, and this assumption is found to not apply to some of the systems that are of interest. At the core of the subject is the experimental fact that some quantities cannot be made arbitrarily small in nature, having then a fundamental and indivisible nature. Examples such as the polarization and interference of photons are discussed in [Dir49], as well as a complete introduction on the subject. The conflict between those systems and classical mechanics is solved by describing the states of the system in a way that allows for so-called superpositions.

We will not go deeper into quantum mechanics in this work, but the interested reader can find extensive treatments of the subject in of course [Dir49] together with [Hal13], [Tak08],[Mac13] and [Jau68].

In this chapter, we will introduce the only ideas about quantum mechanics that we will need during our work, namely the structure that models the states of a quantum system (the projective space of a complex Hilbert space) and the idea of symmetry. In the first section, we will define basic technical concepts about Hilbert spaces that will allow us to move on to the definitions of quantum systems and symmetries in section two. Section two contains one of the central theorems of the work, Wigner's structure theorem for the symmetries of a quantum system.

2.1 Preliminary notions

To talk about the mathematical description of quantum systems we will first review some things about Hilbert spaces that will be used throughout the following discussion.

Definition 2.1.1. *Let $\mathbb{K}$ be a field (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathcal{H}$ be vector space over $\mathbb{K}$. An **inner product** on $\mathcal{H}$ is a map $\langle\, , \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{K}$ that satisfies:*

1. *(Linearity) It is linear in its first argument. That is, $\forall \phi, \psi, \omega \in \mathcal{H}$ and $\forall \lambda, \mu \in \mathbb{K}$:*
$$\langle \lambda\phi + \mu\psi, \omega \rangle = \lambda\langle \phi, \omega \rangle + \mu\langle \psi, \omega \rangle \tag{2.1}$$

2. *(Hermitian symmetry) It is **Hermitian symmetric**. That is, $\forall \phi, \psi \in \mathcal{H}$,*
$$\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle} \tag{2.2}$$
Where the over-line denotes the complex conjugate. Note that if $\mathbb{K} = \mathbb{R}$ then the overline is understood to leave the element of the field invariant.

3. *(Positive definite) It is positive-definite. That is, $\forall \phi \in \mathcal{H}$:*

$$\langle \phi, \phi \rangle \geq 0 \tag{2.3}$$

With equality holding if and only if $\phi = 0$.

*An **inner product space** is then a pair $(\mathcal{H}, \langle\ ,\ \rangle)$ formed by a vector space and an inner product.*

Note that (1.) and (2.) combined imply that an inner product is **conjugate-linear** in its second argument, that is:

$$\langle \phi, \lambda\psi + \mu\omega \rangle = \overline{\lambda}\langle \phi, \psi \rangle + \overline{\mu}\langle \phi, \omega \rangle, \qquad \forall \phi, \psi, \omega \in \mathcal{H},\ \forall \lambda, \mu \in \mathbb{K} \tag{2.4}$$

Also, (2.) implies that the inner product of a vector with itself is always real, since

$$\langle \phi, \phi \rangle = \overline{\langle \phi, \phi \rangle}, \qquad \forall \phi \in \mathcal{H} \tag{2.5}$$

then $\langle \phi, \phi \rangle \in \mathbb{R}$.

We can also remember that:

Definition 2.1.2. *Let $\mathcal{H}$ be a vector space over $\mathbb{K}$. A **norm** on $\mathcal{H}$ is a map $\|\ \| : \mathcal{H} \longrightarrow \mathbb{R}$ that satisfies:*

1. *(Positivity) $\|\phi\| \geq 0$, $\forall \phi \in \mathcal{H}$ and $\|\phi\| = 0$ if and only if $\phi = 0$.*

2. *(Homogeneity) $\|\lambda\phi\| = |\lambda|\|\phi\|$, $\forall \phi \in \mathcal{H}$, $\forall \lambda \in \mathbb{K}$.*

3. *(Triangle inequality) $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$, $\forall \phi, \psi \in \mathcal{H}$.*

*A **normed vector space** is then a pair $(\mathcal{H}, \|\ \|)$ formed by a vector space and a norm.*

Remark 2.1.3. It is easy to see that if $(\mathcal{H}, \langle\ ,\ \rangle)$ is an inner product space, then we can obtain a normed vector space by defining the norm $\|\phi\| = \sqrt{\langle \phi, \phi \rangle}$.

Now, norms also allow us to introduce the notion of limits and therefore of completeness.

Definition 2.1.4. *Let $(\mathcal{H}, \|\ \|)$ be a normed vector space over $\mathcal{K}$. Let $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ be a sequence of elements of $\mathcal{H}$.*

1. *We will say that the sequence **converges** if there exists $\phi \in \mathcal{H}$, so that $\forall \epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ so that $\forall n \geq n_\delta$, $\|\phi_n - \phi\| < \epsilon$.*

2. *We will say that the sequence satisfies the* ***Cauchy condition*** *(or is Cauchy) if* $\forall \epsilon > 0$ *there exists* $n_\epsilon \in \mathbb{N}$ *so that* $\forall n, m \geq n_\epsilon$, $\|\phi_n - \phi_m\| < \epsilon$.

It is easy to see that every convergent sequence is also of Cauchy type. The converse is not always true. Therefore, we will say that a normed vector space $(\mathcal{H}, \| \, \|)$ *is* ***complete*** *if every sequence of Cauchy type is also convergent. It is also common to say that a complete normed vector space is just a* ***Banach space****.*

Now we are in a position to define Hilbert spaces, which are going to be the main object in our formulation of quantum mechanics.

Definition 2.1.5. *Let* $\mathbb{K}$ *be a field (with* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*). A* ***Hilbert space*** *over* $\mathbb{K}$ *is then an inner product space* $(\mathcal{H}, \langle \, , \rangle)$ *so that with the norm induced by the inner product,* $\|\phi\| = \langle \phi, \phi \rangle^{\frac{1}{2}}$, $(\mathcal{H}, \| \, \|)$ *is a complete normed vector space.*

Another very important notion that we will need during the next sections is that of Hermitian operators in Hilbert spaces. We will begin by defining what we mean by an adjoint operator.

Definition 2.1.6. *Let* $(\mathcal{H}, \langle \, , \rangle)$ *be a Hilbert space. Let* $A \in End(\mathcal{H})$*, a linear operator. We will say that* $B \in End(\mathcal{H})$ *is an* ***adjoint operator*** *to* A *if* $\forall \phi, \psi \in \mathcal{H}$,

$$\langle A(\phi), \psi \rangle = \langle \phi, B(\psi) \rangle \tag{2.6}$$

We can prove the uniqueness of adjoint operators given their existence:

Proposition 2.1.7. *Let* $A \in End(\mathcal{H})$ *and let* $B, C \in End(\mathcal{H})$ *be adjoint to* A*. Then,* $B = C$*. Therefore, if there exists and adjoint operator to* A*, we will denote it by* $A^\dagger$.

Proof. Lets just consider $\phi \in \mathcal{H}$ and see that $\|B(\phi) - C(\phi)\|^2 = 0$, meaning that B and C are the same. If we expand:

$$\|B(\phi) - C(\phi)\|^2 = \langle B(\phi), B(\phi) \rangle - \langle B(\phi), C(\phi) \rangle - \langle C(\phi), B(\phi) \rangle + \langle C(\phi), C(\phi) \rangle \tag{2.7}$$

By using the definition of an adjoint operator, that is equal to:

$$\langle A(B(\phi)), \phi \rangle - \langle A(B(\phi)), \phi \rangle - \langle A(C(\phi)), \phi \rangle + \langle A(C(\phi)), \phi \rangle = 0 \tag{2.8}$$

□

Existence, however, does not happen in general. There are some cases where it is guaranteed.

Definition 2.1.8. *A map $A : \mathcal{H} \longrightarrow \mathcal{H}$ of a Hilbert space is said to be* ***bounded*** *if $\exists \lambda \in \mathbb{R}$ so that $\forall \phi \in \mathcal{H}$, $\|A(\phi)\| \leq \lambda \|\phi\|$.*

Theorem 2.1.9. *Let $(\mathcal{H}, \langle \, , \rangle)$ be a Hilbert space and $A \in$End$(\mathcal{H})$ be a bounded linear operator. Then, there exists a unique adjoint operator $A^\dagger$.*

A proof of the above result can be found in [Hal13] (Appendix A, Proposition A.52).

Lastly, we need to define what we mean by a self-adjoint linear operator.

Definition 2.1.10. *Let $(\mathcal{H}, \langle \, , \rangle)$ be a Hilbert space and $A \in$End$(\mathcal{H})$ be a linear operator. We will say that A is* ***self-adjoint*** *if $A^\dagger \in$End$(\mathcal{H})$ exists and $A = A^\dagger$.*

Then, if $A \in \mathrm{End}(\mathcal{H})$ is a self-adjoint operator, it satisfies that $\langle A(\phi), \psi \rangle = \langle \phi, A(\psi) \rangle$ for all $\phi, \psi \in \mathcal{H}$. Lastly:

Lemma 2.1.11. *Let $(\mathcal{H}, \langle \, , \rangle)$ be a Hilbert space and $A \in$End$(\mathcal{H})$ be a self-adjoint linear operator. Then:*

1. *If $\phi \in \mathcal{H}$ is an eigenvector of A, then its eigenvalue is real. That is, $A(\phi) = \lambda\phi$ for some $\lambda \in \mathbb{R}$.*

2. *If $\phi, \psi \in \mathcal{H}$ are eigenvectors of A with different eigenvalues, then $\langle \phi, \psi \rangle = 0$.*

2.2 Quantum systems and symmetries

In this section we will introduce the structure of a quantum system and study the automorphisms of this structure, that is, the maps that preserve its properties. The presentation of this topic will be purely mathematical in its approach and its main concern is to introduce the objects that we will be using further on. Those definitions, however, attempt to provide a model for a broad set of experiments in nature. That means that there is a strong connection between the abstract objects introduced here and the realities they model. References will be provided for the reader interested in the ideas that the following definitions are modeling.

The fundamental object in quantum mechanics is that of a complex Hilbert space $(\mathcal{H}, \langle \, , \rangle)$. It is common in the physics literature to postulate that to any given “quantum mechanical system” there corresponds a Hilbert space that completely characterizes it. Since the notion of a “quantum mechanical system” presents some difficulties when trying to define it with precision, we will avoid it altogether and just define abstractly a "quantum mechanical system" to *be* that Hilbert space. This subtle distinction allows us to treat the subject with consistency at the expense of disconnecting our discussion from the reality that is being modeled. For a less abstract, standard introduction to the subject in the physics literature, [Gri17] or [SN17].

The Hilbert space is not, however, the object whose properties are analogous to those of a quantum system, in the sense that the elements of the Hilbert space do not correspond bijectively to states of the quantum system. Due to some superposition properties of the experiments being modeled (c.f. [Dir49] Chapter 1), there is a degree of redundancy in the Hilbert space that has to be dealt with. Namely, elements that are multiples of one another are needed to correspond to the same state of the system. That brings us to the notion of the projective Hilbert space.

Definition 2.2.1. *Let $\mathcal{H}$ be a complex Hilbert space. The relation $\sim$ in $\mathcal{H} \setminus \{0\}$ given by $\phi \sim \psi$ if and only if $\phi = \lambda\psi$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ is an equivalence relation. The set of equivalence classes is called the* ***projective space*** *of $\mathcal{H}$ and is denoted by $\mathbb{P}\mathcal{H}$.*

The elements of the projective space of a given Hilbert space are going to model the notion of states of the system. Given an element $\phi \in \mathcal{H}$, its equivalence class in $\mathbb{P}\mathcal{H}$ is going to be denoted as $[\phi] \in \mathbb{P}\mathcal{H}$. Any element of the projective space is of this form. Let us note also that the inner product of the Hilbert space induces naturally a real map on its projective space.

Definition 2.2.2. *Let $(\mathcal{H}, \langle\ ,\ \rangle)$ be a Hilbert space and $\mathbb{P}\mathcal{H}$ its projective space. Then, the* ***projective inner product*** *is the map*

$$\langle\ ,\ \rangle_{\mathbb{P}} : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \longrightarrow \mathbb{R}_{\geq 0}$$

defined by

$$\langle [\phi], [\psi] \rangle_{\mathbb{P}} = \frac{\langle \phi, \psi \rangle^2}{\langle \phi, \phi \rangle \langle \psi, \psi \rangle}$$

It is easy to check that this map is well-defined. This projective inner product is usually referred to as the transition probabilities of the system. This is so because it is related to the notion of performing a "measurement" on the system. We will not get into this topic here since it is not relevant to the discussion. It is important to note, however, that this projective inner product is part of the structure of the system and has to be preserved by any notion of automorphism of quantum systems. Let us then define what is meant by a quantum system.

Definition 2.2.3. *A* ***quantum system*** *is a complex Hilbert space* $(\mathcal{H}, \langle\,,\rangle)$*. The* ***states*** *of the system are the elements of the projective space* $\mathbb{P}\mathcal{H}$*. The* ***transition probability*** *of the system is the map*

$$\langle\,,\rangle_{\mathbb{P}} : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \longrightarrow \mathbb{R}$$

induced by the inner product of $\mathcal{H}$*.*

The relevant notion of automorphism of a quantum system is not concerned with the elements of the Hilbert space but with the states themselves, that is, with elements of the projective space. Therefore, the automorphisms of a quantum system are going to be maps that preserve the structure of the space of states.

Definition 2.2.4. *An* ***automorphism*** *of a quantum system* $\mathcal{H}$ *is a bijective map* $\Theta : \mathbb{P}\mathcal{H} \longrightarrow \mathbb{P}\mathcal{H}$ *that preserves the transition probabilities of the system. That is, so that*

$$\langle\Theta([\phi]), \Theta([\psi])\rangle_{\mathbb{P}} = \langle[\phi], [\psi]\rangle_{\mathbb{P}}, \qquad \forall\phi, \psi \in \mathcal{H} \setminus \{0\}.$$

Automorphisms of the system are going to be denoted by $\mathrm{Aut}(\mathbb{P}\mathcal{H})$. We will also call them **symmetries** of the system. They form a group under composition. Although symmetries of a quantum system refer to bijections of the projective space, some symmetries can be seen as coming from maps of the underlying Hilbert space satisfying certain conditions.

Definition 2.2.5. *Given a Hilbert space* $(\mathcal{H}, \langle\,,\rangle)$*, a* ***unitary operator*** *is a map* $U : \mathcal{H} \longrightarrow \mathcal{H}$ *that preserves the inner product. That is,*

$$\langle U(\phi), U(\psi)\rangle = \langle\phi, \psi\rangle, \qquad \forall\phi, \psi \in \mathcal{H} \tag{2.9}$$

We will denote the set of unitary operators as $\mathcal{U}(\mathcal{H})$*.*

Note that any unitary operator is in particular bounded. Now we will prove a couple of properties of unitary operators.

Lemma 2.2.6. *Given a Hilbert space as above, let* $U : \mathcal{H} \longrightarrow \mathcal{H}$ *be a unitary operator. Then,* $U \in \mathrm{End}(\mathcal{H})$*, i.e.,* U *is a linear operator.*

Proof. The proof is elementary and follows easily by considering arbitrary $\phi, \psi \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$ and expanding the expression

$$\|\,(\lambda U(\phi) + \mu U(\psi)) - U(\lambda\phi + \mu\psi)\,\|^2 \tag{2.10}$$

into its expression using the inner product and the definition of a unitary operator. □

Also, we can link the definition we gave with another usual definition of unitary operators.

Lemma 2.2.7. *Consider a Hilbert space as above and a unitary operator U. Since U is linear and bounded, we can consider its adjoint operator $U^\dagger$. Then, U is a bijective map and its inverse is its adjoint. That is:*

$$U^\dagger U = UU^\dagger = I \tag{2.11}$$

Where I denotes the identity operator and the product is understood to be the composition of linear operators.

Proof. Again, the proof is elementary and it suffices to consider two arbitrary vectors $\phi, \psi \in \mathcal{H}$ and see that

$$\langle \phi, (I - U^\dagger U)(\psi) \rangle = \langle \phi, \psi \rangle - \langle U(\phi), U(\psi) \rangle = 0 \tag{2.12}$$

And therefore since this is true for every pair of vectors, we get that $U^\dagger U = I$. Now, if we consider $\langle (I - UU^\dagger)(\phi), \psi \rangle$ and use the fact that $U^\dagger$ is also unitary, we get that $UU^\dagger = I$. Then, since $U^\dagger$ is the left and right inverse of U, we conclude that U is a bijection and that $U^{-1} = U^\dagger$. □

Now, all this discussion was done so that we can provide a very important example of the symmetries of a quantum system. As can be expected, unitary transformations of a quantum system induce symmetries.

Definition 2.2.8. *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a bijective map. We will denote by $[T]$ the map $[T] : \mathbb{P}\mathcal{H} \longrightarrow \mathbb{P}\mathcal{H}$ defined by $[T]([\phi]) = [T(\phi)]$ whenever it is well defined.*

Proposition 2.2.9. *Let $(\mathcal{H}, \langle\, , \rangle)$ be a quantum system and let $U \in \mathcal{U}(\mathcal{H})$ be a unitary operator. Then, $[U] : \mathbb{P}\mathcal{H} \longrightarrow \mathbb{P}\mathcal{H}$ is well defined and is a symmetry of the system, i.e., $[U] \in \mathrm{Aut}(\mathbb{P}\mathcal{H})$.*

Proof. The proof is immediate from the definitions. □

There is also the closely related notion of a conjugate-unitary operator, that also gives rise to symmetries of the quantum system.

Definition 2.2.10. *Given a Hilbert space as above, a* ***conjugate-unitary operator*** *is a map $W : \mathcal{H} \longrightarrow \mathcal{H}$ that satisfies:*

$$\langle W(\phi), W(\psi) \rangle = \overline{\langle \phi, \psi \rangle} \tag{2.13}$$

Note the similarity with the definition of unitary operator. Indeed, the properties of conjugate-unitary operators are closely related to unitary operators. It can be proven in a way similar to the one above that they are conjugate-linear maps instead of linear maps, meaning that

$$W(\lambda\phi + \mu\psi) = \bar{\lambda}W(\phi) + \bar{\mu}W(\psi), \qquad \forall\lambda, \mu \in \mathbb{C}, \qquad \forall\phi, \psi \in \mathcal{H}.$$

Given a conjugate-linear operator W we can also define the notion of **conjugate-adjoint**, $W^{\ddagger}$ as satisfying the equation

$$\langle W(\phi), \psi\rangle = \overline{\langle\phi, W^{\ddagger}(\psi)\rangle}. \tag{2.14}$$

The conjugate-adjoint of a conjugate-unitary map is also conjugate-unitary. We can prove similarly as above that conjugate-linear operators are bijective and have their conjugate-adjoint as their inverse, $WW^{\ddagger} = W^{\ddagger}W = I$.

We will denote the set of conjugate-unitary transformations as $\overline{\mathcal{U}}(\mathcal{H})$. We will also denote by $\hat{\mathcal{U}}(\mathcal{H}) = \mathcal{U}(\mathcal{H}) \cup \overline{\mathcal{U}}(\mathcal{H})$ the set of unitary or conjugate-unitary maps of $\mathcal{H}$. It is a group under composition, with $\mathcal{U}(\mathcal{H})$ being a subgroup.

Corollary 2.2.11. *Let $T \in \hat{\mathcal{U}}(\mathcal{H})$. Then, $[T] \in \mathrm{Aut}(\mathbb{P}\mathcal{H})$, i.e., T induces a symmetry of the system.*

The main result of this section, due to Wigner, is that *all* symmetries of a quantum system in this sense arise from unitary or conjugate-unitary operators in this way and that there is a certain notion of uniqueness in the correspondence between those operators and symmetries.

Theorem 2.2.12. (Wigner) *Let $(\mathcal{H}, \langle\ ,\ \rangle)$ be the Hilbert space representing a given quantum system and let $\Theta \in \mathrm{Aut}(\mathbb{P}\mathcal{H})$ be a symmetry. Let us assume also that $\dim(\mathcal{H}) \geq 2$ (see remark). Then:*

i) There exists $T \in \hat{\mathcal{U}}(\mathcal{H})$ so that $[T(\phi)] = \Theta([\phi])$ for all $\phi \in \mathcal{H}$, $\phi \neq 0$.

ii) If there is $T' \in \hat{\mathcal{U}}(\mathcal{H})$ so that also $[T'] = \Theta$, then there is $z \in \mathbb{T}$ so that $T' = zT$.

Remark 2.2.13. Note that in particular, *ii*) implies that if there is a unitary (resp. conjugate-unitary) map inducing Θ, then any other $T \in \hat{\mathcal{U}}(\mathcal{H})$ inducing Θ is also unitary (resp. conjugate-unitary).

Remark 2.2.14. Note that *ii*) is not implied in *i*). If T and W induce the same symmetry, then $[T(\phi)] = [W(\phi)]$ for all $\phi \neq 0$. All we can deduce from that is that there is a complex function $\omega_{W,T} : \mathcal{H} \setminus \{0\} \longrightarrow \mathbb{T}$ so that

$W(\phi) = \omega_{W,T}(\phi)T(\phi)$ for all $\phi \neq 0$. Wigner's theorem says something stronger, namely that $\omega_{W,T}$ is a constant function.

Remark 2.2.15. If $\dim(\mathcal{H}) = 1$, then $\mathbb{P}\mathcal{H}$ consists of a single element. Therefore, any bijective map $T : \mathcal{H} \longrightarrow \mathcal{H}$ satisfying $T(\phi) \neq 0$ for $\phi \neq 0$ will induce a symmetry of the system. Since there is only one possible symmetry (namely, the identity map in $\mathbb{P}\mathcal{H}$), all such bijections will induce the same symmetry.

This is the main result of the chapter. Proofs of this theorem can be found in [Bar64], [SMCS08], [Mor18] (Theorem 12.11) and of course [Wig12] (appendix to Chapter 20).

Its importance as a central piece of the study of symmetries in quantum mechanics comes from the fact that after seeing that any unitary or conjugate-unitary map induces a symmetry of the projective space by projection, Wigner's theorem tells us that this is the only way in which symmetries of the system can arise. Further, it tells us exactly when two such maps are going to induce the same symmetry.

To see the meaning of this result in another way, we can introduce the notion of an exact short sequence of groups. In this approach, we follow the ideas found in [Sim06] (Chapters 1 and 2).

Definition 2.2.16. *An* ***short sequence of groups*** *is a finite set $\{G_1, \ldots, G_n\}$ of groups together with $\{f_1, \ldots, f_{n-1}\}$ group homomorphisms so that $f_k : G_k \longrightarrow G_{k+1}$ for all $k = 1, \ldots, n-1$ that satisfies the property that $\mathrm{Im} f_k \subseteq \mathrm{Ker} f_{k+1}$. It can be presented as:*

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-2}} G_{n-1} \xrightarrow{f_{n-1}} G_n \tag{2.15}$$

The short sequence is called ***exact*** *if $\mathrm{Im} f_k = \mathrm{Ker} f_{k+1}$ for all $k = 1, \ldots, n-1$.*

Given a complex Hilbert space $\mathcal{H}$, we will denote by $\mathcal{U}(1)$ the set of maps of $\mathcal{H}$ that are unit-norm multiples of the identity map. That is, given $T \in \mathcal{U}(1)$, there is $z \in \mathbb{C}$ with $|z| = 1$ so that $T = z \cdot \mathrm{Id}$. Clearly, $\mathcal{U}(1)$ is a subgroup of $\hat{\mathcal{U}}(\mathcal{H})$.

We can also define the map

$$p : \hat{\mathcal{U}}(\mathcal{H}) \longrightarrow \mathrm{Aut}(\mathbb{P}\mathcal{H})$$

given, for all $\phi \in \mathcal{H} \setminus \{0\}$ by $p(T)([\phi]) = [T(\phi)]$. It is easy to see that it is a group homomorphism. With these notions, we can reformulate Wigner's theorem in the following way:

Corollary 2.2.17. *(Wigner) Let $\mathcal{H}$ be a quantum system with* $\dim(\mathcal{H}) \geq 2$. *Then, the following short sequence is exact:*

$$\{\mathrm{Id}\} \hookrightarrow \mathcal{U}(1) \xhookrightarrow{\mathrm{inc}} \hat{\mathcal{U}}(\mathcal{H}) \xrightarrow{p} \mathrm{Aut}(\mathbb{P}\mathcal{H}) \longrightarrow \{\mathrm{Id}\} \tag{2.16}$$

Proof. The unlabeled homomorphisms on the left and the right are the only possible ones, namely, the inclusion on the left and the map that sends everything to the identity on the right. Also, trivially, $\mathrm{Ker(inc)} = \{1\}$ and $\mathrm{Im(inc)} = \mathcal{U}(1)$. Therefore, the only equalities that need to be proven are $\mathrm{Im}(p) = \mathrm{Aut}(\mathbb{P}\mathcal{H})$ and $\mathrm{Ker}(p) = \mathcal{U}(1)$. Both of them are provided by Wigner's theorem above.

Indeed, *i*) of Wigner's theorem says that $\forall \Theta \in \mathrm{Aut}(\mathbb{P}\mathcal{H})$ there exists $T \in \hat{\mathcal{U}}(\mathcal{H})$ so that $p(T) = \Theta$. This gives $\mathrm{Im}(p) = \mathrm{Aut}(\mathbb{P}\mathcal{H})$.

To see the remaining equality, consider $T \in \hat{\mathcal{U}}(\mathcal{H})$ so that $p(T) = \mathrm{Id}$. Then, since also $p(\mathrm{Id}) = p(T)$, by *ii*) we have that $T = z\mathrm{Id}$ for some $z \in \mathbb{T}$, which is equivalent to saying $T \in \mathcal{U}(1)$. Therefore, $\mathrm{Ker}(p) = \mathcal{U}(1)$ since the other inclusion is trivial.

□

Remark 2.2.18. Let $\mathcal{H}$ be a quantum system and consider the group homomorphism $p : \hat{\mathcal{U}}(\mathcal{H}) \longrightarrow \mathrm{Aut}(\mathbb{P}\mathcal{H})$. Then, the isomorphism theorem for groups gives

$$\mathrm{Aut}(\mathbb{P}\mathcal{H}) \cong \hat{\mathcal{U}}(\mathcal{H})/\mathcal{U}(1).$$

If one considers the restriction $p_{|\mathcal{U}}$ to the subgroup $\mathcal{U}(\mathcal{H}) \subseteq \hat{\mathcal{U}}(\mathcal{H})$, then the image $\mathrm{Im}(p_{|\mathcal{U}})$ corresponds to those symmetries of the system that come from a unitary map. We will denote them by $\mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$. Again by the isomorphism theorem,

$$\mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H}) \cong \mathcal{U}(\mathcal{H})/\mathcal{U}(1).$$

3

The Minkowski spacetime

In the previous chapter, we occupied ourselves with the quantum mechanical aspect of the topic that concerns us, that of the description of an elementary particle. In this chapter, we deal with the relativistic side of the problem. Both sides of the discussion are essential to it since it is desirable for a description of an elementary particle to agree with both quantum mechanics and special relativity.

In this chapter, we will focus on motivating where is the Poincaré group coming from and why it appears as the group of symmetries of special relativity. To do that, we will model the Minkowski spacetime as an affine space with a pseudo-distance function (that is, a not necessarily positive-definite distance). Then, we will see that any map preserving a distance function in an affine space is necessarily an affine map, and see how this allows us to define the Poincaré group. In section two, we will do a brief detour from our study of elementary particles to see the implications of our treatment of Minkowski spacetime and how this allows us to model a change of observer in special relativity. We will link our affine space description with the more standard (but seen to be equivalent) description by using a smooth manifold.

3.1 Affine spaces, Minkowski spacetime and the Poincaré group

As a brief reminder of special relativity, we will introduce here the group of symmetries of Minkowski space. We will introduce it by using non-degenerate symmetric bilinear forms. The interested reader can refer to [Kri03] (Section 1.4) or [Nab03] for a more extensive mathematical introduction to special relativity.

Definition 3.1.1. *Let $\mathbb{V}$ be a finite-dimensional vector space over the real numbers. A non-degenerate symmetric bilinear form is a map $B : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ that satisfies:*

1. $B(v+w, u) = B(v, u)+B(w, u)$ *and* $B(\lambda v, w) = \lambda B(v, w)$ *for all* $u, v, w \in \mathbb{V}$ *and* $\lambda \in \mathbb{R}$.

2. $B(v, w) = B(w, v)$ *for all* $v, w \in \mathbb{V}$*. That is,* B *is symmetric.*

3. *If* $v \in \mathbb{V}$ *is such that* $B(v, w) = 0$ *for all* $w \in \mathbb{V}$*, then* $v = 0$*. That is,* B *is non-degenerate.*

We will state the following well-known results about symmetric non-degenerate bilinear forms, usually known as Sylvester's Law of Inertia (c.f. [Lan12] Chapter XV Theorem 4.1):

Proposition 3.1.2. *Given a finite dimensional real vector space $\mathbb{V}$ and a non-degenerate symmetric bilinear form $B : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$:*

1. *There exists a basis $\{e_1, ..., e_n\}$ of $\mathbb{V}$ so that the matrix of B in this basis is*

$$\mathrm{diag}(\underbrace{-1, \ldots, -1}_{s}, \underbrace{1, \ldots, 1}_{q})$$

2. *Further, the natural numbers (s, q) are unique for a given form B. We will say that (s, q) is the* ***signature*** *of the form B.*

We can now go one step further and consider an affine space $\mathbb{A}$ over the vector space $\mathbb{V}$, with a pseudo-distance function induced by the bilinear form B of $\mathbb{V}$.

Definition 3.1.3. *An* ***affine space*** *$\mathbb{A}$ over the real finite dimensional vector space $\mathbb{V}$ is a triple $(\mathbb{A}, \mathbb{V}, \varphi)$ where $\mathbb{A}$ is a set and $\varphi : \mathbb{A} \times \mathbb{V} \longrightarrow \mathbb{A}$ is a right transitive* *and free action.*

Since the action is transitive, given $p, q \in \mathbb{A}$, there exists $v \in \mathbb{V}$ so that $\varphi(p, v) = q$. Further, since the action is free, this v is unique. Therefore, we will define the "point subtraction map":

$$\mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{E}, \qquad (p, q) \mapsto \overrightarrow{pq} = v$$

mapping each pair p, q to the unique vector v so that $\varphi(p, v) = q$. In this way, we will get that for every $p, q \in \mathbb{A}$, $\varphi(p, \overrightarrow{pq}) = q$.

Given an affine space $(\mathbb{A}, \mathbb{V}, \varphi)$ and a bilinear symmetric non-degenerate form B on $\mathbb{V}$, we can induce a (pseudo) distance function $d : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{R}$ given by $d(p, q)^2 = B(\overrightarrow{pq}, \overrightarrow{pq})$

The reason why we are introducing all these ideas is that the special theory of relativity is formulated in one of those affine spaces, the Minkowski spacetime, and that the pseudo distance function induced by a particular bilinear form (the spacetime interval) turns out to be a fundamental property of this model of nature. We can define a Minkowski spacetime as follows:

Definition 3.1.4. *A **Minkowski spacetime** $\mathcal{M}$ is an affine space $(\mathcal{M}, \mathbb{R}^4, \varphi)$, where the vector space of this affine space is $\mathbb{R}^4$ together with a non degenerate, symmetric, bilinear form $\eta : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ of signature $(1, 3)$.*

That is, there is a basis of $\mathbb{R}^4$ so that the matrix of η is $\operatorname{diag}(-1, +1, +1, +1)$. It can be seen that any two Minkowski spacetimes are isomorphic, so we will refer to the Minkowski spacetime.

For the rest of the section, we will see that the set of maps of an affine space $f : \mathbb{A} \longrightarrow \mathbb{A}$ that preserve a pseudo-distance function induced by a form B on $\mathbb{V}$ are going to be necessarily affine maps. That is, we will see that if f preserves the distance function, then there exists a linear map $\tilde{f} : \mathbb{V} \longrightarrow \mathbb{V}$ that preserves the form B and so that $f(\varphi(p, v)) = \varphi(f(p), \tilde{f}(v))$ for all $p \in \mathbb{A}$, $v \in \mathbb{V}$. This is a rather easy argument if one assumes that B is positive-definite (i.e., $B(v, v) > 0$ for all $v \neq 0$). This case was already proved in Lemma 2.2.6. However, we will be interested in the symmetries of Minkowski space, where the bilinear form in that case is not positive-definite, but only non-degenerate. That makes the proof of the statement a bit less conventional. More on the subject is said in [Vog72]. We will start with the definition of an affine map.

Definition 3.1.5. *Given an affine space $(\mathbb{A}, \mathbb{V}, \varphi)$, a map $f . \mathbb{A} \longrightarrow \mathbb{A}$ is called **affine** if there exists a linear map $\tilde{f} : \mathbb{V} \longrightarrow \mathbb{V}$ so that $f(\varphi(p, v)) = \varphi(f(p), \tilde{f}(v))$ for all $p \in \mathbb{A}$ and $v \in \mathbb{V}$.*

By using the "vector subtraction" map, one can restate the definition of an affine map as satisfying that $\tilde{f}(\overrightarrow{pq}) = \overrightarrow{f(p)f(q)}$. This can be seen by writing $f(q) = \varphi(f(p), \overrightarrow{f(p)f(q)})$ and equating it with $f(q) = f(\varphi(p, \overrightarrow{pq})) = \varphi(f(p), \tilde{f}(\overrightarrow{pq}))$ coming from the definition of affine map.

We will now first show that if a function in a vector space preserves a symmetric, non-degenerate, bilinear form, then it has to be linear.

Lemma 3.1.6. *Let $\mathbb{V}$ be a finite-dimensional vector space and $B : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ be a symmetric bilinear non-degenerate form. Let $f : \mathbb{V} \longrightarrow \mathbb{V}$ be a map that preserves B, that is, so that $B(f(v), f(w)) = B(v, w)$ for all $v, w \in \mathbb{V}$. Then, f is linear.*

Proof. Let $v, w \in \mathbb{V}, \lambda \in \mathbb{R}$. Then we have that by the bilinearity of B and since f preserves the form B:

(i) $B(f(v + w) - f(v) - f(w), f(u)) = B(0, u) = 0, \ \forall u \in \mathbb{V}$

(ii) $B(f(\lambda v) - \lambda f(v), f(u)) = B(0, u) = 0, \ \forall u \in \mathbb{V}$

Therefore, by the non-degeneracy of B we get that $f(v + w) = f(v) + f(w)$ and that $f(\lambda v) = \lambda f(v)$. □

Now, we will finish the proof by seeing that a bijection that preserves the distance function in an affine space has to be an affine map.

Proposition 3.1.7. *Consider $(\mathbb{A}, \mathbb{V}, \varphi)$, an affine space with a (pseudo) distance function $d : \mathbb{A} \longrightarrow \mathbb{A}$ induced by a form B of signature (r, s). Let $f : \mathbb{A} \longrightarrow \mathbb{A}$ be a bijective map that preserves the (pseudo) distance, $d(f(p), f(q)) = d(p, q)$. Then, f is an affine map.*

Proof. Let $f : \mathbb{A} \longrightarrow \mathbb{A}$ be a bijection that preserves the distance function. Choose $o \in \mathbb{A}$ and define $\tilde{f} : \mathbb{V} \longrightarrow \mathbb{V}$ to be the map defined as

$$\tilde{f}(\overrightarrow{op}) = \overrightarrow{f(o)f(p)}$$

This is well defined, since given $v \in \mathbb{V}$, there is a unique $p \in \mathbb{A}$ so that $v = \overrightarrow{op}$ and it is non zero if $p \neq o$ (because of injectivity and surjectivity of f). Further, it preserves the form B of $\mathbb{V}$, in the sense that $B(\tilde{f}(v), \tilde{f}(w)) = B(v, w)$ for all $v, w \in \mathbb{V}$.. Indeed, let $v, w \in \mathbb{V}$ and lets see that $B(\tilde{f}(v), \tilde{f}(w)) = B(v, w)$. Let $p, q \in \mathbb{A}$ be the unique points so that $p = \varphi(o, v)$ and $q = \varphi(o, w)$. That is, $v = \overrightarrow{op}$ and $w = \overrightarrow{oq}$. Then, we can see that the vector $\overrightarrow{pq}$ can be written as just

$\overrightarrow{pq} = \overrightarrow{po} + \overrightarrow{oq} = -v + w$ by just using the transitivity of the map φ. Then:

$$d(p,q)^2 = B(\overrightarrow{pq},\overrightarrow{pq}) = B(v,v)+B(w,w)-2B(v,w) = d^2(o,p)+d^2(o,q)-2B(v,w) \tag{3.1}$$

And similarly:

$$d(f(p),f(q))^2 = B(\overrightarrow{f(p)f(q)},\overrightarrow{f(p)f(q)}) = B(\overrightarrow{f(p)f(o)}+\overrightarrow{f(o)f(q)},\overrightarrow{f(p)f(o)}+\overrightarrow{f(o)f(q)}) =$$
$$= d(f(p),f(o))^2 + d(f(o),f(q))^2 - 2B(\overrightarrow{f(p)f(o)},\overrightarrow{f(o)f(q)}) =$$
$$= d(f(p),f(o))^2 + d(f(o),f(q))^2 - 2B(\tilde{f}(v),\tilde{f}(w))$$

Thus, by using the fact that f preserves the distances, we can equate both expressions and cancel the corresponding terms to get that $B(v,w) = B(\tilde{f}(v),\tilde{f}(w))$ as well. Then by using the previous lemma, we know that a map that preserves a bilinear form B is linear. Therefore, f together with $\tilde{f}$ is an affine map, as we wanted to see. □

Thus, we have proved:

Theorem 3.1.8. *Let $(\mathbb{A},\mathbb{V},\varphi)$ be an affine space. Let B be a symmetric, non-degenerate, bilinear form in $\mathbb{V}$ and d be the corresponding induced pseudo distance on $\mathbb{A}$. let $f : \mathbb{A} \longrightarrow \mathbb{A}$ be a bijective map that preserves the distance function (i.e., $d(f(p),f(q))^2 = d(p,q)^2$ for all $p,q \in \mathbb{A}$). Then, f is an affine map so that its associated linear map $\tilde{f}$ preserves the form B.*

Remark 3.1.9. Let $\mathbb{V}$ be an n-dimensional real vector space. Let B be a symmetric, non-degenerate, bilinear form. Then, let $r,s \geq 0$ be natural numbers so that in some basis of $\mathbb{V}$, the matrix of B is just $\mathrm{diag}(\underbrace{-1,\ldots,-1}_{r},\underbrace{1,\ldots,1}_{s})$.
Going back to Example 1.3.4 we can see that the set of linear maps that leave invariant such a matrix are precisely the set of (r,s) pseudo-orthogonal matrices, $O(r,s) \subseteq \mathrm{GL}(\mathbb{V})$. Further, we saw that it is a Lie group.

We will denote by $\mathrm{Aff}(\mathbb{A})$ the set of affine maps of an affine space, not necessarily preserving any bilinear form on it (we will add that assumption further below). It is not difficult to see that it is a group, the group of affine maps of an affine space. We will now see that this group can be described in a simple way.

Remark 3.1.10. Let $(\mathbb{A},\mathbb{V},\varphi)$ be an affine space and let $o \in \mathbb{A}$. Then, given $p \in \mathbb{A}$ and $\tilde{f} \in \mathrm{GL}(\mathbb{V})$, there exists a unique $f \in \mathrm{Aff}(\mathbb{A})$ so that $f(o) = p$ and so that $f(q) = f(\varphi(o,\overrightarrow{oq})) = \varphi(f(o),\tilde{f}(\overrightarrow{oq}))$, for all $q \in \mathbb{A}$. That is, a choice of a linear map $\tilde{f}$ and a choice of image for one point $f(o)$ completely determines an affine map.

This remark points clearly toward the direction that affinities are no more than just a composition of a linear map and a translation. We will see that in fact, the structure of the group of affine maps is that of a semidirect product of linear maps and translations.

Definition 3.1.11. *Let G, H be groups and let $\rho : G \times H \longrightarrow H$ be an action of G on H that is free and transitive. Then, it induces an homomorphism of groups $\tilde{\rho} : G \longrightarrow \mathrm{Aut}(H)$.*

We will define the semidirect product $G \ltimes H$ as being the set $G \times H$ together with the multiplication given, for $(g, h), (g', h') \in G \times H$, as $(g, h) \ltimes (g', h') = (gg', h\tilde{\rho}_g(h'))$, where the operation in each component of the tuple refers to the corresponding operation of each group. This is a group, with identity element (e_g, e_h).

We will now consider, given a (finite-dimensional, real) vector space $\mathbb{V}$, the group given by the semi-direct product of the translations and the linear maps. That is:

Remark 3.1.12. Let $\mathbb{V}$ be n-dimensional real vector space. Let the group $\mathrm{GL}(\mathbb{V})$ play the role of G in the above definition, and let $\mathbb{V}$ seen as an additive group plays the role of H above. Then, consider:

(i) The action of the group of linear maps on the additive group, $\rho : \mathrm{GL}(\mathbb{V}) \times \mathbb{V} \longrightarrow \mathbb{V}$. It is transitive and free, and therefore it induces the group homomorphism $\tilde{\rho} : \mathrm{GL}(\mathbb{V}) \longrightarrow \mathrm{Aut}(\mathbb{V})$.

(ii) The product $\mathrm{GL}(\mathbb{V}) \times \mathbb{V}$ with the product defined as, for all $A, B \in \mathrm{GL}(\mathbb{V})$ and for all $v, w \in \mathbb{V}$, then $(A, v) \cdot (B, w) = (AB, v + \rho(A, w))$.

This product turns the set into a semi-direct product as defined above, $\mathrm{GL}(\mathbb{V}) \ltimes \mathbb{V}$

Proposition 3.1.13. *Let $(\mathbb{A}, \mathbb{V}, \varphi)$ be an affine space. Then, the affine group is isomorphic to the semi-direct product $\mathrm{Aff}(\mathbb{A}) \cong \mathrm{GL}(\mathbb{V}) \ltimes \mathbb{V}$.*

Proof. We will see that we can construct a pair of group homomorphisms, after choosing $o \in \mathbb{A}$, as follows.

(i) Define $\psi : \mathrm{Aff}(\mathbb{A}) \longrightarrow \mathrm{GL}(\mathbb{V}) \ltimes \mathbb{V}$ as follows. We saw that given an affine map f, it is completely determined by its associated linear map $\tilde{f}$ and by the vector $\overrightarrow{of(o)}$. Then, define $\psi(f) = (\tilde{f}, \overrightarrow{of(o)}) \in \mathrm{GL}(\mathbb{V}) \ltimes \mathbb{V}$.

It is easy to see that ψ is a group homomorphism, this can be checked by seeing that if $h = g \circ f \in \mathrm{Aff}(\mathbb{A})$, then $\overrightarrow{oh(o)} = \overrightarrow{og(o)} + \tilde{g}(\overrightarrow{of(o)})$, which corresponds with the product $\psi(g) \cdot \psi(f)$.

(ii) Define $\phi : \mathrm{GL}(\mathbb{V}) \ltimes \mathbb{V} \longrightarrow \mathrm{Aff}(\mathbb{A})$ as follows. Given $(\tilde{f}, v) \in \mathrm{GL}(\mathbb{V}) \ltimes \mathbb{V}$, define $\phi(\tilde{f}, v) = f$, where f is the affine map determined by $f(o) = \varphi(o, v)$ and the associated linear map $\tilde{f}$. This map is well-defined. It is also easy to see that it is a group homomorphism in a similar way as above.

Finally, one can see that the compositions of those maps are the respective identity maps. As an example, well see that $\phi \circ \psi = \mathrm{Id}$, since given an affine map f determined by $f(o)$ and by $\tilde{f}$, then $\psi(f) = (\tilde{f}, \overrightarrow{of(o)})$. Then, $h = \phi \circ \psi(f)$ is the affine map defined as $h(o) = \varphi(o, \overrightarrow{of(o)}) = f(o)$ and the associated linear map $\tilde{f}$, which is just the original affine map f.

□

So we saw that indeed $\mathrm{Aff}(\mathbb{A})$ is just the semi-direct product of linear maps and translations of the vector space. We are interested in those affine maps that also preserve a distance induced by non-degenerate, symmetric, bilinear form B on $\mathbb{V}$. We will denote them as:

Remark 3.1.14. Let $(\mathbb{A}, \mathbb{V}, \varphi)$ be an affine space, let B be a non-degenerate, symmetric, bilinear form on $\mathbb{V}$ and let d be the distance on $\mathbb{A}$ induced by this form as $d(p,q)^2 = B(\overrightarrow{pq}, \overrightarrow{pq})$. We saw that if $f : \mathbb{A} \longrightarrow \mathbb{A}$ is a bijection that preserves the distance function, then $f \in \mathrm{Aff}(\mathbb{A})$ and its associated linear map $\tilde{f}$ preserves the form B. We will denote the set of such maps as $\mathrm{Aff}_B(\mathbb{A})$.

It is not difficult to see based on Remark 3.1.9 that $\mathrm{Aff}_B(\mathbb{A})$ is a group and that:

Corollary 3.1.15. *Let $(\mathbb{A}, \mathbb{V}, \varphi)$ be an affine space, let B be a non-degenerate, symmetric, bilinear form on $\mathbb{V}$. Then, $\mathrm{Aff}_B(\mathbb{A}) \cong O(r,s) \ltimes \mathbb{V}$*

Now we are in a position to define the Poincaré group, which is precisely the connected component of the group of affine maps that preserve the bilinear form of the Minkowski spacetime. Let $\mathcal{M}$ a Minkowski spacetime. That is, an affine space $(\mathcal{M}, \mathbb{R}^4, \varphi)$ with a symmetric, non-degenerate, bilinear form η of signature $(1,3)$ and the corresponding distance function defined as $d^2(p,q) = \eta(\overrightarrow{pq}, \overrightarrow{pq})$. From our previous discussion, the set of maps that preserve this distance function is the group $\mathrm{Aff}_B(\mathcal{M}) \cong O(1,3) \ltimes \mathbb{R}^4$. Now, this is, in turn, a Lie group, since it is the product of two Lie groups and the semidirect product operation is smooth.

Now, the Minkowski spacetime is the basic underlying structure for most of the physical theories that are accepted nowadays, at least locally. If one considers the effects of gravity, then we move on from a Minkowski spacetime and towards more general kinds of spacetimes. But, and this is a key point too, any of those more general spacetimes have to satisfy the condition of being locally isometric to the Minkowski spacetime.

Another key assumption of the currently accepted physical theories is that they have to be formulated in the same way independently of the observer. In terms of what we have introduced until now, an observer amounts to a choice of a reference in the Minkowski spacetime. And a change of observer is represented as an affine map that preserves the distance function. So this is why the group $\mathrm{Aff}_\eta(\mathcal{M})$ is of such fundamental importance.

Not all of the transformations in $\mathrm{Aff}_\eta(\mathcal{M})$ are going to be physically relevant for the discussion that follows. There are some elements of the group $O(1,3) \ltimes \mathbb{R}^4$ that represent discontinuous transformations, even though it is a smooth (Lie) group. By discontinuous, we mean transformations that cannot happen continuously by deforming the identity transformation. Such transformations involve reflections in space and reflections in time (time-reversing transformations). We will be interested in preserving space orientation and time orientation in the discussion that follows, and therefore we will define the group that will be of interest for us to be the **identity component** of the group of affine transformations of the Minkowski spacetime. We will name this group the Poincaré group.

Corollary 3.1.16. *Let $\mathbb{A}$ be an affine space with bilinear (non-degenerate, symmetric) form B of signature (r,s). Then, the identity component of its affine group is $\mathrm{Aff}_B(\mathbb{A})_0 \cong SO(r,s)_0 \ltimes \mathbb{R}^{r+s}$.*

Definition 3.1.17. *Let $\mathcal{M}$ be a Minkowski spacetime, η its bilinear form. The **Poincaré group** is the identity component of the group $\mathrm{Aff}_\eta(\mathcal{M})$. We will denote it by $\mathcal{P}$. We can see that $\mathcal{P} = SO(1,3)_0 \ltimes \mathbb{R}^4$. We will also denote by $\mathcal{L} = SO(1,3)_0$ and call it the restricted Lorentz group, or just the **Lorentz group** for short.*

Remark 3.1.18. The Poincaré group $\mathcal{P}$ is connected, since it is defined as the identity component of $\mathrm{Aff}_\eta(\mathcal{M})$.

We will end the section with two results on the Poincaré group that we will use in the last chapter.

Proposition 3.1.19. *The universal cover of the connected component of the or-*

thogonal group $O(1,3)$ is the special linear group $SL(2,\mathbb{C})$.

Proof. We will, as anticipated in 1.3.25, provide a sketch of the proof. We begin by denoting by $H = \{A \in \mathrm{M}_2(\mathbb{C}) : A = A^\dagger\}$ the set of self-adjoint matrices of dimension 2. It is a 4-dimensional real vector space, with a basis given by the usual Pauli matrices and the identity:

$$H = \left\langle \sigma_0 = \mathrm{Id}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \tag{3.2}$$

We note that there is a vector space isomorphim $\psi : \mathbb{R}^4 \longrightarrow H$ given by

$$\psi(x_0, x_1, x_2, x_3) = x_3\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_0\sigma_3 = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_0 + x_3 \end{pmatrix} \tag{3.3}$$

It is easy to check that if η is the bilinear form of signature $(1,3)$ given by the diagonal matrix $\mathrm{diag}\{-1,+1,+1,+1\}$, then

$$\|x\|^2 = \eta(x,x) = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = \det(\psi(x)) \tag{3.4}$$

This allows us to define the following Lie group homomorphism:

$$\varphi : \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}_4(\mathbb{R}), \qquad \varphi(A)(x) = \psi^{-1}(A\psi(x)A^\dagger). \tag{3.5}$$

We can see that $\varphi(A)$ preserves the inner product. Indeed, $\|\psi^{-1}(\varphi(A)(x))\|^2 = \det(\varphi(A)(x)) = \det(A\psi(x)A^\dagger)$. Then, since $\det(A) = \det(A^\dagger) = 1$ since $A \in \mathrm{SL}(2,\mathbb{C})$, we have that

$$\|\psi^{-1}(\varphi(A)(x))\|^2 = \det(\psi(x)) = \|x\| \tag{3.6}$$

Therefore, the image of φ is in $\mathrm{O}(1,3)$. Further, since the map is continuous and $\mathrm{SL}(2,\mathbb{C})$ is connected, the image is in the connected component of $\mathrm{O}(1,3)$, the Lorentz group $\mathcal{L}$. It can be proved that $\varphi : \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathcal{L}$ is a surjective Lie group homomorphism and that its kernel is $\mathrm{Ker}(\varphi) = \{\mathrm{Id}, -\mathrm{Id}\}$. Finally, the Lie algebras of $\mathrm{SL}(2,\mathbb{C})$ and of $\mathcal{L}$ are isomorphic (for example, by using Theorem 21.32 of [Lee13]). This proves that $\mathrm{SL}(2,\mathbb{C})$ is the universal cover of the Lorentz group. □

Since we can extend the covering map from the previous proof to the semi-direct product, we have that:

Corollary 3.1.20. *The universal cover of the Poincaré group is* $\tilde{\mathcal{P}} = \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$.

Remark 3.1.21. We will use further on the fact that the covering map given by the universal cover of the Poincaré group, $\varphi : \tilde{\mathcal{P}} \longrightarrow \mathcal{P}$, has $\mathrm{Ker}(\varphi) = \{\mathrm{Id}, -\mathrm{Id}\}$. In particular, this means that $\tilde{\mathcal{P}} = \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$ is a double cover of the Poincaré group.

3.2 References in spacetime

We will start by introducing some extra structure in our description of the Minkowski spacetime of the previous chapters. Remember that a Minkowski spacetime is an affine space $(\mathcal{M}, \mathbb{R}^4, \eta, \varphi)$, where η is a symmetric, non-degenerate, bilinear form on $\mathbb{R}^4$ of signature $(1,3)$. Remember also that this form induces a pseudo distance d^2 on $\mathcal{M}$. Further, remember that we proved that if $f : \mathcal{M} \longrightarrow \mathcal{M}$ is a bijective map that preserves the distance function, then f is an affine map, $f \in \mathrm{Aff}_\eta(\mathcal{M}) \cong O(1,3) \ltimes \mathbb{R}^4$. We will now define, in this context, what we mean by a reference in Minkowski spacetime.

Definition 3.2.1. *Let $(\mathcal{M}, \mathbb{R}^4, \eta, \varphi)$ be a Minkowski spacetime. A **reference frame** is a pair $(O, \{e_i\}_{i=1}^4)$, where $O \in \mathcal{M}$ and $\{e_i\}_i \subseteq \mathbb{R}^4$ is an orthonormal basis (i.e., $\eta(e_0, e_0) = -1$ and $\eta(e_i, e_i) = +1$ for $i = 1, 2, 3$).*

A reference frame on $\mathcal{M}$ induces a map $\phi : \mathbb{R}^4 \longrightarrow \mathcal{M}$ that turns $\mathcal{M}$ into a smooth manifold.

Proposition 3.2.2. *Let $(O, \{e_i\}_i)$ be a reference frame in a Minkowski space $\mathcal{M}$. Then, the map $\phi : \mathbb{R}^4 \longrightarrow \mathcal{M}$ defined as $\phi(x^1, \ldots, x^4) = \varphi(O, \sum_i x^i e_i)$ is a global chart for $\mathcal{M}$. Thus, $(\mathcal{M}, (\mathbb{R}^4, \phi))$ is a smooth manifold as defined in Chapter 1.*

With this smooth structure on the Minkowski space, we can consider such things as the tangent spaces on $\mathcal{M}$ and the vector fields on them. Before that, let's see how a change of reference is treated:

Proposition 3.2.3. *Let $\mathcal{M}$ be a Minkowski spacetime as above, and let $(O, \{e_i\}_i)$ and $(\tilde{O}, \{\tilde{e}_i\}_i)$ be references. Let $\phi, \tilde{\phi} : \mathbb{R}^4 \longrightarrow \mathcal{M}$ be their associated charts. Then, there exists $f \in \mathrm{Aff}_\eta(\mathcal{M})$ so that $\tilde{\phi} = f \circ \phi$.*

Proof. Define $f : \mathcal{M} \longrightarrow \mathcal{M}$ to be the map determined by $f(O) = \tilde{O}$ and the linear map $\Lambda : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined as $\Lambda(e_i) = \tilde{e}_i$. Since both bases are orthonormal with respect to η by definition, we know that Λ preserves the form η and therefore $\Lambda \in O(1,3)$. Then, f is simply the map $f = (\Lambda, v) \in \mathrm{Aff}_\eta(\mathcal{M})$, where $\varphi(O, v) = \tilde{O}$.

Further, it is clear that:

$$\tilde{\phi}(x^i) \equiv \varphi(\tilde{O}, \sum_i x^i \tilde{e}_i) = \varphi(f(O), \sum_i x^i \Lambda(e_i)) = f \circ \varphi(O, \sum_i x^i e_i) = f \circ \phi(x^i) \tag{3.7}$$

□

We can consider the tangent spaces $T_p\mathcal{M}$ as defined in Chapter 2. Further, given a reference frame and its associated chart, we can consider the tangent basis induced by the chart. Due to the affine structure of $\mathcal{M}$, we can write the basis tangent vectors in a different way.

Proposition 3.2.4. *Let $(\mathcal{M}, \mathbb{R}^4, \eta, \varphi)$ be a Minkowski space and let $(O, \{e_i\}_i)$ be a reference frame. Let $\phi : \mathbb{R}^4 \longrightarrow \mathcal{M}$ be the chart induced by the reference. Then, the basis tangent vectors induced by the chart at the point $p \in \mathcal{M}$, $(\partial_i\phi)_p$, are given, for any smooth function $\alpha \in C_p^\infty(\mathcal{M})$ (that is, $\alpha : U \subseteq \mathcal{M} \longrightarrow \mathbb{R}$ smooth at $p \in U$) by:*

$$(\partial_i\phi)_p(\alpha) = \frac{d}{dt}\alpha \circ \varphi(p, te_i)\Big|_{t=0} \tag{3.8}$$

Proof. Remember from Example 1.1.8 that the tangent basis vectors induced by a chart $\phi : \mathbb{R}^4 \longrightarrow \mathcal{M}$ are given by

$$(\partial_i\phi)_p(\alpha) = \frac{d}{dt}\alpha \circ \phi(\phi^1(p), \dots, \phi^i(p) + t, \dots, \phi^n(p))\Big|_{t=0} \tag{3.9}$$

But by using the definition of the chart ϕ, which is induced by the reference frame $(O, \{e_i\}_i)$, we can see that

$$\phi(\phi^1(p), \dots, \phi^i(p) + t, \dots, \phi^n(p)) = \varphi(O, \sum_j \phi^j(p)e_j + te_i) = \varphi(p, te_i) \tag{3.10}$$

since $(\phi^1(p), \dots, \phi^n(p)) = \phi^{-1}(p)$ by construction. □

With this expression for the tangent basis vectors, it is easy to see how different basis induced by different references are related. We will need the following lemma:

Lemma 3.2.5. *Let $D_v = \sum_i v^i(\partial_i\phi)_p \in T_p\mathcal{M}$ be a tangent vector at the point p of a Minkowski spacetime $(\mathcal{M}, \mathbb{R}^4, \eta, \varphi)$, where ϕ is the chart induced by a reference frame $(O, \{e_i\}_i)$. Then, we can write, for a given smooth function α,*

$$D_v(\alpha) = \left(\sum_i v^i(\partial_i\phi)_p\right)(\alpha) = \frac{d}{dt}\alpha \circ \varphi\left(p, t\left(\sum_i v^i e_i\right)\right)\Big|_{t=0} \tag{3.11}$$

Proof. It all comes down to a straightforward although cumbersome application of the chain rule in real analysis. First of all, we define, given $v \in \mathbb{R}^4$, the curve $\gamma_v : \mathbb{R} \longrightarrow \mathbb{R}^4$ given by $\gamma_v(t) = tv = (tv^1, \dots, tv^4)$. We also define, given $\alpha \in$

$C_p^\infty(\mathcal{M})$, the function $\psi_\alpha : \mathbb{R}^4 \longrightarrow \mathbb{R}$ given by $\psi_\alpha(x^1, ..., x^4) = \alpha \circ \varphi(p, \sum_i x^i e_i)$. It looks cumbersome but we've just decomposed:

$$(\partial_i \phi)_p(\alpha) = \frac{d}{dt}\alpha \circ \varphi(p, te_i)\Big|_{t=0} = \frac{d}{dt}\psi_\alpha \circ \gamma_{e_i}(t)\Big|_{t=0} \tag{3.12}$$

With the difference that now both functions γ_{e_i} and ψ_α are smooth functions between real spaces. Therefore, we can use the chain rule to see that:

$$\frac{d}{dt}\psi_\alpha \circ \gamma_v(t)\Big|_{t=0} = \sum_j \frac{\partial \psi_\alpha}{\partial x_j}\Big|_{x=0} \cdot \frac{d(\gamma_v)_i}{dt}\Big|_{t=0} = \sum_j \frac{\partial \psi_\alpha}{\partial x_j}\Big|_{x=0} \cdot v^j \tag{3.13}$$

But using the same expression, we can see that:

$$(\partial_i \phi)_p(\alpha) = \frac{d}{dt}\psi_\alpha \circ \gamma_{e_i}(t)\Big|_{t=0} = \frac{\partial \psi_\alpha}{\partial x_i}\Big|_{x=0} \tag{3.14}$$

And therefore,

$$\frac{d}{dt}\psi_\alpha \circ \gamma_v(t)\Big|_{t=0} = \sum_i v^i (\partial_i \phi)_p(\alpha) \tag{3.15}$$

But by expanding the left-hand side we can see that by construction,

$$\psi_\alpha \circ \gamma_v(t) = \alpha \circ \varphi(p, t \sum_i v^i e_i) \tag{3.16}$$

This finishes the proof, since we got that:

$$D_v(\alpha) = \left(\sum_i v^i (\partial_i \phi)_p\right)(\alpha) = \sum_i v^i \left((\partial_i \phi)_p\right)(\alpha) = \frac{d}{dt}\psi_\alpha \circ \gamma_v(t)\Big|_{t=0} = \frac{d}{dt}\alpha \circ \varphi\left(p, t\left(\sum_i v^i e_i\right)\right)\Big|_{t=0} \tag{3.17}$$

□

Corollary 3.2.6. *Let $(O, \{e_i\})$ and $(\tilde{O}, \{\tilde{e}_i\}_i)$ be reference frames for a Minkowski space $(\mathcal{M}, \mathbb{R}^4, \eta, \varphi)$ with associated charts ϕ and $\tilde{\phi}$. Let $f = (\Lambda, v) \in \mathrm{Aff}_\eta(\mathcal{M})$ be the affine map so that $\tilde{\phi} = f \circ \phi$. Then:*

$$(\partial_i \tilde{\phi})_p = \sum_j \Lambda_i^j (\partial_j \phi)_p \in T_p\mathcal{M} \tag{3.18}$$

Where $\Lambda_i^j \in \mathbb{R}$ are defined as $\tilde{e}_i = \Lambda(e_i) = \sum_j \Lambda_i^j e_j$.

Proof. Let $\alpha \in C_p^\infty(\mathcal{M})$, a real function smooth at a point $p \in \mathcal{M}$. The result follows from the previous lemma. Let's consider the action of the basis tangent vectors of the chart $\tilde{\phi}$:

$$(\partial_i \tilde{\phi})_p(\alpha) = \frac{d}{dt}\alpha \circ \varphi(p, t\tilde{e}_i)\Big|_{t=0} = \frac{d}{dt}\alpha \circ \varphi(p, t\sum_j \Lambda_i^j e_j)\Big|_{t=0} = \left(\sum_j \Lambda_i^j (\partial_j \phi)_p\right)(\alpha) \tag{3.19}$$

□

Therefore, the tangent vectors associated with a chart follow the same change of reference rules that the vectors themselves:

$$e_i \longmapsto \sum_j \Lambda^j_i e_j = \tilde{e}_i \tag{3.20}$$

$$(\partial_i \phi)_p \longmapsto \sum_j \Lambda^j_i (\partial_j \phi)_p = (\partial_i \tilde{\phi})_p \tag{3.21}$$

It can be seen by using the definition of the dual tangent basis that it transforms in the opposite way, that is, involving the inverse matrix of Λ. That is:

$$\begin{aligned} e_i &\longmapsto \sum_j \Lambda^j_i e_j = \tilde{e}_i \\ (d\phi^i)_p &\longmapsto \sum_j (\Lambda^{-1})^i_j (d\phi^j)_p = (d\tilde{\phi}^i)_p \end{aligned} \tag{3.22}$$

To finish off this digression about observers and the action of the Poincaré group on the tangent bundle of the Minkowski spacetime, we can briefly say how this comes into play when modeling fields in classical field theory. This is one of the starting points towards gauge theory and quantum field theory.

Remark 3.2.7. Let $\mathcal{M}$ be a Minkowski spacetime and consider a tensor field $T \in \Gamma(T^{r,s}\mathcal{M}, \mathcal{M})$ (c.f. Definition 1.2.9). Then, given a reference $(\mathcal{O}, \{e_i\})$ with associated chart $\phi : \mathbb{R}^4 \longrightarrow \mathcal{M}$ and associated basis for the tangent spaces $T_p\mathcal{M}$ given by $\{(\partial_i \phi)_p\}_{i=0}^3$, we can write the components of the tensor field with respect to this chart (c.f. Definition 1.2.10) as:

$$T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(p) = T_p\left((d\phi^{i_1})_p, \dots, (d\phi^{i_r})_p, (\partial_{j_1}\phi)_p, \dots, (\partial_{j_s}\phi)_p\right) \in \mathbb{R}, \tag{3.23}$$

where all indices run from 0 to 3.

To make the expressions more manageable, consider instead a rank $(0, 2)$ tensor field $F \in \Gamma(T^{0,2}\mathcal{M}, \mathcal{M})$ and its components in the given reference:

$$F_{ij}(p) = F_p((\partial_i \phi)_p, (\partial_j \phi)_p), \qquad i, j = 0, \dots, 3. \tag{3.24}$$

This allows us to write the tensor field F in terms of the tangent dual basis as:

$$F_p = \sum_{i=0}^{3} \sum_{j=0}^{3} F_{ij}(p)(d\phi^i)_p \otimes (d\phi^j)_p \tag{3.25}$$

Now, and coming to the point of the whole example, if we consider another reference $(\tilde{\mathcal{O}}, \{\tilde{e}_i\}_i)$ with associated chart $\tilde{\phi}$ and affine map $f = (\Lambda, v) \in$

$\mathrm{Aff}_\eta(\mathcal{M})$ so that $\tilde{\phi} = f \circ \phi$, we can see how the components of F change when expressing them in a different basis. Indeed, we can denote by $\tilde{F}_{ij}$ the components of F in the tangent dual basis $\{(d\tilde{\phi}^i)_p\}_{i=0}^3$, defined as:

$$\tilde{F}_{ij}(p) = F_p((\partial_i\tilde{\phi})_p, (\partial_j\tilde{\phi})_p), \qquad i, j = 0, \ldots, 3. \tag{3.26}$$

and write

$$F_p = \sum_{i=0}^{3}\sum_{j=0}^{3} \tilde{F}_{ij}(p)(d\tilde{\phi}^i)_p \otimes (d\tilde{\phi}^j)_p. \tag{3.27}$$

If we use the change of basis rules from eq. 3.22 we can see that indeed the components of F change as:

$$\tilde{F}_{ij}(p) = F_p((\partial_i\tilde{\phi})_p, (\partial_j\tilde{\phi})_p) = \sum_k\sum_l \Lambda_i^k\Lambda_j^l F_p((\partial_k\phi)_p, (\partial_l\phi)_p) = \sum_k\sum_l \Lambda_i^k\Lambda_j^l F_{kl}(p) \tag{3.28}$$

And therefore by using eq. 3.22 together with eq. 3.27 we can see that both choices of references agree when the corresponding changes of basis in the tangent bundles are taken into account.

$$\begin{aligned} F_p &= \sum_{i,j} \tilde{F}_{ij}(p)(d\tilde{\phi}^i)_p \otimes (d\tilde{\phi}^j)_p = \\ &= \sum_{i,j}\left(\sum_{k,l} \Lambda_i^k\Lambda_j^l F_p((\partial_k\phi)_p, (\partial_l\phi)_p)\right)\left(\sum_{m,n}(\Lambda^{-1})_m^i(\Lambda^{-1})_n^j(d\phi^m)_p \otimes (d\phi^n)_p\right) = \\ &= \sum_{k,l,m,n} \delta_m^k\delta_n^l F_p((\partial_k\phi)_p, (\partial_l\phi)_p)(d\phi^m)_p \otimes (d\phi^n)_p = F_p \end{aligned} \tag{3.29}$$

Incidentally, this last equation provides an example of how far can we go without introducing Einstein's summation convention, since the above computation is made substantially clearer when one adopts such a convention.

The point of this last rather cumbersome example is to show how the definitions given about references and how they relate to charts and components of tensor fields agree with their expected behavior. This is a point that is presented in the physics literature with a considerably less degree of formality, leaving in a vague state some definitions such as those of a reference frame. The fact that we can write equations about tensor fields in these two ways (that is, by either using their components in a given reference frame or by using the tensors themselves without referring to any particular reference) is at the core of what is commonly referred in the physics literature as "general covariance" of a theory in physics. Note that any tensor field in a Minkowski spacetime is invariant under the action of the Poincaré group on any reference, as the example above shows. We will leave this discussion here and continue our discussion about elementary particles in the next chapter.

4

Elementary particles and the Poincaré group

After introducing the necessary mathematical machinery in the first chapter, the required notions of quantum mechanics in the second chapter, and the Poincaré group in the third chapter, we are now in a position to culminate our work by putting all the pieces together and being able to define what an elementary particle is understood to be. In this fourth and last chapter, we will introduce the notion that allows us to tie together Lie groups and quantum systems: that of a group representation. We will discuss how quantum symmetries are related to projective unitary representations of Lie groups and discuss the fundamental result by V. Bargmann that allows us to talk about unitary representations instead of projective ones. Finally, the definition of an elementary particle will be given and discussed.

4.1 Unitary and projective unitary representations

We will start by introducing the usual definition of a finite-dimensional representation of a Lie group and a Lie algebra.

Definition 4.1.1. *Let G be a Lie group and V be a finite-dimensional vector space over $\mathbb{K}$. Then, a representation of G on V is a Lie group homomorphism*

$$\Pi : G \longrightarrow \mathrm{Aut}(V). \tag{4.1}$$

That is, it is a smooth group homomorphism. Similarly, we can think of representations of Lie algebras on vector spaces as being Lie algebra homomorphisms between the Lie algebra and the algebra of endomorphisms of the vector space.

Definition 4.1.2. *Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra and V a finite-dimensional vector space. A Lie algebra representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism*

$$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V) \tag{4.2}$$

One of the obvious results of the theory that we have introduced in the previous sections is that

Corollary 4.1.3. *Let $\rho : G \longrightarrow \mathrm{Aut}(V)$ be a Lie group representation. Then, $d_e\rho : \mathrm{Lie}(G) \longrightarrow \mathrm{End}(V)$ is a Lie algebra representation on V. The exponential map of G of $\mathrm{Aut}(V)$ commutes with these representations in an obvious way.*

Example 4.1.4. For a basic example of a finite-dimensional representation of a Lie group, one can think of the real general linear group of dimension n, $\mathrm{GL}_n(\mathbb{R})$, and its obvious action on $\mathbb{R}^n$ by matrix multiplication:

$$\rho : \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \rho(A, x) = A \cdot x \in \mathbb{R}^n. \tag{4.3}$$

Then, for every $A \in \mathrm{GL}_n(\mathbb{R})$, the map $\Pi : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R})$ given by $\Pi(A) = \rho(A, .)$ is a Lie group homomorphism and therefore a representation. Similarly, the restriction of this representation to any matrix Lie group $G \subseteq \mathrm{GL}_n(\mathbb{R})$ will also induce a representation. This representation of a matrix Lie group on $\mathbb{R}^n$ is called the standard representation.

Example 4.1.5. For an example of a representation of a Lie algebra, one can take the differential map of a Lie group representation. Indeed, let $\Pi : G \longrightarrow \mathrm{GL}(V)$ be a representation on the finite-dimensional vector space V. Since it is a smooth map between Lie groups, we can consider the differential of the map at the identity $e \in G$, giving a Lie algebra homomorphism:

$$d_e\Pi : T_eG \cong \mathrm{Lie}(G) \longrightarrow T_{\mathrm{Id}}\mathrm{GL}(V) \cong \mathrm{Aut}(V) \tag{4.4}$$

Most of the representations of groups that we will need in our work are not, however, finite-dimensional as the ones we have introduced so far. That happens because we are interested in representations where the vector space is a quantum system, and we did not restrict our definition of quantum systems to only finite-dimensional ones. This means that Definitions 4.1.1 and 4.1.2 will no longer be of use in those situations.

We will work, instead, with unitary representation in Hilbert spaces.

Definition 4.1.6. *Let G be a Lie group and $\mathcal{H}$ be a Hilbert space. A* ***unitary representation*** *of G on $\mathcal{H}$ is a group homomorphism $\Pi : G \longrightarrow \mathcal{U}(\mathcal{H})$. The representation is said to be* ***continuous*** *if the map $\Pi^{\phi} : G \longrightarrow \mathcal{H}$ given by $\Pi^{\phi}(g) = \Pi(g)(\phi)$ is continuous for all $\phi \in \mathcal{H}$.*

We will usually denote the image of the unitary map $\Pi(g)$ at $\phi \in \mathcal{H}$ as $\Pi_g(\phi)$ instead of as $\Pi(g)(\phi)$.

Definition 4.1.7. *Let G be a Lie group and $(\mathcal{H}, \langle\, ,\rangle)$ a Hilbert space. A* ***projective unitary representation*** *of G on $\mathcal{H}$ is a group homomorphism $\Pi : G \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$. It is said to be* ***continuous*** *if the map $\Pi^{\phi,\psi} : G \longrightarrow \mathbb{R}$ given by $\Pi^{\phi,\psi}(g) = \langle \Pi_g([\phi]), \Pi_g([\psi])\rangle_{\mathbb{P}}$ is continuous for all $\phi, \psi \in \mathcal{H} \neq \{0\}$.*

Remember the group homomorphism $p_{\mathcal{U}} : \mathcal{U}(\mathcal{H}) \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$.

Proposition 4.1.8. *Let G be a Lie group, $\mathcal{H}$ a Hilbert space and $\Pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ be a continuous unitary representation. Then, $\tilde{\Pi} = p_{\mathcal{U}} \circ \Pi : G \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$ is a continuous projective unitary representation.*

Proof. $\tilde{\Pi}$ is a group homomorphism since it is the composition of two group homomorphisms. It remains to be checked that it is continuous in the projective sense. Given $\phi, \psi \in \mathcal{H} \setminus \{0\}$, consider the map $\tilde{\Pi}^{\phi,\psi} : G \longrightarrow \mathbb{R}$ given by

$$\tilde{\Pi}^{\phi,\psi}(g) = \langle \tilde{\Pi}_g([\phi]), \tilde{\Pi}_g([\psi])\rangle_{\mathbb{P}}. \tag{4.5}$$

By the definition of $\langle\, ,\rangle_{\mathbb{P}}$ and by the definition of $\tilde{\Pi}$, we can write it as:

$$\tilde{\Pi}^{\phi,\psi}(g) = \langle [\Pi_g(\phi)], [\Pi_g(\psi)]\rangle_{\mathbb{P}} = \frac{\langle \Pi_g(\phi), \Pi_g(\psi)\rangle^2}{|\Pi_g(\phi)|^2 \cdot |\Pi_g(\psi)|^2} = \frac{\langle \Pi^{\phi}(g), \Pi^{\psi}(g)\rangle^2}{|\Pi^{\phi}(g)|^2 \cdot |\Pi^{\psi}(g)|^2} \tag{4.6}$$

The denominator never vanishes for $\phi, \psi \neq 0$, since Π_g is a unitary map and therefore $\Pi_g(\phi) \neq 0$. Therefore, the norm does not vanish either. Now, this expression is continuous as a function of g by the continuity of Π^{ϕ} for every $\phi \neq 0$. This finishes the proof. □

Definition 4.1.9. *We will say that a unitary representation $\Pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ is* ***irreducible*** *if the only vector subspaces $\mathcal{V} \subseteq \mathcal{H}$ with the property that $\Pi_g(\mathcal{V}) \subseteq \mathcal{V}$ for every $g \in G$ are $\{0\}$ and $\mathcal{H}$.*

Similarly, a projective unitary representation $\Pi : G \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$ *is said to be irreducible if the only vector subspaces* $\mathcal{V} \subseteq \mathcal{H}$ *with the property that* $\Pi_g(\mathbb{P}\mathcal{V}) \subseteq \mathbb{P}\mathcal{V}$ *for every* $g \in G$ *is* $\mathcal{H}$.

Definition 4.1.10. *Let* $\Pi : G \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$ *be a continuous projective unitary representation. A* ***unitary lift*** *of* Π *is a continuous unitary representation* $\tilde{\Pi} : G \longrightarrow \mathcal{U}(\mathcal{H})$ *so that* $\Pi = p_{\mathcal{U}} \circ \tilde{\Pi}$.

Then, the lift can be seen as “inducing” the projective representation. A lift makes the following diagram commute:

$$\begin{tikzcd} G \arrow[d, "\tilde{\Pi}"] \arrow[dr, "\Pi"] & \\ \mathcal{U}(\mathcal{H}) \arrow[r, "p_{\mathcal{U}}"] & \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H}) \end{tikzcd} \tag{4.7}$$

The next result allows us to think of projective unitary representations of a connected Lie group and unitary representations of its universal cover equivalently, under the very strong assumption that every projective unitary representation of the covering group has a unitary lift.

Theorem 4.1.11. *Let* G *be a connected Lie group and assume that its universal cover* $\tilde{G}$ *has the property that every continuous projective unitary representation has a unitary lift. Then:*

(i) Given a Hilbert space $\mathcal{H}$ *and a projective unitary representation* $\Pi : G \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$*, there is a unitary representation* $\tilde{\Pi}$ *of its universal cover so that* $\Pi \circ \varphi = p_{\mathcal{U}} \circ \tilde{\Pi}$.

(ii) Given a Hilbert space $\mathcal{H}$ *and a unitary representation* $\tilde{\Pi} : \tilde{G} \longrightarrow \mathcal{U}(\mathcal{H})$ *so that* $\tilde{\Pi}(\mathrm{Ker}(\varphi)) \subseteq \mathcal{U}(1)$*, then there is a unique projective unitary representation* Π *of* G *on* $\mathcal{H}$ *so that* $\Pi \circ \varphi = p_{\mathcal{U}} \circ \tilde{\Pi}$.

A way in which the above result can be visualized is by realizing that we have the following two short exact sequences of groups, with the dashed arrows standing for a possible unitary or projective unitary representation if there’s any.

$$\begin{tikzcd} \{e\} \arrow[r, hook] & \mathrm{Ker}(\varphi) \arrow[r, hook, "\mathrm{inc}"] & \tilde{G} \arrow[r, "\varphi"] \arrow[d, dashed, "\tilde{\Pi}"] & G \arrow[r] \arrow[d, dashed, "\Pi"] & \{\mathrm{Id}\} \\ \{\mathrm{Id}\} \arrow[r, hook] & \mathcal{U}(1) \arrow[r, hook, "\mathrm{inc}"] & \mathcal{U}(\mathcal{H}) \arrow[r, "p_{\mathcal{U}}"] & \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H}) \arrow[r] & \{\mathrm{Id}\} \end{tikzcd} \tag{4.8}$$

Then, the existence of a projective unitary representation Π of G implies the existence of a unitary representation $\tilde{\Pi}$ of the covering group $\tilde{G}$. Conversely, the existence of a unitary representation $\tilde{\Pi}$ of $\tilde{G}$ with $\mathrm{Ker}(\varphi) \subseteq \mathcal{U}(1)$ implies the existence of a unique projective unitary representation Π of G.

Proof. (i) Let $\Pi : G \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$ be a projective unitary representation. Then, $\Pi \circ \varphi : \tilde{G} \longrightarrow \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H})$ is also a projective unitary representation. By the assumption on $\tilde{G}$, this representation has a unitary lift $\tilde{\Pi} : \tilde{G} \longrightarrow \mathcal{U}(\mathcal{H})$ and by definition of a unitary lift, $\Pi \circ \varphi = p_{\mathcal{U}} \circ \tilde{\Pi}$ as we wanted.

(ii) Let $\tilde{\Pi} : \tilde{G} \longrightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation with $\tilde{\Pi}(\mathrm{Ker}(\varphi)) \subseteq \mathcal{U}(1)$. Then, $\tilde{\Pi}$ defines a group homomorphism on the quotient groups $\Pi : \tilde{G}/\mathrm{Ker}(\varphi) \longrightarrow \mathcal{U}(\mathcal{H})/\mathcal{U}(1)$ given by $\Pi([g]) = [\tilde{\Pi}(g)]$. But since we saw that:

$$\tilde{G}/\mathrm{Ker}(\varphi) \cong G, \qquad \mathcal{U}(\mathcal{H})/\mathcal{U}(1) \cong \mathrm{Aut}_{\mathcal{U}}(\mathbb{P}\mathcal{H}), \tag{4.9}$$

then the induced homomorphism Π is a projective unitary representation and it gives $\Pi \circ \varphi = p_{\mathcal{U}} \circ \tilde{\Pi}$ by construction. The uniqueness of Π can be seen by assuming that there is another projective unitary representation Π' with the same property and with $\Pi(h) \neq \Pi'(h)$ for some $h \in G$. But then for any $g \in \varphi^{-1}(h) \subseteq \tilde{G}$, we have that $\Pi \circ \varphi(g) \neq \Pi' \circ \varphi(g)$, which contradicts the assumption.

□

It now remains to be seen whether that rather strong assumption on the universal cover of a connected Lie group, namely, that every projective unitary representation has a unitary lift, is a reasonable assumption to make for the Lie groups that will be of interest to us. The answer is, surprisingly, a positive one, and it comes under the name of Bargmann's theorem, which gives a condition for the existence of those unitary lifts. This particular form of Bargmann's theorem follows [Mor19] Theorem 7.14.

Theorem 4.1.12. *Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Assume that for every bilinear skew-symmetric map $\varphi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ such that*

$$\varphi([x, y], z) + \varphi([y, z], x) + \varphi([z, x], y) = 0 \tag{4.10}$$

there exists a map $\alpha : \mathfrak{g} \longrightarrow \mathbb{R}$ so that $\varphi(x, y) = \alpha([x, y])$, for all $x, y \in \mathfrak{g}$.

Then, every projective unitary representation of G has a unitary lift.

A proof of this result can be found in [Mor18] Theorem 12.72. The original proof due to Bargmann can be found in [Bar47], where the result that we are going to be using corresponds to Theorem 3.2. Note that several formulations of Bargmann's theorem are stated by using the cohomology theory of Lie algebras, such as [Sim06] (Section 2 Theorem 4).

4.2 Elementary particles

The importance of the definitions and results of the last section comes from the fact that they capture the idea of what is meant by saying that a Lie group is a symmetry group of a quantum system. The key idea is that given a Lie group G and a quantum system $\mathcal{H}$, we will say that G is a symmetry of the system if there is a continuous projective unitary representation of G on $\mathcal{H}$. Intuitively, one can think of the slightly more natural concept of a group action, and define a group G to be a symmetry group of a quantum system whenever there is an action of G on the group of symmetries of $\mathcal{H}$.

Remark 4.2.1. A group action of a Lie group G on the group of symmetries of $\mathcal{H}$ would be a map $\rho : G \times \mathrm{Aut}(\mathbb{P}\mathcal{H}) \longrightarrow \mathrm{Aut}(\mathbb{P}\mathcal{H})$ so that $\rho(e, .)$ is the identity map of $\mathrm{Aut}(\mathbb{P}\mathcal{H})$ and so that $\rho(g, \rho(h, .)) = \rho(gh, .)$. To make it into a continuous symmetry, we would further demand a continuity condition similar to the one in Definition 4.1.7. In summary, such an action would end up being equivalent to a projective unitary representation.

Thus, we define the notion of a Lie group being a symmetry group of a quantum system as follows:

Definition 4.2.2. *Let $\mathcal{H}$ be a quantum system and let G be a Lie group. We will say that G is a* ***symmetry group of the system*** *if there is a projective unitary representation of G on $\mathcal{H}$.*

For an extended discussion about this definition the interested reader can refer to [Wei05] (section 2.2) and the original papers by E.P. Wigner and V. Bargmann, such as [BW88] or [Wig39].

The connection between the different sections of this work, relating the appearance of the Poincaré group with the discussion on quantum mechanics, is as follows. Due to fundamental considerations in theoretical physics, a necessary condition for a quantum system to be consistent with the special theory of relativity is to have the Poincaré group as a group of symmetries. Cf [Wei05] section 2.2 for a discussion about this topic.

Definition 4.2.3. *A quantum system $\mathcal{H}$ is called* ***relativistic*** *if the Poincaré group $\mathcal{P}$ is a symmetry group of the system.*

We want to exploit the results of the previous chapter concerning lifts of projective unitary representations. The first result that is of importance in that direction is Bargmann's theorem. In fact, the universal cover of the Poincaré group satisfies the hypothesis of Bargmann's theorem (Theorem 4.1.12).

Proposition 4.2.4. *Let $\mathcal{P}$ be the Poincaré group and $\tilde{\mathcal{P}}$ be its universal cover. Then, every projective unitary representation of $\tilde{\mathcal{P}}$ has a unitary lift.*

This is equivalent to saying that the Poincaré group satisfies the hypotheses of Bargmann's theorem. A discussion on that can be found in [Bar47] (Section 6). Alternatively, also in [Mor18] (Proposition 12.76).

With this result, we can finally establish the equivalence between relativistic quantum systems and unitary representations of the universal cover of the Poincaré group, as in Theorem 4.1.12.

Corollary 4.2.5. *A quantum system $\mathcal{H}$ is relativistic if and only if there is a unitary representation $\Pi : \tilde{\mathcal{P}} \longrightarrow \mathcal{U}(\mathcal{H})$ of the universal cover of the Poincaré group.*

Proof. We saw in Remark 3.1.21 that the kernel of the covering map φ of the Poincaré group is $\mathrm{Ker}(\varphi) = \{\mathrm{Id}, -\mathrm{Id}\}$. The result follows from Theorem 4.1.11. □

This leads to the last definition of the work, and the one towards which we have been working all the way through.

Definition 4.2.6. *A relativistic quantum system $\mathcal{H}$ will be called an **elementary particle** if the unitary representation of the universal cover of the Poincaré group is irreducible.*

This definition marks the end of our progress. It is, however, the starting point of many other studies and discussions, such as:

(i) Still on the mathematical side, one can start working with this definition and move on towards studying the representation theory of the Poincaré group and its universal cover. The classification of those representations is known as Wigner's classification.

(ii) On the philosophical side, however, Definition 4.2.6 provides a precise definition of what an elementary particle is. Looking for the fundamental constituents of nature has been a relevant question ever since humanity started to think about nature.

Regarding (*i*), a study and eventual classification of the unitary representations of the universal cover of the Poincaré group is far from trivial. Even though this classification was done by Wigner in his original paper [Wig39] (Section 6) by using his "little groups" method, the problem was not solved with generality until G. Mackey published his work on induced representations. The problem of studying the representation theory of Poincaré's group (or of its universal cover) is part of the more general problem of classifying the representations of a semidirect product of Lie groups when one knows the representation theory of the factors of the product. A mathematical treatment of the theory of induced representations of groups, including the study of representations of semidirect products, is out of the scope of this work. The interested reader can find a complete treatment of the subject in G. Mackey's original work, [Mac68], in the more modern work [Var07] (chapter VI) and in [Sim06] (Chapter 6, 7 and 8). [Ste95] (section 3.9) contains a lighter and more practical approach.

Regarding (*ii*), a short discussion on the relevance of Wigner's theorem and classification can be found in [Ste95] (section 3.9, pgs 148-50). Wigner's classification of elementary particles is widely recognized as a central result in mathematical physics. It is also a starting point for the standard model of particle physics, which studies the classification of elementary particles and their interactions through their symmetry groups. Then, the Poincaré group is the first symmetry group that is introduced into the theory, since it is a symmetry that has to be satisfied by any relativistic system, with the resulting Wigner's classification of elementary particles. This classification, though, only classifies particles according to two parameters that label the representation of the Poincaré group on their Hilbert space (these two parameters end up being the "spin" of the particle, a half-integer, and the mass of the particle, a positive real number, with no further restrictions). If more specific symmetry groups are added to the model (specific in the sense that they model the properties of specific particles and are not shared by all of the particles in nature), a finer classification of elementary particles is obtained, giving rise to the notions of "color", "flavour", "charge", etc.

Both of these considerations lead towards the standard model of particle physics. A general study of the standard model requires a large number of prerequisites in a wide range of fields. A starting point to motivate the study of the standard model from the algebraic viewpoint is [BH10] and also [BM94]. A more general but still mathematical treatment, including most of the mathematical prerequisites for gauge theory (mostly differential geometry), can be found in the excellent [Ham17]. Once out of those purely mathematical treatments, even more general accounts exist, but they usually require some knowledge of the theory of quantum fields. Mathematical treatments of quantum field theory can be found in [Tic99] and in [Fol08]. For a more physical approach, [Jos65], the very extensive [Wei05] and lastly [Dir66].

The author of this work hopes that the contents presented inside it, as well as the ideas and works, referred to outside of it, trigger the reader's curiosity to know more about the topics discussed here. The fruits that grow out of the collaboration between mathematics and physics have been providing for a long time some of nature's most precise and beautiful descriptions, and that is reason enough to continue to keep an interest in those subjects.

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