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ORIGINAL PAPER

Some new restricted maximal operators of Fejér means of Walsh–Fourier series

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Abstract

In this paper, we derive the maximal subspace of natural numbers $\{n_k : k \geq 0\}$, such that the restricted maximal operator, defined by $\sup_{k\in\mathbb{N}} \left|\sigma_{n_k} F\right|$ on this subspace of Fejér means of Walsh–Fourier series is bounded from the martingale Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$. The sharpness of this result is also proved.

Keywords Walsh system · Fejér means · Martingale Hardy space · Maximal operators · Restricted maximal operators

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1 Introduction

All symbols used in this introduction can be found in Sect. 2.

In the one-dimensional case, the weak (1,1)-type inequality for the maximal operator σ^* of Fejér means σ_n with respect to the Walsh system, defined by

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$$

was investigated in Schipp [11] and Pál and Simon [7]. Fujii [1] and Simon [13] proved that σ^* is bounded from H_1 to L_1. Weisz [19] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [12] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [3]. Moreover, in [4], he proved that there exists a martingale $F \in H_p$ $(0 < p \leq 1/2)$, such that

$$\sup_{n\in\mathbb{N}} \|\sigma_n F\|_p = +\infty.$$

Weisz [22] proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

To study convergence of subsequences of Fejér means and their restricted maximal operators on the martingale Hardy spaces $H_p(G)$ for $0 < p \leq 1/2$, the central role is played by the fact that any natural number $n \in \mathbb{N}$ can be uniquely expression as $n = \sum_{k=0}^{\infty} n_j 2^j, \quad n_j \in Z_2 \ (j \in \mathbb{N})$, where only a finite numbers of n_j differ from zero and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are defined by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) = |n| - [n]$$

and

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \quad \text{for all } n \in \mathbb{N}.$$

Weisz [21] (see also [20]) also proved that for any $F \in H_p(G)$ $(p > 0)$, the maximal operator $\sup_{n\in\mathbb{N}} |\sigma_{2^n} F|$ is bounded from the Hardy space H_p to the Lebesgue space L_p. Furthermore, in [8] was generalized this result and it proved that if $0 < p \leq 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive numbers, such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) \leq c < \infty, \tag{1.1}$$

then the maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} F = \sup_{k\in\mathbb{N}} \left|\sigma_{n_k} F\right|,$$

is bounded from the Hardy space H_p to the Lebesgue space L_p. Moreover, if $0 < p < 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive numbers, such that $\sup_{k\in\mathbb{N}} \rho(n_k) = \infty$, then there exists a martingale $F \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \left\| \sigma_{n_k} F \right\|_p = \infty.$$

From this fact, it follows that if $0 < p < 1/2$, $f \in H_p$ and $\{n_k : k \geq 0\}$ is any sequence of positive numbers, then $\sigma_{n_k} f$ are uniformly bounded from the Hardy space H_p to the Lebesgue space L_p if and only if the condition (1.1) is fulfilled. Moreover, condition (1.1) is necessary and sufficient condition for the boundedness of subsequence $\sigma_{n_k} f$ from the Hardy space H_p to the Hardy space H_p.

In [18], it was proved some results which in particular, implies that if $f \in H_{1/2}$ and $\{n_k : k \geq 0\}$ is any sequence of positive numbers, then $\sigma_{n_k} f$ are bounded from the Hardy space $H_{1/2}$ to the space $H_{1/2}$ if and only if, for some c,

$$\sup_{k\in\mathbb{N}} V(n_k) < c < \infty.$$

In this paper, we complement the reported research above by investigating the limit case $p = 1/2$. In particular, we derive the maximal subspace of natural numbers $\{n_k : k \geq 0\}$, such that restricted maximal operator, defined by $\sup_{k\in\mathbb{N}} \left|\sigma_{n_k} F\right|$ on this subspace of Fejér means of Walsh–Fourier series is bounded from the martingale Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

This paper is organized as follows: some definitions and notations are presented in Sect. 2. The main result (Theorem 3.1) and some of its consequences can be found in Sect. 3. For the proof of the main result, we need some auxiliary statements, some of them are new and of independent interest. These results are presented in Sect. 4. The detailed proof of Theorem 3.1 is given in Sect. 5.

2 Definitions and notations

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is 1/2.

Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2. The elements of G are represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$, where $x_k = 0 \vee 1$.

It is easy to give a base for the neighborhood of $x \in G$:

$$I_0(x) := G, \; I_n(x) := \{y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \setminus I_n$ and $e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G$, for $n \in \mathbb{N}$. Then, it is easy to prove that

$$\overline{I_M} = \bigcup_{i=0}^{M-1} I_i \backslash I_{i+1} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l) \right) \bigcup \left(\bigcup_{k=0}^{M-1} I_M(e_k) \right). \tag{2.1}$$

If $n \in \mathbb{N}$, then every n can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j 2^j$, where $n_j \in Z_2 \;\; (j \in \mathbb{N})$ and only a finite numbers of n_j differ from zero.

Every $n \in \mathbb{N}$ can be also represented as

$$n = \sum_{i=1}^{r} 2^{n_i}, n_1 > n_2 > \cdots n_r \geq 0.$$

For such a representation of $n \in \mathbb{N}$, we denote numbers

$$n^{(i)} = 2^{n_{i+1}} + \cdots + 2^{n_r}, \quad i = 1, \ldots, r.$$

Let $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}$, $s \in \mathbb{N}$. For such n_{s_j}, which can be written as

$$n_{s_j} = \sum_{i=1}^{r_{s_j}} \sum_{k=l_i^{s_j}}^{t_i^{s_j}} 2^k,$$

where $0 \leq l_1^{s_j} \leq t_1^{s_j} \leq l_2^{s_j} - 2 < l_2^{s_j} \leq t_2^{s_j} \leq \cdots \leq l_{r_j}^{s_j} - 2 < l_{r_{s_j}}^{s_j} \leq t_{r_{s_j}}^{s_j}$, we define

$$\begin{aligned} A_s &:= \bigcup_{j=1}^{r} \left\{ l_1^{s_j}, t_1^{s_j}, l_2^{s_j}, t_2^{s_j}, \ldots, l_{r_{s_j}}^{s_j}, t_{r_{s_j}}^{s_j} \right\} \\ &= \left\{ l_1^s, l_2^s, \ldots, l_{r_s^1}^s \right\} \bigcup \left\{ t_1^s, t_2^s, \ldots, t_{r_s^2}^s \right\} = \left\{ u_1^s, u_2^s, \ldots, u_{r_s^3}^s \right\}, \end{aligned} \tag{2.2}$$

where $u_1^s < u_2^s < \cdots < u_{r_s^3}^s$. We note that $t_{r_{s_j}}^{s_j} = s \in A_s$, for $j = 1, 2, \ldots, r$.

We denote the cardinality of the set A_s by $|A_s|$, that is

$$\mathrm{card}(A_s) := |A_s|.$$

By this definition, we can conclude that $|A_s| = r_s^3 \leq r_s^1 + r_s^2$.

It is evident that $\sup_{s \in \mathbb{N}} |A_s| < \infty$ if and only if the sets $\{n_{s_1}, n_{s_2}, \ldots, n_{s_r}\}$ are uniformly finite for all $s \in \mathbb{N}_+$ and each n_{s_j} has bounded variation

$$V(n_{s_j}) < c < \infty, \quad \text{for each } j = 1, 2, \ldots, r. \tag{2.3}$$

The norms (or quasi-norm) of the spaces $L_p(G)$ and $L_{p,\infty}(G)$, $(0 < p < \infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu \quad \text{and} \quad \|f\|_{L_{p,\infty}(G)}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda) < +\infty.$$

The k-th Rademacher function is defined by

$$r_k(x) := (-1)^{x_k} \quad (x \in G,\ k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [10]).

If $f \in L_1(G)$, then we can define the Fourier coefficients, partial sums of Fourier series, Fejér means, Dirichlet and Fejér kernels in the usual manner:

$$\begin{aligned}
\widehat{f}(n) &:= \int_G f w_n d\mu, \quad (n \in \mathbb{N}), \\
S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \quad (n \in \mathbb{N}_+, S_0 f := 0), \\
\sigma_n f &:= \frac{1}{n} \sum_{k=1}^{n} S_k f, \\
D_n &:= \sum_{k=0}^{n-1} w_k, \\
K_n &:= \frac{1}{n} \sum_{k=1}^{n} D_k, \quad (n \in \mathbb{N}_+).
\end{aligned}$$

Recall that (see [10])

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n \\ 0 & \text{if } x \notin I_n. \end{cases} \tag{2.4}$$

Let $n = \sum_{i=1}^{r} 2^{n_i}$, $\quad n_1 > n_2 > \cdots > n_r \geq 0$. Then, (see [6, 10])

$$nK_n = \sum_{A=1}^{r} \left(\prod_{j=1}^{A-1} w_{2^{n_j}} \right) \left(2^{n_A} K_{2^{n_A}} + n^{(A)} D_{2^{n_A}} \right). \tag{2.5}$$

The σ-algebra, generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n $(n \in \mathbb{N})$. Denote by $F = (F_n, n \in \mathbb{N})$ a martingale with respect to ζ_n $(n \in \mathbb{N})$ (for details see, e.g., [20]). The maximal function F^* of a martingale F is defined by

$$F^* := \sup_{n\in\mathbb{N}} |F_n|.$$

In the case $f \in L_1(G)$, the maximal functions f^* are also given by

$$f^*(x) := \sup_{n\in\mathbb{N}} \left(\frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, \mathrm{d}\mu(u) \right| \right).$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G)$ consist of all martingales, for which

$$\|F\|_{H_p} := \left\|F^*\right\|_p < \infty.$$

A bounded measurable function a is a p-atom, if there exists an interval I, such that

$$\operatorname{supp}(a) \subset I, \quad \int_I a\,\mathrm{d}\mu = 0, \ \|a\|_\infty \le \mu(I)^{-1/p}.$$

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n\to\infty} \int_G F_n(x)\, w_k(x)\, \mathrm{d}\mu(x)$$

exists and it is called the k-th Walsh–Fourier coefficients of F.

The Walsh–Fourier coefficients of $f \in L_1(G)$ are the same as those of the martingale $(S_{2^n} f, n \in \mathbb{N})$ obtained from f.

3 The main result and its consequences

Our main result reads:

Theorem 3.1 (a) *Let $f \in H_{1/2}(G)$ and $\{n_k : k \ge 0\}$ be a sequence of positive numbers and let $\left\{n_{s_i} : 1 \le i \le r\right\} \subset \{n_k : k \ge 0\}$ be numbers such that $2^s \le n_{s_1} \le n_{s_2} \le \cdots \le n_{s_r} \le 2^{s+1}$, $s \in \mathbb{N}$. If the sets A_s, defined by (2.2), are uniformly finite for all $s \in \mathbb{N}$, that is the cardinality of the sets A_s are uniformly finite*:

$$\sup_{s\in\mathbb{N}}|A_s| < c < \infty,$$

then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} F = \sup_{k\in\mathbb{N}} \left|\sigma_{n_k} F\right|, \tag{3.1}$$

is bounded from the Hardy space $H_{1/2}$ *to the Lebesgue space* $L_{1/2}$.

(b) (*Sharpness*) *Let*

$$\sup_{s\in\mathbb{N}}|A_s| = \infty. \tag{3.2}$$

Then, there exists a martingale $f \in H_{1/2}(G)$, *such that the maximal operator, defined by* (3.1), *is not bounded from the Hardy space* $H_{1/2}$ *to the Lebesgue space* $L_{1/2}$.

In particular, Theorem 3.1 implies the following optimal characterization:

Corollary 3.2 *Let* $F \in H_{1/2}(G)$ *and* $\{n_k : k \geq 0\}$ *be a sequence of positive numbers. Then, the restricted maximal operator* $\widetilde{\sigma}^{*,\nabla}$, *defined by* (3.1), *is bounded from the Hardy space* $H_{1/2}$ *to the Lebesgue space* $L_{1/2}$ *if and only if any sequence of positive numbers* $\{n_k : k \geq 0\}$ *which satisfies* $n_k \in [2^s, 2^{s+1})$, *is uniformly finite for each* $s \in \mathbb{N}_+$ *and each* $\{n_k : k \geq 0\}$ *has bounded variation, i.e.,*

$$\sup_{k\in\mathbb{N}} V(n_k) < c < \infty.$$

In order to be able to compare with some other results in the literature (see Remark 3.4), we also state the following:

Corollary 3.3 *Let* $F \in H_{1/2}(G)$. *Then, the restricted maximal operators* $\widetilde{\sigma}_i^{*,\nabla}$, $i = 1, 2, 3$, *defined by*

$$\widetilde{\sigma}_1^{*,\nabla}F = \sup_{k\in\mathbb{N}}\left|\sigma_{2^k}F\right|, \tag{3.3}$$

$$\widetilde{\sigma}_2^{*,\nabla}F = \sup_{k\in\mathbb{N}}\left|\sigma_{2^k+1}F\right|, \tag{3.4}$$

$$\widetilde{\sigma}_3^{*,\nabla}F = \sup_{k\in\mathbb{N}}\left|\sigma_{2^k+2^{[k/2]}}F\right|, \tag{3.5}$$

where $[n]$ *denotes the integer part of* n, *are all bounded from the Hardy space* $H_{1/2}$ *to the Lebesgue space* $L_{1/2}$.

Remark 3.4 In [8], it was proved that if $0 < p < 1/2$, then the restricted maximal operators $\widetilde{\sigma}_2^{*,\nabla}$ and $\widetilde{\sigma}_3^{*,\nabla}$, defined by (3.4) and (3.5), are not bounded from the Hardy space H_p to the Lebesgue space weak $- L_p$.

On the other hand, Weisz [20] (see also [8]) proved that if $0 < p \leq 1/2$, then the restricted maximal operator $\widetilde{\sigma}_1^{*,\nabla}$, defined by (3.3) is bounded from the Hardy space H_p to the Lebesgue space L_p.

4 Auxiliary lemmas and propositions

Lemma 4.1 (Weisz [21] (see also Simon [14])) *A martingale* $F = (F_n, n \in \mathbb{N})$ *is in* H_p $(0 < p \leq 1)$ *if and only if there exists a sequence* $(a_k, k \in \mathbb{N})$ *of p-atoms and a* *sequence* $(\mu_k, k \in \mathbb{N})$ *of a real numbers, such that for every* $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{4.1}$$

Moreover, $\|F\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}$, *where the infimum is taken over all decomposition of* F *of the form* (4.1).

Lemma 4.2 (Weisz [20]) *Suppose that an operator* T *is* σ*-linear and*

$$\int_{\overline{I}} |Ta|^p \, d\mu \le c_p < \infty, \quad (0 < p \le 1)$$

for every p*-atom* a, *where* I *denote the support of the atom. If* T *is bounded from* L_∞ *to* L_∞, *then*

$$\|TF\|_p \le c_p \|F\|_{H_p}.$$

Lemma 4.3 (See, e.g., [2]) *Let* $t, n \in \mathbb{N}$. *Then,*

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_n(e_t), \ n > t, \ x \in I_t \backslash I_{t+1}, \\ (2^n + 1)/2, & \text{if } x \in I_n, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 4.4 (See, e.g., [5, 16]) *Let* $n \ge 2^M$ *and* $x \in I_M^{k,l}$, $k = 0, \ldots, M-1$, $l = k+1, \ldots, M$. *Then,*

$$\int_{I_M} |K_n(x+t)| \, d\mu(t) \le cn2^{k+l-M}.$$

Lemma 4.5 (See [17]) *Let*

$$n = \sum_{i=1}^{s} \sum_{k=l_i}^{t_i} 2^k, \quad \text{where } t_1 \ge l_1 > l_1 - 2 \ge t_2 \ge l_2 > l_2 - 2 > \cdots > t_s \ge l_s \ge 0.$$

Then, for any $i = 1, 2, \ldots, s$,

$$n |K_n(x)| \ge 2^{2l_i - 4}, \quad \text{for } x \in E_{l_i} := I_{l_i+1}\left(e_{l_i-1} + e_{l_i}\right),$$

where $I_1(e_{-1} + e_0) = I_2(e_0 + e_1)$.

We also need the following new statement of independent interest:

Proposition 4.6 *Let*

$$n=\sum_{i=1}^{s}\sum_{k=l_i}^{t_i}2^k,\quad \text{where } t_1\geq l_1>l_1-2\geq t_2\geq l_2>l_2-2>\cdots>t_s\geq l_s\geq 0.$$

Then, for any $i=1,2,\ldots,s$,

$$n\left|K_n(x)\right|\geq 2^{2t_i-2},\quad \text{for } x\in E_{t_i}:=I_{t_i+3}\left(e_{t_i+1}+e_{t_i+2}\right).$$

Proof It is evident that we always have that $t_i+2\leq l_{i-1}$. If $t_i+2=l_{i-1}$, then $E_{t_i}=I_{t_i+3}\left(e_{t_i+1}+e_{t_i+2}\right)=I_{l_{i-1}+1}\left(e_{l_{i-1}-1}+e_{l_{i-1}}\right)=E_{l_{i-1}}$ and if we apply Lemma 4.5, we find that

$$n\left|K_n(x)\right|\geq 2^{2l_{i-1}-4}=2^{2t_i},\quad \text{for } x\in E_{l_{i-1}}=E_{t_i}.$$

Let $t_i+2<l_{i-1}$. By combining (2.4) and Lemma 4.3, for any $n\geq t_i+3$, we get that

$$D_{2^n}(x)=K_{2^n}(x)=0,\quad \text{for } x\in E_{t_i}.$$

From (2.5), for $x\in E_{t_i}$, we can conclude that

$$\begin{aligned}nK_n&=\sum_{r=1}^{s}\sum_{k=l_r}^{t_r}\left(\prod_{j=i+1}^{s}\prod_{q=l_j}^{t_j}w_{2^q}\prod_{j=k+1}^{t_i}w_{2^j}\right)\left(2^kK_{2^k}+\left(\sum_{j=i+1}^{s}\sum_{q=l_j}^{t_j}2^q+\sum_{q=l_i}^{k-1}2^q\right)D_{2^k}\right)\\&=\sum_{r=i}^{s}\sum_{k=l_r}^{t_r}\left(\prod_{j=i+1}^{s}\prod_{q=l_j}^{t_j}w_{2^q}\prod_{j=k+1}^{t_i}w_{2^j}\right)\left(2^kK_{2^k}+\left(\sum_{j=i+1}^{s}\sum_{q=l_j}^{t_j}2^q+\sum_{q=l_i}^{k-1}2^q\right)D_{2^k}\right).\end{aligned}\tag{4.2}$$

Suppose that $l_i<t_i$. Since

$$\sum_{j=i+1}^{s}\sum_{q=l_j}^{t_j}2^q+\sum_{q=l_i}^{t_i-1}2^q\geq 2^{t_i-1}$$

for $x\in E_{t_i}$, we find that

$$\begin{aligned}n\left|K_n\right|&\geq\left|2^{t_i}K_{2^{t_i}}+2^{t_i-1}D_{2^{t_i}}\right|-\sum_{k=0}^{t_i-1}\left|2^kK_{2^k}\right|-\sum_{k=0}^{t_i-1}\left|2^kD_{2^k}\right|\\&:=I_1-I_2-I_3.\end{aligned}\tag{4.3}$$

Moreover, by combining (2.4) and Lemma 4.3, we get that

$$I_1 = 2^{t_i} K_{2^{t_i}}(x) + 2^{t_i-1} D_{2^{t_i}} = \frac{2^{2t_i}}{2} + 2^{t_i-1} + \frac{2^{2t_i}}{2} = 2^{2t_i} + 2^{t_i-1}. \tag{4.4}$$

For I_2, we have that

$$I_2 \leq \sum_{k=0}^{t_i-1} 2^k \frac{\left(2^k+1\right)}{2} = \frac{1}{2}\frac{2^{2t_i}-1}{4-1} + \frac{1}{2}\frac{2^{t_i}-1}{2-1} \leq \frac{2^{2t_i}}{6} + 2^{t_i-1}. \tag{4.5}$$

Moreover, I_3 can be estimated as follows:

$$I_3 \leq \sum_{k=0}^{l_i-1} 4^k = \frac{2^{2t_i}}{3}. \tag{4.6}$$

By combining (4.4)–(4.6) and putting them into (4.3), we obtain that

$$n\left|K_n(x)\right| \geq I_1 - I_2 - I_3 \geq \frac{2^{2l_i}}{2}. \tag{4.7}$$

If $t_i = l_i$, we get that $t_{i+1} \leq l_i - 2 = t_i - 2$. Hence, using (4.2), we find that

$$n\left|K_n\right| \geq \left|2^{t_i} K_{2^{t_i}}\right| - \sum_{k=0}^{t_i-2}\left|2^k K_{2^k}\right| - \sum_{k=0}^{t_i-2}\left|2^k D_{2^k}\right| := I_1 - I_2 - I_3. \tag{4.8}$$

By simple calculations, we get that

$$I_1 \geq 2^{t_i} K_{2^{t_i}}(x) = \frac{2^{2t_i}}{2} + \frac{2^{t_i}}{2}, \tag{4.9}$$

$$I_2 \leq \sum_{k=0}^{t_i-2} 2^k \frac{\left(2^k+1\right)}{2} \leq \frac{1}{2}\frac{2^{2t_i-2}-1}{4-1} + \frac{1}{2}\frac{2^{t_i-1}-1}{2-1} \leq \frac{2^{2t_i}}{24} + 2^{t_i-2} \tag{4.10}$$

and

$$I_3 \leq \sum_{k=0}^{l_i-2} 4^k = \frac{2^{2t_i}}{12}. \tag{4.11}$$

We insert (4.9)–(4.11) into (4.8) and find that

$$n\left|K_n(x)\right| \geq I_1 - I_2 - I_3 \geq \frac{2^{2l_i}}{4}. \tag{4.12}$$

The proof is complete by just combining the estimates (4.7) and (4.12). □

Corollary 4.7 *Let*

$$n=\sum_{i=1}^{s}\sum_{k=l_i}^{t_i}2^k,\quad where\ t_1\geq l_1>l_1-2\geq t_2\geq l_2>l_2-2>\cdots>t_s\geq l_s\geq 0.$$

Then, for any $i=1,2,\ldots,s,$

$$n\left|K_n(x)\right|\geq 2^{2t_i-5},\quad for\ x\in E_{t_i}:=I_{t_i+1}\left(e_{t_i+1}+e_{t_i+2}\right).$$

and

$$n\left|K_n(x)\right|\geq 2^{2l_i-5},\quad for\ x\in E_{l_i}:=I_{l_i+1}\left(e_{l_i-1}+e_{l_i}\right),$$

where $I_1\left(e_{-1}+e_0\right)=I_2\left(e_0+e_1\right).$

Our second auxiliary result of independent interest is the following:

Proposition 4.8 *Let*

$$n=\sum_{i=1}^{s}\sum_{k=l_i}^{t_i}2^k,\quad where\ t_1\geq l_1>l_1-2\geq t_2\geq l_2>l_2-2>\cdots>t_s\geq l_s\geq 0.$$

Then,

$$\left|nK_n\right|\leq c\sum_{A=1}^{s}\left(2^{l_A}K_{2^{l_A}}+2^{t_A}K_{2^{t_A}}+2^{l_A}\sum_{k=l_A}^{t_A}D_{2^k}\right).$$

Proof A proof of the corresponding result in [15] was also here so we omit the details. □

5 Proof of Theorem 3.1

Proof Since σ_n is bounded from L_∞ to L_∞, by Lemma 4.2, the proof of Theorem 3.1 will be complete, if we prove that

$$\int_{\overline{I_M}}\left(\sup_{s_k\in\mathbb{N}}\left|\sigma_{n_{s_k}}a(x)\right|\right)^{1/2}d\mu(x)\leq c<\infty,\tag{5.1}$$

for every 1/2-atom a. We may assume that a is an arbitrary $1/2$-atom, with support I, $\mu(I)=2^{-M}$ and $I=I_M$. It is easy to see that $\sigma_n(a)=0$, when $n<2^M$. Therefore, we can suppose that $n_{s_k}\geq 2^M$.

Let $x \in I_M$ and $2^s \leq n_{s_k} < 2^{s+1}$ for some $n_{s_k} \geq 2^M$. Since $\|a\|_\infty \leq 2^{2M}$, we obtain that

$$\begin{aligned}\left|\sigma_{n_{s_k}} a(x)\right| &\leq \int_{I_M} |a(t)| \left|K_{n_{s_k}}(x+t)\right| \mathrm{d}\mu(t) \\ &\leq \|a\|_\infty \int_{I_M} \left|K_{n_{s_k}}(x+t)\right| \mathrm{d}\mu(t) \\ &\leq 2^{2M} \int_{I_M} \left|K_{n_{s_k}}(x+t)\right| \mathrm{d}\mu(t). \end{aligned} \tag{5.2}$$

Using Proposition 4.8 and (2.2), we get that

$$\begin{aligned}\left|K_{n_{s_k}}\right| &\leq \frac{c}{n_{s_k}} \sum_{A=1}^{s} \left(2^{l_A^{s_k}} K_{2^{l_A^{s_k}}} + 2^{t_A^{s_k}} K_{2^{t_A^{s_k}}} + 2^{l_A^{s_k}} \sum_{k=l_A^{s_k}}^{\infty} D_{2^k}\right) \\ &\leq \frac{c}{2^s} \sum_{A=1}^{s} \left(2^{l_A} K_{2^{l_A}} + 2^{t_A} K_{2^{t_A}} + 2^{l_A} \sum_{k=l_A}^{\infty} D_{2^k}\right)\end{aligned}$$

and

$$\begin{aligned}\left|\sigma_{n_{s_k}} a(x)\right| &\leq \frac{2^M}{2^s} \left(2^M \sum_{A=1}^{r_s^1} \int_{I_M} 2^{l_A^s} K_{2^{l_A^s}}(x+t)\, \mathrm{d}\mu(t)\right) \\ &+ \frac{2^M}{2^s} \left(2^M \sum_{A=1}^{r_s^2} \int_{I_M} 2^{t_A^s} K_{2^{t_A^s}}(x+t)\, \mathrm{d}\mu(t)\right) \\ &+ \frac{2^M}{2^s} \left(2^M \sum_{A=1}^{r_s^1} \int_{I_M} 2^{l_A^s} \sum_{k=l_A^s}^{\infty} D_{2^k}(x+t)\, \mathrm{d}\mu(t)\right).\end{aligned}$$

If we define

$$\begin{aligned}\mathrm{II}^1_{\alpha_A^s}(x) &:= 2^M \int_{I_M} 2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t)\, \mathrm{d}\mu(t), \quad \alpha = l, \quad \text{or} \quad \alpha = t \\ \mathrm{II}^2_{l_A^s}(x) &:= 2^M \int_{I_M} 2^{l_A^s} \sum_{k=l_A^s}^{\infty} D_{2^k}(x+t)\, \mathrm{d}\mu(t),\end{aligned}$$

then, from (5.2), we can conclude that

$$\left|\sigma_{n_{s_k}} a(x)\right| \leq \frac{2^M}{2^s} \left(\sum_{A=1}^{r_s^1} \mathrm{II}^1_{l_A^s}(x) + \sum_{A=1}^{r_s^2} \mathrm{II}^1_{t_A^s}(x) + \sum_{A=1}^{r_s^1} \mathrm{II}^2_{l_A^s}(x)\right)$$

and

$$\sup_{2^s \le n_{s_k} < 2^{s+1}} \left|\sigma_{n_{s_k}} a(x)\right| \le \frac{2^M}{2^s}\left(\sum_{A=1}^{r_s^1} \mathrm{II}^1_{l_A^s}(x) + \sum_{A=1}^{r_s^2} \mathrm{II}^1_{t_A^s}(x) + \sum_{A=1}^{r_s^1} \mathrm{II}^2_{l_A^s}(x)\right).$$

Hence,

$$\begin{aligned}
&\int_{\overline{I_M}} \left(\sup_{2^s \le n_{s_k} < 2^{s+1}} \left|\sigma_{n_{s_k}} a(x)\right|\right)^{1/2} d\mu \\
&\le \frac{2^{M/2}}{2^{s/2}}\left(\sum_{A=1}^{r_s^1} \int_{\overline{I_M}} \left|\mathrm{II}^1_{l_A^s}(x)\right|^{1/2} d\mu + \sum_{A=1}^{r_s^2} \int_{\overline{I_M}} \left|\mathrm{II}^1_{t_A^s}\right|^{1/2} d\mu\right. \\
&\quad \left. + \sum_{A=1}^{r_s^1} \int_{\overline{I_M}} \left|\mathrm{II}^2_{l_A^s}(x)\right|^{1/2} d\mu\right). \qquad (5.3)
\end{aligned}$$

Since $\sup_{s\in\mathbb{N}} r_s^1 < r < \infty, \quad \sup_{s\in\mathbb{N}} r_s^2 < r < \infty$, we obtain that (5.1) holds so that Theorem 3.1(a) is proved if we can prove that

$$\int_{\overline{I_M}} \left|\mathrm{II}^2_{l_A^s}(x)\right|^{1/2} d\mu \le c < \infty, \quad A = 1, \ldots, r_s^1 \qquad (5.4)$$

and

$$\int_{\overline{I_M}} \left|\mathrm{II}^1_{\alpha_A^s}(x)\right|^{1/2} d\mu \le c < \infty, \qquad (5.5)$$

for all $\alpha_A^s = l_A^s, \quad A = 1, \ldots, r_s^1$ and $\alpha_A^s = t_A^s, \quad A = 1, \ldots, r_s^2$. Indeed, if (5.4) and (5.5) hold, from (5.3), we get that

$$\begin{aligned}
&\int_{\overline{I_M}} \left(\sup_{n_{s_k} \ge 2^M} \left|\sigma_{n_{s_k}} a(x)\right|\right)^{1/2} d\mu \\
&\le \sum_{s=M}^{\infty} \int_{\overline{I_M}} \left(\sup_{2^s \le n_{s_k} < 2^{s+1}} \left|\sigma_{n_{s_k}} a(x)\right|\right)^{1/2} d\mu \le \sum_{s=M}^{\infty} \frac{c2^{M/2}}{2^{s/2}} < C < \infty.
\end{aligned}$$

It remains to prove (5.4) and (5.5). Let

$$t \in I_M \quad \text{and} \quad x \in I_{l+1}(e_k + e_l).$$

If $0 \le k < l < \alpha_A^s \le M$ or $0 \le k < l < M < \alpha_A^s$, then $x + t \in I_{l+1}(e_k + e_l)$ and if we apply Lemma 4.3, we obtain that

$$K_{2^{\alpha_A^s}}(x+t) = 0 \quad \text{and} \quad II_{\alpha_A^s}^1(x) = 0. \tag{5.6}$$

Let $0 \le k < \alpha_A^s \le l < M$. Then, $x + t \in I_{l+1}(e_k + e_l)$ and if we use Lemma 4.3, we get that

$$2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \le 2^{\alpha_A^s + k}$$

so that

$$II_{\alpha_A^s}^1(x) \le 2^{\alpha_A^s + k}. \tag{5.7}$$

Analogously to (5.7), we can prove that if $0 \le \alpha_A^s \le k < l < M$, then

$$2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \le 2^{2\alpha_A^s}, t \in I_M, x \in I_{l+1}(e_k + e_l)$$

so that

$$II_{\alpha_A^s}^1(x) \le 2^{2\alpha_A^s}, t \in I_M, x \in I_{l+1}(e_k + e_l). \tag{5.8}$$

Let

$$t \in I_M \quad \text{and} \quad x \in I_M(e_k).$$

Let $0 \le k < \alpha_A^s \le M$ or $0 \le k < M \le \alpha_A^s$. Since $x + t \in x \in I_M(e_k)$ and if we apply Lemma 4.3, we obtain that

$$2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \le 2^{\alpha_A^s + k}$$

and

$$II_{\alpha_A^s}^1(x) \le 2^{\alpha_A^s + k}. \tag{5.9}$$

Let $0 \le \alpha_A^s \le k < M$. Since $x + t \in x \in I_M(e_k)$ and if we apply Lemma 4.3, then we find that

$$2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \le 2^{2\alpha_A^s}$$

and

$$II_{\alpha_A^s}^1(x) \le 2^{2\alpha_A^s}. \tag{5.10}$$

Let $0 \le \alpha_A^s < M$. By combining (2.1) with (5.6)–(5.10) for any $A = 1, \ldots, s$ we have that

$$\begin{aligned}
&\int_{\overline{I_M}} \left|\mathrm{II}^1_{\alpha_A^s}(x)\right|^{1/2} \mathrm{d}\mu \\
&= \sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1} \int_{I_{l+1}(e_k+e_l)} \left|\mathrm{II}^1_{\alpha_A^s}(x)\right|^{1/2} \mathrm{d}\mu + \sum_{k=0}^{M-1} \int_{I_M(e_k)} \left|\mathrm{II}^1_{\alpha_A^s}(x)\right|^{1/2} \mathrm{d}\mu \\
&\le c \sum_{k=0}^{\alpha_A^s-1}\sum_{l=\alpha_A^s}^{M-1} \int_{I_{l+1}(e_k+e_l)} 2^{(\alpha_A^s+k)/2} \mathrm{d}\mu + c \sum_{k=\alpha_A^s}^{M-2}\sum_{l=k+1}^{M-1} \int_{I_{l+1}(e_k+e_l)} 2^{\alpha_A^s} \mathrm{d}\mu \\
&\quad + c \sum_{k=0}^{\alpha_A^s-1} \int_{I_M(e_k)} 2^{(\alpha_A^s+k)/2} \mathrm{d}\mu + c \sum_{k=\alpha_A^s}^{M-1} \int_{I_M(e_k)} 2^{\alpha_A^s} \mathrm{d}\mu \\
&\le c \sum_{k=0}^{\alpha_A^s-1}\sum_{l=\alpha_A^s+1}^{M-1} \frac{2^{(\alpha_A^s+k)/2}}{2^l} + c \sum_{k=\alpha_A^s}^{M-2}\sum_{l=k+1}^{M-1} \frac{2^{\alpha_A^s}}{2^l} \\
&\quad + c \sum_{k=0}^{\alpha_A^s-1} \frac{2^{(\alpha_A^s+k)/2}}{2^M} + c \sum_{k=\alpha_A^s}^{M-1} \frac{2^{\alpha_A^s}}{2^M} \le C < \infty.
\end{aligned}$$

Analogously we can prove that (5.5) holds also for the case $\alpha_A^s \ge M$. Hence, (5.5) holds and it remains to prove (5.4).

Let $t \in I_M$ and $x \in I_i \backslash I_{i+1}$. If $i \le l_A^s - 1$, since $x + t \in I_i \backslash I_{i+1}$, using (2.4), we have that

$$\mathrm{II}^2_{l_A^s}(x) = 0. \tag{5.11}$$

If $l_A^s \le i < M$, then using (2.4), we obtain that

$$\mathrm{II}^2_{l_A^s}(x) \le 2^M \int_{I_M} 2^{l_A^s} \sum_{k=l_A^s}^{i} D_{2^k}(x+t)\, \mathrm{d}\mu(t) \le c 2^{l_A^s+i}. \tag{5.12}$$

Let $0 \le l_A^s < M$. By combining (2.1), (5.11) and (5.12), we get that

$$\begin{aligned}
&\int_{\overline{I_M}} \left|\mathrm{II}^2_{l_A^s}(x)\right|^{1/2} \mathrm{d}\mu \\
&= \left(\sum_{i=0}^{l_A^s-1} + \sum_{i=l_A^s+1}^{M-1}\right) \int_{I_i \backslash I_{i+1}} \left|\mathrm{II}^2_{l_A^s}(x)\right|^{1/2} \mathrm{d}\mu \\
&\le c \sum_{i=l_A^s}^{M-1} \int_{I_i \backslash I_{i+1}} 2^{(l_A^s+i)/2} \mathrm{d}\mu(x) \le c \sum_{i=l_A^s}^{M-1} 2^{(l_A^s+i)/2} \frac{1}{2^i} \le C < \infty. \quad (5.13)
\end{aligned}$$

If $M \le l_A^s$, then $i < M \le l_A^s$ and apply (5.11), we get that

$$\int_{\overline{I_M}} \left| \mathrm{II}^2_{l_A^s}(x) \right|^{1/2} d\mu = 0, \tag{5.14}$$

and also (5.4) is proved by just combining (5.13) and (5.14) so part (a) is complete and we turn to the proof of (b).

Under condition (3.2), there exists an increasing sequence $\{\alpha_k : k \ge 0\} \subset \{n_k : k \ge 0\}$ of positive integers, such that

$$\sum_{k=1}^{\infty} 1/|A^2_{|\alpha_k|} \le c < \infty. \tag{5.15}$$

Let

$$F_A := \sum_{\{k; |\alpha_k| < A\}} \lambda_k a_k,$$

where

$$\lambda_k := \frac{1}{\left|A_{|\alpha_k|}\right|} \quad \text{and} \quad a_k := 2^{|\alpha_k|} \left(D_{2^{|\alpha_k|+1}} - D_{2^{|\alpha_k|}}\right).$$

Since $\operatorname{supp}(a_k) = I_{|\alpha_k|}, \quad \|a_k\|_\infty \le 2^{2|\alpha_k|} = \mu(\operatorname{supp} a_k)^{-2}$ and

$$S_{2^A} a_k = \begin{cases} a_k & |\alpha_k| < A, \\ 0 & |\alpha_k| \ge A, \end{cases}$$

if we apply Lemma 4.1 and (5.15), we can conclude that $F = (F_1, F_2, \ldots) \in H_{1/2}$.

It is easy to prove that

$$\widehat{F}(j) = \begin{cases} 2^{|\alpha_k|} / \left|A_{|\alpha_k|}\right|, & j \in \left\{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\right\}, k = 0, 1, \ldots \\ 0, & j \notin \bigcup\limits_{k=0}^{\infty} \left\{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\right\}. \end{cases} \tag{5.16}$$

Let $2^{|\alpha_k|} < j < \alpha_k$. Using (5.16), we get that

$$S_j F = S_{2^{|\alpha_k|}} F + \sum_{v=2^{|\alpha_k|}}^{j-1} \widehat{F}(v) w_v = S_{2^{|\alpha_k|}} F + \frac{\left(D_j - D_{2^{|\alpha_k|}}\right) 2^{|\alpha_k|}}{\left|A_{|\alpha_k|}\right|}. \tag{5.17}$$

Let $2^{|\alpha_k|} \le \alpha_{s_n} \le 2^{|\alpha_k|+1}$. Then, using (5.17), we find that

$$
\begin{aligned}
\sigma_{\alpha_{s_n}} F &= \frac{1}{\alpha_{s_n}} \sum_{j=1}^{2^{|\alpha_k|}} S_j F + \frac{1}{\alpha_{s_n}} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}} S_j F \\
&= \frac{\sigma_{2^{|\alpha_k|}} F}{\alpha_{s_n}} + \frac{\left(\alpha_k - 2^{|\alpha_k|}\right) S_{2^{|\alpha_k|}} F}{\alpha_{s_n}} + \frac{2^{|\alpha_k|}}{\left|A_{|\alpha_k|}\right| \alpha_{s_n}} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}} \left(D_j - D_{2^{|\alpha_k|}}\right) \\
&:= \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3. \qquad (5.18)
\end{aligned}
$$

Since

$$
D_{j+2^m} = D_{2^m} + w_{2^m} D_j, \quad \text{when } j < 2^m,
$$

we obtain that

$$
\begin{aligned}
|\mathrm{III}_3| &= \frac{2^{|\alpha_k|}}{\left|A_{|\alpha_k|}\right| \alpha_{s_n}} \left| \sum_{j=1}^{\alpha_{s_n} - 2^{|\alpha_k|}} \left(D_{j+2^{|\alpha_k|}} - D_{2^{|\alpha_k|}}\right) \right| \\
&= \frac{2^{|\alpha_k|}}{\left|A_{|\alpha_k|}\right| \alpha_{s_n}} \left| \sum_{j=1}^{\alpha_{s_n} - 2^{|\alpha_k|}} D_j \right| \\
&= \frac{2^{|\alpha_k|}}{\left|A_{|\alpha_k|}\right| \alpha_{s_n}} \left(\alpha_{s_n} - 2^{|\alpha_k|}\right) \left| K_{\alpha_{s_n} - 2^{|\alpha_k|}} \right| \\
&\geq \frac{1}{2\left|A_{|\alpha_k|}\right|} \left(\alpha_{s_n} - 2^{|\alpha_k|}\right) \left| K_{\alpha_{s_n} - 2^{|\alpha_k|}} \right|. \qquad (5.19)
\end{aligned}
$$

By combining the well-known estimates (see [9])

$$
\left\| S_{2^k} F \right\|_{H_{1/2}} \leq c_1 \left\| F \right\|_{H_{1/2}} \quad \text{and} \quad \left\| \sigma_{2^k} F \right\|_{H_{1/2}} \leq c_2 \left\| F \right\|_{H_{1/2}}, \qquad k \in \mathbb{N},
$$

we obtain that

$$
\|\mathrm{III}_1\|_{1/2} \leq C \quad \text{and} \quad \|\mathrm{III}_2\|_{1/2} \leq C.
$$

Let $2^{|\alpha_k|} \leq \alpha_{s_1} \leq \alpha_{s_2} \leq \cdots \leq \alpha_{s_r} \leq 2^{|\alpha_k|+1}$ be natural numbers which generates the set

$$
A_{|\alpha_k|} = \left\{ l_1^{|\alpha_k|}, l_2^{|\alpha_k|}, \ldots, l_{r^1_{|\alpha_k|}}^{|\alpha_k|} \right\} \bigcup \left\{ t_1^{|\alpha_k|}, t_2^{|\alpha_k|}, \ldots, t_{r^2_{|\alpha_k|}}^{|\alpha_k|} \right\}
$$

and choose number $\alpha_{s_n} = \sum_{i=1}^{r_n} \sum_{k=l_i^n}^{t_i^n} 2^k$, where

$$
t_1^{|\alpha_k|} \geq l_1^{|\alpha_k|} > l_1^{|\alpha_k|} - 2 \geq t_2^{|\alpha_k|} \geq l_2^{|\alpha_k|} > l_2^{|\alpha_k|} - 2 \geq \cdots \geq t_{|\alpha_k|}^{|\alpha_k|} \geq l_{|\alpha_k|}^{|\alpha_k|} \geq 0,
$$

for some $1 \leq n \leq r$, such that $l_u^{|\alpha_k|} = l_i$, for some $1 \leq u \leq r^1_{|\alpha_k|}$, $1 \leq i \leq r^1_{|\alpha_k|}$.

Since $\mu \left\{E_{l_i}\right\} \geq 1/2^{l_i+1}$, using Corollary 4.7, we get that

$$\begin{aligned}\int_{E_{l_i}} \left|\widetilde{\sigma}^{*,\nabla} F\right|^{1/2} \mathrm{d}\mu &\geq \int_{E_{l_i}} \left|\sigma_{\alpha_{s_n}} F(x)\right|^{1/2} \mathrm{d}\mu \\ &\geq \frac{2^{(2l_i-6)/2}}{\sqrt{2}\left(\left|A_{|\alpha_k|}\right|\right)^{1/2}} \frac{1}{2^{l_i+1}} \geq \frac{1}{2^5\left(\left|A_{|\alpha_k|}\right|\right)^{1/2}}. \end{aligned} \tag{5.20}$$

On the other hand, we can also choose number α_{s_n}, for some $1 \leq n \leq r$, such that $t_u^{|\alpha_k|} = t_i$, for some $1 \leq u \leq r^2_{|\alpha_k|}$, $1 \leq i \leq r^2_{|\alpha_k|}$. According to the fact that $\mu \left\{E_{t_i}\right\} \geq 1/2^{t_i+3}$, using again Corollary 4.7 for some α_k and $1 \leq i \leq r_s^2$, we also get that

$$\begin{aligned}\int_{E_{t_i}} \left|\widetilde{\sigma}^{*,\nabla} F\right|^{1/2} \mathrm{d}\mu &\geq \int_{E_{t_i}} \left|\sigma_{\alpha_{s_n}} F(x)\right|^{1/2} \mathrm{d}\mu \\ &\geq \frac{1}{\sqrt{2}\left(\left|A_{|\alpha_k|}\right|\right)^{1/2}} 2^{(2t_i-6)/2} \frac{1}{2^{t_i+3}} \\ &\geq \frac{1}{2^7\left(\left|A_{|\alpha_k|}\right|\right)^{1/2}}. \end{aligned} \tag{5.21}$$

By combining (5.18)–(5.21) with Proposition 4.6 for sufficiently big α_k, we get that

$$\begin{aligned}&\int_G \left|\widetilde{\sigma}^{*,\nabla} F\right|^{1/2} \mathrm{d}\mu \\ &\geq \|\mathrm{III}_3\|_{1/2}^{1/2} - \|\mathrm{III}_2\|_{1/2}^{1/2} - \|\mathrm{III}_1\|_{1/2}^{1/2} \\ &\geq \sum_{i=1}^{r^1_{|\alpha_k|}-1} \int_{E_{l_i}} \left|\widetilde{\sigma}^{*,\nabla} F\right|^{1/2} \mathrm{d}\mu + \sum_{i=1}^{r^2_{|\alpha_k|}-1} \int_{E_{t_i}} \left|\widetilde{\sigma}^{*,\nabla} F\right|^{1/2} \mathrm{d}\mu - 2C \\ &\geq \frac{1}{2^7\left(\left|A_{|\alpha_k|}\right|\right)^{1/2}} (r^1_{|\alpha_k|} + r^2_{|\alpha_k|}) - 2C \geq \frac{\left(\left|A_{|\alpha_k|}\right|\right)^{1/2}}{2^8} \to \infty, \quad \text{as } k \to \infty, \end{aligned}$$

so also part (b) is proved and the proof is complete. □

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