Mathematical
Inequalities
& Applications
Volume 26, Number 4 (2023), 887–903 doi:10.7153/mia-2023-26-54

SOME NEW MULTIDIMENSIONAL COCHRAN–LEE AND HARDY TYPE INEQUALITIES

MARKOS FISSEHA YIMER, LARS ERIK PERSSON* AND TSEGAYE GEDIF AYELE

(*Communicated by S. Varošanec*)

Abstract. A multidimensional Cochran-Lee operator is introduced and investigated in the frame of Hardy-type inequalities with parameters $0 < p \leqslant q < \infty$. Moreover, for the case $p = q$ and power weights even the sharp constant is derived, thus generalizing the original Cochran-Lee inequality to a multidimensional setting. As applications both several known but also new inequalities are pointed out.

1. Introduction

In 1928, G. H. Hardy proved the following first weighted form of his famous inequality from 1925 (the case $\alpha = 0$, see [4]):

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p x^\alpha \,\mathrm{d}x \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha \,\mathrm{d}x, \tag{1}$$

whenever $p \geqslant 1$ and $\alpha < p-1$. He also proved that the constant $\left(\dfrac{p}{p-1-\alpha}\right)^p$ is sharp (see [5]). After this, a lot of generalizations and complementary results have been published see e.g. [13], [15], [18] and the references therein. For some early contributions see also the classical book [6]. Recently, the discrete version of (1) with $\alpha = 0$ was discussed in [3]. Next we note that by replacing $f(x)$ with $(f(x))^{1/p}$ and letting $p \to \infty$ in (1), we obtain the following Pólya-Knopp's weighted inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right) x^\alpha \,\mathrm{d}x \leqslant e^{(1+\alpha)} \int_0^\infty f(x)x^\alpha \,\mathrm{d}x, \tag{2}$$

for $\alpha > -1$ and f is a positive and measurable function on $(0,\infty)$. Moreover, the constant $e^{(1+\alpha)}$ is sharp. Concerning the name Pólya-Knopp's inequality see our Remark 3.

Mathematics subject classification (2020): Primary 26D15; Secondary 26D10, 26D20.
Keywords and phrases: Multidimensional inequalities, Hardy-type operators, Hardy-type inequalities, Cochran-Lee's inequality, weights, sharp constants, scales of conditions.

* Corresponding author.

Paper MIA-26-54

For the purpose of this paper we just mention the following generalization of (2) by J. A. Cochran and C. S. Lee (see [2]):

$$\int_0^\infty x^a \exp\left[\beta x^{-\beta}\int_0^x t^{\beta-1}\log f(t)\,\mathrm{d}t\right]\mathrm{d}x \leqslant e^{(a+1)/\beta}\int_0^\infty x^a f(x)\,\mathrm{d}x, \tag{3}$$

where $\beta > 0$, $a \in \mathbb{R}$ and the constant $e^{(a+1)/\beta}$ is sharp. This means that the geometric mean operator G, defined by

$$(Gf)(x) := \exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right)$$

is replaced by the more general weighted geometric mean operator G_β, defined by

$$\left(G_\beta f\right)(x) := \exp\left(\beta x^{-\beta}\int_0^x t^{\beta-1}\log f(t)\,\mathrm{d}t\right)$$

for any $\beta > 0$. Later on a number of results are proved concerning more general weighted versions of (2):

$$\left(\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \log f(t)\,\mathrm{d}t\right)\right]^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}$$

for various parameters p and q, weights $u(x), v(x)$ and some constant $C > 0$ (by a weight we as usual mean a measurable and nonnegative function). See e.g. [1], [9], [10], [11], [14] and the references therein.

Moreover, some results for the corresponding two-dimensional cases are also known, see e.g. [7], [8], [19], [20] and the references therein. In particular, the following result in [19] (see also [20]) is of special interest for this paper since also good estimates are given of the sharp constant C in the inequality.

THEOREM 1. (See [19, Theorem 4.1]) *Let $0 < p \leqslant q < \infty$, and let u, v and f be positive functions on $\mathbb{R}^2_+$. If $0 < b_1, b_2 \leqslant \infty$, then*

$$\begin{aligned}&\left(\int_0^{b_1}\int_0^{b_2}\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln f(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2\right)\right]^q u(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{q}}\\ &\leqslant C\left(\int_0^{b_1}\int_0^{b_2} f^p(x_1,x_2)v(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{p}}\end{aligned} \tag{4}$$

if and only if

$$D_W(s_1,s_2,p,q) := \sup_{\substack{y_1\in(0,b_1)\\ y_2\in(0,b_2)}} y_1^{\frac{s_1-1}{p}}\, y_2^{\frac{s_2-1}{p}}\left(\int_{y_1}^{b_1}\int_{y_2}^{b_2} x_1^{-\frac{s_1q}{p}}\, x_2^{-\frac{s_2q}{p}}\, w(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{1}{q}} < \infty, \tag{5}$$

where $s_1, s_2 > 1$ *and*

$$w(x_1,x_2) = \left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\ln v^{-1}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2\right)\right]^{\frac{q}{p}} u(x_1,x_2),$$

and the best possible constant C *in* (4) *can be estimated in the following way:*

$$\begin{aligned}&\sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}} D_W(s_1,s_2,p,q)\\ &\leqslant C \leqslant \inf_{s_1,s_2>1} e^{\frac{s_1+s_2-2}{p}} D_W(s_1,s_2,p,q).\end{aligned} \tag{6}$$

Concerning the general n-dimensional case ($n \in \mathbb{Z}_+$) the only published results so far seems to be those in the recent paper by M. F. Yimer (see [21]).

In Section 2 of this paper we prove a generalization of Theorem 1 (see Theorem 2) general n-dimensional case ($n \in \mathbb{Z}_+$) but we do it in a more general frame where the standard n-dimensional geometric operator is replaced by a more general geometric mean operator G_β (see (7)) so we can cover also the Cochran-Lee situation.

By using this result for the power weighted case (and $p = q$) we prove in Section 3 a general new n-dimensional Cochran-Lee inequality with sharp constant (see Theorem 4). This result generalizes several results in the literature including one in [21].

Finally, in Section 4 we give some concluding remarks and applications, including both well-known and new Hardy-type inequalities. In particular, we point out that Theorem 2 and its proof can be used to formulate the first example where a multidimensional Hardy-type inequality can be characterized not only by one condition but by infinite many (equivalent) conditions, even by a scale of condition (see Theorem 5). For the one-dimensional case this fairly new idea in the theory of Hardy-type inequalities is described and applied in [13, Section 7.3].

2. A new multidimensional Hardy-type inequality of Cochran-Lee type

First we introduce the following multidimensional geometric mean operator G_β:

$$\left(G_{\boldsymbol{\beta}}f\right)(\mathbf{x}) := \exp\left(\prod_{i=1}^n \beta_i x_i^{-\beta_i}\int_0^{x_1}\cdots\int_0^{x_n}\prod_{i=1}^n t_i^{\beta_i-1}\ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right), \tag{7}$$

for any nonnegative and measurable function $f(t)$ on $\mathbb{R}_+^n := (0,\infty)^n$, where $\beta_i > 0$ $(i = 1,\dots,n)$, $\mathrm{d}\mathbf{t} := \mathrm{d}t_1\cdots\mathrm{d}t_n$ and $\mathbf{t} := (t_1,\dots,t_n)$.

We investigate the multidimensional weighted geometric mean inequality

$$\left(\int_0^{b_1}\cdots\int_0^{b_n}\left[\left(G_{\boldsymbol{\beta}}f\right)(\mathbf{x})\right]^q u(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^{b_1}\cdots\int_0^{b_n} f^p(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}, \tag{8}$$

where u and v are weight functions, $0 < p \leqslant q < \infty$, $0 < b_i \leqslant \infty$, $\beta_i > 0$ $(i = 1,\dots,n)$, and C is a positive constant independent of f.

Here, and in the sequel we use the following notations in the respective variables and parameters. For $n \geqslant 2$,

$$\mathbf{xt} = (x_1t_1,\ldots,x_nt_n),\ \mathbf{1/t} = (1/t_1,\ldots,1/t_n),\ \mathbf{x}^{\frac{1}{\boldsymbol{\beta}}} = (x_1^{\frac{1}{\beta_1}},\ldots,x_n^{\frac{1}{\beta_n}}),$$
$$J_n = \{1,\ldots,n\},\ \mathbb{N}_0 = \mathbb{N}\cup\{0\},\ \mathbf{y}\leqslant\mathbf{x} \Leftrightarrow y_i \leqslant x_i\ (i=1,\ldots,n),$$

and

$$\int_{\mathbf{y}^{\boldsymbol{\eta}}}^{\mathbf{b}^{\boldsymbol{\beta}}} = \int_{y_1^{\eta_1}}^{b_1^{\beta_1}}\cdots\int_{y_n^{\eta_n}}^{b_n^{\beta_n}},\ \int_{\mathbb{R}^n_+} = \underbrace{\int_0^\infty\cdots\int_0^\infty}_{n \text{ times}},\ \int_{\mathbf{0}}^{\mathbf{x}} = \int_0^{x_1}\cdots\int_0^{x_n},\ \int_{\mathbf{0}}^{\mathbf{1}} = \underbrace{\int_0^1\cdots\int_0^1}_{n \text{ times}},$$

where $0 \leqslant y_i < b_i \leqslant \infty\ \ (i=1,\ldots,n)$.

CONVENTION. Throughout this paper we assume that f is a positive and measurable function defined on $\mathbb{R}^n_+$ and $\mathbf{b}=(b_1,\ldots,b_n)$, $n\in\mathbb{Z}_+$, $0<b_i\leqslant\infty$, $i=1,\ldots,n$.

Our main theorem in this Section reads:

THEOREM 2. *Let $n\in\mathbb{Z}_+$, $0<p\leqslant q<\infty$, $\beta_i>0\ \ (i=1,\ldots,n)$, and let u,v and f be positive and measurable functions on $\mathbb{R}^n_+$. Then, the inequality (8) holds for some finite constant C if and only if for any $\alpha_i>0\ \ (i=1,\ldots,n)$,*

$$A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) := \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}} \prod_{i=1}^n t_i^{\frac{\alpha_i+\beta_i-1}{p}} \left(\int_{\mathbf{t}}^{\mathbf{b}} \prod_{i=1}^n x_i^{-(\alpha_i+\beta_i)\frac{q}{p}} w(\mathbf{x})\,d\mathbf{x}\right)^{\frac{1}{q}} < \infty, \tag{9}$$

where

$$w(\mathbf{x}) = u(\mathbf{x})\left[G_{\boldsymbol{\beta}}v^{-1}(\mathbf{x})\right]^{\frac{q}{p}}. \tag{10}$$

Moreover, if C is the best possible constant in (8), then

$$\sup_{\substack{\alpha_i>0\\ i\in J_n}} \prod_{i=1}^n \left(\frac{(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}{1+(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}\right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \leqslant C \leqslant \inf_{\substack{\alpha_i>0\\ i\in J_n}} \prod_{i=1}^n \left(\beta_i \exp\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}). \tag{11}$$

Proof. Sufficiency. Let $g(\mathbf{x}) = f^p(\mathbf{x})v(\mathbf{x})$. Then the inequality (8) is equivalent to the inequality

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} \left[(G_{\boldsymbol{\beta}}g)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,d\mathbf{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})\,d\mathbf{x}\right)^{\frac{1}{p}}, \tag{12}$$

where $w(x)$ is defined by (10).

Let $t_i = x_i y_i \ \ (i=1,\dots,n)$. Then the inequality (12) becomes

$$\left(\int_{\mathbf{0}}^{\mathbf{b}}\left[\exp\left(\prod_{i=1}^{n}\beta_i\int_{\mathbf{0}}^{\mathbf{1}}\prod_{i=1}^{n}y_i^{\beta_i-1}\ln g(\mathbf{xy})\,\mathrm{d}\mathbf{y}\right)\right]^{\frac{q}{p}}w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}\leqslant C\left(\int_{\mathbf{0}}^{\mathbf{b}}g(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}. \tag{13}$$

For $\alpha_i > 0 \ \ (i=1,\dots,n)$, we trivially have that

$$\exp\left(\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\right)\exp\left(\prod_{i=1}^{n}\beta_i\int_{\mathbf{0}}^{\mathbf{1}}\prod_{i=1}^{n}y_i^{\beta_i-1}\log\prod_{i=1}^{n}y_i^{\alpha_i}\,\mathrm{d}\mathbf{y}\right)=1. \tag{14}$$

By applying the identity (14) and then using Jensen's inequality, we find that the left hand side of (13) can be written and estimated as follows:

$$\begin{aligned}
&\exp\left(\sum_{i=1}^{n}\frac{\alpha_i}{p\beta_i}\right)\left(\int_{\mathbf{0}}^{\mathbf{b}}\left[\exp\left(\prod_{i=1}^{n}\beta_i\int_{\mathbf{0}}^{\mathbf{1}}\prod_{i=1}^{n}y_i^{\beta_i-1}\log g(\mathbf{xy})\prod_{i=1}^{n}y_i^{\alpha_i}\,\mathrm{d}\mathbf{y}\right)\right]^{\frac{q}{p}}w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}\\
&\leqslant\left(\prod_{i=1}^{n}\beta_i\exp\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}}\left(\int_{\mathbf{0}}^{\mathbf{b}}\left(\int_{\mathbf{0}}^{\mathbf{1}}\prod_{i=1}^{n}y_i^{\beta_i+\alpha_i-1}g(\mathbf{xy})\,\mathrm{d}\mathbf{y}\right)^{\frac{q}{p}}w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}\\
&=\left(\prod_{i=1}^{n}\beta_i\exp\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}}\left(\int_{\mathbf{0}}^{\mathbf{b}}\left(\int_{\mathbf{0}}^{\mathbf{x}}g(\mathbf{t})\prod_{i=1}^{n}t_i^{\beta_i+\alpha_i-1}\,\mathrm{d}\mathbf{t}\right)^{\frac{q}{p}}\prod_{i=1}^{n}x_i^{-(\beta_i+\alpha_i)\frac{q}{p}}w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}:=I.
\end{aligned} \tag{15}$$

Therefore, by using Minkowski's integral inequality when $p<q$ or Fubini's theorem when $p=q$ (c.f. [19, Remark 5.2]), we have that

$$I\leqslant\left(\prod_{i=1}^{n}\beta_i\exp\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}}A_{\boldsymbol{\beta}}(\boldsymbol{\alpha})\left(\int_{\mathbf{0}}^{\mathbf{b}}g(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)^{\frac{1}{p}}, \tag{16}$$

where $A_\beta(\alpha)$ is defined by (9)–(10).

By combining (15)–(16) we find that (13) and, thus, (12) holds. Moreover, since (8) is equivalent to (12), we conclude that (8) holds and that the best constant C in (8) satisfies

$$C\leqslant\inf_{\substack{\alpha_i>0\\ i\in J_n}}\prod_{i=1}^{n}\left(\beta_i\exp\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}}A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}). \tag{17}$$

Necessity. Assume that (8), or equivalently (13), holds. In order to prove that (13) implies (9)–(10), we define the test function g by

$$g(\mathbf{x}):=\prod_{i=1}^{n}\left(t_i^{-1}\chi_{[0,t_i]}(x_i)+\frac{t_i^{\beta_i+\alpha_i-1}}{x_i^{(\beta_i+\alpha_i)}}\exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\chi_{(t_i,b_i)}(x_i)\right),$$

for fixed t_i, $0 < t_i < b_i$ $(i = 1, \ldots, n)$. Then

$$\begin{aligned}
\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{0}}^{\mathbf{b}} \prod_{i=1}^{n} \left(t_i^{-1}\chi_{[0,t_i]}(x_i) + \frac{t_i^{\beta_i+\alpha_i-1}}{x_i^{(\beta_i+\alpha_i)}} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\chi_{(t_i,b_i)}(x_i)\right) \mathrm{d}\mathbf{x} \\
&= \prod_{i=1}^{n} \left(\int_0^{t_i} t_i^{-1}\,\mathrm{d}x_i + \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right) t_i^{\beta_i+\alpha_i-1} \int_{t_i}^{b_i} x_i^{-(\beta_i+\alpha_i)}\,\mathrm{d}x_i \right) \\
&= \prod_{i=1}^{n} \left(1 + \frac{1}{\beta_i+\alpha_i-1} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right) \left(1 - \left(\frac{t_i}{b_i}\right)^{\beta_i+\alpha_i-1}\right)\right) \\
&\leqslant \prod_{i=1}^{n} \left(\frac{1+(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)}{(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)} \right).
\end{aligned}$$

This implies that

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \leqslant \prod_{i=1}^{n} \left(\frac{1+(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)}{(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)} \right)^{\frac{1}{p}}. \tag{18}$$

Trivially, for $\mathbf{0} \leqslant \mathbf{t} < \mathbf{b}$, we have that

$$\left(\int_{\mathbf{t}}^{\mathbf{b}} \left[\left(G_{\boldsymbol{\beta}} g\right)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leqslant \left(\int_{\mathbf{0}}^{\mathbf{b}} \left[\left(G_{\boldsymbol{\beta}} g\right)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}. \tag{19}$$

Moreover, for $\mathbf{t} \leqslant \mathbf{x} < \mathbf{b}$, we find that

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{x}} \prod_{j=1}^{n} y_j^{\beta_j-1} \ln g(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \sum_{i=1}^{n} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{j=1}^{n} y_j^{\beta_j-1} \ln \left(t_i^{-1}\chi_{[0,t_i]}(y_i) + \frac{t_i^{\beta_i+\alpha_i-1}}{y_i^{(\beta_i+\alpha_i)}} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\chi_{(t_i,b_i)}(y_i)\right) \mathrm{d}\mathbf{y} \\
&= \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j^{\beta_j}}{\beta_j} \int_0^{x_i} y_i^{\beta_i-1} \ln \left(t_i^{-1}\chi_{[0,t_i]}(y_i) + \frac{t_i^{\beta_i+\alpha_i-1}}{y_i^{(\beta_i+\alpha_i)}} \exp\left(-\frac{\beta_i+\alpha_i}{\beta_i}\right)\chi_{(t_i,b_i)}(y_i)\right) \mathrm{d}y_i \\
&= \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j^{\beta_j}}{\beta_j} \left(\frac{x_i^{\beta_i}}{\beta_i} \ln\left(t_i^{\beta_i+\alpha_i-1} x_i^{-(\beta_i+\alpha_i)}\right)\right) \\
&= \left(\prod_{j=1}^{n} \frac{x_j^{\beta_j}}{\beta_j}\right) \sum_{i=1}^{n} \ln\left(t_i^{\beta_i+\alpha_i-1} x_i^{-(\beta_i+\alpha_i)}\right),
\end{aligned}$$

and, hence,

$$\left(\int_{\mathbf{t}}^{\mathbf{b}} \left[\left(G_{\boldsymbol{\beta}} g\right)(\mathbf{x})\right]^{\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} = \prod_{i=1}^{n} t_i^{\frac{\beta_i+\alpha_i-1}{p}} \left(\int_{\mathbf{t}}^{\mathbf{b}} \prod_{i=1}^{n} x_i^{-(\beta_i+\alpha_i)\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}. \tag{20}$$

It follows from (13) and (18)–(20) that

$$
\begin{aligned}
A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) &= \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}} \prod_{i=1}^{n} t_i^{\frac{\beta_i+\alpha_i-1}{p}} \left(\int_{\mathbf{t}}^{\mathbf{b}} \prod_{i=1}^{n} x_i^{-(\beta_i+\alpha_i)\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \\
&\leqslant C \prod_{i=1}^{n} \left(\frac{1+(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)}{(\beta_i+\alpha_i-1)\exp\left(\frac{\beta_i+\alpha_i}{\beta_i}\right)}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus, since $C<\infty$, we conclude that indeed (9)–(10) holds and that the left hand side inequality of (11) holds. Thus, also the necessity part is proved. The proof is complete including the fact that (11) holds. The proof is complete. $\square$

REMARK 1. Note that for the case $n=2$, $\beta_i=1$, $\alpha_i=s_i-1$, $(i=1,2)$, we obtain Theorem 1 so in particular, we can conclude that Theorem 1 holds also in a general n-dimensional setting ($n\in\mathbb{Z}_+$).

As in the classical situation, by doing suitable substitutions, we can also derive a dual version of Theorem 2 (where integrals $\int_0^t$ are replaced by $\int_t^\infty$).

THEOREM 3. *Let $0<p\leqslant q<\infty$, $\beta_i>0$ $(i=1,\dots,n)$, and let u,v and f be positive and measurable functions on $\mathbb{R}_+^n$. Then, for $n=2,3,\dots$,*

$$
\begin{aligned}
&\left(\int_{\mathbb{R}_+^n}\left[\exp\left(\prod_{i=1}^{n}\beta_i x_i^{\beta_i}\int_{\mathbf{x}}^{\infty}\prod_{i=1}^{n} t_i^{-(\beta_i+1)}\ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q u(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \\
&\leqslant C\left(\int_{\mathbb{R}_+^n} f^p(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}
\end{aligned}
\tag{21}
$$

holds for some finite C if and only if for any $\alpha_i>0$ $(i=1,\dots,n)$,

$$
B_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) := \sup_{\substack{t_i>0\\ i\in J_n}} \prod_{i=1}^{n} t_i^{\frac{\beta_i+\alpha_i-1}{p}} \left(\int_{\mathbf{t}}^{\infty} \prod_{i=1}^{n} x_i^{-(\beta_i+\alpha_i)\frac{q}{p}} W(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} < \infty, \tag{22}
$$

where

$$
W(\mathbf{x}) = U(\mathbf{x})\left[G_{\boldsymbol{\beta}} V^{-1}(\mathbf{x})\right]^{\frac{q}{p}}, \tag{23}
$$

and

$$
U(\mathbf{x}) = u(\mathbf{1}/\mathbf{x})\prod_{i=1}^{n} x_i^{-2} \text{ and } V(\mathbf{x}) = v(\mathbf{1}/\mathbf{x})\prod_{i=1}^{n} x_i^{-2}. \tag{24}
$$

Moreover, if C is the best possible constant in (21), then

$$\sup_{\substack{\alpha_i>0\\ i\in J_n}}\prod_{i=1}^{n}\left(\frac{(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}{1+(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}\right)^{\frac{1}{p}} B_{\boldsymbol{\beta}}(\boldsymbol{\alpha})\leqslant C\leqslant \inf_{\substack{\alpha_i>0\\ i\in J_n}}\prod_{i=1}^{n}\left(\beta_i\exp\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{p}} B_{\boldsymbol{\beta}}(\boldsymbol{\alpha}). \tag{25}$$

Proof. First we note that by using the substitutions first $y_i=1/t_i$, and then $z_i=1/x_i\ (i=1,\dots,n)$ and $g(\mathbf{t})=f(\mathbf{1}/\mathbf{t})$, elementary calculations show that (21) is equivalent to the inequality

$$\begin{aligned}&\left(\int_{\mathbb{R}^n_+}\left[\exp\left(\prod_{i=1}^{n}\beta_i z_i^{-\beta_i}\int_{\mathbf{0}}^{\mathbf{z}}\prod_{i=1}^{n}t_i^{\beta_i-1}\ln g(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^{q}U(\mathbf{z})\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{q}}\\ &\leqslant C\left(\int_{\mathbb{R}^n_+}g^p(\mathbf{z})V(\mathbf{z})\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{p}},\end{aligned} \tag{26}$$

where $U(x)$ and $V(x)$ are defined by (24).

In view of Theorem 2, the inequality (26) holds for some finite C if and only if for any $\alpha_i>0\ (i=1,\dots,n)$,

$$B_{\boldsymbol{\beta}}(\boldsymbol{\alpha})=\sup_{\substack{t_i>0\\ i\in J_n}}\prod_{i=1}^{n}t_i^{\frac{\beta_i+\alpha_i-1}{p}}\left(\int_{\mathbf{t}}^{\infty}\prod_{i=1}^{n}x_i^{-(\beta_i+\alpha_i)\frac{q}{p}}W(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}<\infty,$$

where

$$W(\mathbf{x})=U(\mathbf{x})\left[\left(G_{\boldsymbol{\beta}}V^{-1}\right)(\mathbf{x})\right]^{\frac{q}{p}},$$

with $U(x)$ and $V(x)$ are defined by (24).

Since the inequality (8) is equivalent to (21) with $U(x)$ and $V(x)$ are defined by (24), we can, by Theorem 2, conclude that (21) holds if and only if (22)–(24) holds. Moreover, for the sharp constant in (21) we have the estimates (25). The proof is complete. $\square$

3. Multidimensional Cochran-Lee inequalities with sharp constants

First we note that Theorem 2 implies the following inequality for power weights:

PROPOSITION 1. Let $0<p\leqslant q<\infty$, and let $\beta_i>0$, $\eta_i,\gamma_i>-1\ (i=1,\dots,n)$. Then, the inequality

$$\left(\int_{\mathbf{0}}^{\mathbf{b}}\left[\left(G_{\boldsymbol{\beta}}f\right)(\mathbf{x})\right]^{q}\prod_{i=1}^{n}x_i^{\eta_i}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}}\leqslant C\left(\int_{\mathbf{0}}^{\mathbf{b}}f^p(\mathbf{x})\prod_{i=1}^{n}x_i^{\gamma_i}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \tag{27}$$

holds if and only if

$$\frac{1+\eta_i}{q} = \frac{1+\gamma_i}{p}, \tag{28}$$

for all $i = 1,\dots,n$.

Proof. By applying Theorem 2 with these power weights we easily obtain the proof since in this case $A_\beta(\alpha)$ (see (9)) is of the form

$$A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{p\beta_i}\right) \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}} \prod_{i=1}^{n} t_i^{\frac{1+\eta_i}{q}-\frac{1+\gamma_i}{p}} \left[\frac{1-\left(\frac{t_i}{b_i}\right)^{(\alpha_i+\gamma_i+\beta_i)\frac{q}{p}-(1+\eta_i)}}{(\alpha_i+\gamma_i+\beta_i)\frac{q}{p}-(1+\eta_i)}\right]^{\frac{1}{q}}$$

$$\leqslant \exp\left(\sum_{i=1}^{n} \frac{\gamma_i}{p\beta_i}\right) \sup_{\substack{t_i\in(0,b_i)\\ i\in J_n}} \prod_{i=1}^{n} \frac{t_i^{\frac{1+\eta_i}{q}-\frac{1+\gamma_i}{p}}}{\left[(\alpha_i+\gamma_i+\beta_i)\frac{q}{p}-(1+\eta_i)\right]^{\frac{1}{q}}},$$

provided that $\beta_i > \frac{p}{q}(1+\eta_i)-(\alpha_i+\gamma_i)$ $(i=1,\dots,n)$. We omit the details. □

REMARK 2. For the case $p<q$ we judge that it is a difficult and open question to find the sharp constant C in (27). See Remark 5. However, in our next main theorem we will derive the sharp constant for the case $p=q$ and thus obtain a genuine generalization of the Cochran-Lee inequality to a multidimensional setting. Moreover, since $\left(G_\beta f^p\right)(x) = \left[\left(G_\beta f\right)(x)\right]^p$ it is sufficient to prove this fact for the case $p=q=1$. See also Remark 4.

THEOREM 4. *Let $\beta_i>0$, $\eta_i>-1$ $(i=1,\dots,n)$, and let f be a positive and measurable function defined on $\mathbb{R}^n_+$, $n\in\mathbb{Z}_+$. Then the inequality*

$$\int_{\mathbb{R}^n_+} \left(G_{\boldsymbol{\beta}} f\right)(\mathbf{x}) \prod_{i=1}^{n} x_i^{\eta_i}\,\mathrm{d}\mathbf{x} \leqslant \exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \prod_{i=1}^{n} x_i^{\eta_i}\,\mathrm{d}\mathbf{x} \tag{29}$$

holds and the constant $\exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right)$ *is sharp.*

Proof. In view of Proposition 1, the inequality (29) holds with $\exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right)$ replaced by some finite $C>0$. Now, we prove that the best constant $C = \exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right)$. From (11) in Theorem 2, it follows that the best constant C satisfies

$$C \leqslant \exp\left(\sum_{i=1}^{n} \frac{\eta_i}{\beta_i}\right) \inf_{\substack{\alpha_i>0\\ i\in J_n}} \left(\prod_{i=1}^{n} \frac{\exp\left(\frac{\alpha_i}{\beta_i}\right)}{\left(\frac{\beta_i+\alpha_i-1}{\beta_i}\right)}\right).$$

The infimum in the above inequality is attained at $\alpha_i = 1 \ (i = 1,\ldots,n)$. Hence, we find that

$$C \leqslant \exp\left(\sum_{i=1}^{n} \frac{1+\eta_i}{\beta_i}\right). \tag{30}$$

It only remains to prove that the inequality (30) holds also in the reversed direction. Consider the test function

$$f(\mathbf{x}) = \prod_{i=1}^{n}\left(\chi_{[0,e^{\frac{1}{\beta_i}}]}(x_i) + x_i^{-\gamma_i}\chi_{(e^{\frac{1}{\beta_i}},\infty)}(x_i)\right),$$

where $\gamma_i > 1+\eta_i \ (i = 1,\ldots,n)$. Next we note that then the integral part on the right hand side of (29) becomes

$$\begin{aligned}\int_{\mathbb{R}_+^n} f(\mathbf{x})\prod_{i=1}^{n} x_i^{\eta_i}\,\mathrm{d}\mathbf{x} &= \prod_{i=1}^{n}\int_{\mathbb{R}_+} x_i^{\eta_i}\left(\chi_{[0,e^{\frac{1}{\beta_i}}]}(x_i) + x_i^{-\gamma_i}\chi_{(e^{\frac{1}{\beta_i}},\infty)}(x_i)\right)\mathrm{d}x_i \\ &= \prod_{i=1}^{n}\exp\left(\frac{1+\eta_i}{\beta_i}\right)\left(\frac{1}{1+\eta_i} + \frac{\exp\left(-\frac{\gamma_i}{\beta_i}\right)}{\gamma_i-(1+\eta_i)}\right).\end{aligned} \tag{31}$$

Moreover, the left hand side of (29) is equal to

$$\begin{aligned}&\prod_{i=1}^{n}\int_{\mathbb{R}_+} x_i^{\eta_i}\exp\left(\beta_i x_i^{-\beta_i}\int_0^{x_i} t_i^{\beta_i-1}\ln\left(\chi_{[0,e^{\frac{1}{\beta_i}}]}(t_i) + t_i^{-\gamma_i}\chi_{(e^{\frac{1}{\beta_i}},\infty)}(t_i)\right)\mathrm{d}t_i\right)\mathrm{d}x_i \\ &\qquad = \prod_{i=1}^{n}\exp\left(\frac{1+\eta_i}{\beta_i}\right)\left(\frac{\gamma_i}{(1+\eta_i)(\gamma_i-(1+\eta_i))}\right)\end{aligned} \tag{32}$$

It follows from (29) with best constant C, (31) and (32) that

$$\prod_{i=1}^{n}\frac{\exp\left(\frac{\gamma_i}{\beta_i}\right)}{\left(\frac{1+\eta_i}{\gamma_i} + \left(1-\frac{1+\eta_i}{\gamma_i}\right)\exp\left(\frac{\gamma_i}{\beta_i}\right)\right)} \leqslant C.$$

By letting $\gamma_i \to (1+\eta_i)^+ \ (i = 1,\ldots,n)$, we find that

$$\prod_{i=1}^{n}\exp\frac{1+\eta_i}{\beta_i} = \exp\left(\sum_{i=1}^{n}\frac{1+\eta_i}{\beta_i}\right) \leqslant C. \tag{33}$$

Therefore, the sharpness of the constant in (29) is proved by just combining (30) and (33). The proof is complete. $\square$

Next we consider the special case $\beta_i = a$ and $\eta_i = c \ (i = 1,\ldots,n)$. First we point out the following immediate consequence of Theorem 4:

COROLLARY 1. *Let $n \in \mathbb{Z}_+$, let $a > 0$, $c > -1$, and let f be positive and measurable function defined on $\mathbb{R}^n_+$. Then, the inequality*

$$\int_{\mathbb{R}^n_+} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t}) \mathrm{d}\mathbf{t}\right) \prod_{i=1}^n x_i^c \,\mathrm{d}\mathbf{x} \leqslant \exp\left(n\frac{1+c}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \prod_{i=1}^n x_i^c \,\mathrm{d}\mathbf{x}$$

holds and the constant $\exp\left(n\frac{1+c}{a}\right)$ *is sharp.*

But Theorem 4 also implies the following less obvious weighted multidimensional generalization of the Cochran-Lee inequality:

COROLLARY 2. *Let $n \in \mathbb{Z}_+$, let $a > 0$, $c \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $ck > -1$, and let f be a positive and measurable function defined on $\mathbb{R}^n_+$. Then, the inequality*

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t}) \,\mathrm{d}\mathbf{t}\right) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x} \\ &\leqslant \exp\left(\frac{n+ck}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x} \end{aligned} \tag{34}$$

holds and the constant $\exp\left(\frac{n+ck}{a}\right)$ *is sharp.*

Proof. In view of Theorem 4 and by applying the multinomial theorem twice, we have that

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t}) \,\mathrm{d}\mathbf{t}\right) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x} \\ &= \sum_{\substack{m_1+\cdots+m_n=k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1,\ldots,m_n} \int_{\mathbb{R}^n_+} \exp\left(a^n \prod_{i=1}^n x_i^{-a} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^n t_i^{a-1} \ln f(\mathbf{t}) \,\mathrm{d}\mathbf{t}\right) \prod_{i=1}^n x_i^{cm_i} \,\mathrm{d}\mathbf{x} \\ &\leqslant \sum_{\substack{m_1+\cdots+m_n=k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1,\ldots,m_n} \exp\left(\sum_{i=1}^n \frac{1+cm_i}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \prod_{i=1}^n x_i^{cm_i} \,\mathrm{d}\mathbf{x} \\ &= \exp\left(\frac{n+ck}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \sum_{\substack{m_1+\cdots+m_n=k \\ m_i \in \mathbb{N}_0}} \binom{k}{m_1,\ldots,m_n} \prod_{i=1}^n x_i^{cm_i} \,\mathrm{d}\mathbf{x} \\ &= \exp\left(\frac{n+ck}{a}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) \left(\sum_{i=1}^n x_i^c\right)^k \mathrm{d}\mathbf{x}, \end{aligned}$$

where

$$\binom{k}{m_1,\ldots,m_n} = \frac{k!}{m_1!\cdots m_n!}$$

is a multinomial coefficient. Since the constant in Theorem 4 is sharp, then the sharpness of the constant $\exp\left(\frac{n+ck}{a}\right)$ in (34) is guaranteed so the proof is complete. $\square$

We conclude this Section by pointing out that the fact that Theorem 4 also implies the following multidimensional Cochran-Lee type inequality, which, in particular, generalizes a result in [11] (see Example 1 and Remark 7):

COROLLARY 3. *Let $n \in \mathbb{Z}_+$, let $\beta_i > 0$ $(i = 1,\dots,n)$, let f be positive and measurable functions on $\mathbb{R}^n_+$ and let the weights $u(x)$ and $v(x)$ be related by $u(x) = \left(G_\beta v\right)(x)$. Then, the inequality*

$$\int_{\mathbb{R}^n_+} \left[\left(G_{\boldsymbol{\beta}} f\right)(\mathbf{x})\right] u(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \exp\left(\sum_{i=1}^{n} \frac{1}{\beta_i}\right) \int_{\mathbb{R}^n_+} f(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{35}$$

holds and the constant $\exp\left(\sum_{i=1}^{n} \frac{1}{\beta_i}\right)$ *is sharp.*

Proof. Let $g(\mathbf{x}) = f(\mathbf{x})v(\mathbf{x})$. Then, the inequality (35) is equivalent to

$$\int_{\mathbb{R}^n_+} \left[\left(G_{\boldsymbol{\beta}} g\right)(\mathbf{x})\right]\,\mathrm{d}\mathbf{x} \leqslant \exp\left(\sum_{i=1}^{n} \frac{1}{\beta_i}\right) \int_{\mathbb{R}^n_+} g(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{36}$$

In view of Theorem 4 with $\eta_i = 0$ $(i = 1,\dots,n)$, we have that indeed the inequality (36) holds and the constant $\exp\left(\sum_{i=1}^{n} \frac{1}{\beta_i}\right)$ is sharp. Therefore, from the equivalence of (35) and (36), we can conclude that (35) holds and the constant $\exp\left(\sum_{i=1}^{n} \frac{1}{\beta_i}\right)$ is sharp. The proof is complete. $\square$

4. Concluding Remarks and results

REMARK 3. Some authors referred early to the inequality (2) with $\alpha = 0$ as Knopp's inequality with reference to [12] but it was later on discovered that G. H. Hardy in his famous 1925 paper [4] mentioned that his friend G. Pólya had pointed out to him that this inequality is just a limit case of his original inequality. By applying our results in Sections 2 and 3 with $\beta_i = 1$, $i = 1,\dots,n$, we obtain as special cases most of us known multidimensional Pólya-Knopp's inequalities, and especially all concerning sharp constants, see especially the recent paper [21], and the references therein.

REMARK 4. Let $0 < p \leqslant q < \infty$. If the condition (28) in Proposition 1 holds, then the best possible constant C in (27) satisfies

$$C \leqslant \exp\left(\frac{n}{q} - \frac{n}{p}\right) \prod_{i=1}^{n} \beta_i^{\frac{1}{p} - \frac{1}{q}} \exp\left(\frac{1+\gamma_i}{p\beta_i}\right). \tag{37}$$

In particular, if $p=q$, $0<p<\infty$, and $b_i=\infty$ $(i=1,\dots,n)$, then by replacing $f(x)$ by $f^p(x)$ in Theorem 4, we obtain that the inequality

$$\left(\int_{\mathbb{R}^n_+}\left[\left(G_{\boldsymbol{\beta}}f\right)(\mathbf{x})\right]^p\prod_{i=1}^n x_i^{\eta_i}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}\leqslant C\left(\int_{\mathbb{R}^n_+}f^p(\mathbf{x})\prod_{i=1}^n x_i^{\eta_i}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}, \tag{38}$$

holds with the sharp constant $C=\prod_{i=1}^n\exp\left(\frac{1+\gamma_i}{p\beta_i}\right)$. Hence, (38) is a formal generalization of (29) which is the case $p=1$.

OPEN QUESTION. Find the sharp constant in the inequality (27) for the case $0<p<q<\infty$.

REMARK 5. We believe that this open question is not so easy to solve. Our motivation for that is that the corresponding question in the theory of Hardy-type inequalities (with our geometric mean operator G_β replaced by the corresponding Hardy arithmetic mean operator H) was an especially long lasting question even in the one dimensional case. It was finally solved only in 2015 in the paper [16] by L. E. Persson and S. Samko.

REMARK 6. For $\beta=1$ an one dimensional analogue of the estimate (27) in Proposition 1 was stated in [17, Example on page 744]. However, the inequality (37) gives a better estimate of the sharp constant than that in [17].

The next result follows from Theorem 4 and, in particular, it generalizes the result in [14, Theorem C]:

COROLLARY 4. *Let $\beta_i>0$, $\eta_i<-1$ $(i=1,\dots,n)$, and let f be a positive measurable function defined on $\mathbb{R}^n_+$. Then, the inequality*

$$\int_{\mathbb{R}^n_+}\exp\left(\prod_{i=1}^n\beta_i x_i^{\beta_i}\int_{\mathbf{x}}^{\infty}\prod_{i=1}^n t_i^{-(\beta_i+1)}\ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\prod_{i=1}^n x_i^{\eta_i}\,\mathrm{d}\mathbf{x}\leqslant C\int_{\mathbb{R}^n_+}f(\mathbf{x})\prod_{i=1}^n x_i^{\eta_i}\,\mathrm{d}\mathbf{x} \tag{39}$$

holds for some finite $C>0$ and the constant $C=\exp\left(\sum_{i=1}^n\frac{-(1+\eta_i)}{\beta_i}\right)$ is sharp.

Proof. By using the substitutions first $y_i=1/t_i$, and then $z_i=1/x_i$ $(i=1,\dots,n)$ and $g(\mathbf{t})=f(\mathbf{1}/\mathbf{t})$, elementary calculations show that (39) is equivalent to the inequality

$$\int_{\mathbb{R}^n_+}\exp\left(\prod_{i=1}^n\beta_i z_i^{-\beta_i}\int_{\mathbf{0}}^{\mathbf{z}}\prod_{i=1}^n y_i^{(\beta_i-1)}\ln g(\mathbf{y})\,\mathrm{d}\mathbf{y}\right)\prod_{i=1}^n z_i^{-\eta_i-2}\,\mathrm{d}\mathbf{z}\leqslant C\int_{\mathbb{R}^n_+}g(\mathbf{z})\prod_{i=1}^n z_i^{-\eta_i-2}\,\mathrm{d}\mathbf{z}. \tag{40}$$

In view of Theorem 4, the inequality 40 holds with sharp constant $C=\exp\left(\sum_{i=1}^n\frac{-(1+\eta_i)}{\beta_i}\right)$. Therefore, from the equivalence of (39) and (40), we can conclude that (39) holds for some finite $C > 0$ and the constant $C = \exp\left(\sum\limits_{i=1}^{n} \frac{-(1+\eta_i)}{\beta_i}\right)$ is sharp. The proof is complete. $\square$

In order to relate our results to another result in the literature (see Remark 7) we present the following consequence of Corollary 3:

EXAMPLE 1. Let $n \in \mathbb{Z}_+$, let $\beta_i > 0$, and let η_i and γ_i be real numbers such that $\beta_i + \gamma_i > 0$ $(i = 1, \ldots, n)$. Then, the inequality

$$\int_{\mathbb{R}_+^n} \left[\left(G_{\boldsymbol{\beta}} f \right)(\mathbf{x}) \right] \exp\left(\sum_{i=1}^{n} \frac{\beta_i \eta_i}{\beta_i + \gamma_i} x_i^{\gamma_i} \right) \mathrm{d}\mathbf{x} \leqslant \exp\left(\sum_{i=1}^{n} \frac{1}{\beta_i} \right) \int_{\mathbb{R}_+^n} f(\mathbf{x}) \exp\left(\sum_{i=1}^{n} \eta_i x_i^{\gamma_i} \right) \mathrm{d}\mathbf{x} \tag{41}$$

holds and the constant $\exp\left(\sum\limits_{i=1}^{n} \frac{1}{\beta_i}\right)$ is sharp.

REMARK 7. The one-dimensional analogue of Example 1 was discussed in [11, Corollary 1.6] for the case $\beta_i = \gamma_i = 1$ $(i = 1, \ldots, n)$ but without the estimate of the sharp constant.

Next we note that it is possible to derive also reversed Cochran-Lee type inequalities on the cone of non-increasing functions. This fact follows from the following elementary fact:

If a function f, defined on $\mathbb{R}_+^n$, is nonnegative and nonincreasing in all the variables, then

$$\prod_{i=1}^{n} \beta_i x_i^{-\beta_i} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^{n} t_i^{\beta_i - 1} f(t) \, \mathrm{d}t \geqslant f(\mathbf{x}), \tag{42}$$

where $\beta_i > 0$ $(i = 1, \ldots, n)$.

REMARK 8. Several of the results in this paper can be given also in the reversed direction on the cone of nondecreasing functions but in this case it is not always clear that is the constant 1 is sharp. Next, we present such a reversed Cochran-Lee inequality where indeed the constant is sharp.

PROPOSITION 2. Let $n \in \mathbb{Z}_+$, let $\beta_i > 0$ and $\eta_i > -1$ $(i = 1, \ldots, n)$. Then, the inequality

$$\int_{\mathbb{R}_+^n} \exp\left(\prod_{i=1}^{n} \beta_i x_i^{-\beta_i} \int_{\mathbf{0}}^{\mathbf{x}} \prod_{i=1}^{n} t_i^{\beta_i - 1} \ln f(t) \mathrm{d}t \right) \prod_{i=1}^{n} x_i^{\eta_i} \, \mathrm{d}\mathbf{x} \geqslant 1 \cdot \int_{\mathbb{R}_+^n} f(\mathbf{x}) \prod_{i=1}^{n} x_i^{\eta_i} \, \mathrm{d}\mathbf{x} \tag{43}$$

holds for all nonnegative and non-increasing functions f defined on $\mathbb{R}_+^n$ and the constant 1 is sharp.

Proof. Clearly, (42) implies (43). The sharpness follows by considering the test function $f_{\boldsymbol{\delta}}$, defined as

$$f_{\boldsymbol{\delta}}(\mathbf{x}) = \prod_{i=1}^{n}\left(\chi_{[0,1]}(x_i) + e^{-\frac{\delta_i}{\beta_i}} x_i^{-\delta_i}\chi_{(1,\infty)}(x_i)\right),$$

where $\delta_i > 1$ and letting $\delta_i \to \infty$ $(i = 1,\dots,n)$. We omit the details. $\square$

REMARK 9. The one and two-dimensional analogues of inequality (43) with $\beta_i = 1$ $(i = 1,\dots,n)$ are given in [9, Example 5.1] and [8], respectively.

Next, we prove that inequalities (8) and inequality (10) in [21] are equivalent. This equivalence in one and two dimensions were proved in [9] and [8], respectively.

PROPOSITION 3. Let $0 < p \leqslant q < \infty$, $0 < b_i \leqslant \infty$, $\beta_i > 0$ $(i = 1,\dots,n)$, and let u, v and f be positive functions on $\mathbb{R}^n_+$. Then, the inequality

$$\left(\int_{\mathbf{0}}^{\mathbf{b}}\left[\exp\left(\prod_{i=1}^{n}\beta_i x_i^{-\beta_i}\int_{\mathbf{0}}^{\mathbf{x}}\prod_{i=1}^{n} t_i^{\beta_i-1}\ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q u(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbf{0}}^{\mathbf{b}} f^p(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \tag{44}$$

is equivalent to the inequality

$$\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}}\left[\exp\left(\prod_{i=1}^{n} x_i^{-1}\int_{\mathbf{0}}^{\mathbf{x}}\ln g(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q u_{\boldsymbol{\beta}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}} g^p(\mathbf{x})v_{\boldsymbol{\beta}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}, \tag{45}$$

where C is a finite constant,

$$u_{\boldsymbol{\beta}}(\mathbf{x}) = u(\mathbf{x}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{x_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i},\quad v_{\boldsymbol{\beta}}(\mathbf{x}) = v(\mathbf{x}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{x_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i},$$

and

$$g(\mathbf{x}) = f(\mathbf{x}^{\frac{1}{\boldsymbol{\beta}}}).$$

Proof. By making the substitution $z_i = x_i^{\beta_i}$ $(i = 1,\dots,n)$, we find that the inequality (44) is equivalent to

$$\begin{aligned}&\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}}\left[\exp\left(\prod_{i=1}^{n}\beta_i z_i^{-1}\int_{\mathbf{0}}^{\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}}}\prod_{i=1}^{n} t_i^{\beta_i-1}\ln f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\right]^q u(\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{z_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i}\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{q}}\\ &\leqslant C\left(\int_{\mathbf{0}}^{\mathbf{b}^{\boldsymbol{\beta}}} f^p(\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}})v(\mathbf{z}^{\frac{1}{\boldsymbol{\beta}}})\prod_{i=1}^{n}\frac{z_i^{\frac{1-\beta_i}{\beta_i}}}{\beta_i}\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{p}}.\end{aligned} \tag{46}$$

Moreover, by making the variable transformation $t_i = y_i^{1/\beta_i}$, $(i = 1,\dots,n)$ in (46) we can conclude that (44) is equivalent to (45). The proof is complete. $\square$

Our final remark is related to a fairly new development in the theory of Hardy-type inequalities, namely the following: In the classical situation a Hardy-type inequality was usually characterized by using one condition (e.g. the famous Muckenhoupt condition for the case $1 < p \leqslant q < \infty$). However, it was discovered that this condition is not unique and can be replaced by infinite many equivalent conditions, even by scales of conditions. See Section 7.3 of the book [13]. However, for multidimensional Hardy-type inequalities no such scales of characterizing conditions are known. But correctly interpreted our proof of Theorem 2 shows that we indeed have such scale of conditions to characterize the multidimensional limit Hardy-type inequality (8). Indeed, Theorem 2 can be reformulated as follows:

THEOREM 5. *Let* $n \in \mathbb{Z}_+$, *let* $0 < p \leqslant q < \infty$, $\beta_i > 0$ $(i = 1,\dots,n)$ *and let* u, v *and* f *be positive and measurable functions. Then the inequality* (8) *holds if and only if any condition on the scale of conditions* $(0 < \alpha_i < \infty)$ $(i = 1,\dots,n)$

$$A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) := \sup_{\substack{t_i \in (0,b_i) \\ i \in J_n}} \prod_{i=1}^{n} t_i^{\frac{\alpha_i+\beta_i-1}{p}} \left(\int_{\mathbf{t}}^{\mathbf{b}} \prod_{i=1}^{n} x_i^{-(\alpha_i+\beta_i)\frac{q}{p}} w(\mathbf{x})\,\mathrm{d}\mathbf{x} \right)^{\frac{1}{q}} < \infty,$$

holds. Moreover, the sharp constant in (8) *can be estimated as follows:*

$$\sup_{\substack{\alpha_i>0 \\ i\in J_n}} \prod_{i=1}^{n} \left(\frac{(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)}{1+(\beta_i+\alpha_i-1)\exp\left(1+\frac{\alpha_i}{\beta_i}\right)} \right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \leqslant C \leqslant \inf_{\substack{\alpha_i>0 \\ i\in J_n}} \prod_{i=1}^{n} \left(\beta_i \exp\frac{\alpha_i}{\beta_i} \right)^{\frac{1}{p}} A_{\boldsymbol{\beta}}(\boldsymbol{\alpha}).$$

Acknowledgement. The first and third authors thank International Science Program (ISP), Uppsala, Sweden for financial and material support. We thank the referee for some good suggestions which have improved the final version of this paper.

REFERENCES

[1] A. ČIŽMEŠIJA, J. PEČARIĆ AND L.-E. PERSSON, *On strengthened Hardy and Pólya-Knopp's inequalities*, J. Approx. Theory, 125 (2003), 74–84.
[2] J. A. COCHRAN AND C. S. LEE, *Inequalities related to Hardy's and Heinig's*, Math. Proc. Camb. Phil. Soc. **96**, (1984), 1–7.
[3] I. GADJEV AND V. GOCHEV, *On the constant in the Hardy inequality for finite sequences*, Math. Inequal. Appl. **26** (2), (2023), 493–498.
[4] G. H. HARDY, *Notes on some points in the integral calculus, LX*, An inequality between integrals, Messenger of Math. **54**, (1925), 150–156.
[5] G. H. HARDY, *Notes on some points in the integral calculus, LXIV*, Messenger of Math. **57**, (1927–1928), 12–16.
[6] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, 1952 (1934).

[7] H. P. HEINIG, R. KERMAN AND M. KRBEC, *Weighted exponential inequalities*, Georgian Math. J. **8** (1), (2001), 69–86.
[8] P. JAIN AND R. HASSIJA, *Some Remarks on Two-Dimensional Knopp Type Inequalities*, Appl. Math. Lett. **16**, (2003), 459–464.
[9] P. JAIN, L.-E. PERSSON AND A. WEDESTIG, *Carleman-Knopp type inequalities via Hardy inequalities*, Math. Inequal. Appl. **4** (3), (2001), 343–355.
[10] P. JAIN AND A. P. SINGH, *A characterization for the boundedness of geometric mean operator*, Appl. Math. Lett. **13**, (2000) 63–67.
[11] S. KAIJSER, L. NIKOLOVA, L.-E. PERSSON AND A. WEDESTIG, *Hardy-type inequality via convexity*, Math. Inequal. Appl. **8** (3), (2005), 403–417.
[12] K. KNOPP, *Über Reihen mit positivern Gliedern*, J. London Math. Soc. **3**, (1928), 205–211.
[13] A. KUFNER, L.-E. PERSSON AND N. SAMKO, *Weighted Inequalities of Hardy Type, Second Edition*, World Scientific Publishing Co. New Jersey, 2017.
[14] E. G. KWON AND M. J. JO, *A new proof of Pólya-Knopp's inequality with an extension*, J. Math. Inequal. **15** (1), (2001), 9–16.
[15] N. A. DAO, *Hardy operators and commutators on generalized central function spaces*, Math. Inequal. Appl. **25** (4), (2022), 963–983.
[16] L. E. PERSSON AND N. SAMKO, *A note on the best constants in some Hardy inequalities*, J. Math. Inequal. **9** (2), (2015), 437–447.
[17] L.-E. PERSSON AND V. D. STEPANOV, *Weighted integral inequalities with the Geometric Mean Operator*, J. Inequal. Appl. **7** (5), (2002), 727–746.
[18] V. D. STEPANOV AND E. P. USHAKOVA, *Weighted Hardy inequality with two-dimensional rectangular operator: the case $q < p$*, Math. Inequal. Appl. **26** (1), (2023), 267–288.
[19] A. WEDESTIG, *Some new Hardy type inequalities and their limiting inequalities*, J. Inequal. Pure Appl. Math. **4** (3), (2003), Article 61.
[20] A. WEDESTIG, *Weighted inequalities of Hardy-type and their limiting inequalities*, Doctoral thesis 2003:17. Department of Mathematics, Luleå University of Technology, 2003.
[21] M. F. YIMER, *Multidimensional weighted Pólya-Knopp inequalities with sharp constants*, Math. Inequal. Appl. **25** (3), (2022), 869–880.

(Received February 3, 2023)

Markos Fisseha Yimer
Deparment of Mathematics
Addis Ababa University
1176 Addis Ababa, Ethiopia
and
Department of Mathematics and Computer Science
Karlstad University
SE-651 88 Karlstad, Sweden
e-mail: markos.fisseha@aau.edu.et

Lars Erik Persson
Department of Mathematics and Computer Science
Karlstad University
SE-651 88 Karlstad, Sweden
and
Department of Computer Science and Computational Engineering
UiT The Arctic University of Norway, N-8505, Narvik, Norway
e-mail: larserik6pers@gmail.com

Tsegaye Gedif Ayele
Deparment of Mathematics
Addis Ababa University
1176 Addis Ababa, Ethiopia
e-mail: tsegaye.ayele@aau.edu.et