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REFINEMENTS OF BENNETT TYPE INEQUALITIES

J. A. OGUNTUASE*, L.-E. PERSSON AND E. O. ADELEKE

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Abstract. In this paper we discuss, complement and improve some Bennett type inequalities.In particular, we prove a new refinement of a Bennett type inequality using superquadracity argument.

1. Introduction

In a note [5] published in 1920, Hardy announced, and then proved, in [6] the following famous important classical integral inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1}$$

so-called Hardy's inequality, where $p>1$ and $f\in L_p(0,\infty)$ is a nonnegative function. In 1927, Hardy [7] obtained a weighted generalization of (1) as follows:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^a dx \leqslant \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p x^a dx, \tag{2}$$

where f is a nonnegative measurable function on $(0,\infty)$, $p>1$, $a<p-1$ and the constant $\left(\frac{p}{p-1-a}\right)^p$ is sharp.

Since Hardy discovered inequalities (1) and (2), several researchers have either reproved them using various techniques or obtained its variants, extensions in many directions. For instance, the readers may consult [8, 9, 10, 11] and the references cited therein. Bennett [4] in 1973 derived the following results as an important tool when describing the intermediate spaces between L and $L\log^+ L$:

THEOREM 1. *Let* $\alpha>0$, $1\leqslant p\leqslant\infty$, *and* f *be a nonnegative and measurable function on* $[0,1]$. *Then*

$$\int_0^1 \left[\log\frac{e}{x}\right]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x} \leqslant \alpha^{-p}\int_0^1 x^p\left[\log\frac{e}{x}\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x} \tag{3}$$

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* Corresponding author.

and

$$\int_0^1 \left[\log\frac{e}{x}\right]^{-\alpha p-1}\left(\int_x^1 f(y)dy\right)^p\frac{dx}{x} \leqslant \alpha^{-p}\int_0^1 x^p\left[\log\frac{e}{x}\right]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x} \tag{4}$$

with the usual modification if $p=\infty$.

REMARK 1. Note that (2) is without sense for the limit case but by involving logarithms and working with finite interval we can obtain the similar result (3) for this limit case.

REMARK 2. The constants α^{-p} in both (3) and (4) are sharp. This observation was not pointed out by Bennett [4] but was later proved in Barza et al. [3].

In 2014, Barza et al. [3] obtained some refinements and extensions of inequality (3). In particular, the following inequality was derived and proved:

$$\begin{aligned}\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p\int_0^1\left[\log\frac{e}{x}\right]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p\frac{dx}{x}\\ \leqslant \int_0^1 x^p\left[\log\frac{e}{x}\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x},\end{aligned} \tag{5}$$

where $p\geqslant 1$ and f is a nonnegative measurable function on $[0,1]$. Both constants α^{p-1} and α^p in (5) are sharp.

REMARK 3. In the paper [3] it was also proved that inequality (5) holds in the reversed direction when $0<p\leqslant 1$ and both constants α^{p-1} and α^p are sharp in this case too. In particular, we have equality for $p=1$.

In 2017, Oguntuase et al. [13] used mainly the concept of superquadracity introduced by Abramovich et al. [1] to prove some results that show that if the "turning point" is $p=2$ instead of $p=1$, that it is possible to add, yet, another refinement term to the left hand side of inequality (4). In particular, the following result was obtained:

THEOREM 2. *Let* $\alpha>0$, $p>1$ *and* f *a nonnegative and measurable function on* $[0,1]$.

(1) If $p\geqslant 2$, *then*

$$\begin{aligned}&\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p\int_0^1\left[\log\frac{e}{x}\right]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p\frac{dx}{x}\\ &\quad+\int_0^1\left|x\log\frac{e}{x}f(x)-\alpha\int_0^x f(y)dy\right|^p\left(\log\frac{e}{x}\right)^{\alpha p-1}\frac{dx}{x}\\ &\leqslant \int_0^1 x^p\left[\log\frac{e}{x}\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}.\end{aligned} \tag{6}$$

All constants α^{p-1},α^p *and* 1 *in front of the integrals on the left-hand side in* (6) *are sharp.*

(2) If $1 < p \leqslant 2$, then (6) holds in the reverse direction and the constants on the inequality are sharp.

(3) If $p = 2$ we have equality in (6) for any measurable function f and any $\alpha > 0$.

In a recent paper, Kwon [12], generalized inequality (3) with weights. Specifically, the following results was stated and proved:

THEOREM 3. *Let $1 \leqslant p < \infty$ and $a < b < \infty$. Let H be an increasing function having continuous derivative on $[a,b]$. Then the following inequality is valid for all nonnegative measurable functions f on $[a,b]$:*

$$\begin{aligned} &pe^{-H(b)}\left(\int_a^b f(t)dt\right)^p + \int_a^b \left(\int_a^x f(t)dt\right)^p \left|\left(e^{-H(x)}\right)'\right| dx \\ &\leqslant p^p \int_a^b f^p(x) \left|\left(e^{-H(x)}\right)'\right| \left|H'(x)\right|^{-p} dx. \end{aligned} \tag{7}$$

In Section 2 of this paper, we discuss, complement and sharpen Theorem 3. However, our most important new contribution is that we obtain some results that show that if the "turning point" is $p = 2$, instead of $p = 1$ in (7), that it is possible to add, yet, another refinement term to the left hand side of the inequality; thereby obtained its refinement. Moreover, the method of proof employed in this paper is elementary as well as different from those in [12] and [13]. In Section 3, we give some definitions and results needed to prove our main result (Theorem 8) which is stated, proved and applied in Section 4. In particular, it gives some new refined Bennett-type inequalities, which provide generalizations of the inequalities (5) and (6).

2. Some Remarks, complements and sharpenings of Theorem 3

REMARK 4. It was claimed but not proved in [12] that (7) implies (5). For the readers convenience and for later purposes (see Example 2) we include a proof.

Proof. Let $a = 0, b = 1$, $p \geqslant 1$, $\alpha > 0$ and $e^{-H(x)} = \frac{1}{\alpha p}\left(\log \frac{e}{x}\right)^{\alpha p}$. Then

$$e^{-H(1)} = \frac{1}{\alpha p}, \quad (e^{-H(x)})' = -\frac{1}{x}\left(\log \frac{e}{x}\right)^{\alpha p - 1}$$

and

$$H(x) = -\log \frac{1}{\alpha p} - \alpha p \log\left(\log \frac{e}{x}\right)$$

so that

$$H'(x) = \alpha p \left(\log \frac{e}{x}\right)^{-1} \frac{1}{x}.$$

Hence,

$$\left(H'(x)\right)^{-p} = (\alpha p)^{-p}\left(\log \frac{e}{x}\right)^p x^p.$$

Substitute into (7) and we get that

$$\frac{p}{\alpha p}\left(\int_0^1 f(t)dt\right)^p + \int_0^1 \left(\int_0^x f(t)dt\right)^p \left(\log\frac{e}{x}\right)^{\alpha p-1}\frac{dx}{x}$$
$$\leqslant p^p \int_0^1 f^p(x)\left(\log\frac{e}{x}\right)^{\alpha p-1}(\alpha p)^{-p}\left(\log\frac{e}{x}\right)^p x^p\frac{dx}{x}.$$

i.e. that

$$\alpha^{p-1}\left(\int_0^1 f(t)dt\right)^p + \alpha^p\int_0^1 \left(\int_0^x f(t)dt\right)^p \left(\log\frac{e}{x}\right)^{\alpha p-1}\frac{dx}{x}$$
$$\leqslant \int_0^1 f^p(x)\left(\log\frac{e}{x}\right)^{(\alpha+1)p-1}x^p\frac{dx}{x}. \quad \square$$

REMARK 5. Kwon [12] did not discuss the sharpness at all of (7) but by using also the sharpness in (5) (proved in [3] cf. Remark 2) we can state the following more sharp version of Theorem 3.

THEOREM 4. *Let $1 \leqslant p < \infty$, $0 \leqslant a < b \leqslant \infty$ and let H be an increasing function having continuous derivatives on $[a,b]$.*

1. *For all nonnegative measurable functions f on $[a,b)$, it yields that*

$$pe^{-H(b)}\left(\int_a^b f(t)dt\right)^p + \int_a^b \left(\int_a^x f(t)dt\right)^p \left|\left(e^{-H(x)}\right)'\right| dx$$
$$\leqslant p^p \int_a^b f^p(x)\left|\left(e^{-H(x)}\right)'\right|\left|H'(x)\right|^{-p} dx. \tag{8}$$

2. *The inequality (2.1) is sharp in the sense that it does not hold for all increasing functions $H(x)$ on $[a,b)$ with any of the constants $pe^{-H(b)}$ or p^p replaced by some smaller number.*

Proof. (1). The statement follows from Theorem 3 for any $b < \infty$ but it follows also for $b = \infty$ by an obvious limit argument.

(2). This follows from the fact that if we choose

$$H(x) = -\log\frac{1}{\alpha p}\left(\log\frac{e}{x}\right)^{\alpha p},$$

then according to the proof above, (8) coincides with (5) and both constants in this inequality are sharp so that also both constants in (8) are sharp for this function and when $[a,b] = [0,1]$. The same holds for the case $[a,b]$ for any $0 \leqslant b < \infty$, which can be seen by just making a variable substitution and in the case $b = \infty$ we just complement with a limit argument. $\square$

REMARK 6. In the same paper [12] it was also stated that under the assumptions in Theorem 3 but with $H(x)$ decreasing the following inequality holds:

$$\begin{aligned} & pe^{-H(a)}\left(\int_a^b f(t)dt\right)^p + \int_a^b \left(\int_x^b f(t)dt\right)^p \left|\left(e^{-H(x)}\right)'\right| dx \\ & \leqslant p^p \int_a^b f^p(x)\left|\left(e^{-H(x)}\right)'\right| \left|H'(x)\right|^{-p} dx. \end{aligned} \tag{9}$$

EXAMPLE 1. By applying (9) with $a=0$, $b=1$ and

$$e^{-H(x)} = \frac{1}{\alpha p}\left(\log\frac{e}{x}\right)^{-\alpha p}$$

we can step by step calculate as in Remark 4 and arrive at the inequality

$$\begin{aligned} & \alpha^p \int_0^1 \left(\log\frac{e}{x}\right)^{\alpha p-1} \left(\int_x^1 f(y)dy\right)^p \frac{dx}{x} \\ & \leqslant \int_0^1 x^p \left(\log\frac{e}{x}\right)^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}, \end{aligned}$$

which coincides with (4). The crutial observation here is that $e^{-H(0)}=0$.

Our next goal is to prove that the two statements: Theorem 2.1 in paper [12] are in fact equivalent.

THEOREM 5. *The statements in Theorem 3 and Remark 4 are in fact equivalent.*

Proof. We first assume that H is decreasing and apply (7) on the increasing function $H_1(x) = H(\frac{1}{x})$ on the interval $(\frac{1}{b}, \frac{1}{a})$ where $0<a<b$ and with a function g we define later on. We have that

$$\int_{\frac{1}{b}}^{\frac{1}{a}} g(s)ds = \left[s = \frac{1}{t}\right] = \int_a^b g\left(\frac{1}{t}\right)t^{-2}dt \tag{10}$$

Now we define $f(t) = g(\frac{1}{t})t^{-2}$. Moreover

$$\begin{aligned} & \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{\frac{1}{b}}^{y} g(s)\right)^p \left|\left(e^{-H(\frac{1}{y})}\right)'\right| dy \\ & = \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{\frac{1}{b}}^{y} g(s)ds\right)^p \left|\left(e^{H(\frac{1}{y})}\right)\right| \left|H'\left(\frac{1}{y}\right)\right| \frac{dy}{y^2} \\ & = \left[s = \frac{1}{t}\right] = \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{\frac{1}{y}}^{b} g\left(\frac{1}{t}\right)t^{-2}dt\right)^p \left|e^{-H(\frac{1}{y})}\right| \left|H'\left(\frac{1}{y}\right)\right| \frac{dy}{y^2} \\ & = \left[x = \frac{1}{y}\right] = \int_a^b \left(\int_x^b f(t)dt\right)^p \left|\left(e^{-H(x)}\right)'\right| dx \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\int_{\frac{1}{b}}^{\frac{1}{a}} g^p(y)\left|\left(e^{-H(\frac{1}{y})}\right)'\right|\left|\left(e^{H(\frac{1}{y})}\right)'\right|^{-p} dy \\ &= \int_{\frac{1}{b}}^{\frac{1}{a}} g^p(y)\left|\left(e^{-H(\frac{1}{y})}\right)\right|\left|\left(e^{H(\frac{1}{y})}\right)'\right|^{-p+1} \frac{dy}{y^2} \\ &= \int_{\frac{1}{b}}^{\frac{1}{a}} \left(g(y)y^2\right)^p \left|e^{-H(\frac{1}{y})}\right|\left|\left(H'\left(\frac{1}{y}\right)\right|\frac{1}{y^2}\right)^{-p} \frac{dy}{y^2} \\ &= \left[\frac{1}{y}=x\right] = \int_a^b \left(g\left(\frac{1}{x}\right)x^{-2}\right)^p \left|\left(e^{-H(x)}\right)'\right| \mid (H'(x)) \mid^{-p} dx \\ &= \int_a^b f^p(x)\left|\left(e^{-H(x)}\right)'\right| \mid H'(x)) \mid^{-p} dx. \quad \square \end{aligned} \tag{12}$$

Since also $H_1(\frac{1}{a}) = H(a)$ we find, by combining (10)–(12), that (9) holds with the decreasing function H. The implication in the reversed direction follows in the same way by just doing the calculations "backwards". In particular, Theorem 5 means that our Theorem 4 can be reformulated in the following equivalent way and then, in particular we get also an improvement of the second inequality in Theorem 2.1 of the paper [12] (c.f. Remark 4).

THEOREM 6. *Let $1 \leqslant p < \infty$ and $0 \leqslant a < b \leqslant \infty$. Let H be a decreasing function having continuous derivative on $[a,b]$.*

(a). For each nonnegative measurable function f on $[a,b]$ we have that

$$\begin{aligned} &pe^{-H(a)}\left(\int_a^b f(t)dt\right)^p + \int_a^b \left(\int_x^b f(t)dt\right)^p \left|\left(e^{-H(x)}\right)'\right| dx \\ &= p^p \int_a^b f^p(x) \left|\left(e^{-H(x)}\right)'\right| \left| H'(x)\right|^{-p} dx. \end{aligned} \tag{13}$$

(b). The inequality (13) is sharp in the sence that it does not hold for all decreasing functions $H(x)$ on $[a,b]$ with any of the constants $pe^{-H(a)}$ or p^p replaced by some smaller number.

3. Preliminaries

In this Section, we present some basic definitions and results on superquadratic and subquadratic functions needed in the proof of our main result in the next Section.

DEFINITION 1. ([1]) A function $\varphi : [0,\infty) \to \mathbb{R}$ is said to be superquadratic provided for each $x \geqslant 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) - C_x(y-x) \geqslant 0 \tag{14}$$

for all $y \in [0,\infty)$. φ is subquadratic if $-\varphi$ is superquadratic.

LEMMA 1. ([1]) *Let Let* $\phi : [0,\infty) \to \mathbb{R}$ *be a superquadratic function with* C_x *as in Definition* 1. *Then*

1. $\phi(0) \leqslant 0$.
2. *If* $\phi(0) = \phi'(0) = 0$, *then* $C_x = \phi'(x)$ *whenever* ϕ *is differentiable at* $x > 0$.
3. *If* $\phi \geqslant 0$, *then* ϕ *is convex and* $\phi(0) = \phi'(0) = 0$.

Next we present the refined Jensen's inequality for superquadratic and subquadratic functions.

THEOREM 7. ([1]) *Let* (Ω, Σ, μ) *be a probability measure space. Then the inequality*

$$\varphi\left(\int_\Omega f(x)d\mu(x)\right) \leqslant \int_\Omega \left[\varphi(f(x)) - \varphi\left(\left|f(x) - \int_\Omega f(y)d\mu(y)\right|\right)\right] d\mu(x) \tag{15}$$

holds for all probability measures μ *and all non-negative* μ*-integrable functions* f *if and only if* $\varphi : [0,\infty) \to \mathbb{R}$ *is superquadratic. Moreover,* (15) *holds in the reversed direction if and only if* φ *is subquadratic.*

PROPOSITION 1. ([13]) *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be differentiable function such that* $\phi(0) = \phi'(0)$. *Then*

$$\phi(y) - \phi(1) - \phi'(1)(y-1) - \phi(|\,y-1\,|) \begin{cases} \geqslant 0 \text{ if } \phi \text{ is superquadratic}, \\ \leqslant 0 \text{ if } \phi \text{ is subquadratic}, \end{cases} \tag{16}$$

holds for all $y \geqslant 0$. *If* $\phi(x) = x^p$, *then equality in* (16) *holds for all* y *if and only if* $p = 2$.

A direct consequence of Proposition 1 or Theorem 7 yields the following refinement of the well-known Bernoulli's inequality, which plays a central role in the proof of our main result.

LEMMA 2. ([13]) *Let* $h > 0$ *then,*

$$h^p - p(h-1) - 1 - |h-1|^p \begin{cases} \geqslant 0 \text{ if } p \geqslant 2, \\ \leqslant 0 \text{ if } 1 < p \leqslant 2. \end{cases}$$

Equality holds for all $h > 0$ *if and only if* $p = 2$ *and also when* $p \neq 2$ *if and only if* $h = 1$.

4. The main result

Our main result reads as follows:

THEOREM 8. *Let $1 \leqslant p < \infty$, $0 \leqslant a < b \leqslant \infty$ and f be nonnegative and measurable functions on $[a,b]$. Assume that H is a nonnegative function having continuous derivative on $[a,b]$. If $p \geqslant 2$, and H is increasing, then*

$$\begin{aligned} & pe^{-H(b)}\left(\int_a^b f(t)dt\right)^p + \int_a^b \left(\int_a^x f(t)dt\right)^p \left|\left(e^{-H(x)}\right)'\right| dx \\ & \quad + \int_a^b \left|\left(e^{-H(x)}\right)'\right| \left|\frac{pf(x)}{H'(x)} - \int_a^x f(t)dt\right|^p dx \\ & \leqslant p^p \int_a^b f^p(x) \left|\left(e^{-H(x)}\right)'\right| \left|H'(x)\right|^{-p} dx; \end{aligned} \tag{17}$$

However, if $1 < p \leqslant 2$, then (17) holds in the reverse direction.

Proof. Let $b < \infty$, $p \geqslant 2$ and $\eta = exp(-H)$. By the hypothesis, $(-\eta'(x))$ is nonnegative. Suppose that f is a continuous and nonnegative function on $[a,b]$, then define for $x \in [a,b]$ the function G by

$$\begin{aligned} G(x;\alpha,p) := {} & \frac{p^p}{p}[\eta(x)]^p[-\eta'(x)]^{-p+1} f^p(x) + \left[-\eta'(x)\right]\left(\int_a^x f(t)dt\right)^p \\ & - pf(x)[\eta(x)]\left(\int_a^x f(t)dt\right)^{p-1} - \frac{1}{p}\left[-\eta'(x)\right]\left(\int_a^x f(t)dt\right)^p \\ & - \frac{1}{p}[-\eta'(x)]\left|\frac{p\eta(x)f(x)}{-\eta'(x)} - \int_a^x f(t)dt\right|^p \end{aligned} \tag{18}$$

and, by rewriting the terms in (18), we find that

$$\begin{aligned} G(x;\alpha,p) = {} & \frac{1}{p}\left[-\eta'(x)\right]\left(\int_a^x f(t)dt\right)^p \\ & \times \left[\left(\frac{p\eta(x)f(x)}{-\eta'(x)\int_a^x f(t)dt}\right)^p - p\left(\frac{p\eta(x)f(x)}{-\eta'(x)\int_a^x f(t)dt} - 1\right) - 1\right] \\ & - \frac{1}{p}\left[-\eta'(x)\right]\left(\int_a^x f(t)dt\right)^p \left|\frac{p\eta(x)f(x)}{-\eta'(x)\int_a^x f(t)dt} - 1\right|^p. \end{aligned} \tag{19}$$

Hence, by putting

$$h(x,\alpha) := \frac{p\eta(x)f(x)}{-\eta'(x)\int_a^x f(t)dt}$$

in view of Lemma 2, we obtain that

$$\begin{aligned} G(x;\alpha,p) & = \frac{1}{p}[-\eta'(x)\left(\int_a^x f(t)dt\right)^p [h^p(x,\alpha) - p(h(x,\alpha)-1) - 1 - |\, h(x,\alpha) - 1\,|^p] \\ & \geqslant 0. \end{aligned} \tag{20}$$

Now, by using a (20), integrate and rearrange the terms, we obtain that

$$p\int_a^b f(x)[\eta(x)]\left(\int_a^x f(t)dt\right)^{p-1}dx+\frac{1}{p}\int_a^b[-\eta^{'}(x)]\left|\frac{p\eta(x)f(x)}{-\eta^{'}(x)}-\int_a^x f(t)dt\right|^p dx$$
$$\leqslant\frac{p-1}{p}\int_a^b\left(-\eta^{'}(x)\left(\int_a^x f(t)dt\right)^p\right)dx+\frac{p^p}{p}\int_a^b[\eta(x)]^p[-\eta^{'}(x)]^{-p+1}f^p(x)dx. \tag{21}$$

Further, by virtue of integration by parts, we have that

$$\int_a^b\left(-\eta^{'}(x)\left(\int_a^x f(t)dt\right)^p\right)dx=-\eta(b)\left(\int_a^b f(t)dt\right)^p$$
$$+p\int_a^b f(x)[\eta(x)]\left(\int_a^x f(t)dt\right)^{p-1}dx. \tag{22}$$

Hence, by substituting (22) into (21) we obtain after simple calculation, that

$$\eta(b)\left(\int_a^b f(t)dt\right)^p+\frac{1}{p}\int_a^b\left(-\eta^{'}(x)\left(\int_a^x f(t)dt\right)^p\right)dx$$
$$+\frac{1}{p}\int_a^b[-\eta^{'}(x)]\left|\frac{p\eta(x)f(x)}{-\eta^{'}(x)}-\int_a^x f(t)dt\right|^p dx \tag{23}$$
$$\leqslant-\frac{p^p}{p}\int_a^b f^p(x)\eta^{'}(x)\left[\left(\log\frac{1}{\eta(x)}\right)^{'}\right]^{-p}dx.$$

Multiply by p and after substituting $H=-\log\eta$ into (23) we obtain that

$$pe^{-H(b)}\left(\int_a^b f(t)dt\right)^p+\int_a^b\left(\int_a^x f(t)dt\right)^p\left|\left(e^{-H(x)}\right)^{'}\right|dx$$
$$+\int_a^b\left|\left(e^{-H(x)}\right)^{'}\right|\left|\frac{pf(x)}{H^{'}(x)}-\int_a^x f(t)dt\right|^p dx \tag{24}$$
$$\leqslant p^p\int_a^b f^p(x)\left|\left(e^{-H(x)}\right)^{'}\right|\left|H^{'}(x)\right|^{-p}dx.$$

The proof of the case $b<\infty$ is complete. The proof of the case $b=\infty$ is obtained by letting $b\to\infty$ in what we just proved. The proof of the case $1<p\leqslant 2$ is similar to the proof above except that the signs of the inequalities are reversed. The proof is complete. □

REMARK 7. Put

$$I_1=pe^{-H(b)}\left(\int_a^b f(t)dt\right)^p+\int_a^b\left(\int_a^x f(t)dt\right)^p\left|\left(e^{-H(x)}\right)^{'}\right|dx$$
$$I_2=\int_a^b\left|\left(e^{-H(x)}\right)^{'}\right|\left|\frac{pf(x)}{H^{'}(x)}-\int_a^x f(t)dt\right|^p dx \tag{25}$$
$$\text{and}\quad I_3=p^p\int_a^b f^p(x)\left|\left(e^{-H(x)}\right)^{'}\right|\left|H^{'}(x)\right|^{-p}dx;$$

in (17), then it shows that the generalized Bennett's inequality (7), i.e.,

$$I_1 \leqslant I_3,$$

can be refined to

$$I_1 + I_2 \leqslant I_3$$

for $p \geqslant 2$ while for $1 < p \leqslant 2$ we have the following two-sided estimate

$$I_1 \leqslant I_3 \leqslant I_1 + I_2.$$

Hence, our Theorem 8 gives some essential refinements of (7).

EXAMPLE 2. Assume that $\alpha > 0$, $p \geqslant 2$, $a = 0$, $b = 1$ and $e^{-H(x)} = \frac{1}{\alpha p}\left(\log\frac{e}{x}\right)^{\alpha p}$ in (17). Then, we have the following inequality

$$\begin{aligned}
&\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p \int_0^1 \left[\log\frac{e}{x}\right]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x} \\
&\quad + \int_0^1 \left| x\log\frac{e}{x} f(x) - \alpha\int_0^x f(y)dy\right|^p \left(\log\frac{e}{x}\right)^{\alpha p-1}\frac{dx}{x} \\
&\leqslant \int_0^1 x^p\left[\log\frac{e}{x}\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x},
\end{aligned}$$

which coincides with (6). The proof of this fact can be performed step by step as in our Remark 4 so we leave out the details.

REMARK 8. By making step by step calculations as in the proof of Theorem 5, we can write our Theorem 8 in an equivalent form for decreasing functions H and thus, in particular, obtain a further refinement of also the second inequality in Theorem 2.1 in [12].

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J. A. Oguntuase
Department of Mathematics
Federal University of Agriculture
P.M.B. 2240, Abeokuta, Ogun State, Nigeria
e-mail: oguntuase@yahoo.com

L.-E. Persson
Department of Computer Science and Computational Engineering
UiT The Artic University of Norway
and
Department of Mathematics and Computer Science
Karlstad University
Sweden
e-mail: larserik6pers@gmail.com

E. O. Adeleke
Department of Mathematics
Federal University of Agriculture
P.M.B. 2240, Abeokuta, Ogun State, Nigeria
e-mail: yemi376@yahoo.com