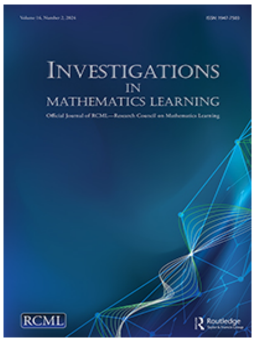

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Pre-Service teachers' Knowledge of and Beliefs About Direct and Indirect Proofs

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ABSTRACT

Teachers have difficulty integrating proof in their mathematics instruction due to both narrow beliefs about proofs and limited understanding of proofs. Indirect proofs seem to be a particular cause for concern. In this exploratory study, we contribute to the research area by reporting on an empirical study of Norwegian pre-service teachers' knowledge of and beliefs about direct and indirect proofs. Inspired by situativity theory, we investigated pre-service teachers' knowledge of and beliefs about proofs both professed generally and out-of-context and in situation-specific circumstances. Our initial findings are in line with much of the previous literature. First, for situation-specific beliefs and knowledge, we found that indirect proofs seem to be more challenging than direct proofs. Second, for general beliefs and knowledge, we found pre-service teachers' views about proofs in general are narrow and rigid. However, we also investigated possible patterns between general and situation-specific beliefs and knowledge. We found that participants who empirically validated proofs also professed views that a good mathematical argument is an argument that is simply convincing, and not necessarily rigorous. Second, participants who professed preferences for direct proofs, also struggled with the logical conditions of indirect proofs. Implications are discussed.

KEYWORDS

Beliefs; indirect proof; knowledge; teachers

Recent reform efforts across the world have elevated the status and importance of proofs and deductive argumentation in school mathematics. According to Mariotti (2006), there is a consensus that proof is a key element of mathematics education and there seems to be a general trend of including proofs in national curricula. Implementing this in the classroom is often up to the teacher, and as the National Council of Teachers of Mathematics [NCTM] (2000) states, "students learn mathematics through the experiences that teachers provide" (p.16). It is therefore vital that teachers both understand mathematical proofs and value mathematical proofs as a key component of school mathematics – as teachers' ability and willingness to teach specific mathematical ideas depend on their knowledge and beliefs (Philipp, 2007). A. J. Stylianides et al. (2016) describes a case from a third-grade class that illustrates this point. After working on ideas related to even and odd numbers for several days, the students had formulated the conjecture: "An odd number plus an odd number equals an even number." Instead of simply telling the students the answer, or limit the argumentation to a few examples, the students investigated, tested, and attempted both to refute and justify the conjecture – with the teacher's help. Building on the students' work, the class were able to move from empirical to more general arguments. Not only were the students provided with an

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opportunity to work on authentic mathematics (Lampert, 1992), but they also had an opportunity to develop a better understanding of the definitions of even and odd numbers (A. J. Stylianides et al., 2016).

However, research has shown that teachers have in general difficulty integrating proof into their mathematics instruction. Several explanations for this have been proposed, ranging from teachers' limited beliefs about the role of proofs and what constitutes proof, to acceptance of empirical or invalid arguments as proofs (Harel & Sowder, 2007; Knuth, 2002; Ko, 2010; Morris, 2007). It should be noted that this is not a problem limited to teachers. There has been a long-standing concern that students at all levels and both in general and in teacher education programs struggle with mathematical proofs (G. J. Stylianides et al., 2017). Students tend to accept logically invalid deductions, confuse empirical evidence for proof, focus too much on superficial properties, concentrate on algebraic manipulations, believe proofs are appropriate only for a minority of students, and hold a very limited view of the role of proof etc (e.g. Hodds et al., 2014; Inglis & Alcock, 2012; Ko, 2010). More generally, Raman (2003) explains that while proofs are essentially about key ideas, many students are not able to see a connection between a privately held idea and the corresponding formal, public proof. Raman (2003) explains that while mathematicians clearly link heuristic aspects of proofs and rigorous language of proofs, students may see proofs as creating something out of nothing.

These tendencies highlight the importance of investigating preservice teachers' knowledge of and beliefs about proofs and addressing the corresponding concerns in teacher preparation programs (A. J. Stylianides, 2007). However, despite the previously mentioned research findings, there is still much work to be done. Most extant studies have focused on secondary (in-service or preservice) teachers, not specific methods of proofs, or contextual mediators that affect understanding of or beliefs about proofs (Knuth, 2002; Movshovitz-Hadar, 1993; Simon & Blume, 1996; A. J. Stylianides, 2007; G. J. Stylianides et al., 2017). In this exploratory study, we contribute to the research area by reporting on an empirical study on Norwegian preservice middle school teachers' knowledge of and beliefs about direct and indirect proofs. The rationale behind the focus on direct and indirect proofs will be expanded on later. For now, we will simply clarify the meaning of the two types of proofs. Direct proof is a straightforward way of establishing the truth of a statements by known facts and logical principles. Indirect proofs, on the other hand, generally demonstrates the truth of a proposition by assuming the opposite is true and showing that this assumption leads to a contradiction (Courant & Robbins, 1996).

Although most authors generally distinguish between knowledge and beliefs, from a psychological perspective they are closely related constructs. Both determine, in close interaction, students' understanding of mathematical ideas, problems and situations (Op't Eynde et al., 2002). For the purpose of this study, we do not draw a clear distinction between knowledge and beliefs. Instead, we approach the issue of pre-service teachers' knowledge of and beliefs about direct and indirect proofs from the perspective of situativity theory, which points out that knowledge and beliefs are sensitive to circumstances and context and often held in clusters (Op't Eynde et al., 2002). The importance of contextual factors for learning has been well known ever since the end of the 1980's, when the situated cognition and learning paradigm emerged in reaction to the mentalistic view of learning and thinking (J. S. Brown et al., 1989). Beliefs and knowledge are formed in the socio-cultural environment one lives and works in. What we hear, perceive and comprehend in one specific situation can be accepted as true in that particular context, but untrue, irrelevant or even meaningless in a different context (Bogdan, 1986). As such, research have often reported a discrepancy between knowledge and beliefs professed generally and out of context, and knowledge and beliefs conveyed in situation-specific and context-dependent circumstances (Philipp, 2007). To the outsider, this might suggest that individuals sometimes hold contradictory beliefs. However, clusters of beliefs and knowledge are organized in a subjective, quasi-logical manner that is rarely consciously known to the individual. To the individual, there is no discrepancy or contradiction – only different circumstances and context (Philipp, 2007). In this study, we investigate pre-service teachers' knowledge of and beliefs about direct and indirect proofs on the basis of this distinction. More specifically, we set out to answer the following research questions:

RQ1: What characterizes pre-service teachers' situation-specific knowledge of and beliefs about direct and indirect proofs?

RQ2: What characterizes pre-service teachers' general knowledge of and beliefs about direct and indirect proofs?

RQ3: How are pre-service teachers' situation-specific knowledge of and beliefs about direct and indirect proofs related to their general knowledge of and beliefs about direct and indirect proofs?

As noted earlier, beliefs and knowledge affect teachers' ability and willingness to teach specific mathematical ideas. However, little is known about how pre-service teachers' knowledge of and beliefs about proofs can differ, or align, in different contexts. In this study, we therefore investigate pre-service teachers' knowledge of and beliefs about proofs both tied to specific proof situations and professed more generally out of context. To distinguish between the two constructs, we will in this paper use the term *proof views* regarding knowledge and beliefs professed generally and out of context, and the term *proof understanding* regarding knowledge and beliefs in situation-specific contexts.

Furthermore, few studies on knowledge of and beliefs about proofs have focused on specific methods of proof (A. J. Stylianides, 2007). In this study we focus on indirect proofs, as researchers have argued that they are a particular cause for concern. In the research literature it is claimed that mathematics learners often dislike indirect proofs and generally find them unconvincing (e.g. Harel & Sowder, 1998; Leron, 1985). In this study, we therefore also investigate if and how the nature of the proofs themselves – i.e. direct vs. indirect – mediate middle school preservice teachers' knowledge of and beliefs about proofs.

Finally, there is a need for more research on proofs and proving in primary and middle school. There are several reasons, both empirical and pedagogical, for this. Here, we'll mention two particularly salient reasons. First, although both researchers and national documents (e.g. KD, 2019; NCTM, 2000) have called for proof and proving to be central aspects of mathematics for students of all ages, most of the research on the teaching and learning of proofs has focused on secondary and tertiary mathematics (Campbell et al., 2020; A. J. Stylianides et al., 2016). Second, the act of proving, as in removing doubts of an assertion and its related acts such as reasoning and justification (Harel, 2008), is essential to working with authentic mathematics and cultivate deeper understandings of ideas and relationships in mathematics (Walkington & Woods, 2022). Proving can allow students to engage in mathematics as a sense-making activity and explore why things "work" in mathematics rather than just memorize superficially – at all ages and levels (A. J. Stylianides et al., 2016).

Theoretical Background

Proofs in Mathematics and Mathematics Education

From the perspective of mathematical logic, a proof is a formal sequence of mathematical statements written in a formal language with specific characteristics (Gowers et al., 2008, p. 70). This view is somewhat of an idealization of what the real practice of mathematical proof looks like. A purely formal proof would be very long and very hard to read. Hersh (1993) therefore distinguishes between what mathematical proof means in principle and in practice. In principle, mathematical proof is a formal sequence of steps according to strict logical rules. In practice, however, a proof is what mathematicians do to make other mathematicians believe their theorems. It is an argument that convinces the qualified skeptical expert. In practice, proofs omit routine logical steps, an enormous amount of context is assumed by the reader, and in many areas, proofs rely on intuitive arguments that could be translated into arguments that are more rigorous (Auslander, 2008). Mathematical proof is thus not just about the establishment of mathematical truth (Rav, 1999). Mathematical proofs also display fresh methods, tools, strategies and concepts that are of wider applicability in mathematics and open up new mathematical direction. Proofs themselves are indispensable to the broadening of mathematical knowledge and are in fact "'the heart of mathematics, the royal road to creating analytic tools and catalysing growth'" (Rav, 1999, p. 6). Because proofs have the potential of conveying important elements such as concepts and methods, Hanna and Barbeau (2008) argue that proofs should also be a primary focus of interest in mathematics education. In other words, mathematical proofs can not only help students see why statements are true, but also enhance their understanding of mathematical concepts and promote mathematical proficiency and reasoning (Hanna, 2000).

Indirect Proofs

There are two types of indirect proofs: proof of the contrapositive and proof by contradiction. Although the two types of proof are somewhat different, they both start by assuming in some manner the denial of the conclusion. This can be illustrated with the following: to prove p => Q by proof of contradiction we can assume P and ¬Q and derive a contradiction. To prove p => Q by proof of the contrapositive we assume ¬Q and show ¬P. Historically, indirect proofs have been highly controversial and doubted by anti-realists as they depend on "logical tricks" and do not address causality (Elitzur et al., 2018). Although indirect proofs are nowadays accepted without qualms, mathematicians do recognize that indirect proofs may ask for additional mental effort (Antonini & Mariotti, 2008). This has also been observed in mathematics education settings, as mathematics learners often find indirect proofs more troublesome than direct proofs (e.g. Harel & Sowder, 1998; Leron, 1985; Quarfoot & Rabin, 2022). There is currently no agreed upon cause for learners' difficulties with indirect proofs, and in the literature we find several proposed hypotheses. Leron (1985), for example, noted that indirect proofs were non-constructive and that when we work on them, we "enter a false, impossible world" (p. 323). This can create the sense that the proof is simply a trick and not a "real" proof. Others, such as Antonini and Mariotti (2008), have claimed that issues regarding indirect proofs are tied to what they refer to as meta-theorems and understanding logically equivalent relationships such as $(\neg q \rightarrow \neg p) \equiv (p \rightarrow q)$.

Knowledge and Beliefs

From an epistemological perspective, beliefs and knowledge are different. While beliefs generally refer to what an individual considers to be true, knowledge must satisfy some truth condition (Op't Eynde et al., 2002). Richardson (1996, p.103), after reviewing definitions in the fields of psychology, anthropology and philosophy, concluded that beliefs are: " . . . psychologically held understandings, premises, or propositions about the world that are felt to be true." In other words, beliefs can be described as subjective knowledge, rather than objective knowledge, even though there are varying definitions found in the literature (Furinghetti & Pehkonen, 2002). Knowledge, on the other hand, is "beliefs held with certainty or justified true belief" (Philipp, 2007, p. 259). Knowledge refers to things that we "more than believe." We have a sense of certainty of their truth value.

Nevertheless, from a psychological perspective, beliefs and knowledge are closely related concepts. Problem-solving behavior, for example, is always guided by what the solver believes to be true, referring to both knowledge and beliefs (Op't Eynde et al., 2002). It is therefore not easy to identify and separate an individual's beliefs and knowledge from observed behavior. Furthermore, an important finding in the literature is that beliefs and knowledge are context-dependent (Philipp, 2007). Since the late 1980s, researchers have pointed out that knowledge and beliefs are structured in clusters around specific contexts. What we believe to be true – referring to both knowledge and beliefs –

depend on the specific circumstances in particular situations (Green, 1971). Professed beliefs do not therefore always resonate with actions (Thompson, 1992; Diego-Mantecón et al., 2019).

Teachers' Knowledge and Beliefs About Proofs

In recent years, several literature reviews on teachers' knowledge and beliefs about proofs have been published (e.g. Cabassut et al., 2011; Ko, 2010; Lin et al., 2012; G. J. Stylianides et al., 2017, 2024). A key observation is that although the research has been broadly conceptualized as either teachers' beliefs about proofs or teachers' knowledge of proof, most of the literature do not clearly delineate beliefs and knowledge about proof. It is beyond the scope of this paper to provide an exhaustive re-conceptualization of the relevant literature. Instead, we will here provide a short summary on the relevant literature, and separate findings according to two common approaches to research on teachers' beliefs and knowledge: 1) professed knowledge and beliefs stated explicitly in response to interview questions and 2) knowledge and beliefs attributed to and inferred via observations in task-specific situations (Putnam & Borko, 2000).

Regarding the first approach, much of the research have focused on the role and nature of proofs. A common theme in the literature is that many teachers and pre-service teachers profess limited knowledge of and beliefs about the role and nature of proofs (e.g. Cabassut et al., 2011; Ko, 2010; A. J. Stylianides et al., 2016; G. J. Stylianides et al., 2024). Knuth (2002), for example, found that teachers view proof as appropriate for the mathematics education of a minority of students. Furthermore, teachers tended to view proof in a pedagogically limited way, namely, as a topic of study rather than as a tool for communicating and studying mathematics. Harel and Sowder (2007) later concluded from a review of the literature that teachers do not seem to value other important roles of proof, most noticeably its explanatory role. In another more recent study, Aaron and Herbst (2019 found that teachers preferred to separate conjecturing and proving in their teaching, as they perceived them as having different purposes.

However, the research is not unambiguous. A few studies have noticed some variation. For example, in a survey with 30 pre-service elementary school teachers and 21 secondary mathematics teachers, Mingus and Grassl (1999) found that the secondary teachers emphasized the explanatory power of proofs, while the elementary pre-service teachers focused on verification. Ko (2010), in a review of the literature, similarly concluded that although the verification role of proofs is widespread, a minority of teachers also seem to value other roles of proof, in particular the explanatory role of proofs.

As for teachers' and pre-service teachers' knowledge of and beliefs about proofs inferred from task-specific contexts, most of the research have focused on asking respondents to construct proofs or assess the validity of proofs (G. J. Stylianides et al., 2017). Findings from this research suggest that teachers and pre-service teachers emphasize algebraic manipulations and surface properties, while ignoring structural properties (Hodds et al., 2014; Knuth, 2002; Selden & Selden, 2017; Weber, 2010). Several explanations for this have been proposed. Selden and Selden (2003) speculated that this is caused by a focus on local calculations within a proof, rather than its global structure. Other possible explanations might be that teachers seem to be most convinced by arguments that include concrete features such as examples or visual references (Knuth, 2002), or a lack of knowledge of the limitations of empirical evidence (Inglis & Alcock, 2012).

In summary, the existing research literature indicates that teachers' and pre-service teachers are often unable to distinguish valid proofs from invalid proofs. However, this conclusion is based on an assumption that that the types of arguments teachers produce or evaluate are indicative of their standards of mathematical conviction. According to G. J. Stylianides et al. (2017), there is evidence challenging this assumption. For example, solvers may be well aware of the limitations of their constructed proofs, but unable to produce better ones (e.g. Weber & Mejia-Ramos, 2015). Teachers may also evaluate arguments differently based on the context of the argument (Morris, 2007).

These findings raise several methodological issues that need to be taken into account in future research on knowledge of and beliefs about proofs. First, participants in these types of studies should be allowed to explain and qualify their responses. By allowing this, researchers can see if participants are absolutely convinced by empirical arguments or if they simply believe a claim is probably true (G. J. Stylianides et al., 2017). Second, according to Mejia-Ramos et al. (2012), proofs should be assessed both at a local technical level and at a holistic intuitive level. This allows researchers to investigate how participants both view proofs as a chain of logical assertions, and as a big idea behind the structure of a proof.

Conceptual Framework: Knowledge and Beliefs About Proofs

As mentioned, there are two main approaches to research on teachers' beliefs and knowledge: 1) beliefs and knowledge professed and stated explicitly and out of context, and 2) beliefs and knowledge inferred through observations of behavior in specific situations (Philipp, 2007; Putnam & Borko, 2000). To clearly separate the two facets of knowledge and beliefs, we use the term proof views about professed general and out of context, and the term proof understanding about situation-specific knowledge.

Within the context of mathematical proofs, the former approach has usually focused on the nature and role of proofs in both mathematics and school, while the latter approach has usually focused on which type of arguments one finds convincing. In this study, we conceptualize general knowledge of and beliefs about proofs, or proof views, as pre-service teachers professed and explicit knowledge and beliefs regarding the nature and role of proofs. As for situation-specific knowledge of and beliefs about proofs inferred from behavior, we take into account the previously mentioned methodological issues, and rely on the theoretical perspectives of proof validation and proof comprehension – which together form the basis of proof understanding (Weber & Mejia-Ramos, 2011).

Proof Validation

According to Weber and Mejia-Ramos (2011), understanding proofs consists of two related processes: proof validation and proof comprehension. Proof validation is about evaluating and judging mathematical arguments, and the purpose is to determine whether a proof is correct (Selden & Selden, 2017). For the purpose of this study, we limit our conceptualization of proof validation to what we can infer from direct observations. We also draw on the concept proof scheme, which refers to "what constitutes ascertaining and persuading" for a particular person (Harel & Sowder, 1998, p. 244). Meaning, in this study proof validation is the evaluation of the correctness of a proof or argument, and the justification and explanation for said evaluation.

Proof Comprehension

Unlike proof validation, the purpose of proof comprehension is not to evaluate the correctness of a proof. Instead, proof comprehension is about understanding and learning from the content of the proof (Weber & Mejia-Ramos, 2011). In this study, we use a model proposed by Mejia-Ramos et al. (2012) for conceptualizing proof comprehension. The main reason for this choice is that a) the model builds on previous models of proof comprehension (e.g. Yang & Lin, 2008), b) the model isn't limited to particular mathematical topics, and c) it takes into account both what the authors refer to as local and holistic dimensions of proofs (Mejia-Ramos et al., 2012). Local dimensions of proof comprehension refer to aspects of proofs limited to one or a small numbers of statements in the proof. Holistic dimensions, on the other hand, deal with aspects of proof related to big ideas and the proof as a whole. There are three local dimensions of proof comprehension (Mejia-Ramos et al., 2012):

(1) Meaning of terms and statements: Understand the meaning of key terms and individual statements of the proof. (L1)

(2) Justification of claims: Understand how and why an assertion made in the proof follows from previous statements and other proven or assumed statements. (L2)
(3) Logical status of statements and proof framework: Understand the logical status of statements in the proof and the logical relationship between this statement and other statements in the proof. (L3)

The four holistic dimensions are (Mejia-Ramos et al., 2012):

(1) Identifying the modular structure: Understand how a proof can be broken into different modules or sub-proofs, and the logical relationship between them. (H1)
(2) Illustrating with examples: Understand how the proof relates to specific examples and follow a sequence of inferences for specific examples. (H2)
(3) Summarizing via high-level ideas: Understand the main idea of the proof and its principal approach. (H3)
(4) Transferring the general ideas or methods to another context: ability to adopt the ideas and approaches of the proof to other proofs. (H4)

Materials and Methods

Sample and Procedure

There were 18 Norwegian pre-service teachers who participated in the study. The participants were all enrolled in a five-year teacher education program directed at middle and lower secondary school. All participants were in their final and fifth year of training, and would graduate with a Masters degree in mathematics education. All 18 participants, ten female and eight male, were aged 23–27, and had a similar educational background from the Norwegian education system – 13 years of primary and secondary education. None of the participants can be considered experts in mathematics, as none of them had any mathematical training beyond basic mathematics courses the five-year teacher education program. More specifically, all participants had completed: 1) one course on fundamental concepts and procedures in mathematics central to middle and lower secondary school mathematics such as numbers, fractions, equations, functions etc, 2) one course on the teaching and learning of mathematics, and 3) one course on systematic and scientific development of mathematics teaching in school. The first course, offered in the pre-service teachers' first and second year, consists of 120 hours of instruction. The second and third course, offered in the pre-service teachers' third and fifth year respectively, consists of 60 hours of instruction each. In total, the three courses make up one full year of the five-year teacher education program. Although the participants are not mathematical experts, these courses provide some opportunities to learn about proofs. In the first course, both direct and indirect proofs are studied explicitly in relation to algebra, geometry and elementary number theory. In the second course, proofs are implicitly studied in relation to students' mathematical reasoning and argumentation. More specifically, the act of proving is situated in a broader context of students' mathematical reasoning and argumentation within proof and argumentation frameworks (e.g. Balacheff, 1988). In the third course, the pre-service teachers are exposed to several examples of research where proof has been implemented in classrooms (e.g. Stylianou et al., 2009).

The data were collected within a larger study that focused on integrating mathematical proofs in mathematics teacher education programs (Haavold, 2021). Here, we report on analyses of data collected in two separate interview sessions. The first session consisted of individual semi-structured interviews designed to explore participants proof views. Each interview lasted about 25–30 minutes. The interview guide (Appendix 1) consisted of 20 questions on: 1) the general nature of mathematics, and 2) the role and nature of proofs and argumentation in mathematics. The second session, designed to explore participants proof understanding, consisted of pairwise task-based interviews that lasted about 60–70 minutes. We decided to make use of group-based protocols as they are particularly appropriate for observing decision-making and participants' real problem solving behavior in social environments (Schoenfeld, 1985). At the beginning of each interview, the pair of participants was presented with two mathematical theorems and three proofs corresponding with each of the theorems (see Table 1). The participants were then told that the purpose of the session was to determine whether the proofs were correct or incorrect, and also present a justification for their conclusion. After answering questions from the participants regarding more practical issues, such as length of the interview, we asked the participants to work on the proofs as they normally would on their own. After the participants' work on the proof tasks came to an end, we also asked them a series of supplemental questions based on our conceptualization of proof comprehension and proof validation. The purpose was to further investigate the pre-service teachers' explicit thoughts on issues related to proof validation and proof comprehension, which might not have come up during the participants' uninterrupted work. This meant that we first asked whether the proofs were correct and if the pre-service teachers could provide a justification for their answer. We then asked the participants questions formed straightforward on the basis of the proof comprehension categories described in the framework that could reveal the pre-service teachers' local and holistic comprehension of proofs (Mejía-Ramos et al., 2017). For example, the first category in the proof comprehension framework is *meaning of terms and statements*. According to Mejía-Ramos et al. (2017), assessing learners understanding of specific terms in the proof can involve asking for examples and definitions.

Table 1. Proof validation tasks-.

Proof Task 1	Proof Task 2
Theorem: Suppose n is an integer. If n^2 is even, then n is even.	Theorem: $\sqrt{2}$ is an irrational number.
Proof 1A: If n is even, then we can write $n = 2k$. We then see that $n^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$. Therefore, n^2 is even.	Proof 2A: Suppose that $\sqrt{2}$ is a rational number. Then we can write it as an irreducible fraction $\frac{a}{b} = \sqrt{2}$. We square both sides and see that $\frac{a^2}{b^2} = 2a^2 = 2b^2$. $2b^2$ is even, and therefore a^2 must also be even. It follows that a is even, and we can write a as $a = 2k$. We substitute $a = 2k$ into $a^2 = 2b^2$, and see that $(2k)^2 = 2b^2$. We then see that $4k^2 = 2b^2 2a^2 = b^2$. Therefore, b^2 must be even, and it follows that b must be even as well. We now have that both a and b are even. But that means $\frac{a}{b}$ is not irreducible, which contradicts our assumption. We therefore have to conclude that $\sqrt{2}$ is irrational.
Proof 1B: Suppose n is not even. Then it is odd, and we can write $n = 2k + 1$. We then see that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + k) + 1$. That means n^2 is also odd. We therefore have to conclude the statement in task 1 is correct.	Proof 2B: We can write $\sqrt{2} = 1 + \left(\sqrt{2} - 1\right) = 1 + \frac{1}{1+\sqrt{2}}$. Because $\frac{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}{1\left(\sqrt{2}+1\right)} = \frac{1}{\left(1+\sqrt{2}\right)}$. It follows then that $\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}} = 1 + \frac{1}{1+\left(1+\frac{1}{1+\sqrt{2}}\right)} = 1 + \frac{1}{2+\frac{1}{1+\sqrt{2}}}$, However, we can again make the same substitution, and this expression is an infinite continued fraction: $1 + \frac{1}{2+\frac{1}{1+\sqrt{2}}} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{1+\sqrt{2}}}} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{1+\sqrt{2}}}}} =$ So $\sqrt{2}$ is therefore irrational.
Proof 1C: $n^2 + n = n(n+1)$. The right hand side is even. Since n^2 is even, then n must also be even.	Proof 2C: Suppose that $\sqrt{2}$ is a rational number. Then we can write it as an irreducible fraction $\frac{a}{b} = \sqrt{2}$. We square both sides and see that $\frac{a^2}{b^2} = 2a^2 = 2b^2$. Every integer can be factored into primes, and we suppose this has been done for a and b. Thus in a^2 there are certain number of primes doubled up. And in b^2 there are a certain number of doubled-up primes. But, in $2b^2$ there is a 2 that has no partner. We have a contradiction and must conclude that $\sqrt{2}$ is irrational.

We therefore asked the participants if they could exemplify and/or define terms like integer, even and odd, irrational numbers etc.

Proof Tasks

The proof tasks used in this study are also part of a larger study, and their use have been explained and justified earlier (Haavold, 2021). Nevertheless, we will here describe and explain the proofs used for the task-based interviews. We used both direct and indirect proofs for each of the two theorems, as we wanted to explicitly investigate how the logical structure of proofs themselves mediated proof comprehension and validation. Furthermore, learners' understanding of proofs can be influenced by numerous factors. We therefore employed multiple proofs, both direct and indirect, that were comparable in terms of length, content and complexity.

The proof tasks included here were proofs that pre-service teachers with little exposure to advanced mathematics could understand (see also Haavold, 2021). This meant that we chose mathematical theorems within elementary number theory that were appropriate for the pre-service teachers' academic level, and could be worked on and comprehended within the limits of a task-based interview. Each of the six proofs also present the participants with challenges related to local and/or holistic aspects.

Proof 1A is a direct proof. However, it confirms the converse relationship of theorem 1. The next proof, 1B, is an indirect and contrapositive proof. This proof is logically correct, but contain a small algebraic error. Both proof 1A and 1B are flawed, however the errors are in the former case related to local aspects of the proof while in the latter case related to holistic aspects of the proof (Mejia-Ramos et al., 2012). The last proof in task 1 (1C) is a compact and abstract proof that is both correct and direct. We chose this proof because we wanted to see how the participants comprehended both the holistic overarching idea of the proof, and how each part of the proof is related locally and algebraically.

Proof 2A is a well-known indirect proof of the irrationality of $\sqrt{2}$. However, there is a small mistake in line eleven. As proof 1B, the purpose was to see if the participants' focus was on the holistic or local aspects of the proof. The next proof 2B is a direct proof of theorem 1 that uses what is known as infinite continued fractions. The last proof, 2C, is similar to proof 2A. However, unlike proof 2A it is correct, and it uses a slightly different method of establishing a contradiction. Again the purpose was to see if the participants focused on the local or holistic aspects of the proof.

Data Analysis

To answer our three research questions, we drew on qualitative content analysis (Mayring, 2015). For RQ1, and participants' proof understanding, we first took a deductive approach. After transcribing each of the task-based interviews, we ran through the material line by line, and identified, extracted and coded each text component that corresponded with one of the categories in our proof comprehension framework (Mejia-Ramos et al., 2012). This was done according to the recommendations of Mayring (2015) related to structuring procedures: 1) A trial-run was first carried out in which we attempted to mark all text-extracts according to the proof comprehension categories defined by Mejia-Ramos et al. (2012), p. 2) based on the trial-run we discussed and resolved unclear text-components, and created anchor examples and specific coding rules for ambiguous text components; 3) we ran through the entire material once more, coding and grouping text components according to the category definitions, anchor examples, and coding rules. As an example of coding rules, both category L1 and H2 can refer to the use of specific numerical examples. However, the main difference is that while L1 can refer to the use of specific examples to illustrate individual terms and statements of the proof, H2 refer to the use of specific examples which are used throughout the entire proof. Applying this framework to material allowed us to identify which proof comprehension dimension each participants employed for each proof task. We then took a more inductive approach, with a focus on reducing procedures (Mayring, 2015), and went through the material once more to identify how the participants validated the proofs. This was done in the following manner. First, we established that the primary themes of the analysis were to determine whether or not each participant considered a proof correct or incorrect, and what the explicit corresponding justification was. We then went through ca. 30% of the data material, and coded the participants' utterances related to the correctness of the proofs and the corresponding justification. The former theme was simply labeled as correct or incorrect, while the latter theme was coded as short labels close to the data material. The codes related to the corresponding justifications were then grouped together and more general categories were formed. Finally, we went through all the data material, coding utterances according to the newly constructed categories.

To answer our second research question, regarding participants proof views, we analyzed the individual interviews through an inductive and constant comparison approach, similar to the inductive approach described previously. During this process, we extracted and coded statements related to three questions: 1) what is a proof; 2) what is the purpose and role of proofs; and 3) what is a good argument in mathematics. We then summarized the text excerpts into specific categories that captured each pre-service teacher's views related to the three questions.

Finally, to answer our third research question, we investigated possible patterns between the participants' proof views and proof understanding. Here, we applied the principles of frequency analysis within qualitative content analysis (Mayring, 2015). For each category of proof views, we counted the number of different proof comprehension dimensions and proof validation categories employed by this sub sample of participants. Possible patterns were considered based on numerical trends in occurrences of proof comprehension dimensions or validation justifications along indirect and direct proofs or individual proofs (Mayring, 2015). Once we had identified possible patterns, we then investigated more closely if participants' utterances during the task-based interviews could be linked thematically or theoretically to the proof view category. The main purpose was to identify possible hypotheses that could explain how and why proof understanding and proof views might be related. A short example illustrates this process. 10 out of 18 participants professed views that we referred to as proofs are direct. To determine whether this particular proof view was associated with understanding of proofs, we then counted the occurrences of how these ten participants comprehended and validated each of the six proofs. As we explain later, we noted a clear pattern in how these ten participants comprehended and validated the indirect proofs.

To ensure a consistent and reliable coding procedure and analysis, immediately after the initial run-through of the material, each of the authors of this paper analyzed and coded ca. 30% of the data material individually. The results of these coding procedures were then compared and disparities resolved.

Limitations

There are at least two important limitations in this study that need to be mentioned. First, this was a small exploratory study in a particular cultural context. Norwegian pre-service teachers' educational, and otherwise, background will of course influence their beliefs and knowledge about proofs. It is therefore important to exercise caution when generalizing these findings across cultural contexts. Second, situation-specific beliefs and knowledge was investigated in pairwise interviews. The participants may have influenced each other, and introduced some bias in our analyses. However, we wanted this setting to be as natural as possible, with minimal interference from the interviewers during the participants' work on the proofs. We decided therefore it was better to let the participants work and talk among themselves, instead of replying to direct questions from the interviewers. Furthermore, we noted several instances where pairs of participants disagreed and employed different approaches. This indicates that although the participants influenced each other, we were still able to observe important individual variation.

Results

Pre-Service Teachers' Proof Understanding

As previously mentioned, proof understanding consists in this study of proof validation and proof comprehension. The participants justified their proof validation in four different ways. The first category, *because of logical conditions*, refers to justifications based on the logical status of statements in the proof. The second category, *because of empirical verification*, refers to justifications based on numerical examples. The third category, *because of algebraic verification*, refers to justifications based on algebraic manipulations and verifications. The fourth category, *proof is incomprehensible*, was different as it referred to occurrences where the participants indicated that they could not make sense of the proof. For each of the four categories of proof validation, we also considered whether the participants' assessment of the proof was correct or incorrect. For example, only four participants concluded proof 1A was incorrect (see Table 2).

Using the theoretical framework related to proof comprehension (Mejia-Ramos et al., 2012), we were also able to identify how the participants attempted to comprehend each of the six proofs. This was done by determining whether or not each participant used a particular comprehension dimension for a particular proof. We then counted the number of participants who employed each proof comprehension dimension at least once for each of the six proofs.[1] Table 2 displays how many participants used each proof comprehension dimension for each of the six proofs.

Each of the 18 participants worked on three direct and three indirect proofs. There were therefore 54 proof validation attempts of direct and indirect proofs respectively (Table 2). Not considering the participants' justifications, they were right about the validity of the indirect proofs 34 times and the

Table 2. Number of participants employing each proof comprehension dimension by each proof validation category.

	N	L1	L2	L3	H1	H2	H3	H4
Direct proofs								
Proof 1A								
Incorrect – Logical	4	4	4	4		4	4	
Correct – Empirical	8	8	4			4		
Correct – Algebraic	6	6	6			2		
Proof 2B								
Correct – Algebraic	10	10	10			4	2	
Correct – Empirical	8	8	6			8		
Proof 1C								
Correct – Logical	8	8	4	8		8	6	
Incomprehensible	10	10	10					
Total	54	54	44	12		30	12	
Indirect proofs								
Proof 1B								
Correct – Logical	2	2		2		2	2	
Incorrect – Algebraic	6	6	6					
Incorrect – Logical	10	10						
Proof 2A								
Correct – Logical	2	2		2		2	2	
Incorrect – Algebraic	6	6	6					
Incorrect – Logical	10	10						
Proof 2C								
Correct – Logical	2	2		2		2	2	2
Incorrect – Logical	16	16	6					
Total	54	54	18	6		6	6	2

Note: L1-L3 and H1-H4 refer to the three local and four holistic proof comprehension dimensions respectively.

[1]For a more detailed explanation of how the proof comprehension framework can be used in these proof tasks we refer to Haavold (2021).

direct proofs 30 times.[2] For example, proof 2B is a correct proof of theorem 2, and all 18 participants concluded the proof was correct. Proof 1A, on the other hand, is an incorrect proof and only 4 participants concluded that the proof was incorrect.

However, there were several noticeable differences in terms of proof validation and proof comprehension. While L2 was used a total of 44 times in the participants' work on the direct proof, it was used only 18 times in the work on the indirect proofs. Similarly, H2 was used 30 times for the direct proofs and only 6 times for the indirect proofs. We can also see from Table 2 that L3 and H3 were rarely used, but somewhat more prevalent for direct proofs than indirect proofs.

As for the participants' proof validation of indirect proofs, 42 justifications were based on the logical conditions and 12 justifications were based on algebraic verification. For the direct proofs, on the other hand, only 12 justifications were based logical conditions, while algebraic verification and empirical verification were referenced 26 and 16 times respectively.

These findings indicate some differences in how the participants proof understanding of direct and indirect proofs. For direct proofs, they mostly justified their validation based on algebraic or empirical verification. This is also seen in the widespread use of comprehension dimensions L1 and L2, which refers to understanding of individual terms and statements in the proof and how claims follow each other in the proof. For indirect proofs on the other hand, the participants justified their validation mostly on logical conditions of the proof. Although this is not clear from Table 2, this was also seen in how the participants attempted to comprehend the indirect proofs. The reason is that Table 2 only shows correct uses of comprehension dimensions. On all of the indirect proofs, at least ten of the participants attempted to use L3, but they failed to understand the logical relation between the assumptions and conclusions in the proofs. For example, on proof 1B, ten of the pre-service teachers explicitly said that the proof proved a different statement than what was given in theorem 1. Based on these observations, it seems logical structure of indirect proofs caused particular problems for the pre-service teachers. Regarding the direct proofs, on the other hand, the participants tended to verify them by going through the proofs line by line, either algebraically or empirically.

Pre-Service Teachers' Proof Views

Our analysis resulted in three categories regarding the nature of proofs (What is a proof?), two categories regarding the role of proofs (What is the purpose and role of proofs?), and two categories regarding arguments in mathematics in more general (What is a good argument in mathematics?). The purpose of the last question was to allow the participants to profess their views about arguments in mathematics in more general terms and not limited to proofs.

What is a Proof

The participants professed mixed and somewhat uncertain views about what a proof actually is and its nature. Most of the participants hesitated to provide definitions or defining characteristics of proofs. Nevertheless, we were able to identify three recurring themes.

Proofs are Based on Foundations. Although only two participants mentioned axioms explicitly, 14 participants said that a proof is based or built upon on some set of foundations. Furthermore, these foundations are agreed-upon truths, and statements we don't need to prove. Participant 1 articulated this as: "Proofs are built on some basic foundation . . . some statements we think are true. And then we build other statements on top of that." The excerpt indicates that proofs start out with some set of statements we assume to be true, or axioms, and then other statements are "used to show what we need to show" as Participant 4 explained. Although none of the participants delved further into the nature

[2]Here we mean that the participants' assessment of the proof was aligned with the actual truth value of the proof. In Table 2, on the other hand, we refer to correct and incorrect simply as the participants' assessment of the truth value of each proof.

of these foundations, the participants' statements indicate proof views related to an axiomatic proof scheme (Harel & Sowder, 2007).

Proofs are Formal and Abstract. Another common idea throughout the interviews, mentioned by 12 participants, was that abstraction, rigor and correct use of mathematical language and notation are very important to mathematical proofs. As Participant 3 said: "mathematical proofs are formal and abstract ... you have to use a very specific and correct type of language and symbols, and one statement comes after another and so on." Similarly, Participant 7 commented that: "proofs are different from other things in mathematics. They are more general and they are very condensed, and they follow strict rule for what is allowed." In both excerpts mentioned here, the participants explicitly highlight how proofs are distinctly characterized by abstraction and correct use of mathematical notation. Here, the participants' proof views seem to related to the formalities of proofs (Hersh, 1993).

Proofs are Direct. A third recurring theme was that mathematical proofs are direct. Although none of the participants explicitly mentioned the term direct proof, 10 participants mentioned that proofs should take the assumption P and show that the conclusion Q is true through a sequence of deductions. Participant 6, for instance, said that: "When we prove we need to show that it works ... that it is true. We have to prove that a claim is true. So we take the assumption and work step by step until we end up at the conclusion ... and the conclusion is then true." Here, the participant highlights how proofs are stepwise and straight-forward demonstrations of truth, from an assumption to a conclusion. Nine other participants made similar statements that are were in alignment with the structure of direct proofs. The fact that none of the participants mentioned other logical structures, such as indirect proofs, can indicate that the participants' proof views are limited to proofs that are more straightforward. In the research literature, mathematics learners are often reported to not only dislike indirect proofs, but also experience a lack of conviction from them (S. A. Brown, 2018).

What is the Purpose and Role of Proofs

Although proofs have numerous functions – Hanna (2000) lists for example eight different functions – we identified only two recurring categories of the participants' views about purpose or role of proofs.

Proofs as Verification. All 18 participants stated that the primary function of proofs was to demonstrate or verify the truth of a mathematical statement. Similar to what Knuth (2002) found, the preservice teacher's views could be further separated into two groups. Six participants suggested that proofs establish truth through deductive arguments, while 12 participants said that proofs established truth through more general convincing arguments. The former group can be exemplified by Participant 11 who said that: "Proofs show that something is true. They demonstrate it ... by deduction. One statement follows logically from the previous statement." For the latter group, a statement from Participant 1 exemplifies the verification aspect more generally: "Proofs are used to show that some things in math is correct or true ... like, you have a hypothesis, and then a proof can verify it."

Proofs as Explanation. Twelve participants mentioned explicitly the potential for proofs to provide an understanding of the underlying mathematical relationships, and not just an explanation of why a statement is true. Participant 11, for instance, said: "Proofs are important even in middle school. The formulas there have a background. A reason. They are not just there. Students need to understand why for instance two odd numbers equals an even number. They have to get the main idea behind it." Here, Participant 11 explains that proofs can add to our understanding of important ideas in mathematics, and the relationship between them. The key word in the statement is "why," which refers to a causal relationship that can explain the relationship between the sum of odd numbers and even numbers.

What is a Good Argument in Mathematics

We identified two recurring themes of what is a good argument in mathematics

A Good Argument is Convincing.

All 18 participants mentioned that a good argument is convincing. There were however two slight nuances of this idea. The first referred to the argument itself. Ten of the participants pointed to the argument as an object in itself, and said that a good argument is true. It is a statement, or sequence of statements, that are technically true as in the form of logical deduction. The other eight participants focused more on the recipient of the argument, and said that a good argument is persuasive and removes doubt about the truth of an assertion. Participant 12 for instance said that: "a good argument shows why something is true. It removes doubt … like if you have an assertion and you make a good argument, then it should convince you it's correct."

A Good Argument Can be Less Certain than a Proof.

16 of the participants said that a good argument is convincing. However, they also said that a good argument could be both convincing and, at the same time, less certain than a proof. Only a few participants provided an explanation for this. Participant 11 for instance said that: "A proof is a good argument, but a good argument is less certain than a proof. An argument is more individual and less formal and rigorous." In other words, a majority of the participants made a distinction between proofs and good arguments, while at the same time expressing that convincing arguments do not have to be as certain as proofs – indicating a certain intuitionist inclination (Auslander, 2008).

Patterns Between Pre-Service Teachers' Proof Views and Proof Understanding

We identified two possible connections between the participants' proof views and proof understanding. First, ten participants professed views we categorized as proofs are direct. These were the same ten participants who concluded that the indirect proofs 1B and 2A were incorrect based on logical conditions, and ten of the 16 participants who concluded that proof 2C was incorrect based on logical conditions. This subgroup of participants used proof comprehension dimension L1 for all three indirect proofs. We can see in Table 2 that this subgroup of participants also differs from the rest of the overall sample, as the other eight participants validated and/or comprehended the indirect proofs differently. This indicates a possible relationship between the participants' proof views and their proof understanding. Statements made by the participants during their work on the indirect proofs might shed some light on this possible pattern. A majority of the ten participants stated that they did not understand how the indirect proofs could assume conditions that challenged the theorem statements. For example, working on proof 2C, Participant 14 said that "(it) assumes that the square root of two is rational, but here (points to statement in task 2) it says that it is irrational… how can we assume something that is incorrect?."

Second, eight participants professed views we categorized as a good argument is convincing and removes doubt about the truth of an assertion. These were also the same, and only, eight participants who concluded that proof 1A and 2B were correct based on empirical verification. Although the data cannot demonstrate definitely that the eight participants hold what is known as an empirical proof scheme (Harel & Sowder, 2007), this structural pattern between proof views and proof understanding indicate that empirical verification is sufficient to convince them of the truth of an assertion. Some of the participants' statements during work on proof 1A and 2B lend support to this hypothesis. For example, working on proof 1A, Participant 2 and Participant 3 both used proof comprehension dimensions L1, L2, and H2. Nevertheless, validating the proof, Participant 2 explicitly stated that "proof 1A is correct, because it worked for all numbers we tried."

Discussion

In this exploratory study, we posed three research questions. First, we investigated what characterizes Norwegian pre-service teachers' situation-specific knowledge of and beliefs about direct and indirect proofs, which we throughout this paper has referred to as proof understanding. As much of the previous literature (e.g. Antonini & Mariotti, 2008), we found that indirect proofs seem to be more challenging than direct proofs. We also noted that the participants seemed to emphasize the logical structure of the indirect proofs, while the direct proofs were evaluated through algebraic and empirical verification.

Based on our analyses, it seems the meta-logical structure of indirect proofs caused particular problems for the pre-service teachers, while the direct proofs were evaluated fairly straight-forward by checking them line-by-line either empirically or algebraically. In the context of the proof comprehension framework used in this study (Mejia-Ramos et al., 2012, the participants focused almost exclusively on local dimensions of proofs, and less on the holistic dimensions related to big ideas of the proofs. Although it is difficult to point to one particular cause of this apparent trend, statements made by the participants during the interview may shed some light on this issue. During their work on the indirect proofs, most participants said at some point that the proofs were unclear and even puzzling, as the assumptions of the proofs contradicted the statements in the theorem. Although several explanations for this observation is found in the literature, Sierpinska's (2007) consistency hypothesis offers a particularly pertinent explanation. The question of contradiction, as in the case of indirect proofs, depend on conceptual meaning. This means that if a particular statement is considered meaningless, then the issue of consistency and contradiction will be meaningless as well. In this study, it seemed several participants considered the assumptions made by the indirect proofs to be unfounded and without cause. As such, the participants seemed to be unwilling to further consider both the big idea of the proof and the correctness of each line-by-line statement of the proof.

In our second research question, we asked what characterizes pre-service teachers' proof views, which we conceptualized as general professed knowledge of and beliefs about proofs. Our findings indicate that the participants view proofs as formal and abstract, that proofs are based on foundations, and all 18 participants mentioned that the purpose of proofs was to establish the truth of a statement. Furthermore, the analysis indicate that most of the participants view good arguments as less certain and formal than mathematical proofs. This suggests the participants' general knowledge of and beliefs about mathematical proofs, or proof views, are somewhat narrow and rigid, as much of similar research indicate (Knuth, 2002; Szydlik et al., 2003). However, unlike many previous studies that have concluded that teachers do not value the explanatory role of proofs (e.g. Harel & Sowder, 2007), we found that a majority of the participants in this study explicitly mentioned the potential for proofs to provide an understanding of the underlying mathematical relationships. Similarly to what Mingus and Grassl (1999) and Ko (2010) concluded, the results of this study therefore show that teachers, including pre-service teachers, do not necessarily have homogenous proof views related to the explanatory role of proofs.

Our third research question asked how proof understanding is related to proof views. In other words, how situation-specific knowledge and beliefs about proofs are related to more general and out-of-context knowledge and beliefs about proofs. The analysis indicated two apparent patterns. First, we noticed that the participants who empirically validated proofs were the same who also professed views that a good argument is an argument that is convincing. We cannot conclusively say that the participants hold an empirical proof scheme (Harel & Sowder, 2007), and consider empirical validation strictly correct. However, this pattern indicates that these participants were convinced by empirical arguments, and that an empirical argument is sufficient for removing doubt about the truth of an assertion. Second, more than half the participants professed views that proofs should be direct. These were the same participants who struggled with the logical conditions of the indirect proofs. As mentioned earlier, the research literature generally indicate that mathematics learners often dislike indirect proofs and generally find them unconvincing. In this study, we found that many of the participants did not just struggle to understand indirect proofs, they also professed views that are contradictory to indirect proofs on a meta-logical level. These two apparent patterns indicate that some beliefs related to proofs and indirect proofs seem to be particularly central to some of the participants.

As we noted earlier, knowledge and beliefs are organized in clusters around specific situations and contexts (Green, 1971). According to Rokeach (1968), central and important clusters tend to be more consistent across settings and situations. These are the clusters for which the individual has complete consensus. Less central, and important, clusters tend be accompanied by more disagreement and inconsistencies. Based on this we can speculate that particular beliefs and knowledge that are consistent across situations and contexts tend to be more central and important to the individuals. In this study, we noted that there were two sets of beliefs and knowledge about proofs that were aligned across both a situation-specific context and a more general context. We suggest therefore that the beliefs and knowledge related to the direct nature of proofs and empirical proof scheme, respectively, might be particularly central and important to some pre-service teachers. If these clusters of knowledge and beliefs are central to the individual, they might be held more strongly than others and therefore also more difficult to challenge and modify (Op't Eynde et al., 2002).

Somewhat surprisingly, the analysis did not reveal any other patterns between the participants' situation-specific knowledge and beliefs about proofs (proof understanding) and out-of-context knowledge and beliefs about proofs (proof views). For example, 12 participants professed views that highlighted the explanatory purpose of proofs. It would not have been surprising if these 12 participants had displayed a propensity for using the holistic comprehension dimensions in their situation-specific work on proofs. However, we were unable to identify any such or other pattern between this particular proof view and proof understanding. Although such a finding may seem unexpected at first, the literature does provide plausible explanations. Due to the complexity of beliefs and knowledge systems, such as clustering and centrality, research often reports a discrepancy between general out of context expressed beliefs and actual practice (Philipp, 2007). We may therefore speculate that the participants' other professed proof views – i.e. their general and out-of-context beliefs and knowledge – were more peripheral and less important. Here, it may be possible to suggest that the participants were not able to recognize the big ideas (Mejia-Ramos et al., 2012) and the explanatory aspects of the proofs. However, although we did not investigate this issue further, none of the proofs in this study relied on methods or concepts outside the scope of the known contents of the participants' previous mathematical training.

These findings do not tell us why some clusters of belief and knowledge seem to be more central and important to the participants. However, according to much of the literature, there are two primary sources for knowledge and, in particular, beliefs: emotion packed experiences and cultural transmission (Pajares, 1992). In both cases, they are a result of learners' experiences. For pre-service teachers, and in-service teachers, it is particularly the second source, cultural transmission, that is relevant for teacher education practices. Cultural transmission contributes to the formation of beliefs and knowledge that may be held at a subconscious level and can be thought of as resulting from the "hidden curricula" of our everyday lives. People tend to be unaware of the culturally transmitted beliefs they hold and limitations of knowledge, taking them for granted because they have neither examined nor discussed them (Pajares, 1992). The lack of attention to deductive reasoning and the issue of learners' empirical proof schemes has, in this context, received a lot of attention in the research literature (e.g. G. J. Stylianides et al., 2017). However, the issue of indirect proofs is less recognized. Given the importance of cultural transmission and hidden curricular, it may be the case that indirect proofs are not more complex or hard than direct proofs. Instead, learners' difficulties related to indirect proofs may be a result of a lack of attention to indirect proofs in learners' mathematical education. There is some evidence for this in the research literature. For example, in a somewhat recent well-designed survey study, S. A. Brown (2018) found that length, complexity and familiarity were more important for students' successful proof validation than proof type – i.e. direct vs. indirect proof. In other words, adjusting for the scope and depth of proofs, previous exposure to the different types of proofs was the primary factor that impacted whether students were able to validate and understand proofs. In this study, we used indirect and direct proofs of comparable length and complexity, meaning that a lack of familiarity with indirect proofs could be the main cause of the participants' difficulties with indirect proofs.

Implications

Although the focus in this study was observational patterns in pre-service teachers' proof views and proof understanding, it is possible to propose certain implications for the teaching and learning of indirect proofs. First, it seems exposure and familiarity are key factors for the development of learners' beliefs about and knowledge of proofs. This was also seen in a somewhat recent teaching experiment. Amit and Portnov-Neeman (2017) observed that explicit attention to the meaning and structure of indirect proofs was highly effective even with young students. Therefore, explicit attention to indirect proofs is arguably needed in mathematics instruction if we want learners to become better at understanding, using and appreciating indirect proofs. Second, as Sierpinska (2007) argues, sensitivity to contradictions requires meaning. This implies that if we want learners to develop consistent and productive beliefs and knowledge about indirect proofs, mathematics instruction must also focus explicitly on the purpose and meaning of contradictions in indirect proofs.

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Appendix A: Translated Interview Guide

General nature of mathematics

1. What is mathematics? How would you describe it?
2. Is mathematics the same today as it was 500 years ago? Does mathematics evolve?
3. Where can we find mathematics?
4. Do you have any thoughts on why humans started doing mathematics?
5. Is mathematics something humans invent or discover? Why do you think so?
6. Does mathematics exist independent of humans? Why?
7. Can you say something about your own experience of mathematics as a student in school?

Proofs and argumentation in mathematics

1. What is a proof in mathematics?
2. How would you describe proofs in mathematics?
3. What is the purpose/role of proofs in mathematics?
4. What is a proof in school mathematics?
5. What is the purpose/role of proofs in school mathematics?
6. If you compare proofs in mathematics with proofs in other fields (such as law), is there a difference? If so, what is the difference?
7. How does one make/create proofs?
8. What does the word axiom mean?
9. What does logical validity mean?
10. What is a good argument in mathematics?
11. Is there any difference between a good argument and proofs in mathematics? If so, what?
12. What are the roles of intuition and logic in mathematics? In proofs?
13. Can you say something about your own experience of proofs as a student in school?