UiT The Arctic University of Norway

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Combinatorics and semi-algebraic geometry of orbit spaces

Hyperbolic and stable polynomials, the degree principle and few inequalities defining orbit spaces

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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of **Philosophiae Doctor** at UiT The Arctic University of Norway. The research presented here was conducted under the supervision of professor Cordian Riener and professor Philippe Moustrou. This work has been supported by the Tromsø Research Foundation (grant agreement 17matteCR).

The thesis is a collection of four papers, presented in an order which relates to their topics. The common theme to them are symmetries in real algebraic geometry, in particular descriptions and properties of orbit spaces of the symmetric group and other finite groups. The papers are preceded by an introductory chapter that relates them to each other and provides background information and motivation for the work. The first paper is joint work with Arne Lien. The second paper is joint work with Cordian Riener. The third paper is joint work with Sebastian Debus and Cordian Riener. The fourth paper is joint work with Philippe Moustrou and Cordian Riener.

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Tromsø, July 2024

List of Papers

Paper I

Arne Lien and Robin Schabert. “Shellable slices of hyperbolic polynomials and the degree principle”. preprint on arXiv: 2402.05702, submitted for publication.

Paper II

Cordian Riener and Robin Schabert. “Linear slices of Hyperbolic polynomials and positivity of symmetric polynomial functions”. In: **Journal of Pure and Applied Algebra**. May 2024, volume 228, issue 5, article 107552. DOI: 10.1016/j.jpaa.2023.107552.

Paper III

Sebastian Debus, Cordian Riener and Robin Schabert. “Stable and Hurwitz slices, a degree principle and a generalized Grace-Walsh-Szegő theorem”. preprint on arXiv: 2402.05905, submitted for publication.

Paper IV

Philippe Moustrou, Cordian Riener and Robin Schabert. “Constructively describing orbit spaces of finite groups by few inequalities”.

Contents

List of Figures

Chapter 1

Introduction

Algebraic geometry studies the set of common zeros of a system of polynomials in one or several variables - most commonly over algebraically closed fields. Real algebraic geometry studies semialgebraic sets, i.e. subsets of $\mathbb{R}^n$ defined by a finite system of polynomial equalities and inequalities. A starting point for modern real algebraic geometry can be traced back to Hilbert: In 1888, Hilbert showed the existence of nonnegative polynomials which are not sums of squares of polynomials. In 1900, he posed his famous 23 problems and, in particular, the 17th can be stated as follows: *Is every nonnegative polynomial a sum of squares of rational functions?* Artin's solution to Hilbert's 17th problem can be seen as a kick-off for real algebraic geometry. This thesis deals with real symmetric polynomials and the results are closely related to an answer of Hilbert's 17th problem for symmetric polynomials: A characterization of all symmetric nonnegative polynomials.

A symmetric polynomial f can be uniquely written as a polynomial in the elementary symmetric polynomials, say $f = g(e_1, \ldots, e_n)$. So instead of studiyng f on $\mathbb{R}^n$, one can study g on $(e_1, \ldots, e_n)(\mathbb{R}^n)$. It turns out that this real image of the *Vieta map* has several nice geometric and combinatorial properties and by studying it one obtains several results reducing the complexity of problems in real algebraic geometry and computer algebra involving symmetric polynomials.

In the following, the background and preliminaries of the papers are presented with a historical view. The necessary background on symmetric polynomials is evolved in Section 1.1 around Gauss' proof from 1816 of the fundamental theorem of algebra [Gau16] and the more general setting of invariant theory in Section 1.2 is based on Hilbert's results from 1890 [Hil90] and 1893 [Hil70]. The foundations of real algebraic geometry presented in Section 1.3 can also be traced back to Hilbert [Hil88] and in particular to Artin's solution to Hilbert's 17th problem [Art27]. Moreover, one of the papers of this thesis is based on more recent works about the semialgebraic geometry of real orbit spaces by Procesi and Schwarz from 1985 [PS85] and by Bröcker from 1998 [Brö98] and the other papers are based on articles by Arnold, Kostov, and Meguerditchian ([Arn86],[Kos89], [Meg92]) and recently Riener [Rie12] and Lien [Lie23] using the connection between univariate polynomials and elementary symmetric polynomials to get a deeper understanding of the real orbit space for the symmetric group and therefore of symmetric real polynomial functions.

1.1 The fundamental theorem of algebra

The fact that every univariate polynomials of degree n has n complex roots (counted with multiplicities) was already conjectured in 1629 by Girard [Gir10].

This classical statement, known as the fundamental theorem of algebra, was proven multiple times. The first rigorous proof was published by Argand in 1806 [Arg74], but had almost no impact due to being self-published with a minimal number of copies. The second and more known rigorous proof is due to Gauss in 1816 [Gau16].

Theorem 1.1.1 (Fundamental theorem of algebra)**.** *Let $f \in \mathbb{C}[T]$ be a univariate monic polynomial of degree n. Then there are $x_1, \ldots, x_n \in \mathbb{C}$ such that*

$$f = (T - x_1) \cdots (T - x_n). \tag{1.1}$$

The fundamental theorem of algebra does not hold over the reals, since $\mathbb{R}$ is not algebraically closed. Real polynomials with only real roots are called *hyperbolic*. Throughout this chapter, we fix $n \in \mathbb{N}$ and denote by $\mathcal{H}$ the set of monic hyperbolic polynomials of degree n. By a straightforward computation of the right hand side in Equation 1.1 and comparing coefficients, one obtains the so called *Vieta's relations*.

Corollary 1.1.2 (Vieta's relations)**.** *The coefficients of a monic polynomial $f = T^n - a_1 T^{n-1} + \cdots + (-1)^n a_n \in \mathbb{C}[T]$ with roots $x_1, \ldots, x_n \in \mathbb{C}$ are given by $a_i = e_i(x_1, \ldots, x_n)$, where*

$$e_i := \sum_{1 \le j_1 < j_2 < \cdots < j_i \le n} X_{j_1} \cdots X_{j_i}$$

denotes the i-th elementary symmetric polynomial.

In particular, by introducing the *Vieta map*

$$\begin{array}{rcc} \nu_{\mathbb{K}} : \mathbb{K}^n & \longrightarrow & \mathbb{K}^n \\ x & \longmapsto & (e_1(x), \ldots, e_n(x)) \end{array}, \quad (\mathbb{K} = \mathbb{C} \text{ or } \mathbb{K} = \mathbb{R})$$

one can interpret the fundamental theorem of algebra as the statement, that $\nu_{\mathbb{C}}$ is surjective. In this light, the set of monic hyperbolic polynomials $\mathcal{H}$ can be identified with the image of $\nu_{\mathbb{R}}$.

The elementary symmetric polynomials are examples of symmetric polynomials, i.e. polynomials that are invariant under all permutations of variables. Gauss showed in his proof from 1816 of the fundamental theorem of algebra [Gau16] that every symmetric polynomial can be written uniquely as a polynomial in the elementary symmetric polynomials. Moreover, by analyzing his proof, one gets the following for symmetric polynomials of relatively small degree.

Theorem 1.1.3 (Gauss' fundamental theorem of symmetric polynomials)**.** *Let $f \in \mathbb{K}[\underline{X}]$ be a polynomial of degree $d \le n$. Then f can be written as*

$$f = g_0(e_1, \ldots, e_{\lfloor \frac{d}{2} \rfloor}) + \sum_{i=1}^{d} g_i(e_1, \ldots, e_{\lfloor \frac{d}{2} \rfloor}) e_i$$

where $g_0, \ldots, g_d \in \mathbb{K}[Z_1, \ldots, Z_d]$.

1.2 Hilbert's foundation of invariant theory

Invariant theory generalizes the notion of symmetric polynomials by studying the ring of polynomials invariant under some linear group action.

Throughout this section, let K be a field of characteristic 0 and denote by $K[\underline{X}] := K[X_1, \ldots, X_n]$ the polynomial ring in n variables. Furthermore let $G \subseteq \mathrm{GL}(K, n) := \{A \in K^{n \times n} \mid A \text{ invertible}\}$ be a subgroup of the general linear group. Then G acts in a canonical way on K^n by matrix multiplication and on $K[\underline{X}]$ by

$$A \cdot f := f\Big(A \cdot \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}\Big)$$

for $A \in G$. Now $f \in K[\underline{X}]$ is called G-*invariant*, if $A \cdot f = f$ for all $A \in G$ and we denote the ring of all G-invariant polynomials as $K[\underline{X}]^G$. Note that by writing the symmetric group S_n in terms of permutation matrices, the notion of invariant polynomials is really a generalization of symmetric polynomials.

In this setting, Hilbert proved in 1890 [Hil90] that the ring of G-invariant polynomials is finitely generated for many groups G. Hilbert's paper is even more known for his main tool, known as Hilbert's Basissatz. In order to state this result, we have to symmetrize polynomials by averaging over the group action, which leads to the notion of *Reynold operators*.

Definition 1.2.1. A $K-$linear operator

$$R_G : \ K[\underline{X}] \longrightarrow K[\underline{X}]^G$$

with properties

(i) $R_G|_{K[\underline{X}]^G} = \mathrm{id}_{K[\underline{X}]^G}$

(ii) $R_G(fg) = f R_G(g)$ for all $f \in K[\underline{X}]^G$ and $g \in K[\underline{X}]$

is called *Reynolds operator* of G.

For finite groups, one gets the unique Reynolds operator by averaging over the group action, i.e.

$$R_G : \ K[\underline{X}] \longrightarrow K[\underline{X}]^G, \ f \mapsto \frac{1}{|G|} \sum_{a \in G} a \cdot f.$$

Now we can state Hilbert's Fundamental Theorem.

Theorem 1.2.2 (Hilbert's Fundamental Theorem)**.** *Let $G \subseteq \mathrm{GL}(K, n)$ be a group that admits a Reynolds operator R_G. Then $K[\underline{X}]^G$ is generated by finitely many homogenous polynomials - called fundamental invariants -, i.e. there are $\pi_1, \ldots, \pi_m \in K[\underline{X}]^G$ homogenous, such that*

$$K[\underline{X}]^G = K[\pi_1, \ldots, \pi_m].$$

Theorem 1.2.2 holds also under more general assumptions, e.g. also for arbitrary characteristics (for a proof see [DK15]). Moreover, the proof of Theorem 1.2.2 is not constructive. If G is finite, there is also a constructive proof o Theorem 1.2.2: Noether showed that generators of the invariant ring can be obtained by applying the Reynolds operator to all monomials up to a certain degree (see [Cox+94, 7. §3 Theorem 5]).

Theorem 1.2.3 (Noether)**.** *Let G be finite. For $\alpha \in \mathbb{N}_0^n$ we define $\underline{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Then*

$$K[\underline{X}]^G = K[R_G(\underline{X}^\alpha) \mid \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq |G|],$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$ is the multi-index notation. In particular, $K[\underline{X}]^G$ is generated by $m := \binom{|G|+n}{n}$ homogenous polynomials.

Note that the expression of invariant polynomials in terms of fundamental invariants is not necessarily unique in contrast to Theorem 1.1.3. This leads to the notion of the *ideal of relations*

$$I_\Pi := \{h \in K[Y_1, \ldots, Y_m] \mid h(\pi_1, \ldots, \pi_m) = 0\}$$

and its corresponding variety $V(I_\Pi) := \{x \in \mathbb{C}^m \mid \forall f \in I_\Pi :\ f(x) = 0\}$.

As a generalization of the Vieta map, we study the polynomial map defined by the fundamental invariants of other invariant rings, known as the *Hilbert map*

$$\Pi :\ K^n \longrightarrow V(I_\Pi),\ x \mapsto \Pi(x) := (\pi_1(x), \ldots, \pi_m(x)).$$

The Hilbert map is obviously constant on *orbits* $G \cdot x := \{g \cdot x \mid g \in G\}$ of G and induces therefore a map $\overline{\Pi}$ from the *orbit space* $K^n/G := \{G \cdot x \mid x \in K^n\}$ to $V(I_\Pi)$.

We saw at the end of Section 1.1 that the fundamental theorem of algebra can be interpreted as the statement that the complex Vieta map is surjective. Similarly, Hilbert showed in his second paper on invariant theory in 1893 [Hil70] that the complex Hilbert map is surjective.

Theorem 1.2.4 (Hilbert 1893)**.** *Let I_Π be the ideal of relations of the fundamental invariants $\Pi := (\pi_1, \ldots, \pi_m)$. Furthermore let K be algebraically closed. Then the Hilbert map Π is surjective and Π induces a bijection*

$$\overline{\Pi} :\ K^n/G \longrightarrow V(I_\Pi),\ G \cdot x \mapsto \Pi(x).$$

We have seen in Gauss' fundamental theorem 1.1.3 that every polynomial has a unique representation in the elementary symmetric polynomials, i.e. the ideal of relations for the elementary symmetric polynomials $I_E = (0)$ and therefore $V_\mathbb{C}(I_E) = \mathbb{C}^n$. So Theorem 1.2.4 can really be interpreted as a generalization of the fundamental theorem of algebra.

The Hilbert map over the reals is not surjective, as we have already noticed in Section 1.1, but we will see that we still get a nice characterization for its image in Section 1.4.

1.3 Positive polynomials: From Hilbert's 17th problem to real algebra

While algebraic geometry deals with algebraic sets, i.e. solution sets of systems of polynomial equations, real algebraic geometry studies semialgebraic sets, i.e. solution sets of systems of finitely many polynomial equations and inequalities. Closely related to this is the question whether a polynomial is nonnegative on $\mathbb{R}^n$ or, more general, on a semialgebraic set. While testing positivity is very difficult, checking whether a polynomial is a sum of squares of polynomials can be done by semidefinite programming and is therefore much easier.

Obviously, every sum of squares of polynomials is non-negative on $\mathbb{R}^n$ and Hilbert showed in 1888 [Hil88], that the reverse is only true for univariate polynomials, for polynomials of degree 2 and for polynomials in 2 variables and degree 4.

The first explicit example of a non-negative polynomial, that is not a sum of squares, is due to Motzkin in 1967 [Mot67].

In 1900, Hilbert posed his famous 23 problems and, in particular, the 17th can be stated as follows: *Is every non-negative polynomial a sum of squares of rational functions?* Artin's solution [Art27] to Hilbert's 17th problem in the affirmative can be seen as a kick-off for real algebraic geometry.

More recently, Krivine [Kri64] and Stengle [Ste74] generalized independently Artin's solution of Hilbert's 17th problem, characterizing positivity on a *basic closed* semialgebraic set.

Theorem 1.3.1 (Krivine-Stengle Positivstellensatz)**.** *Let $f, g_1, \dots, g_m \in \mathbb{R}[\underline{X}]$ and*

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$$

the basic closed semialgebraic set defined by $g_1, \dots, g_m$. Then

$$f \geq 0 \text{ on } S \iff \exists p, q \in T(g_1, \dots, g_m) : \exists k \in \mathbb{N}_0 : pf = f^{2k} + q,$$

where

$$T(g_1, \dots, g_m) := \left\{ \sum_{\alpha \in \mathbb{N}_0^k} \sigma_\alpha g_1^{\alpha_1} \cdots g_k^{\alpha_k} \mid \sigma_\alpha \in \sum(\mathbb{R}[\underline{X}]^2) \right\}$$

is the smallest preorder of $\mathbb{R}[\underline{X}]$ containing $g_1, \dots, g_m$.

This can also be seen as a real analogue of Hilbert's Nullstellensatz.

The Krivine-Stengle Positivstellensatz holds for general real closed fields R, i.e. $R \neq R(i)$ and $R(i)$ is algebraically closed. Some of the arguments in Artin's solution of Hilbert's 17th problem can be seen as a special case of Tarski's transfer principle.

Theorem 1.3.2. *[Tarski's transfer principle] Let $\tilde{\mathcal{R}}|\mathcal{R}$ be a field extension of real closed fiels and $f, g_1, \dots, g_r \in \mathcal{R}[\underline{X}]$. Then, the semialgebraic set*

$$\{x \in \mathcal{R}^n \mid f(x) = 0, g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$$

is non-empty if and only if the semialgebraic set

$$\left\{x \in \tilde{\mathcal{R}}^n \;\middle|\; f(x) = 0, g_1(x) \geq 0, \ldots, g_r(x) \geq 0\right\}$$

is non-empty.

Tarski proved this in 1948 [Tar98] by showing that quantifier elemination is possible over the reals. This can also be formulated in the following way.

Theorem 1.3.3. *[Tarski-Seidenberg] Images of semialgebraic sets under polynomial maps are semialgebraic.*

1.4 Semialgebraic geometry of real orbit space

As a corollary of the Tarski-Seidenberg theorem, we get that the image of the real Hilbert map is a semialgebraic set. For the symmetric group, this is a well known result: As mentioned at the end of Section 1.1, the image of the real Vieta map can be identified with the set of hyperbolic polynomials and Hermite showed in 1853 [Her09] that a polynomial $f := T^n + a_1 T^{n-1} + \cdots + a_n$ is hyperbolic if and only if

$$H_f := (\mathrm{trace}(C_f^{1+j-2}))_{1 \leq i,j \leq d}$$

is positive semidefinite, where

$$C_f := \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & -a_d \\ 1 & 0 & \ddots & & \vdots & -a_{d-1} \\ 0 & 1 & \ddots & \ddots & \vdots & -a_{d-2} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{pmatrix}$$

is the *Frobenius companion matrix*.

If one equips $\mathbb{R}^n$ and $\mathbb{R}^m$ with the Euclidean topology and $\mathbb{R}^n/G$ with the induced topology, then Schwarz [Sch75] showed that the Hilbert map on $\mathbb{R}^n/G$ is actually a homeomorphism. I.e. one can view the description of hyperbolic polynomials as a description of the real orbit space for the symmetric group.

This description of the real orbit space as a basic closed semialgebraic set was generalized by Procesi and Schwarz [PS85]: One can define a G-invariant inner product on $\mathbb{R}^n$ by

$$\langle v, w \rangle_G := \sum_{A \in G} \langle Av, Aw \rangle.$$

The *Procesi-Schwarz*-matrix M_Π of the Hilbert map is then obtained by writing the entries of the matrix $\tilde{M}_\Pi \in (\mathbb{R}[\underline{X}]^G)^{m \times m}$ defined by

$$(\tilde{M}_\Pi)_{i,j} := \langle \nabla \pi_i, \nabla \pi_j \rangle_G \in \mathbb{R}[\underline{X}]^G$$

as polynomials in the generators. Procesi and Schwarz showed that this matrix gives a description of the image of the Hilbert map as a basic closed semi-algebraic set.

Theorem 1.4.1 (Procesi, Schwarz)**.**

$$\Pi(\mathbb{R}^n) = \{z \in V_{\mathbb{R}}(I_\Pi) \mid M_\Pi(z) \textit{ is positive semidefinite.}\}$$

Bröcker showed in [Brö98] that one can get a description of the image of the Hilbert map in terms of a number of inequalities that just depends on the structure of the group G.

Theorem 1.4.2 (Bröcker)**.** *Let G be a finite group and let k be the maximal number for which G contains an elementary abelian subgroup of order 2^k. Then the orbit space $\Pi(\mathbb{R}^n)$ is of the form*

$$\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid f_1(z) > 0, \dots, f_k(z) > 0\} \cup T,$$

where $\dim(T) < \dim(\Pi(\mathbb{R}^n))$ *and* $f_1, \dots, f_k \in \mathbb{R}[Z_1, \dots, Z_m]$.

Bröcker's and Procesi and Schwarz' description of the real orbit space together with the Krivine-Stengle Positivstellensatz allows in particular for an equivariant solution of Hilbert's 17th problem, i.e. a characterization of all non-negative invariant polynomials.

Corollary 1.4.3 (Equivariant solution of Hilbert's 17th problem)**.** *Let* $\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid g_1(z) \geq 0, \dots, g_k(z) \geq 0\}$ *and* $f_i := g_i(\pi_1, \dots, \pi_m)$. *If* $f \in \mathbb{R}[\underline{X}]^G$ *is invariant and non-negative, then there is* $s \in \sum \left(\mathbb{R}[\underline{X}]^G\right)^2$ *such that* sf *is in the preorder generated by* $f_1, \dots, f_k$.

1.5 Vandermonde varieties, hyperbolic slices and the degree principle

In the case of symmetric polynomials of degree $d \leq 2n$, checking nonnegativity is easier as shown by Timofte in 2003 [Tim03].

Theorem 1.5.1 (half-degree principle)**.** *Let* $f \in \mathbb{R}[\underline{X}]$ *be a symmetric polynomial of degree* $d \leq n$. *Then* f *is nonnegative, if and only if*

$$\inf_{x \in \mathbb{R}^n} f(x) = \inf_{x \in \mathcal{A}_k} f(x),$$

where $k := \max\left\{2, \lfloor \frac{d}{2} \rfloor\right\}$ *and* $\mathcal{A}_k \subseteq \mathbb{R}^n$ *denotes the set of points with at most* k *distinct coordinates. In particular is* f *is nonnegative, if and only if*

$$f^\lambda := f\left(\underbrace{X_1, \dots, X_1}_{\lambda_1 - times}, \dots, \underbrace{X_k, \dots, X_k}_{\lambda_k - times}\right)$$

is nonnegative for every partition λ *of* n *into* k *parts.*

This result is due to the unique representation of symmetric polynomials in terms of the first d elementary seymmetric polynomials introduced at the end of Section 1.1 and due to especially nice geometric and combinatorial properties of the real orbit space of the symmetric group.

There is a canonical identification of the orbit space $\mathbb{R}^n/S_n$ with the *canonical Weyl chamber*

$$\mathcal{W}_n := \{x \in \mathbb{R}^n \mid x_1 \leq \cdots \leq x_n\}.$$

So the Vieta map induces a bijection between $\mathcal{W}_n$ and the set of hyperbolic polynomials $\mathcal{H}$. Arnold, Giventhal, Kostov, Meguerditchian and Riener studied hyperbolic polynomials with fixed k coordinates or - equivalently - Vandermonde varieties.

Definition 1.5.2. For $k \in \{1, \ldots, n\}$ and $a \in \mathbb{R}^k$, we call

$$\mathcal{V}_k(a) := \{x \in \mathbb{R}^n \mid e_1(x) = a_1, \ldots, e_k(x) = a_k\}$$

the **Vandermonde variety** of a.

Arnold showed in [Arn86] that the Vieta map also maps Vandermonde varieties homeomorphically onto their images. The images of Vandermonde varieties under the Vieta map are one instance of *hyperbolic slices*.

Definition 1.5.3. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map and $a \in \mathbb{R}^k$. Then

$$\mathcal{H}_L(a) := \mathcal{H} \cap L^{-1}(a)$$

is called a *hyperbolic slice*. If L is the projection to the first k coordinates, then we call $\mathcal{H}_k(a) := \mathcal{H}_L(a)$ a *canonical* hyperbolic slice. In this case $\mathcal{H}_k(a)$ is the image of the Vandermonde variety $\mathcal{V}_k(a)$ under the real Vieta map $\nu_{\mathbb{R}}$.

One can think of a canonical hyperbolic slice as the set of hyperbolic polynomials $\mathcal{H}$ with fixed first k coefficients. We want to distinguish polynomials by the order and multiplicities of the roots, which we will encode by using compositions.

Definition 1.5.4. A sequence of positive integers $\mu = (\mu_1, \ldots, \mu_l)$ which sum up to n is called a **composition of** n **into** l **parts** and we call $\ell(\mu) := l$ the **length** of λ. Furthermore, we say μ is **alternate odd** if $\mu_l = \mu_{l-2} = \cdots = 1$ and **alternate even** if $\mu_{l-1} = \mu_{l-3} = \cdots = 1$. For $x \in \mathcal{W}_n$ we denote by $c(x)$ the multiplicity composition of x and for a hyperbolic polynomial f with roots $(x_1, \ldots, x_n) \in \mathcal{W}_n$ we write $c(f) := c(x)$.

Meguerditchian [Meg92] proved the following based on works of Arnold [Arn86] and Kostov [Kos89].

Theorem 1.5.5. *A non-empty Vandermonde variety $\mathcal{V}_k(a)$ has exactly one minimizer (maximizer) $x \in \mathcal{W}_n$ of the $(k+1)$-th Newton power sum $p_{k+1} := X_1^{k+1} + \cdots + X_n^{k+1}$. A point $x \in \mathcal{W}_n$ is a minimizer (maximizer) if and only if $c(x)$ is smaller or equal to an alternate odd (respectively even) composition of length k.*

In particular, if one minizes (maximizes) the first free coefficient in a canonical hyperbolic slice, one obtains a polynomial with at most k distinct roots. This holds also for optimization in other directions as Riener showed in [Rie12]. In particular, his theorem can be reformulated in the following way.

Theorem 1.5.6. *The extreme points of the convex hull of a canonical hyperbolic slice $\mathcal{H}_k(a)$ correspond to polynomials with at most k distinct roots and the canonical hyperbolic slice $\mathcal{H}_k(a)$ is compact for $k \geq 2$.*

These results allow for a simple proof of Timofte's half-degree principle as presented in [Rie12]. Since proofs of some of the results in the Papers I, II and III are based on ideas of Riener's proof, we include it here:

Proof of the half-degree principle 1.5.1. $\inf_{x\in\mathbb{R}^n} f(x) \leq \inf_{x\in\mathcal{A}_k} f(x)$ is clear. To show "$\geq$", let $x \in \mathbb{R}^n$. By Gauss' fundamental theorem of symmetric polynomials, $f = g(e_1, \dots, e_d)$ for some $g \in \mathbb{R}[Z_1, \dots, Z_d]$ which is linear in $Z_{k+1}, \dots, Z_d$. Since g is linear on the compact canonical hyperbolic slice

$$\mathcal{H}_k(e_1(x), \dots, e_k(x)),$$

there is a minimizer z of g, which is an extreme point of the convex hull of the canonical slice and corresponds therefore to a polynomial with at most k distinct roots by Theorem 1.5.6. ∎

As a direct corollary one obtains also Timofte's degree principle.

Theorem 1.5.7 (degree principle)**.** *Let $f_1, \dots, f_m \in \mathbb{R}[\underline{X}]$ be symmetric polynomials of degree at most $d \leq n$. Then*

$$V(f_1, \dots, f_m) \neq \emptyset \iff V(f_1, \dots, f_m) \cap \mathcal{A}_d \neq \emptyset,$$

where $\mathcal{A}_d \subseteq \mathbb{R}^n$ denotes the set of points with at most d distinct coordinates.

Proof. Consider the sum of squares $f := \sum_{i=1}^m f_i^2$ of degree $2d$ and use the half-degree principle. ∎

Note that the polynomial f in the proof can already be written by using only the first d elementary symmetric polynomials, i.e. $f = g(e_1, \dots, e_d)$ for some $g \in \mathbb{R}[Z_1, \dots, Z_d]$. This means that g is even constant on every canonical hyperbolic slice $\mathcal{H}_d(a)$ and not only linear as used in the proof of the half-degree principle. In particular, one needs only that every canonical hyperbolic slice $\mathcal{H}_d(a)$ contains a bit with at most d distinct roots.

1.6 Summary of Papers

Paper I Shellable slices of hyperbolic polynomials and the degree principle

One might try to strengthen the degree principle 1.5.7 by not considering all points with at most d distinct coordinates, but only points with specific orbit types. These orbit types correspond to partitions of n into d parts and we call such a set of partitions a (n, d)-*Vandermonde covering.* By inspecting the proof of the degree principle and by taking into account the remark at the end of Section 1.5, a subset $\mathcal{P}$ of the set of partitions of n into d parts is a Vandermonde covering, if and only if every generic canonical hyperbolic slice $\mathcal{H}_d(a)$ contains a polynomial with root multiplicity corresponding to a partition in P. Theorem 1.5.5 shows that the set of partitions corresponding to an alternate odd composition of length d is a Vandermonde covering.

In order to use this characterization of Vandermonde coverings to improve the degree principle even further, one needs to understand which compositions and therefore, which partitions can appear in a canonical hyperbolic slice. To this end we stratify canonical hyperbolic slices with respect to their root multiplicities. To state this stratification we define a partial order $\leq$ on the set of compositions of n, by $\mu \leq \lambda$ if there is a composition ν of $\ell(\lambda)$ of length $l = \ell(\mu)$ such that

$$\mu = (\lambda_1 + \cdots + \lambda_{\nu_1}, \ldots, \lambda_{\ell(\lambda)-\nu_l+1} + \cdots + \lambda_{\ell(\lambda)}).$$

In other words $\mu \leq \lambda$ if one can obtain μ from λ by replacing some of the commas in λ with plus signs.

Definition 1.6.1. Let μ be a composition of n and $\mathcal{H}_k(a)$ a canonical hyperbolic slice. We define the stratum

$$\mathcal{H}_k^{\mu}(a) := \{z \in \mathcal{H}_k(a) \mid c(T^n - z_1T^{n-1} + \cdots \pm z_n) \leq \mu\}$$

and we call the poset of strata of $\mathcal{H}_k(F)$, partially ordered by inclusion, a **hyperbolic poset** and denote it by $\mathcal{L}_k(F)$.

We show that these *hyperbolic posets* share some nice properties with the face lattice of a polytope: The boundary complex of the dual of a generic canonical hyperbolic slice is a shellable simplicial complex and therefore a combinatorial sphere. From this, we obtain the same bounds and relations on the number of i-dimensional strata as for polytopes. Namely, we obtain a "g-theorem" for generic canonical hyperbolic slices and an "upper bound theorem" for the general case.

Corollary 1.6.2 (Upper Bound Theorem)**.** *Let $(f_0, \ldots, f_{n-s})$ be the f-vector of $\mathcal{L}_s(F)$. If c_i is the number of i-dimensional faces of the $(n-s)$-dimensional cyclic polytope with f_{n-s-1} vertices then*

$$f_{n-s-i} \leq c_{i-1} \ \forall \ i \in [n-s].$$

In particular, this gives a bound on the number f_0 of polynomials with compositions of length at most k appearing in $\mathcal{H}_k(a)$ and therefore a bound on the number of extreme points of the convex hull of $\mathcal{H}_k(a)$:

$$f_0 \leq \begin{cases} \binom{(n+s)/2-1}{s-1} + \binom{(n+s)/2-2}{s-1}, & \text{if } n-s \text{ is even} \\ 2\binom{(n+s-3)/2}{s-1}, & \text{if } n-s \text{ is odd} \end{cases}.$$

A central tool to prove these results is a generalization of Theorem 1.5.5 of Meguerditchian and Arnold to strata of canonical hyperbolic slices:

Theorem 1.6.3. *A non-empty stratum $\mathcal{H}_k^\mu(a)$ of a canonical hyperbolic slice has a unique polynomial with minimal (maximal) first free coefficient. A polynomial $h \in \mathcal{H}_k^\mu(a)$ has minimal (maximal) first free coefficient if and only if $c(h)/\mu$ is smaller or equal to an alternate odd (respectively even) composition of length k, where $c(h)/\mu$ is the composition λ, such that*

$$c(h) = (\mu_1 + \cdots + \mu_{\lambda_1}, \ldots, \mu_{\nu_{l-1}+1} + \cdots + \mu_{\lambda_l}).$$

This theorem implies a strong condition on the compositions that can appear in a hyperbolic poset and gives rise to our definition of a *potential hyperbolic poset*.

Definition 1.6.4. Let S be a set of compositions of n of length k. We call the upward closure of S

$$\mathcal{L}(S) := \{\lambda \mid \text{there is a } \mu \in S \text{ with } \mu \leq \lambda\} \cup \{(n)\}$$

the **poset** of S. We say that $\mathcal{L}(S)$ is a **potential hyperbolic poset**, if for every $\lambda \in \mathcal{L}(S)$ there are unique $\mu_{\min}, \mu_{\max} \in S$, such that

1. $\mu_{\min}/\lambda$ is alternate odd and
2. $\mu_{\max}/\lambda$ is alternate even.

Furthermore, we say that $\mathcal{L}(S)$ is a **realizable hyperbolic poset**, if it is isomorphic to a hyperbolic poset.

By computing for fixed small n and d all potential hyperbolic posets, we can find smaller Vandermonde coverings. By realizing some potential hyperbolic slices, we can show that those Vandermonde coverings are actually optimal. In particular, if our conjecture that all potential hyperbolic posets are realizable holds, then this approach always gives optimal Vandermonde coverings. Furthermore, we show that the size of a (n,d)-Vandermonde covering is bounded from below by $\left\lceil \frac{2M}{\lceil \frac{d-1}{2} \rceil \lceil \frac{d+1}{2} \rceil} \right\rceil$, where M denotes the number of partitions of $n - \lceil \frac{d}{2} \rceil$ into $\lfloor \frac{d}{2} \rfloor$ many parts.

Paper II Linear slices of hyperbolic polynomials and positivity of symmetric polynomial functions

Instead of strengthening the degree principle by studying canonical hyperbolic slices, one can also generalize it by studying general hyperbolic slices, i.e. we fix arbitrary linear combinations of coefficients instead of the first. To this end, we generalize Theorem 1.5.6, which is the main tool for the proof of the degree and half-degree principle, to general hyperbolic slices. Here, we show that the local extreme points of general hyperbolic slices of codimension k correspond to polynomials with at most k distinct roots. Furthermore, we give criteria for the existence of local extreme points and for compactness of a hyperbolic slice.

Those results are then applied in a similar way as in the proof of the degree and half-degree principle 1.5.7 and 1.5.1 to show that symmetric polynomials that can be written as polynomials in few linear combinations of elementary symmetric polynomials obtain all their values already on points with few distinct coordinates.

Definition 1.6.5. Let $l_1, \ldots, l_k \in \mathbb{R}[Z_1, \ldots, Z_n]$ be linearly independent linear forms. Furthermore, let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial and write f in terms of elementary symmetric polynomials, say $f = g(e_1, \ldots, e_n)$ for some $g \in \mathbb{R}[Z_1, \ldots, Z_n]$.

1. We say that f is $(l_1, \ldots, l_k)$**-sufficient** if $g \in \mathbb{R}[l_1, \ldots, l_k]$.
2. We say that f is $(l_1, \ldots, l_k)$**-quasi-sufficient** if f admits a representation of the form
$$f = f_0 + f_1 e_1 + \cdots + f_n e_n$$
for some $(l_1, \ldots, l_k)$-sufficient polynomials $f_0, \ldots, f_n$.

Moreover, we say that a symmetric semi-algebraic set $S \subseteq \mathbb{R}^n$ is $(l_1, \ldots, l_k)$**-sufficient**, if it can be described by $(l_1, \ldots, l_k)$-sufficient polynomials.

With this notion, Gauss fundamental theorem of symmetric polynomials says, that symmetric polynomials of degree d are $(Z_1, \ldots, Z_d)$-sufficient and $(Z_1, \ldots, Z_{\lfloor \frac{d}{2} \rfloor})$-quasi-sufficient. In this light, the following theorem is a generalization of the degree and half-degree principle.

Theorem 1.6.6. *Let $S \subseteq \mathbb{R}^n$ be a symmetric $(l_1, \ldots, l_k)$-sufficient semi-algebraic set and let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial.*

1. *If f is $(l_1, \ldots, l_k)$-sufficient, then $f(S) = f(S \cap \mathcal{A}_{k+1})$.*
2. *If f is $(l_1, \ldots, l_k)$-quasi-sufficient, then* $\inf_{x \in S} f(x) = \inf_{x \in S \cap \mathcal{A}_{k+2}} f(x)$.

Furthermore, we get very similar results by the same techniques for even symmetric polynomials, i.e. polynomials that are invariant under the natural action of the hyperoctahedral group B_n.

We use these results to provide new proofs for several known inequalities, such as the Maclaurin's inequality. Furthermore, as mentioned in Section 1.3, checking whether a polynomial is non-negative is difficult, while checking if it is a sum of squares can be done by semi-definite programming and is therefore relatively easy. Obviously, checking non-negativity of homogeneous polynomials, which take their infimum on points with at most 2 distinct coordinates, can be done by checking if the corresponding binary forms are non-negative. This can be done by semi-definite programming, since every binary non-negative form is a sum of squares by Hilbert's result in 1888. Such convex sets which are projections of feasibility regions of semi-definite programs are also called **spectrahedral shadows**. Here, our results give new families of cones of (even) symmetric non-negative homogeneous polynomials which are spectrahedral shadows.

Proposition 1.6.7. *Let $\mathcal{P}_{2d}$ denote the convex cone of positive semi-definite n-ary forms of degree $2d$ and $2 \leq j \leq n$. Then, the subcones of all (Z_1, Z_j)-sufficient and (Z_1, Z_2)-quasi-sufficient symmetric forms are spectrahedral shadows. Similarly, the subcone of all (Z_1, Z_j)-quasi-sufficient even-symmetric forms is a spectrahedral shadow.*

Paper III **Stable and Hurwitz slices, a degree principle and a generalized-Walsh-Szegő theorem** The theory of hyperbolic slices can be even more generalized to stable slices: If $C \subseteq \mathbb{C}$ is a *circular domain*, i.e. a disk, the complement of a disk or an affine half-plane, then a univariate polynomial is called *C-stable* if all its roots lie in C. Note that hyperbolic polynomials are just polynomials that are stable with respect to the *upper half-plane* $\mathbb{H}_+$. We fix $n \in \mathbb{N}$ and denote the set of monic C-stable polynomials of degree n by $\S_C$. For the upper half-plane we write $\S := \S_{\mathbb{H}_+}$

Let $L : \mathbb{C}^n \to \mathbb{C}^k$ be a surjective linear map, $a \in \mathbb{C}^k$ and $\mathbb{H} \subset \mathbb{C}$ a half-plane. Analogous to Definition 1.5.3 we call

$$\S_{\mathbb{H}} \cap L^{-1}(a)$$

an $\mathbb{H}$-*stable slice* and if L is the projection to the first k coordinates, we call the slice *canonical*.

Similar to Paper II, we show that the local extreme points of an upper half-plane stable slice $\S \cap L^{-1}(a)$ correspond to polynomials with at most $2k$ distinct real roots and at most k roots in the open upper half-plane. Here we can replace $2k$ by k for canonical upper half-plane stable slices. If L fixes the first two coefficients, then we get again that $\S \cap L^{-1}(a)$ is compact. By using Möbius transformations, we get the following for general half-plane stable slices.

Theorem 1.6.8. *Every non-empty $\mathbb{H}$-stable slice $\S_{\mathbb{H}} \cap L^{-1}(a)$ contains a polynomial with at most $2(k+2)$ distinct roots on the boundary of $\mathbb{H}$ and at most $k+2$ roots in the interior of $\mathbb{H}$.*

We also study affine slice of Hurwitz polynomials, i.e. real left half-plane stable polynomials and show that also here the local extreme points of these *Hurwitz slices* correspond to polynomials with at most $2k$ distinct roots on the boundary of the left half-plane and at most k roots in the open left half-plane. Similar to Paper I, we introduce a possible stratification of canonical Hurwitz slices in terms of root multiplicities.

The results on stable slices are then used in the same way as in Paper II to investigate symmetric polynomials, that can be written as polynomials in few linear combinations of elementary symmetric polynomials. Here we show that such $(l_1, \ldots, l_k)$-sufficient polynomials have a common zero with all coordinates in some half-plane, if and only if they have one with few distinct coordinates.

Theorem 1.6.9. *Let $f_1, \ldots, f_s \in \mathbb{C}[X_1, \ldots, X_n]$ be symmetric $(l_1, \ldots, l_k)$-sufficient polynomials. Then*

$$V(f_1, \ldots, f_s) \cap \mathbb{H}^n = \emptyset \iff V(f_1, \ldots, f_s) \cap \mathbb{H}_{2(k+2),k+2} = \emptyset,$$

where $\mathbb{H}_{2(k+2),k+2} \subset \mathbb{H}^n$ denotes the set of points with at most $2(k+2)$ distinct coordinates on the boundary of $\mathbb{H}$ and at most $k+2$ coordinates in the interior of $\mathbb{H}$.

This can be seen as a generalization of the degree principle.

Corollary 1.6.10 (degree principle for half-planes)**.** *Let $f_1, \ldots, f_s \in \mathbb{C}[X_1, \ldots, X_n]$ be symmetric polynomials of degree at most $d \leq n$. Then*

$$V(f_1, \ldots, f_s) \cap \mathbb{H}^n = \emptyset \iff V(f_1, \ldots, f_s) \cap \mathbb{H}_{2(d+2),d+2} = \emptyset.$$

If $\mathbb{H}$ is the upper half-plane, one can replace $2(d+2)$ and $d+2$ by d in Corrolary 1.6.10. Theorem 1.6.9 generalizes also another classical result: Grace-Walsh-Szegő's coincidence theorem [Gra02; Sze22; Wal22] for half-planes states that if f is a symmetric and multiaffine, i.e. l-sufficient, polynomial and $(x_1, \ldots, x_n) \in \mathbb{H}^n$, then there is a $x \in \mathbb{H}$ such that $f(x_1, \ldots, x_n) = f(x, \ldots, x)$.

Corollary 1.6.11 (Generalized Grace-Walsh-Szegő's coincidence theorem)**.** *Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric and $(l_1, \ldots, l_k)$ sufficient polynomial and $x \in \mathbb{H}^n$. Then there is $y \in \mathbb{H}_{2(k+2),k+2}$ with $f(x) = f(y)$.*

Furthermore one can give an analogue of the half-degree principle for minimizing (maximizing) real or imaginary parts of symmetric polynomials on the upper half-plane.

Theorem 1.6.12 (Half-degree principle for the upper half-plane)**.** *Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric polynomial of degree $d \leq n$ and $\lambda, \mu \in \mathbb{R}$. Then*

$$\inf_{x \in \mathbb{H}_+^n} \lambda \Re(f(x)) + \mu \Im(f(x)) = \inf_{x \in \mathbb{H}_{+\,k,k}} \lambda \Re(f(x)) + \mu \Im(f(x)),$$

where $k = \max\{\lfloor \frac{d}{2} \rfloor, 2\}$.

Paper IV **Constructively describing orbit spaces of finite groups by few inequalities** Here we want to revisit the more general setting of a finite group G and their real orbit space from Section 1.4. So denote again the generators of $\mathbb{R}[\underline{X}]^G$ be $\pi_1, \ldots, \pi_m$ and denote by Π the Hilbert map.

Here we usually view points $x \in V_{\mathbb{R}}(I_\Pi)$ as homomorphisms $\phi_x : \mathbb{R}[\underline{X}]^G \to \mathbb{R}, g(\Pi) \mapsto g(x)$ and therefore finding a preimage of x under the Hilbert map corresponds to extending ϕ_x to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}] \to \mathbb{R}$. This allows us to extend homomorphisms stepwise to invariant rings of subgroups of G.

We say that the orbit space $\mathbb{R}^n/G$ is *described* by $f_1, \ldots, f_k \in \mathbb{R}[\underline{X}]^G$, if

$$\Pi(\mathbb{R}^n) = \{x \in V_{\mathbb{R}}(I_\Pi) \mid \phi_x(f_1) \geq 0, \ldots, \phi_x(f_1) \geq 0\}.$$

We say $\mathbb{R}^n/G$ is *generically described* by $f_1, \ldots, f_k$, if

$$\Pi(\mathbb{R}^n) = \{x \in V_{\mathbb{R}}(I_\Pi) \mid \phi_x(f_1) > 0, \ldots, \phi_x(f_1) > 0\} \cup T,$$

where $\dim(T) < \dim(\Pi(\mathbb{R}^n))$.

First we give an elementary proof of the fact that the image of the real Hilbert map is basic closed semialgebraic, by giving a new explicit description of the real orbit space: $\mathbb{R}[\underline{X}]$ is integral over $\mathbb{R}[\underline{X}]^G$ and therefore a finitely generated $\mathbb{R}[\underline{X}]^G$-module, say $b_1, \ldots, b_m \in \mathbb{R}[\underline{X}]$ are generators of $\mathbb{R}[\underline{X}]$ over $\mathbb{R}[\underline{X}]^G$. Then

$$\Pi(\mathbb{R}^n) = \{z \in V_{\mathbb{R}}(I_\Pi) \mid \phi_z(B) \succeq 0\},$$

where $B \in (\mathbb{R}[\underline{X}]^G)^{m \times m}$ is defined by $B_{ij} := R_G(b_i b_j)$.

We also give a new short proof of the theorem of Procesi and Schwarz using techniques from real algebraic geometry: Let $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ be a homomorphism. First we note that ϕ can be extended to $\mathbb{R}[\underline{X}]$, if and only if ϕ is nonnegative on all G-invariant sum of squares. The theorem of Procesi and Schwarz states that ϕ can be extended to $\mathbb{R}[\underline{X}]$, if and only if $\phi(\langle \nabla f, \nabla f \rangle) \geq 0$ for all G-invariant f, i.e. it suffices to check that ϕ is nonnegative on some invariant sum of squares. It is very easy to see that those statements are the same for the cyclic group $G = C_2$ and we reduce the general case to this case: If ϕ can not be extended to $\mathbb{R}[\underline{X}]$, then G has a cyclic subgroup C of even order (generated by a complex conjugation of ϕ) such that ϕ can be extended to $\mathbb{R}[\underline{X}]^C$ but not to $\mathbb{R}[\underline{X}]^{C_2}$.

Our description of the orbit space and also the one given by Procesi and Schwarz 1.4.1 are explicit but need sometimes way too many inequalities, e.g. for odd groups the Hilbert map is surjective. The description by Bröcker 1.4.2 needs fewer inequalities but is not constructive. Our aim in the last part of the article is to construct few inequalities describing the orbit space.

We also answer a question raised by Bröcker: We find examples of orbit spaces where the lower dimensional T (see Theorem 1.4.2) is really needed:

Example 1.6.13. Consider a cyclic group C_m acting on $\mathbb{R}^n$ and write $m = 2^k q$, where q is odd. Furthermore consider the cyclic subgroups

$$\{\mathrm{id}\} = C_1 \subset C_2 \subset C_4 \subset \cdots \subset C_{2^k} \subset C_m$$

and the C_m-invariant polynomials

$$f_i := R_{C_m}\left(\sum_{j=1}^{n}\left(R_{C_{2^{i-1}}}(X_j) - R_{C_{2^i}}(X_j)\right)^2\right)$$

Then the orbit space $\mathbb{R}^n/C_m$ is generically described by f_1. Furthermore, $\mathbb{R}^n/C_m$ is described by $f_1, \ldots, f_k$ and can in general not be described by less than k polynomials.

We also give a generic description of the orbit space $\mathbb{R}^n/G$ when $k = 1$ in Bröcker's Theorem 1.4.2.

Theorem 1.6.14. *Let G be a group such that all maximal elementary abelian 2-subgroups of G are of order 2. Then $\mathbb{R}^n/G$ is generically described by some $g \in \mathbb{R}[\underline{X}]^G$.*
More precisely, if $|G| = q2^l$ with q odd and

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_l \subseteq G$$

with $|H_i| = 2^i$, then one can choose

$$g = \prod_{\sigma H_l \in G/H_l}\left(\sigma R_{H_l}\left(\sum_{i=1}^{n}\left(X_i - R_{H_1}(X_i)\right)^2\right)\right).$$

Furthermore, we give a constructive proof of Bröcker's theorem for abelian groups.

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Papers

Paper I

I

Shellable slices of hyperbolic polynomials and the degree principle

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Abstract

We study a natural stratification of certain affine slices of univariate hyperbolic polynomials. We look into which posets of strata can be realized and show that the dual of the poset of strata is a shellable simplicial complex and in particular a combinatorial sphere. From this we obtain a g-theorem and an upper bound theorem on the number of strata. We use these results to design smaller test sets to improve upon Timofte's degree principle and give bounds on how much the degree principle can be improved.

Univariate polynomials with only real roots are called **hyperbolic polynomials**. We will study **canonical hyperbolic slices**, that is, sets of hyperbolic polynomials that share the same first few coefficients. We stratify these sets in terms of the arrangements and multiplicities of the roots of the hyperbolic polynomials and then we study the combinatorial structure of the poset of strata and its implications for the study of real symmetric varieties.

These canonical hyperbolic slices have a rich geometric structure that has been studied by several authors. For instance, [Arn86], [Giv87] and [Kos89] studied **Vandermonde varieties**, that is, varieties given by the first few weighted power sums. By Newton identities, varieties given by the first few elementary symmetric polynomials are a special case of Vandermonde varieties. Since the set of monic hyperbolic polynomials can be viewed as the orbit space of the symmetric group and the elementary symmetric polynomials generate the ring of symmetric polynomials, Vandermonde varieties are the fibers of canonical hyperbolic slices. Thus, canonical hyperbolic slices are deeply connected to Vandermonde varieties.

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More generally, canonical hyperbolic slices are not just connected to Vandermonde varieties, but to the study of any symmetric variety. In [Rie12] and [RS24] this connection is exploited to prove and generalize Timofte's degree and half-degree principle for the symmetric group. The degree principle implies that symmetric polynomials of degree at most d have a common real root if and only if they have a common real root with at most d distinct coordinates, thus it allows one to show nonemptyness of symmetric varieties much faster than arbitrary varieties. We study the poset of strata of canonical hyperbolic slices in order to make improvements on this degree principle.

The poset of strata of canonical hyperbolic slices was already studied by the first author in [Lie23] and the question was raised if it is polytopal. We are able to show the weaker statement in Theorem I.3.6 that the dual of the poset of strata is generically a shellable simplicial complex. To prove this, we will first prove a generalization of a result by Arnold [Arn86] and Meguerditichian [Meg92]. They show that every canonical hyperbolic slice has a unique minimal and maximal polynomial with respect to the first free coefficient and that these polynomials are generically uniquely characterized by alternating single and multiple roots. We show in Theorem I.2.12 that an analogous result is true for every stratum of a canonical hyperbolic slice. Then we use this to show that the dual poset is generically a shellable simplicial complex and therefore a combinatorial sphere (Corollary I.3.7). From this, we obtain the same bounds and relations on the number of i-dimensional strata as for certain polytopes. Namely, we obtain a “g-theorem" (Corollary I.3.11) for generic canonical hyperbolic slices and an “upper bound theorem" (Corollary I.3.16) for the general case.

With the connection between canonical hyperbolic slices and real symmetric varieties, we can use these combinatorial results to improve upon Timofte's degree principle. The degree principle allows one to show nonemptyness of real symmetric varieties by reducing the number of variables needed to the minimal amount, and so our improvement lies in reducing the number of orbit types needed to check. Thus we improve the degree principle by considering test sets that have smaller sizes. These test sets, which we call **Vandermonde coverings**, are therefore characterized by certain orbit types of the symmetric group. We give a lower and an upper bound on the size of an optimal Vandermonde covering (Theorem I.4.4 and I.4.8) and outline a computational approach on how to get better and maybe optimal Vandermonde coverings for real symmetric varieties given by polynomials in few variables and low degrees.

We conclude with several open questions and conjectures on the stratification of canonical hyperbolic slices.

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I.1 Preliminaries

I.1.1 Simplicial complexes, shellings and spheres

Definition I.1.1. A **poset** $(P, \leq)$, or **partially ordered set**, is a set P equipped with a partial order $\leq$.

We usually just write P if the partial order is clear from context. Also, we say that an element a, of a poset, P, **covers** $b \in P$ if $b \leq a$ and for any $c \in P$ with $b \leq c \leq a$, we have $c = a$ or $c = b$. So we see that the partial order on P is **generated** by its covering relations in the following sense: let $a, b \in P$, then $a \leq b$ if there is a sequence of elements $c_1, \ldots, c_m$ with $c_1 = a$, $c_m = b$ and where c_i is covered by c_{i+1} for any $i \in [m-1]$.

Next, we say that two posets $(P, \leq)$ and $(Q, \leq^*)$ are **isomorphic** if there exists an order-preserving bijection between P and Q. Lastly, we say that the poset $(P, \geq)$ is the **dual** poset of $(P, \leq)$.

An important subclass of posets are simplicial complexes.

Definition I.1.2. A **simplicial complex** is a family of finite sets that is closed under taking subsets. A **geometric simplicial complex** is a family of simplices, S in $\mathbb{R}^m$, such that each face of a simplex in S is also in S and such that the intersection of two simplices is a face of each simplex.

Thus any simplicial complex may be identified with a family, C, of subsets of $[m] := \{1, 2, \ldots, m\}$ for some nonnegative integer m, such that if $A \subset B \in C$, then $A \in C$. A **geometric realization** of C is a geometric simplicial complex S whose poset of simplices is isomorphic to C. Since all simplicial complexes have a geometric realization, we will usually not distinguish between a geometric realization and the simplicial complex. Instead, it should always be clear from the context which object we are referring to.

We can construct a geometric realization of a simplicial complex C by identifying the smallest nonempty sets of C with the points $e_1, \ldots, e_m \in \mathbb{R}^m$, where e_i is the i-th standard basis vector, and then take the convex hull of $e_{i_1}, \ldots, e_{i_k}$ whenever $\{i_1, \ldots, i_k\}$ is an element of C.

Just like for a geometric realization of C, the elements of C are called **faces**. The **dimension** of a face is defined as the dimension of the corresponding face in a geometric realization and the dimension of C is the dimension of its highest-dimensional faces. Also, the faces that are maximal with respect to inclusion are called **facets**, the second largest are called **ridges** and the smallest nonempty faces are called **vertices**. When all the facets have the same dimension, the simplicial complex is called **pure**.

We also need this natural generalization of a geometric simplicial complex:

Definition I.1.3. A **polytope complex** is a family of polytopes C, in $\mathbb{R}^m$, such that each face of a polytope is in C and such that the intersection of two polytopes is a face of each.

As with simplicial complexes, we will usually not distinguish between a polytope complex and its abstract poset of polytopes.

Lastly, we need to talk about a particular class of polytope complexes that are similar to spheres from a combinatorial point of view. Note that a **simplicial sphere** is a geometric simplicial complex which is homeomorphic to a sphere. But showing that a simplicial complex is a simplicial sphere can be difficult and thus we introduce the so-called “combinatorial spheres".

Definition I.1.4. A **subdivision** of a polytope complex C is a polytope complex S such that

$$\bigcup_{I\in S} I = \bigcup_{J\in C} J \subset \mathbb{R}^m$$

and such that each face of S is contained in a face of C. Moreover, we say a subdivision S is **simplicial** if S is a geometric simplicial complex.

Definition I.1.5. A **combinatorial (or PL) m-sphere** is a polytope complex for which there exists a simplicial subdivision which is isomorphic to a simplicial subdivision of the boundary of a $(m+1)$-dimensional simplex.

To determine if a simplicial complex is a combinatorial sphere, we need the notion of shellability.

Definition I.1.6. A **shelling** of a pure simplicial complex, C, is an ordering of the facets, $F_1, \dots, F_k$, such that for any $i \in \{2, \dots, k\}$, the simplicial complex

$$\bigcup_{j=1}^{i-1} F_j \cap F_i$$

is pure of dimension $\dim(C) - 1$. If there exists a shelling of C, then C is called **shellable**.

Then from Proposition 1.2 in [DK74] we have the following result:

Proposition I.1.7. *A shellable simplicial complex of dimension m, whose ridges are all contained in exactly two facets, is a combinatorial m-sphere.*

I.1.2 Symmetric polynomials and Vandermonde varieties

Throughout the article, we denote by $\mathcal{S}\mathrm{ym}(n)$ the symmetric group on the set $[n]$, $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$ the polynomial ring in n variables over $\mathbb{R}$ and by $\mathbb{R}[\underline{X}]^{\mathcal{S}\mathrm{ym}(n)}$ the subring of symmetric polynomials.

Definition I.1.8. For $i \in [n]$, we denote by

$$E_i := \sum_{1\leq j_1 < \cdots < j_i \leq n} X_{j_1} \cdots X_{j_i}$$

the i-th **elementary symmetric polynomial** and by

$$P_i := \sum_{j=1}^{n} X_j^i$$

the i-th **power sum**.

The Fundamental Theorem of Symmetric Polynomials states, that every polynomial can be uniquely written in terms of the elementary symmetric polynomials. Furthermore, we have the following:

Theorem I.1.9 (Fundamental Theorem of Symmetric Polynomials)**.** *Any symmetric polynomial $F \in \mathbb{R}[\underline{X}]^{\mathcal{S}ym(n)}$ of degree s, with $s \leq n$, can be uniquely written as*

$$H = G(E_1, \dots, E_s),$$

where G is a polynomial in $\mathbb{R}[Z_1, \dots, Z_s]$.

Proof. Proposition 2.3 in [Rie12]. ∎

Theorem I.1.9 is a key tool in the proof of the degree principle in [Rie12].

Theorem I.1.10 (Degree principle)**.** *Let $f_1, \dots, f_k \in \mathbb{R}[\underline{X}]^{\mathcal{S}ym(n)}$ be symmetric polynomials of degree at most $d < n$. Then the real variety*

$$V_{\mathbb{R}}(f_1, \dots, f_k)$$

is nonempty if and only if it contains a point with at most d distinct coordinates.

Definition I.1.11. A sequence of positive integers $\mu = (\mu_1, \dots, \mu_l)$ which sum up to n is called a **composition of n into l parts** and we call $\ell(\mu) := l$ the **length** of λ.

Next, we introduce Vandermonde varieties and the Weyl chamber:

Definition I.1.12. For $s \in [n]$ and $a \in \mathbb{R}^s$, we call

$$\mathcal{V}(a) := \{x \in \mathbb{R}^n \mid -E_1(x) = a_1, \dots, (-1)^s E_s(x) = a_s\}$$

the **Vandermonde variety** of a. For a monic polynomial

$$F = T^n + F_1 T^{n-1} + \cdots + F_n$$

and $s \leq n$, we define

$$\mathcal{V}_s(F) := \mathcal{V}(F_1, \dots, F_s).$$

Furthermore, for a composition μ of n and a polynomial $Q \in \mathbb{R}[\underline{X}]$ we define

$$Q^\mu := Q^\mu(\underbrace{X_1, \dots, X_1}_{\mu_1-\text{times}}, \underbrace{X_2, \dots, X_2}_{\mu_2-\text{times}}, \dots, \underbrace{X_s, \dots, X_s}_{\mu_s-\text{times}}) \in \mathbb{R}[X_1, \dots, X_l]$$

and

$$\mathcal{V}_s^\mu(F) := \{x \in \mathbb{R}^l \mid (-1)^i E_i^\mu(x) = F_i \ \forall\, i \in [s]\}$$

the **Vandermonde variety** of F with respect to μ and s.

Note that in [Arn86] Vandermonde varieties are defined by the first few (weighted) power sums instead, which in the case of integer weights is equivalent to our definition by Newton identities.

Definition I.1.13. For $l \in \mathbb{N}$, we denote by

$$\mathcal{W}_l := \left\{ x \in \mathbb{R}^l \;\middle|\; x_1 \leq \cdots \leq x_l \right\}$$

the l-dimensional **Weyl chamber**.

I.2 Canonical hyperbolic slices and posets

Throughout the article, we will denote by $\mathcal{H} \subset \mathbb{R}[T]$ the set of monic hyperbolic polynomials, that is, the monic polynomials with only real roots. Furthermore, we fix a monic hyperbolic polynomial $F \in \mathcal{H}$ of degree $n \in \mathbb{N}$ and an integer $s \in \mathbb{N}$, with $s \leq n$. Then the sets of hyperbolic polynomials that we will study are the following.

Definition I.2.1. We call the affine slice

$$\mathcal{H}_s(F) = \{T^n + H_1 T^{n-1} + \cdots + H_n \in \mathcal{H} \mid H_i = F_i \;\forall\; i \in [s]\},$$

where $F = T^n + F_1 T^{n-1} + \cdots + F_n$, a **canonical hyperbolic slice** or short a *canonical slice*.

Note that the definition above is a special case of the more general definition of a hyperbolic slice in [RS24].

First, we recall some previously established results on canonical hyperbolic slices and provide examples of canonical hyperbolic slices and their stratifications. In particular, we will see that the strata are contractible and we see a characterization of the strata's relative interior and the closure of their relative interior.

Then we introduce a generalization of the main theorem in [Meg92]. In that article, they investigate the following question: for which monic hyperbolic polynomials H, of degree n, is $H + c_0 T^k + \cdots + c_k$ not hyperbolic for any $c_0, \ldots, c_k \in \mathbb{R}$ with $c_0 > 0$ (resp. $c_0 < 0$) and $k < n$? They call such polynomials "k-maximal" (resp. "k-minimal") and characterize which polynomials are k-minimal and k-maximal. Thus they characterize which polynomials in $\mathcal{H}_s(F)$ have a minimal first free coefficient and which polynomials have a maximal one. We extend this question to the strata of canonical slices and prove an analogous result.

I.2.1 Stratification of canonical hyperbolic slices

We will study a particular stratification of $\mathcal{H}_s(F)$ and in order to define this stratification, we need to introduce a partial order on compositions.

Definition I.2.2. For two compositions of n, μ and λ, we let $\mu \leq \lambda$ if there is a composition ν of $\ell(\lambda)$ of length $l = \ell(\mu)$ such that

$$\mu = (\lambda_1 + \cdots + \lambda_{\nu_1}, \ldots, \lambda_{\ell(\lambda)-\nu_l+1} + \cdots + \lambda_{\ell(\lambda)}).$$

In other words $\mu \leq \lambda$ if one can obtain μ from λ by replacing some of the commas in λ with plus signs. For a hyperbolic polynomial H with distinct roots $b_1 < \cdots < b_l$ and respective multiplicities $m_1, \ldots, m_l$ we will let $c(H) = (m_1, \ldots, m_l)$ denote the **composition of** H.

Definition I.2.3. Let μ be a composition of n. Then we define the stratum

$$\mathcal{H}_s^\mu(F) := \{H \in \mathcal{H}_s(F) \mid c(H) \leq \mu\},$$

of $\mathcal{H}_s(F)$, and we call the poset of strata of $\mathcal{H}_s(F)$, partially ordered by inclusion, a **hyperbolic poset** and denote it by $\mathcal{L}_s(F)$.

We commonly identify monic polynomials of degree n in $\mathbb{R}[T]$ with points in $\mathbb{R}^n$. Thus we will be equipping $\mathcal{H}_s^\mu(F)$ with the subspace topology of the Euclidean topology on $\mathbb{R}^n$.

Remark I.2.4. The set $\mathcal{H}$ of hyperbolic polynomials can be seen as the image of the **Vieta map**

$$\begin{array}{rcl} \mathcal{E} : \mathbb{R}^n & \longrightarrow & \mathcal{H} \\ x & \longmapsto & (-E_1(x), \ldots, (-1)^n E_n(x)). \end{array}$$

Moreover, $\mathcal{E}$ maps the Vandermonde variety intersected with the Weyl chamber $\mathcal{V}(F_1, \ldots, F_s) \cap \mathcal{W}_n$ homeomorphically (see Lemma 2.1 in [Lie23]) to the canonical slice $\mathcal{H}_s(F)$. So a stratum $\mathcal{H}_s^\mu(F)$ is homeomorphic to

$$\big\{(\underbrace{x_1, \ldots, x_1}_{\mu_1\text{-times}}, \ldots, \underbrace{x_l, \ldots, x_l}_{\mu_l\text{-times}}) \mid (x_1, \ldots, x_l) \in \mathbb{R}^l\big\} \cap \mathcal{V}(F_1, \ldots, F_s) \cap \mathcal{W}_n$$

under the Vieta map.

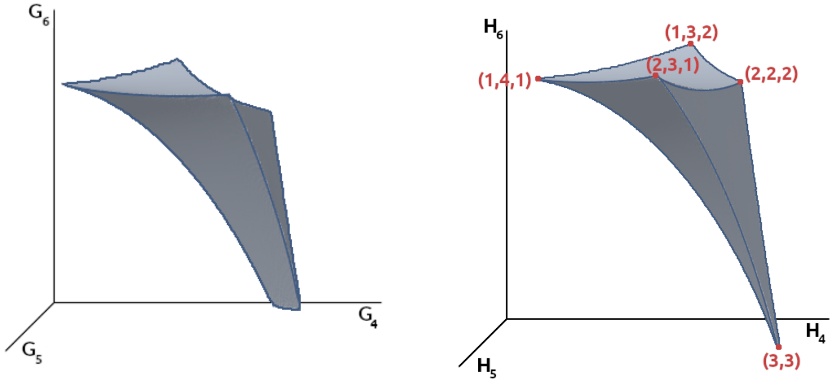
Since $\mathcal{H}_s^\mu(f)$ is the image of a polyhedron intersected with a real algebraic set defined by s polynomials, then in accordance with the terminology in real algebraic geometry, we call $\mathcal{H}_s^\mu(F)$ **generic** if it contains no polynomial with at most $s-1$ distinct roots.

Note that not all compositions need to occur in $\mathcal{H}_s(F)$ and two distinct compositions do not necessarily give rise to distinct strata as can be seen in the following examples:

Example I.2.5. Let $n = 6$, $s = 3$ and let

$$\begin{aligned} G &:= T^6 - \frac{21}{4}T^4 + T^3 + \frac{21}{4}T^2 - 1 \quad \text{and} \\ H &:= T^6 - \frac{21}{4}T^4 \qquad + \frac{21}{4}T^2 - 1. \end{aligned}$$

Consider the canonical slices $\mathcal{H}_3(G)$ and $\mathcal{H}_3(H)$. One can label the strata of these canonical slices by the corresponding compositions as exemplified for the 0-dimensional strata of $\mathcal{H}_3(H)$ in Figure I.1b. The other strata of $\mathcal{H}_3(H)$ can be labeled similary, e.g. the polynomials on the blue curve between $(1,4,1)$ and $(3,3)$ have corresponding composition $(1,2,2,1)$. Note that $\mathcal{H}_3(H)$ is non-generic while $\mathcal{H}_3(G)$ is generic.



(a) $\mathcal{H}_3(G)$ (b) $\mathcal{H}_3(H)$

Figure I.1

From the examples, it looks like the strata have some nice geometric and combinatorial properties. We will present some of these geometric properties in a moment, but first note how the pictures are reminiscent of polytopes except that the strata are not convex. Thus it is natural to ask if this stratification of canonical hyperbolic slices is always polytopal, that is, whether or not the hyperbolic poset is isomorphic to the face lattice of a polytope. We will not be able to answer this question, but we leave it as a conjecture.

Conjecture I.2.6. *Hyperbolic posets are polytopal.*

From the example above we see that the poset of strata $\mathcal{L}_s(H)$ is isomorphic to the face lattice of a pyramid. However, one can check that there is no hyperplane containing the four polynomials with composition $(1,4,1), (2,3,1), (1,3,2)$ and $(2,2,2)$ even though they are all contained in a two-dimensional stratum. Thus $\mathcal{L}_s(H)$ is not poset isomorphic to the face lattice of the convex hull and the convex hull is therefore not the right candidate to show polytopality in general.

As mentioned we will not be answering Conjecture I.2.6 in this article. Instead we will show that hyperbolic posets possess certain traits that are similar to polytopes. For instance, we will show in the next section that the dual of $\mathcal{L}_s(F)$ satisfies the Upper Bound Theorem in general and the g-Theorem in the generic case.

Lemma I.2.7. *The stratum $\mathcal{H}_s^\mu(F)$ is contractible or empty and when $s \geq 2$ it is also compact.*

Proof. See Theorem 1.1 in [Kos89] which was rephrased to our setting in [Lie23], see Proposition 2.2 and Lemma 3.2. ∎

The fact that the strata are contractible has some useful implications on how the compositions are distributed in $\mathcal{H}_s(F)$. To talk about these, note that as a consequence of Remark I.2.4 $\mathcal{H}_s^\mu(F)$ is a semi-algebraic set, thus when we speak about the dimension of $\mathcal{H}_s^\mu(F)$, it is its dimension as a semi-algebraic set.

Definition I.2.8. Let $\mathcal{H}_s^\mu(F)$ be a nonempty stratum of dimension d, then

1. the **relative interior** of $\mathcal{H}_s^\mu(F)$ is the set of polynomials $H \in \mathcal{H}_s^\mu(F)$ such that an open neighbourhood of H is homeomorphic to an open set in $\mathbb{R}^d$ and
2. the **relative boundary** of $\mathcal{H}_s^\mu(F)$ is the set of polynomials $\mathcal{H}_s^\mu(F)$ that are not in the relative interior.

Proposition I.2.9. *Suppose the stratum $\mathcal{H}_s^\mu(F)$ contains a polynomial with at least s distinct roots, then*

1. *the dimension of $\mathcal{H}_s^\mu(F)$ is $l - s$,*
2. *its relative interior is $\{H \in \mathcal{H}_s^\mu(f) \mid c(H) = \mu\}$ and*
3. *it equals the closure of its relative interior.*

If it contains no polynomial with at least s distinct roots, then the stratum is either a single polynomial or empty.

Proof. See Proposition 2.2, Theorem 2.6, Theorem 2.7 and Corollary 2.8 in [Lie23]. ∎

I.2.2 Escaping hyperbolic strata

In this subsection, we ask which polynomials of a stratum $\mathcal{H}_s^\mu(F)$ have a minimal (resp. maximal) first free coefficient. This was asked and answered for $\mathcal{H}_s(F)$ in [Meg92] and it turned out that the question could be fully answered by looking at the composition of the minimal (resp. maximal) polynomials. Thus they classified which polynomials in $\mathcal{H}_s(F)$ have the maximal first free coefficient and which have the minimal (when such polynomials exist). We shall give a similar classification, except we will restrict the domain to be any of the strata of $\mathcal{H}_s(F)$.

Definition I.2.10. We call $H = T^n + H_1T^{d-1} + \cdots + H_d \in \mathcal{H}_s^\mu(F)$ a **minimal** (resp. **maximal**) polynomial of the stratum $\mathcal{H}_s^\mu(F)$ if $H_{s+1} \leq G_{s+1}$ (resp. $H_{s+1} \geq G_{s+1}$) for all $G = T^n + G_1T^{d-1} + \cdots + G_d \in \mathcal{H}_s^\mu(F)$.

As all the polynomials in $\mathcal{H}_s^\mu(F)$ will have an i-th root of multiplicity at least μ_i, it will be useful to mod out these multiplicities. Also, note that if a composition λ is less than or equal to μ, there is a unique composition ν such that $\lambda = (\mu_1 + \cdots + \mu_{\nu_1}, \ldots, \mu_{l-\nu_{l-1}+1} + \cdots + \mu_l)$. Thus we define the following compositions:

Definition I.2.11. If $\lambda \leq \mu$, let λ/μ denote the composition ν such that $\lambda = (\mu_1 + \cdots + \mu_{\nu_1}, \ldots, \mu_{l-\nu_{l-1}+1} + \cdots + \mu_l)$.

To state the result, note that we say a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ is **alternate odd** if $\mu_l = \mu_{l-2} = \cdots = 1$ and **alternate even** if $\mu_{l-1} = \mu_{l-3} = \cdots = 1$.

Theorem I.2.12. *Let λ be the composition of $H \in \mathcal{H}_s^\mu(F)$ and let $s \geq 2$, then*

1. *there is a unique minimal (resp. maximal) polynomial in $\mathcal{H}_s^\mu(F)$ and*

2. *the polynomial H is minimal (resp. maximal) if and only if λ/μ is less than or equal to an alternate odd (resp. even) composition of length s.*

When $s = 1$ there is also a maximal polynomial for all strata, but no minimal polynomial for any strata. The maximal polynomial is then the unique polynomial with only one distinct root and it thus follows from [Meg92].

Note that in the generic case, one can replace λ/μ being "less than or equal" by "equal" in the above theorem since no two compositions of the same length are comparable. Theorem I.2.12 was stated already in [BR22, Theorem 8] without a proof and is based on similar ideas as in [Meg92] and [Arn86], however some of their techniques do not work in this general setting and others need to be adjusted. Furthermore, a similar statement and proof for the hyperoctahedral group can be found in [Ros23, Lemma 6.2.2] and in Proposition 2.10 in [Ace+23].

Since we couldn't find a full proof of Theorem I.2.12, we decided to provide one, but to relocate it to the appendix. However, since the theorem is an important tool for this article, the interested reader may wish to skip ahead to the appendix. Moreover, we will need the following lemma, which will also play an important role in the proof of Theorem I.2.12.

Lemma I.2.13. *Let $l \geq s+2$, then the polynomial $H \in H_s^u(f)$ is minimal (resp. maximal) if and only if it is minimal (resp. maximal) for all strata that contain H and that are strictly contained in $\mathcal{H}_s^\mu(F)$.*

I.3 Shellability of the dual poset

We start in the first subsection by showing that in the generic case, the boundary complex of the dual of $\mathcal{L}_s(F)$ is a simplicial complex. Next, we use the results from the previous section to imitate a line shelling for polytopes thus showing that, in the generic case, the boundary complex of the dual poset is shellable and therefore a combinatorial $(d-s)$-sphere.

This has several consequences for both generic and non-generic canonical slices. Thus in the second subsection, we can make use of the Upper Bound Theorem (UBT) and the g-theorem for simplicial spheres to get bounds on the number of i-dimensional strata in our poset.

I.3.1 Shelling the dual

For this subsection, we restrict to generic canonical slices, that is, the canonical slices where no polynomial has strictly less than s distinct roots. Also note that when $s \leq 1$, hyperbolic posets are simplices (see the proof of Theorem 3.10 in [Lie23]), thus we will only consider the cases when $s \geq 2$.

Recall that $\mathcal{L}_s(F)$ denotes the poset of strata of $\mathcal{H}_s(F)$ partially ordered by inclusion and so we let $\mathcal{L}_s^{\Delta}(F)$ denote the dual poset. That is, $\mathcal{L}_s^{\Delta}(F)$ is the set of strata of $H_s(F)$ partially ordered by reverse inclusion. Also, we call the poset $\partial(\mathcal{L}_s^{\Delta}(F)) := \mathcal{L}_s^{\Delta}(F)\backslash\emptyset$, the **boundary complex** of $\mathcal{L}_s^{\Delta}(F)$.

Lemma I.3.1. *The boundary complex of $\mathcal{L}_s^{\Delta}(F)$ is a pure simplicial complex of dimension $(n-s-1)$.*

Proof. Let ϕ be the mapping defined by $\mu \mapsto \{\mu_1, \mu_1+\mu_2, \dots, n\}$ from the poset of compositions of n to the poset of subsets of $[n]$, partially ordered by inclusion. One can easily check that ϕ is a poset isomorphism and since the poset of subsets of $[n]$ is a simplex, then so is the poset of compositions.

The poset $\mathcal{L}_s(F)$ can be identified with the poset of compositions that occur in $\mathcal{H}_s(F)$, thus the boundary complex of the dual poset can be thought of as the set

$$\{c(H) \mid \ H \in \mathcal{H}_s(F)\},$$

partially ordered by the reverse of our partial order on compositions.

From Proposition I.2.9, we know that if a polynomial $H \in \mathcal{H}_s(F)$ has at least s distinct roots, then all the compositions greater than $c(H)$ occur in $\mathcal{H}_s(F)$. Thus, the set of compositions $\{\mu \mid c(H) \leq \mu\}$, is a downwardly closed subposet of the dual poset of compositions. Thus it is a simplex and so $\partial(\mathcal{L}_s^{\Delta}(F))$ is a simplicial complex. Lastly, from Proposition 3.3 in [Lie23], we have that $\mathcal{L}_s^{\Delta}(F)$ is pure and of dimension $n-s-1$. ■

Remark I.3.2. The restriction to the generic case in I.3.1 is sufficient, but not necessary. That is, there are examples of non-generic canonical slices where the boundary complex, $\partial(\mathcal{L}_s^{\Delta}(F))$, is a simplicial complex and examples where it is not. However, the same kind of argument as in Lemma I.3.1 can be used to show that if we remove the empty set and the 0-dimensional strata from $\mathcal{L}_s(F)$, then the dual poset is a simplicial complex even for non-generic cases.

We will construct a shelling of $\partial(\mathcal{L}_s^{\Delta}(F))$ and to do so we shall use a partial order on the zero-dimensional strata of $\mathcal{H}_s(F)$. So let $\gamma_1, \dots, \gamma_k$ be the compositions of length s that occur in $\mathcal{H}_s(F)$, then $F_1 := \mathcal{H}_s^{\gamma_1}(F), \dots, F_k := \mathcal{H}_s^{\gamma_k}(F)$ are the facets of $\partial(\mathcal{L}_s^{\Delta}(F))$.

Definition I.3.3. Let "$\leq_p$" denote the partial order on $F_1, \dots, F_k$ that is generated by the covering relations $\{H\} = F_i <_p F_j = \{G\}$ if there is a one-dimensional stratum R of $\mathcal{H}_s(F)$ for which H is minimal and G is maximal.

Lemma I.3.4. *Let S be a stratum of $\mathcal{H}_s(F)$. If $H \in F_i$ is the minimal (resp. maximal) polynomial of the stratum S and $F_j \subseteq S$, then $F_i \leq_p F_j$ (resp. $F_i \geq_p F_j$).*

Proof. Since H is minimal in S, then either $F_i = F_j$ or there is a one-dimensional stratum, $R_1 \subseteq S$, for which $G \in F_j$ is maximal. Otherwise G would be minimal in S by Lemma I.2.13. By Theorem I.2.12, the stratum R_1 also contains a minimal polynomial $Q \in F_m$ for some m and therefore $F_m <_p F_j$.

And by the same argument as above, either $Q = H$ or there must be a one-dimensional stratum $R_2 \subseteq S$, for which Q is maximal. We see that by continuing this process we must eventually end up at H and so $F_i \leq_p F_j$. The argument for maximal polynomials is analogous. ∎

Definition I.3.5. Let $\leq$ and $\leq^*$ be partial orders on a set P. Then we say $\leq$ is **finer** than $\leq^*$ if $a \leq^* b$, for some $a, b \in P$, implies $a \leq b$.

Theorem I.3.6. *Let $\leq$ be a total order on $\{F_1, \ldots, F_k\}$ that is finer than $\leq_p$, then the total order (and its reverse order) induces a shelling of $\partial(\mathcal{L}_s^{\Delta}(F))$.*

Proof. We can assume by relabelling that $F_1 < \cdots < F_k$. As we are shelling the boundary complex of the dual poset we will first rephrase Definition I.3.3 to suit our setting:

$F_1, \ldots, F_k$ is a shelling of $\partial(\mathcal{L}_s^{\Delta}(F))$ if for any $i \in \{2, \ldots, k\}$ and any $j \in [i-1]$, there is an $r \in [i-1]$ such that the minimal stratum containing both F_i and F_j also contains a one-dimensional stratum, R, which contains both F_i and F_r. Note that this guarantees that in the dual poset, the intersection of the facets F_i and F_j is contained in the ridge R, which again is contained in the facets F_i and F_r.

By Lemma I.3.1, the boundary complex of the dual poset is simplicial, thus there is a smallest stratum, S, containing both F_i and F_j. The polynomial $H \in F_i$ cannot be the minimal polynomial of S, otherwise $F_j <_p F_i$ by Lemma I.3.4, which would contradict $\leq$ being finer than $\leq_p$.

So by Lemma I.2.13, H is maximal for a one-dimensional stratum $R \subset S$. Let $G \in F_r$ be the minimal polynomial of R. then $F_r <_p F_i$ by Lemma I.3.4 and therefore $F_r < F_i$ since $\leq$ refines $\leq_p$ and so $r \in [i-1]$. ∎

Corollary I.3.7. *The boundary complex $\partial(\mathcal{L}_s^{\Delta}(F))$ is a combinatorial $(n-s-1)$-sphere.*

Proof. Any ridge of $\partial(\mathcal{L}_s^{\Delta}(F))$ corresponds to an edge $\mathcal{H}_s^{\mu}(F) \in \mathcal{L}_s(F)$. By Lemma I.2.7, $\mathcal{H}_s^{\mu}(F)$ is compact and thus has two endpoints. By Proposition I.2.9, those endpoints are polynomials with s distinct roots and they have distinct compositions. Thus there are exactly two vertices in $\mathcal{H}_s^{\mu}(F)$, that is, any ridge in $\partial(\mathcal{L}_s^{\Delta}(F))$ is contained in exactly two facets. So from Proposition I.1.7, $\partial(\mathcal{L}_s^{\Delta}(F))$ is a combinatorial $(n-s-1)$-sphere. ∎

I.3.2 UBT and g-theorem

Due to Corollary I.3.7, we can make use of some previously established results for simplicial spheres to say something about the number of i-dimensional strata in $L_s(F)$.

Definition I.3.8. Let $d = \dim(\mathcal{H}_s(F))$ and for $i \in \{0, 1, \dots, d\}$, let f_i denote the number of i-dimensional strata of $\mathcal{H}_s(F)$. Then $(f_0, \dots, f_d)$ is the **f-vector** of $\mathcal{L}_s(F)$.

As we are looking at the dual poset of $\mathcal{L}_s(F)$, note that generically f_i is the number of $(d - i - 1)$-dimensional simplices in $\mathcal{L}_s^{\Delta}(F)$ (we consider the empty set to have dimension -1). Thus $(f_d, \dots, f_0)$ is the f-vector of the simplicial complex $\partial(\mathcal{L}_s^{\Delta}(F))$. Although the f-vector has an easy interpretation, it is often more convenient to work with the **h-vector**, $(h_0, \dots, h_d)$, of $\partial(\mathcal{L}_s^{\Delta}(F))$, where

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{d-j}.$$

Note that when $\mathcal{H}_s(F)$ is generic, then this definition is the same as the usual definition for simplicial complexes (see Definition 8.18 in [Zie12]) since the simplicial complex $\partial(\mathcal{L}_s^{\Delta}(F))$ has the f-vector $(f_d, \dots, f_0)$. We can pass from the h-vector to the f-vector by using the following relations (see page 249 of [Zie12]):

$$f_{d-i} = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j.$$

In our setting the h-vector has the following interpretation:

Corollary I.3.9. *Let $(h_0, \dots, h_d)$ be the h-vector of $\partial(\mathcal{L}_s^{\Delta}(F))$. Then h_i is the number of polynomials in $\mathcal{H}_s(F)$ that are maximal for exactly i one-dimensional strata. Similarly, h_i is also the number of polynomials in $\mathcal{H}_s(F)$ that are minimal for exactly i one-dimensional strata.*

Proof. Let again $\leq$ be a total order on $\{F_1, \dots, F_k\}$ that is finer than $\leq_p$ and assume that $F_1 < \cdots < F_k$, then by Theorem I.3.6, $F_1, \dots, F_k$ is a shelling of $\partial(\mathcal{L}_s^{\Delta}(F))$. We denote by V_j the set of vertices of F_j and by $R_j \subseteq V_j$ the **restriction** of F_j, which is defined as the subset of vertices of F_j, such that for every $v \in R_j$ the set $V_j \setminus \{v\}$ lies in F_m for some $m < j$. Then from the first part of section 8.3 in [Zie12] we have that h_i is equal to

$$|\{j : |R_j| = i\}|.$$

Let $v \in \mathbb{R}_j$ and let $m < j$, such that $V_j \setminus \{v\} \subset F_m$. Then F_m and F_j are joined by a one-dimensional stratum E of $\mathcal{H}_s(F)$ and since $F_m < F_j$, then $H \in F_j$ is maximal in E. Conversely, for any one-dimensional stratum E' of $\mathcal{H}_s(F)$ such that $H_j \in F_j$ is maximal and $H_r \in F_r$ is minimal in E', we have that $F_r < F_j$ and $V_j \setminus \{v\} \subset F_r$ for some $v \in V_j$.

Thus $|R_j|$ counts the number of one-dimensional strata of $\mathcal{H}_s(F)$ for which $H \in F_j$ is maximal. And so h_i counts the number of zero-dimensional strata that are maximal for exactly i one-dimensional strata. If we now take the reverse order (which by Theorem I.3.6 is also a shelling), then with an analogous argument we find that h_i is equal to the number of vertices that are minimal for exactly i one-dimensional strata. ∎

If a polynomial is maximal for i one-dimensional strata, it must be minimal for the other $n-s-i$ one-dimensional strata that contain it. Thus Corollary I.3.9 implies that the h-vector of $\partial(\mathcal{L}_s^{\Delta}(F))$ must be palindromic. That is, it satisfies the **Dehn-Sommerville equations**:

$$h_i = h_{d-i} \quad \text{for all} \quad i \in \lfloor d/2 \rfloor.$$

Moreover, since $\partial(\mathcal{L}_s^{\Delta}(H))$ is a combinatorial sphere, we can obtain further properties of its h-vector from the g-conjecture for simplicial spheres that was recently proven in [Adi18]. In order to state those results, we have to introduce some notation.

Firstly, for $k, i \in \mathbb{N}$ there are unique integers $a_i \geq \cdots \geq a_1 \geq 0$ such that

$$k = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_1}{1} \text{ (see page 265 in [Zie12]).}$$

Definition I.3.10. We say that $g = (g_0, \ldots, g_r) \in \mathbb{N}_0^r$ is a **Macaulay (or M-) vector**, if $g_0 = 1$ and for any $i \in [r-i]$

$$g_{i+1} \leq \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_1+1}{1+1},$$

where

$$g_i = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_1}{1}$$

is the unique representation of g_i introduced above.

Corollary I.3.11 (“g-theorem"). *Let $\mathcal{H}_s(F)$ be generic, then the h-vector $(h_0, \ldots, h_{n-s})$ of $\partial(\mathcal{L}_s^{\Delta}(F))$ satisfies*

1. $h_i = h_{n-s-i}$ *for all* $i \leq \lfloor (n-s)/2 \rfloor$ *(Dehn-Sommerville),*
2. $h_i \geq h_{i-1}$ *for all* $i \leq \lfloor (n-s)/2 \rfloor$ *(lower bound) and*
3. $(h_0, h_1 - h_0, \ldots, h_{\lfloor (n-s)/2 \rfloor} - h_{\lfloor (n-s)/2 \rfloor - 1})$ *is a Macaulay vector.*

Since we have situations where $\mathcal{L}_s^{\Delta}(F)$ is isomorphic to non-simplicial polytopes where the g-theorem does not hold, we cannot extend the theorem in its entirety to the general setting. See for instance Example I.2.5, where the h-vector is not palindromic. However, the third condition in Corollary I.3.11 can be used to deduce the Upper Bound Theorem for polytopes (see Section 3 in [McM71]) and this is a bound that we can extend to the general case.

To extend the generic bound, we show that the component-wise maximal f-vector of hyperbolic posets is attained in some generic case. In the following we identify $\mathcal{L}_s(F)$ with the poset of compositions that occur in $\mathcal{H}_s(F)$.

Proposition I.3.12. *Suppose F has no repeated roots and $n-s>0$, then there is a $\delta > 0$ such that for all ϵ with $0 < \epsilon < \delta$,*

1. $\mathcal{H}_s(F+\epsilon T^{n-s})$ *is generic,*
2. $\lambda \in \mathcal{L}_s(F+\epsilon T^{n-s}) \implies \lambda \geq \mu$ *for some* $\mu \in \mathcal{L}_s(F)$*,*
3. $\mu \in \mathcal{L}_s(F)$ & $\ell(\mu) \geq s \implies \mu \in \mathcal{L}_s(F+\epsilon T^{n-s})$ *and*
4. *for any* $\mu \in \mathcal{L}_s(F)$ *with* $\ell(\mu) < s$*, there is a* $\lambda \in \mathcal{L}_s(F+\epsilon T^{n-s})$ *of length* s *such that* $\lambda \geq \mu$ *and* λ *is incomparable with all other compositions of length at most* s *in* $\mathcal{L}_s(F)$*.*

Proof. By Proposition I.2.9, $\mathcal{H}_s(F)$ is of dimension $n-s>0$ and $\mathcal{H}_{s-1}(F)$ is of dimension $n-s+1$. Since F is in the interior of $\mathcal{H}_{s-1}(F)$, we can choose a $\delta > 0$ such that $B_\delta(F) \subset \mathcal{H}_s(F)$. Since there are finitely many polynomials in $\mathcal{H}_{s-1}(F)$ with at most $s-1$ distinct roots we can choose a δ such that for all ϵ with $0<\epsilon<\delta$, $\mathcal{H}_s(F+\epsilon T^{n-s})$ contains only polynomials with at least s distinct roots.

For the second statement, let $\lambda \in \mathcal{L}_s(F+\epsilon T^{n-s})$ and let H be the minimal polynomial of $\mathcal{H}^{\lambda}_{s-1}(F)$. By Theorem I.2.12, H has at most $s-1$ distinct roots. Thus we either have $H \in \mathcal{H}_s(F)$ and $c(H) \leq \lambda$ or $H \notin \mathcal{H}_s(F)$ and $\lambda \in \mathcal{L}_s(F)$.

For the third statement, let Q be a polynomial in $\mathcal{H}_s(F)$ with at least s distinct roots and composition μ. By Proposition I.2.9, $\mathcal{H}^{\mu}_{s-1}(F)$ is of dimension $\ell(\mu)-s+1>0$. By Theorem I.2.12, $\mathcal{H}^{\mu}_{s-1}(F)$ has a maximal polynomial, G, with at most $s-1$ distinct roots. Thus the s-th coefficient of G is at least as large as the s-th coefficient of F plus δ. Since $\mathcal{H}^{\mu}_{s-1}(F)$ is contractible the intersection of $\mathcal{H}^{\mu}_{s-1}(F)$ and $\mathcal{H}_s(F+\epsilon T^{n-s})$ is nonempty. So $\mathcal{H}^{\mu}_{s}(F+\epsilon T^{n-s})$ is nonempty and contains no polynomial with strictly less than s distinct roots. Thus, by Proposition I.2.9, $\mathcal{H}^{\mu}_{s}(F+\epsilon T^{n-s})$ contains a polynomial with composition μ.

For the last statement, let P be a polynomial with at most $s-1$ distinct roots and composition μ. Since P is neither the minimal nor the maximal polynomial of $\mathcal{H}_{s-1}(F)$, then $s-1>1$ by the main theorem in [Meg92] and so by Lemma I.2.13, there is a one-dimensional stratum $\mathcal{H}^{\lambda}_{s-1}(F)$ for which P is the minimal polynomial. Similar to the argument above, $\mathcal{H}^{\lambda}_{s}(F+\epsilon T^{n-s})$ must therefore contain a polynomial with composition λ. Also, by Proposition I.2.9, $\ell(\lambda)=s$ since $\mathcal{H}^{\lambda}_{s}(F+\epsilon T^{n-s})$ is generic and zero-dimensional. Lastly, by Theorem I.2.12, P is the unique minimal polynomial of $\mathcal{H}^{\lambda}_{s-1}(F)$, thus $c(P)$ is the only composition in $\mathcal{L}_s(F)$ that is smaller than or equal to λ. ∎

Remark I.3.13. We see in Proposition I.3.12 that a non-generic $\mathcal{H}_s(F)$ can be obtained from some generic $\mathcal{H}_s(H)$ by “contracting" some of the strata of $\mathcal{H}_s(H)$ to points. This corresponds to merging some of the faces of $\partial(\mathcal{L}^{\Delta}_s(H))$. In other words if $\partial(\mathcal{L}^{\Delta}_s(F))$ is a polytopal complex, then the simplicial complex $\partial(\mathcal{L}^{\Delta}_s(H))$ is a simplicial subdivision of $\partial(\mathcal{L}^{\Delta}_s(F))$. Thus whenever $\partial(\mathcal{L}^{\Delta}_s(F))$ is a polytopal complex it is also a combinatorial sphere. However, we do not know if $\partial(\mathcal{L}^{\Delta}_s(F))$ is a polytopal complex in general and thus we have restricted ourselves to the generic case.

Due to the preceding remark, we have the following weaker conjecture than Conjecture I.2.6.

Conjecture I.3.14. *The boundary complex $\partial(\mathcal{L}_s^{\Delta}(F))$ is a polytope complex and thus by Remark I.3.13, a combinatorial sphere.*

To state the bound for the general case we need another definition.

Definition I.3.15. We define

$$\begin{aligned} \phi_d : \mathbb{R} &\longrightarrow \mathbb{R}^d \\ x &\longmapsto (x, x^2, \dots, x^d) \end{aligned}$$

to be the d-th **moment curve**. If $x_1, \dots, x_m \in \mathbb{R}$ are distinct, we say that the convex hull of $\phi_d(x_1), \dots, \phi_d(x_m)$ is the d-dimensional **cyclic polytope** on m vertices.

Corollary I.3.16 (Upper Bound Theorem)**.** *Let $(f_0, \dots, f_{n-s})$ be the f-vector of $\mathcal{L}_s(F)$. If c_i is the number of i-dimensional faces of the $(n-s)$-dimensional cyclic polytope with f_{n-s-1} vertices then*

$$f_{n-s-i} \leq c_{i-1} \ \forall \ i \in [n-s].$$

Proof. By Proposition I.2.9, we may assume $H_s(F)$ is $(n-s)$-dimensional where $n-s>0$ and we may assume F has no repeated roots. Then, by Proposition I.3.12, there is an $\epsilon > 0$ such that $\mathcal{H}_s(F + \epsilon T^{d-s})$ is generic and whose f-vector is component-wise an upper bound on the f-vector of $\mathcal{H}_s(F)$. Thus we can reduce to the case when $\mathcal{H}_s(F)$ is generic.

When $\mathcal{H}_s(F)$ is generic we know that the h-vector of $\partial(\mathcal{L}_s^{\Delta}(F))$ is palindromic. From this, it can be shown that the upper bound on the f-vector is obtained by establishing the following upper bound on the h-vector (see chapter 8.4 in [Zie12]):

$$h_i \leq \binom{f_{n-s-1} - n + s - 1 + i}{i}.$$

The claim now follows directly from the Upper Bound Theorem for simplicial spheres (Cor. 5.3 in [Sta75]) since $\partial(\mathcal{L}_s^{\Delta}(F))$ is a combinatorial sphere for generic $\mathcal{H}_s(F)$ by Corollary I.3.7. ∎

Remark I.3.17. In [Rie12] (Theorem 4.2) it was shown that the extremal points of the convex hull of $\mathcal{H}_s(F)$ are contained in the subset of polynomials of $\mathcal{H}_s(F)$ with at most s distinct roots. And since Corollary I.3.16 together with Exercise 0.9 in [Zie12] gives us an explicit upper bound on the number of polynomials in $\mathcal{H}_s(F)$ with at most s distinct roots, it also gives us an upper bound on the number of local extremal points. This improves the bound given in Theorem 2.14 and Remark 2.15 in [RS24] to the following

$$\begin{aligned} f_0 &\leq \begin{cases} \binom{n-1-(n-s)/2}{(n-s)/2} + \binom{n-2-(n-s)/2}{(n-s)/2-1}, & \text{if } n-s \text{ is even} \\ 2\binom{n-2-(n-s-1)/2}{(n-s-1)/2}, & \text{if } n-s \text{ is odd} \end{cases} \\ &= \begin{cases} \binom{(n+s)/2-1}{s-1} + \binom{(n+s)/2-2}{s-1}, & \text{if } n-s \text{ is even} \\ 2\binom{(n+s-3)/2}{s-1}, & \text{if } n-s \text{ is odd} \end{cases}. \end{aligned}$$

We have computationally verified that the bound in Remark I.3.17 can be attained when $n \leq 8$ and $s \leq n$ and one can also use Proposition I.2.9 to argue that the bound is attained when $s \leq 2$ and when $s \geq n-1$. Therefore we have the following conjecture:

Conjecture I.3.18. *The bound stated in Remark I.3.17 is sharp.*

I.4 Improving Timofte's Degree principle

Throughout the section, we denote by $\mathcal{C}(n,s)$ and by $\mathcal{P}(n,s)$ the set of all compositions and partitions, respectively, of n into s parts and by $\mathcal{C}_{\min}(n,s)$ and $\mathcal{P}_{\min}(n,s)$ the compositions and partitions that correspond to a minimal polynomial in some generic canonical slice.

Timofte showed in [Tim03] the so-called "degree principle": Symmetric polynomials of degree at most s have a common real root if and only if they have a common real root with at most s distinct coordinates. We want to improve this result by considering subsets of the set of points with at most s distinct coordinates. To this end, we introduce some notation:

Definition I.4.1. Let $P \subseteq \mathcal{P}(n,s)$. We say that P is a (n,s)**-Vandermonde covering**, if for every canonical slice $\mathcal{H}_s(F)$ there is a partition $q \in P$ and a polynomial $G \in \mathcal{H}_s(f)$ with corresponding partition $p(G)$ such that $q \geq p(G)$.

Since we are interested in symmetric polynomials, the roots of the polynomials are closed under permutations. So we identify the orbit types of points in R^n by partitions. Instead of considering all points with at most s distinct coordinates in the degree principle, we want to consider only points with orbit types corresponding to a partition in a Vandermonde covering.

Definition I.4.2. Let $P \subseteq \mathcal{P}(n,s)$. We denote by

$$A_P := \left\{ (\underbrace{x_1,\ldots,x_1}_{q_1-\text{times}},\underbrace{x_2,\ldots,x_2}_{q_2-\text{times}},\ldots,\underbrace{x_s,\ldots,x_s}_{q_s-\text{times}}) \in \mathbb{R}^n \;\middle|\; q \in P \right\}$$

the set of points with coordinate multiplicities corresponding to a partition in P.

The following theorem motivates the name "Vandermonde covering" and can also be seen as a strenghtening of the degree principle presented in [Rie12].

Theorem I.4.3. *Let $P \subseteq \mathcal{P}(n,s)$. The following are equivalent:*

1. *$P \subseteq \mathcal{P}(n,s)$ is a (n,s)-Vandermonde covering.*

2. *For all $k \in \mathbb{N}$ and all symmetric polynomials $F_1,\ldots,F_k \in \mathbb{R}[\underline{X}]$ of degree at most s*

$$V_{\mathbb{R}}(F_1,\ldots,F_k) \neq \emptyset \Leftrightarrow V_{\mathbb{R}}(F_1,\ldots,F_k) \cap A_P \neq \emptyset.$$

3. *For all* $a \in \mathbb{R}^s$*, the Vandermonde variety*

$$\mathcal{V}(a) \neq \emptyset \Leftrightarrow \mathcal{V}(a) \cap A_P \neq \emptyset.$$

Proof. (1)⇒(2): Let $P \subseteq \mathcal{P}(n,s)$ be a (n,s)-Vandermonde covering and let $x \in V_{\mathbb{R}}(F_1,\ldots,F_k)$. Consider

$$F := T^n - E_1(x)T^{n-1} + \cdots + (-1)^n E_n(x)$$

with roots $x_1,\ldots,x_n$. Then there is a partition $q \in P$ and a polynomial $g \in \mathcal{H}_s(F)$ with corresponding partition $p(g) \leq q$ and roots

$$\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n) \in A_P,$$

because P is a (n,s)-Vandermonde covering. Since $F_1,\ldots,F_k$ are polynomials of degree at most s, we can write

$$F_1 = G_1(E_1,\ldots,E_s),\ldots,F_k = G_k(E_1,\ldots,E_s)$$

for some $G_1,\ldots,G_k \in \mathbb{R}[Y_1,\ldots,Y_s]$ by Lemma I.1.9. Now

$$0 = F_i(x) = G_i(E_1(x),\ldots,E_s(x)) = G_i(E_1(\tilde{x}),\ldots,E_s(\tilde{x})) = F_i(\tilde{x})$$

and therefore $\tilde{x} \in V_{\mathbb{R}}(F_1,\ldots,F_k)$.
(2)⇒(3): This is clear, because $E_i - a_i$ is symmetric of degree i.
(3)⇒(1): Assume (3) holds. Let $F = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$ be a hyperbolic polynomial with roots $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$. Then the Vandermonde variety $\mathcal{V}(-c_1,\ldots,(-1)^s c_s)$ contains x by construction and is therefore nonempty. By (3) there is an $\tilde{x} \in \mathcal{V}(-c_1,\ldots,(-1)^s c_s) \cap A_P$, i.e.

$$E_1(\tilde{x}) = c_1,\ldots,E_s(\tilde{x}) = c_s.$$

Now

$$G := T^n - E_1(\tilde{x})T^{n-1} + \cdots + (-1)^n E_n(\tilde{x})$$

is a polynomial in $\mathcal{H}_s(F)$ with corresponding partition $p(G) \leq q$ for some $q \in P$. ■

In the light of Theorem I.4.3, the degree principle can be interpreted as thefact that $\mathcal{P}(n,s)$ is a Vandermonde covering which follows from for example Theorem I.2.12.

I.4.1 General bounds on Vandermonde coverings

Since every generic canonical slice has a unique minimal polynomial with a corresponding alternate odd composition, we get the following Vandermonde covering:

Theorem I.4.4. *The set* $\mathcal{P}_{\min}(n,s)$ *is a* (n,s)*-Vandermonde covering of size* $\left|\mathcal{P}\left(n - \lceil \frac{s}{2} \rceil, \lfloor \frac{s}{2} \rfloor\right)\right|$.

Proof. Follows directly from Theorem I.2.12 or from the less general version presented in [Meg92]. ∎

We show below that $\mathcal{P}_{\min}(n,s)$ is in general not the smallest Vandermonde covering. In order to estimate how good this Vandermonde covering is, we want to get lower bounds on the size of Vandermonde coverings. To this end, we need the following definition and some properties of the set of minimal and maximal partitions.

Definition I.4.5. We denote by $\mathcal{P}(n)$ the set of all partitions of n. The partial order on the set of all compositions of n induces a partial order $\leq$ on $\mathcal{P}(n)$: For $p,q \in \mathcal{P}(n)$ we write $p \leq q$ if p can be obtained from q by summing some of the parts in q and then reordering. Additionaly, if $\ell(q) = \ell(p)+1$, then we say q **covers** p.

Note that for two partitions p and q, $p \leq q$ if and only if there are permutations σ and τ, such that $\sigma p \leq \tau q$ as compositions.

Lemma I.4.6.

1. $\mathcal{P}_{\min}(n,s-1) \subseteq \mathcal{P}_{\max}(n,s-1)$.
2. $|\mathcal{P}_{\min}(n,s)| = |\mathcal{P}_{\max}(n-1,s-1)|$.
3. *Let $P \subseteq \mathcal{P}(n,s)$ be a (n,s)-Vandermonde covering, then P has to cover $\mathcal{P}_{\max}(n,s-1)$.*
4. *Every partition in $\mathcal{P}(n,s)$ covers at most*

$$\frac{\left\lceil \frac{s-1}{2} \right\rceil^2 + \left\lceil \frac{s-1}{2} \right\rceil}{2} = \frac{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil}{2}$$

   partitions in $\mathcal{P}_{\max}(n,s-1)$.

Proof.

1. Let $p \in \mathcal{P}_{\min}(n,s-1)$. Then p is of the form

$$p = (p_1,\ldots,p_{\lfloor \frac{s-1}{2} \rfloor}, \underbrace{1,\ldots,1}_{\lceil \frac{s-1}{2} \rceil - \text{times}})$$

   since it corresponds to an alternate odd composition $(\mu_1,\ldots,\mu_{s-1})$ by Theorem I.2.12. Now p corresponds also to the alternate even composition $(\mu_{s-1},\mu_1,\mu_2,\ldots,\mu_{s-2})$ and therefore $p \in \mathcal{P}_{\max}(n,s-1)$.

2. Follows directly from the bijection

$$\begin{array}{rccc} \phi : & \mathcal{C}_{\max}(n-1,s-1) & \longrightarrow & \mathcal{C}_{\min}(n,s) \\ & (\mu_1,\ldots,\mu_{s-1}) & \longmapsto & (\mu_1,\ldots,\mu_{s-1},1) \end{array}.$$

3. Let $\mu \in \mathcal{C}_{\max}(n, s-1)$ and let F be a polynomial with root multiplicities corresponding to μ. Then by Theorem I.2.12 and Proposition I.2.9, F is the maximal polynomial of $\mathcal{H}_{s-1}^{\mu}(F)$ and $\mathcal{H}_{s-1}^{\mu}(F)$ is of dimension $n-s+1$. Now for $\epsilon > 0$ small enough there is some monic polynomial H of degree $n-s$, such that $\mathcal{H}_s(F-\epsilon H)$ is $(n-s)$-dimensional with zero-dimensional strata corresponding to all compositions that cover μ by Proposition I.2.9. Since P is a (n,s)-Vandermonde covering, there has to be a $q \in P$ such that $q \geq p(G)$ for some $G \in \mathcal{H}_s(F-\epsilon T^s)$ and so we have $q \geq p(G) > p(F)$.

4. In order for $p \in \mathcal{P}(n,s)$ to cover a partition in $\mathcal{P}_{\max}(n, s-1)$ there can be at most $\left\lfloor \frac{s-1}{2} \right\rfloor + 1$ entries different from 1 in p. One can now obtain all partitions in $\mathcal{P}_{\max}(n, s-1)$ that are covered by p by summing two of the first $\left\lfloor \frac{s-1}{2} \right\rfloor + 1$ entries in p. So p covers at most

$$\binom{\left\lfloor \frac{s-1}{2} \right\rfloor + 1}{2} = \frac{\left\lceil \frac{s-1}{2} \right\rceil^2 + \left\lceil \frac{s-1}{2} \right\rceil}{2} = \frac{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil}{2}$$

partitions in $\mathcal{P}_{\max}(n, s-1)$.

■

From this lemma, we get the following lower bounds on the size of any Vandermonde covering:

Proposition I.4.7. *Let $P \subseteq \mathcal{P}(n,s)$ be a (n,s)-Vandermonde covering, then*

$$|P| \geq \left\lceil \frac{2\left|\mathcal{P}\left(n+1-\left\lceil \frac{s}{2} \right\rceil, \left\lfloor \frac{s}{2} \right\rfloor\right)\right|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil.$$

Proof. By Lemma I.4.6 (3), P has to cover $\mathcal{P}_{\max}(n, s-1)$. Every partition in $\mathcal{P}(n,s)$ covers at most

$$\frac{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil}{2}$$

partitions in $\mathcal{P}_{\max}(n, s-1)$ by Lemma I.4.6 (4). From the pigeonhole principle, we get that we need at least

$$\left\lceil \frac{2|\mathcal{P}_{\max}(n, s-1)|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil = \left\lceil \frac{2|\mathcal{P}_{\min}(n+1, s)|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil = \left\lceil \frac{2\left|\mathcal{P}\left(n+1-\left\lceil \frac{s}{2} \right\rceil, \left\lfloor \frac{s}{2} \right\rfloor\right)\right|}{\left\lceil \frac{s-1}{2} \right\rceil \left\lceil \frac{s+1}{2} \right\rceil} \right\rceil$$

partitions to have at least one partition from every generic slice. ■

This lower bound can be improved by considering recursively those maximal partitions that have i entries different from 1, which is the main idea behind the following theorem.

Theorem I.4.8. *Let $P \subseteq \mathcal{P}(n,s)$ be a (n,s)-Vandermonde covering. Then*

$$|P| \geq \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} B_i,$$

where $B_0 := 0$, $B_1 := 1$ and

$$B_i := \left\lceil 2\frac{|\mathcal{P}(n-s+1,i)| - iB_{i-1} - B_{i-2}}{i^2+i} \right\rceil$$

for all $i \in \left\{2, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\right\}$.

Proof. Denote by

$$P_i := \left\{ q \in \mathcal{P}_{\max}(n,s-1) \;\middle|\; |\{j \in [n] \mid q_j \neq 1\}| = i \right\}$$

the partitions in $\mathcal{P}_{\max}(n,s-1)$ that have exactly i entries different from 1. Note the following:

1. $|P_i| = |\mathcal{P}(n-s+1,i)|$.
2. Every partition in $\mathcal{P}(n,s)$ covers at most $\binom{i+1}{2} = \frac{i^2+i}{2}$ partitions in P_i by a similar argument as in the proof of Lemma I.4.6 (4).
3. A partition in $\mathcal{P}(n,s)$ that covers a partition in P_i, covers at most $i+1$ partitions in P_{i+1} and at most one partition in P_{i+2}.

Now, in order to cover all partitions in $\mathcal{P}_{\max}(n,s-1)$, we have to cover all partitions in P_i for all $i \in \left[\left\lfloor \frac{s}{2} \right\rfloor\right]$. Combining (1), (2) and (3) we get recursively: We need $B_1 = 1$ partition in $\mathcal{P}(n,s)$ to cover P_1. It covers at most $(1+1)B_1$ partitions in P_2 and at most B_1 partitions in P_3. To cover the at least $P_2 - 2B_1$ remaining many partitions in P_2 we need by the pigeonhole principle at least

$$B_2 = \left\lceil \frac{|P_2| - 2B_1 - B_0}{(2^2+2)/2} \right\rceil = \left\lceil 2\frac{|\mathcal{P}(n-s+1,2)| - 2B_1 - B_0}{2^2+2} \right\rceil$$

additional partitions in $\mathcal{P}(n,s)$. Those partitions cover again at most $(2+1)B_2$ partitions in P_3 and at most B_2 partitions in P_4. To cover at least the $P_3 - 3B_2 - B_1$ remaining partitions in P_3 we need by the pigeonhole principle at least

$$B_3 = \left\lceil \frac{|P_3| - 3B_2 - B_1}{(3^2+3)/2} \right\rceil = \left\lceil 2\frac{|\mathcal{P}(n-s+1,3)| - 3B_2 - B_1}{3^2+3} \right\rceil$$

additional partitions in $\mathcal{P}(n,s)$. In general, if B_i denotes the number of additional partitions needed to cover the remaining partitions in P_i, then

$$B_i := \left\lceil 2\frac{|\mathcal{P}(n-s+1,i)| - iB_{i-1} - B_{i-2}}{i^2+i} \right\rceil.$$

In total, we need at least $\sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} B_i$ partitions in $\mathcal{P}(n,s)$ to cover all partitions in $\mathcal{P}_{\max}(n,s-1)$. ■

I.4.2 Algorithmic improvements of Vandermonde coverings.

In the following we want to present an algorithmic approach on how to obtain smaller - possibly optimal - Vandermonde coverings for small s and n. To this end, we try to characterize if a set of compositions $S \subset \mathcal{C}(n, s)$ corresponds to the set of zero-dimensional strata of some canonical slice.

Definition I.4.9. Let $S \subseteq \mathcal{C}(n, s)$. We call the upward closure of S

$$\mathcal{L}(S) := \{\lambda \mid \text{there is a } \mu \in S \text{ with } \mu \leq \lambda\} \cup (n)$$

the **poset** of S. We say that $\mathcal{L}(S)$ is a **potential hyperbolic poset**, if for every $\lambda \in \mathcal{L}(S)$ there are unique $\mu_{\min}, \mu_{\max} \in S$, such that

1. $\mu_{\min}/\lambda$ is alternate odd and
2. $\mu_{\max}/\lambda$ is alternate even.

Furthermore, we say that $\mathcal{L}(S)$ is a **realizable hyperbolic poset**, if it is isomorphic to a hyperbolic poset $\mathcal{L}_s(F)$.

Note that Theorem I.2.12 states that every realizable hyperbolic poset is a potential hyperbolic poset.

Remark I.4.10. One can also consider more general potential hyperbolic posets, where S is a set of compositions of n into at most s parts. For this we construct $\mathcal{L}(S)$ analagous to Algorithm 3.12 in [Lie23], that is, by first taking the join of pairwise distinct elements of S and then the upward closure of these joins.

Since the poset of compositions is a simplex, the upward closure of a composition ν in some potential hyperbolic poset $\mathcal{L}(S)$, is also a simplex. Thus, by the argument in Proposition I.A.10, analogously to Lemma I.2.13 potential hyperbolic posets have the following property:

Lemma I.4.11. *Let $\mathcal{L}(S)$ be a potential hyperbolic poset and let $\nu \in S$ and $\nu < \mu$ for some $\mu \in \mathcal{L}(S)$ with $\ell(\mu) \geq s + 2$. Then we have that ν/μ is alternate odd (resp. even) if and only if ν/λ is alternate odd (resp. even) for all $\lambda \in \mathcal{L}(S)$ with $\nu \leq \lambda < \mu$.*

One can see that the arguments in the proof of shellability in Section 2 only uses the fact that the boundary complex of the dual poset is a pure simplicial complex along with Theorem I.2.12 and Lemma I.2.13. Thus, since the boundary complex of the dual $\partial(\mathcal{L}^{\Delta}(S))$ of a potential hyperbolic poset $\mathcal{L}(S)$ is a pure simplicial complex then by the defining property of potential hyperbolic posets and Lemma I.4.11, we get the following:

Theorem I.4.12. *Let $\mathcal{L}(S)$ be a potential hyperbolic poset and denote by $\partial(\mathcal{L}^{\Delta}(S))$ the boundary complex of the dual poset of $\mathcal{L}(S)$. Then*

1. *$\partial(\mathcal{L}^{\Delta}(S))$ is a shellable simplicial complex and therefore a combinatorial sphere.*

2. *The h-vector of $\partial(\mathcal{L}^{\Delta}(S))$ satisfies the "g-theorem", i.e. the inequalities stated in Corollary I.3.11.*

3. $|S| \leq \begin{cases} \binom{(n+s)/2-1}{s-1} + \binom{(n+s)/2-2}{s-1}, & \text{if } n-s \text{ is even} \\ 2\binom{(n+s-3)/2}{s-1}, & \text{if } n-s \text{ is odd} \end{cases}.$

Since all the known combinatorial properties of the poset of a generic canonical slice hold for all potential hyperbolic posets, we don't know any combinatorial way to distinguish potential from realizable hyperbolic posets. Moreover, by computationally realizing all hyperbolic posets up to $s \leq n \leq 6$, we state the following conjecture:

Conjecture I.4.13. *Every potential hyperbolic poset is realizable.*

Since it is easy to check if a set of compositions has a potential hyperbolic poset, one can compute better Vandermonde coverings for small n and s.

Example I.4.14. For $n = 6$ and $s = 4$ there are 10 compositions of 6 into 4 parts. One can check that out of the 2^{10} subsets only 17 have potential hyperbolic posets. Up to symmetry - we identify S with $\tilde{S} := \{(\mu_4, \ldots, \mu_1) \mid \mu \in S\}$ - we get the 11 subsets

$$\begin{gathered}
\{(1,1,1,3),(1,1,2,2),(1,1,3,1)\},\\
\{(1,1,3,1),(1,2,2,1),(1,3,1,1)\},\\
\{(1,1,1,3),(2,1,1,2),(2,1,2,1)\},\\
\{(1,1,2,2),(1,1,3,1),(1,2,1,2),(1,2,2,1)\},\\
\{(1,2,1,2),(1,2,2,1),(2,1,1,2),(2,1,2,1)\},\\
\{(1,1,1,3),(1,2,2,1),(2,1,1,2),(3,1,1,1)\},\\
\{(1,1,1,3),(1,1,2,2),(2,1,2,1),(2,2,1,1)\},\\
\{(1,1,1,3),(1,1,3,1),(2,1,1,2),(2,2,1,1)\},\\
\{(1,1,2,2),(1,2,1,2),(1,2,2,1),(2,1,2,1),(2,2,1,1)\},\\
\{(1,1,1,3),(1,1,2,2),(1,2,2,1),(2,2,1,1),(3,1,1,1)\} \text{ and}\\
\{(1,1,3,1),(1,2,1,2),(1,2,2,1),(2,1,1,2),(2,2,1,1).\}
\end{gathered}$$

From this we get that $\{(2,2,1,1)\}$ is a $(6,4)$-Vandermonde covering, which is also optimal in this case.

Example I.4.14 generalizes in the following way:

Proposition I.4.15. $\{(2,2,1,\ldots,1)\}$ *is a* $(n, n-2)$*-Vandermonde covering.*

Proof. Suppose it is not a Vandermonde covering. Then there would be a canonical slice $\mathcal{H}_s(F)$ with all zero-dimensional strata corresponding to compositions with one entry equal to 3 and the other entries equal to 1. By Theorem I.2.12 all of these compositions correspond to minimal or maximal polynomials in $\mathcal{H}_s(F)$ and therefore $\mathcal{H}_s(F)$ contains at most two zero-dimensional strata. But by Proposition I.2.9, $\mathcal{H}_s(f)$ is two-dimensional and thus have at least three extremal points and by Theorem 2.8 in [RS24], the extremal point of $\mathcal{H}_s(f)$ have at most s distinct roots. This is a contradiction to $\mathcal{H}_s(F)$ having at most two zero dimensional strata ■

Since there are $k = \binom{n-1}{s-1}$ compositions of n into s parts, the procedure in Example I.4.14 becomes too computationally expensive to apply directly when n and s are large since it involves considering 2^k subsets. However, we can use some weaker conditions to cut down this big set into a more managable set and that makes it easier to apply our previous method. For example, since we know that every potential hyperbolic poset contains $(1, \ldots, 1)$, we just have to check all the subsets of compositions with exactly one alternate even and one alternate odd composition. Furthermore, we can apply the bounds stated in Theorem I.4.12 and we also know that we need at least $n - s + 1$ compositions of length s by the argument in the proof of Theorem I.4.15. This allows computations of all potential hyperbolic posets up to $s, n \leq 8$ on a standard computer with no more than a few hours running time.

Example I.4.16. For $n = 8$ and $s = 4$, we get from Theorem I.4.4 that there is a Vandermonde covering with 3 partitions and from Theorem I.4.8 we know that we need at least 1 partition. By computing all the potential hyperbolic posets we get several Vandermonde coverings with two elements, e.g.

$$\{(3, 2, 2, 1), (4, 2, 1, 1)\},$$

and one can show that there is no Vandermonde covering with only one partition by realizing appropriate potential hyperbolic posets.

I.5 Conclusion

We studied the rich geometric and combinatorial structure of canonical slices. Although we could not show the conjectured polytopality, we were able to establish the weaker result that dual posets of generic hyperbolic posets are combinatorial spheres. We conjectured in I.3.14 that this is true for general hyperbolic posets. Moreover, we obtained an upper bound theorem for hyperbolic posets from the sphericity of the boundary of the dual posets. We have some computational evidence that this bound is sharp for the number of vertices and maybe also in general. It could be interesting to try to construct and study such “cyclic canonical hyperbolic slices".

It is well known, that every polytope can be obtained as an affine slice of a higher-dimensional simplex. Since a generic hyperbolic poset $L_s(F)$ is a simplex for $s = 2$ (see the proof of Theorem 3.10 in [Lie23]), we can see canonical hyperbolic slices as certain affine slices of “hyperbolic simplices". So we ask the following, which is even stronger then the conjectured sharpness of the Upper Bound Theorem:

Question I.5.1. *For any f-vector of a simple polytope, there are $n, s \in \mathbb{N}$ and a polynomial F such that $\mathcal{L}_s(F)$ has the same f-vector.*

In the second part of the paper, we introduced and studied Vandermonde coverings which allow us to strengthen Timofte's degree principle. We showed how to compute better Vandermonde coverings for small n and s by introducing potential hyperbolic posets and conjectured that potential hyperbolic posets are realizable. Such computations might be used to find patterns for Vandermonde coverings for bigger n and s.

We suspect that many of our results can be translated to other finite reflection groups, at least to the hyperoctahedral group.

Appendix I.A Proof of Theorem I.2.12

In this section we prove the following theorem:

Theorem I.2.12. *Let λ be the composition of $H \in \mathcal{H}_s^\mu(F)$ and let $s \geq 2$, then*

1. *there is a unique minimal (resp. maximal) polynomial in $\mathcal{H}_s^\mu(F)$ and*
2. *the polynomial H is minimal (resp. maximal) if and only if λ/μ is less than or equal to an alternate odd (resp. even) composition of length s.*

We start by proving the first item and as $\mathcal{H}_s^\mu(F)$ is either empty or a point if $l \leq s$ according to Proposition I.2.9, we will let $l = \ell(\mu) > s$ for this section.

Lemma I.A.1. *The map*

$$\begin{array}{cccc} P^{n-l}: & \mathcal{H}_s^\mu(F) & \longrightarrow & \mathbb{R}^{l-s} \\ & T^n + H_1T^{n-1} + \cdots + H_n & \longmapsto & (H_{s+1}, \ldots, H_l) \end{array}$$

is a homeomorphism onto its image and the image is closed in $\mathbb{R}^{l-s}$.

Proof. See Proposition 2.5 in [Lie23]. ∎

Proof of Item 1 from Theorem I.2.12. The statement is clear when $\mathcal{H}_s^\mu(F)$ is just a point so we will assume $\mathcal{H}_s^\mu(F)$ is $(l-s)$-dimensional. By Lemma I.2.7, $\mathcal{H}_s^\mu(F)$ is compact so the existence of minimal and maximal polynomials is clear.

Let $H \in \mathcal{H}_s^\mu(F)$ be a minimal polynomial. To show uniqueness, we assume that $\mathcal{H}_{s+1}^\mu(H)$ contains another polynomial, i.e. it is of dimension $l-s-1$. By Proposition I.2.9, it contains a polynomial G with composition μ. By Lemma I.A.1 and Proposition I.2.9, $P^{n-l}(\mathcal{H}_s^\mu(F))$ is full-dimensional with interior points corresponding to the image of the polynomials with composition μ. This contradicts G being minimal in $\mathcal{H}_s^\mu(F)$ as interior points of $P^{n-l}(\mathcal{H}_s^\mu(F))$ cannot have a minimal first coordinate if it is at least one dimensional. The argument for maximal polynomials is analogous. ∎

We will need some more tools before we get started with the proof of the second part of Theorem I.2.12. As we will need to use some local arguments we need a local definition of minimality and maximality:

Definition I.A.2. We call $H = T^n + H_1T^{d-1} + \cdots + H_d \in \mathcal{H}_s^\mu(F)$ a **locally minimal** (resp. **locally maximal**) polynomial of the stratum $\mathcal{H}_s^\mu(F)$ if $H_{s+1} \leq G_{s+1}$ (resp. $H_{s+1} \geq G_{s+1}$) for all $G = T^n + G_1T^{d-1} + \cdots + G_d \in N$, where $N \subset \mathcal{H}_s^\mu(F)$ is some open neighbourhood of H.

Lemma I.A.3. *A locally minimal or locally maximal polynomial in $\mathcal{H}_s^\mu(F)$ has at most s distinct roots.*

Proof. Assume $\mathcal{H}_s^\mu(F)$ is at least one-dimensional since the other cases follow from Proposition I.2.9 and let $l = \ell(\mu)$. By Lemma I.A.1, $P^{n-l} : \mathcal{H}_s^\mu(F) \to \mathbb{R}^{l-s}$ is a homeomorphism onto its image which is closed in $\mathbb{R}^{l-s}$. So by Proposition I.2.9, the image of the polynomials whose composition is strictly smaller than μ make up the boundary of $P^{n-l}(\mathcal{H}_s^\mu(F))$. Thus a locally minimal or locally maximal polynomial lies in the relative boundary and therefore has strictly less than l roots and so the statement follows inductively. ∎

We will let $B_\epsilon(a)$ denote the open ball about $a \in \mathbb{R}^{n-s}$ of radius ϵ.

Lemma I.A.4. *A polynomial $H \in \mathcal{H}_s^\mu(F)$ is locally minimal (resp. locally maximal) if and only if it is minimal (resp. maximal).*

Proof. One implication is clear, so suppose $H \in \mathcal{H}_s^\mu(F)$ is locally minimal but not minimal. If $\mathcal{H}_{s+1}^\mu(H)$ is at least one-dimensional then by Proposition I.2.9, for any $\epsilon > 0$ there is a polynomial $G \in \mathcal{H}_{s+1}^\mu(F) \cap B_\epsilon(H)$ with composition μ. Thus, by Lemma I.A.1, there is a δ with $0 < \delta < \epsilon$ such that $P^{n-l}(\mathcal{H}_s^\mu(F)) \cap B_\delta(P^{n-l}(G))$ lies in the interior of $P^{n-l}(\mathcal{H}_s^\mu(F))$. So there is a polynomial in $\mathcal{H}_s^\mu(F) \cap B_\epsilon(H)$ whose first free coefficient is smaller than the first free coefficient of H contradicting the local minimality of H.

Thus, by Proposition I.2.9, $H_{s+1}^\mu(H)$ must be a point. Since $H_s^\mu(F)$ is contractible, there is a path, $\Phi : [0,1] \to H_s^\mu(F)$, where $[0,1]$ is the unit interval, from H to the minimal polynomial. Since $H_{s+1}^\mu(H)$ is a point we may assume that the first free coefficient of $\Phi(y)$ is strictly smaller than the first free coefficient of H for all $y \in (0,1]$. But this is a contradiction since H was assumed to be locally minimal. Thus if H is locally minimal, it must also be minimal. The proof for locally maximal polynomials works analogously. ∎

To prove the second part of Theorem I.2.12 we will first consider generic strata of hyperbolic slices and then extend the result to the general setting. For the generic case we will do an induction on the dimension of the strata and so we start by considering a generic one-dimensional stratum $\mathcal{H}_s^\mu(F)$.

Since $s \geq 2$ we know from Lemma I.2.7, that $\mathcal{H}_s^\mu(F)$ is compact and contractible. Thus there are two polynomials, H and G, in the relative boundary of $\mathcal{H}_s^\mu(F)$ which, by Lemma I.A.3, are the minimal and maximal polynomials of $\mathcal{H}_s^\mu(F)$. Since the stratum is generic, H and G have s distinct roots and thus the compositions $c(H)/\mu$ and $c(G)/\mu$ have one part equal to 2 and the rest equal to 1. Thus they are both alternate and we just have to show that $c(H)/\mu$ is alternate odd if H is minimal and that $c(G)/\mu$ is alternate even.

For this we use Lagrange multipliers and since it will be useful to work with power sums instead of elementary symmetric polynomials, we need the following lemma:

Lemma I.A.5. *Let $a, b \in \mathbb{R}^n$ and suppose $E_i(a) = E_i(b)$ for all $i \in [s]$, then $P_{s+1}(a) > P_{s+1}(b)$ if and only if $(-1)^{s+1}E_{s+1}(a) < (-1)^{s+1}E_{s+1}(b)$.*

Proof. This is straightforward to show using Newtons identities, see for instance the proof of Proposition 9 in [Meg92]. ■

Due to Lemma I.A.5, instead of looking at the minimizers (resp. maximizers) of $(-1)E^{\mu}_{s+1}$, we will look at the maximizers (resp. minimizers) of P^{μ}_{s+1}. Recall that we can define the Vandermonde variety $\mathcal{V}^{\mu}_s(F)$ as

$$\mathcal{V}^{\mu}_s(F) = \{x \in \mathbb{R}^{s+1} \mid P^{\mu}_i(x) = c_i \ \forall\ i \in [s]\},$$

for some $c_1, \ldots, c_s \in \mathbb{R}$. The Jacobian matrix of $(P^{\mu}_1(x), \ldots, P^{\mu}_s(x))$, where $x = (x_1, \ldots, x_{s+1})$, is the following Vandermonde matrix with columns and rows scaled by positive integers:

$$J(x) := \begin{pmatrix} \mu_1 & 2\mu_1 x_1 & \cdots & s\mu_1 x_1^{s-1} \\ \mu_2 & 2\mu_2 x_2 & \cdots & s\mu_2 x_2^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{s+1} & 2\mu_{s+1} x_{s+1} & \cdots & s\mu_{s+1} x_{s+1}^{s-1} \end{pmatrix}.$$

If we let $x = (x_1, \ldots, x_{s+1}) \in \mathcal{V}^{\mu}_s(F) \cap \mathcal{W}_{s+1}$ be the tuple of the roots of either the minimal or maximal polynomial then we know x has s distinct coordinates. Suppose $x_{j_1}, \ldots, x_{j_s}$ are the distinct coordinates of x, then the $s \times s$ submatrix of $J(x)$ consisting of the rows $j_1, \ldots, j_s$ has the determinant $c\prod_{i=1}^{s-1}(x_{j_i} - x_{j_{i+1}})$, for some positive integer c. Since the x_{j_i}'s are distinct, the determinant does not vanish and the column vectors are linearly independent. That is, $\nabla P^{\mu}_1(x), \ldots, \nabla P^{\mu}_s(x)$ are linearly independent.

Similarly any $(s+1) \times (s+1))$ submatrix of the Jacobian of $P^{\mu}_1(x), \ldots, P^{\mu}_{s+1}(x)$ has a vanishing determinant since x has only s distinct roots. Thus the vectors $\nabla P^{\mu}_1(x), \ldots, \nabla P^{\mu}_{s+1}(x)$ are linearly dependent and there are therefore $a_1, \ldots, a_s \in \mathbb{R}$ such that $\nabla L(x) = 0$, where

$$L(x) = P^{\mu}_{s+1}(x) - \sum_{i=1}^{s} a_i P^{\mu}_i(x).$$

The gradient of L at equals

$$\nabla L(x) = \nabla P^{\mu}_{s+1}(x) - \sum_{i=1}^{s} a_i \nabla P^{\mu}_i(x) = (\mu_1 Q(x_1), \ldots, \mu_{s+1} Q(x_{s+1}))),$$

where $Q(T) = (s+1)T^s - \sum_{i=1}^{s} a_i i T^{i-1}$. The univariate polynomial $Q(t)$ has s distinct roots, since it is of degree s and vanishes at x_j for any j. With this we are ready to prove the second part of Theorem I.2.12 for generic one-dimensional hyperbolic strata.

Proposition I.A.6. *Let $\mathcal{H}_s^\mu(F)$ be generic and one-dimensional. Then $H \in \mathcal{H}_s^\mu(F)$ is the minimal (resp. maximal) polynomial if and only if $\ell(c(H)) = s$ and $c(H)/\mu$ is alternate odd (resp. even).*

Proof. We continue with the notation above and we let $x_1 \leq \cdots \leq x_{s+1}$ be the roots of a polynomial H in the relative boundary of $\mathcal{H}_s^\mu(F)$. We just saw that H has s distinct coordinates since $\mathcal{H}_s^\mu(f)$ is generic and that H is either the minimal or the maximal polynomial. Thus x has s distinct roots and so we will let Q be defined as above.

If we let

$$HL(x) := \nabla^2 L(x) = \begin{pmatrix} \mu_1 Q'(x_1) & 0 & \cdots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ 0 & \cdots & 0 & & \mu_{s+1} Q'(x_{s+1}) \end{pmatrix},$$

then by Theorem 5.4 in [Sun96], x is a local maximizer (resp. minimizer) of P_{s+1}^μ if $v^t HL(x) v < 0$ (resp. $v^t HL(x) v > 0$) for all nonzero vectors $v \in \mathbb{R}^{s+1}$ in the kernel of $J(x)$. Suppose $x_k = x_{k+1}$ are the two equal coordinates of x and $v \in \mathbb{R}^{s+1}$. If $v_k + v_{k+1} = 0$ and all other coordinates of v are zero, then v lies in the kernel of $J(x)$. Also, since the set of such vectors is a one dimensional subspace of $\mathbb{R}^{s+1}$, they make up the kernel of $J(x)$.

So we have that

$$v^t H(x) v = \sum_j \mu_j Q'(x_j) v_j^2 = Q'(x_k)(\mu_k v_k^2 + \mu_{k+1} v_{k+1}^2)$$

for all $v \in \ker(J(x))$. Thus $v^t H(x) v$ is negative (resp. positive) for all nonzero $v \in \ker(J(x))$ if and only if $Q'(x_k)$ is negative (resp. positive). The univariate polynomial Q has only the simple roots $x_1 < \ldots < x_k < x_{k+2} < \ldots < x_{s+1}$, so by Rolle's Theorem the roots of Q' strictly interlace the roots of Q. Also, since the leading coefficient of Q is positive, $Q'(x_{s+1})$ is positive and thus

$$Q'(x_s) < 0, Q'(x_{s-1}) > 0, \ldots$$

Thus x is a local maximizer (resp. minimizer) of P_{s+1}^μ if $k = s + 1 - 2m$ (resp. $k = s - 2m$) for some nonnegative integer m. So by Lemma I.A.5, x is a local minimizer (resp. maximizer) of $(-1)^{s+1} E_{s+1}^\mu$ if $c(h)/\mu$ is alternate odd (resp. even).

Lastly, since the Vieta map is a homeomorphism by Remark I.2.4, the image of a local minimum (resp. maximum) is a locally minimal (resp. maximal) polynomial. And due to Lemma I.A.4, a locally minimal (resp. maximal) polynomial is a minimal (resp. maximal) polynomial, so the proposition follows. ∎

Having settled the initial step of our induction, we need to establish some tools for the inductive step. Firstly we need something on the combinatorial side and we start by rephrasing Definition I.2.11: so if $\lambda \leq \mu$ and $r = \ell(\lambda)$, then there is an increasing sequence of integers $n_0, \ldots, n_r$, with $n_0 = 0$ and $n_r = l = \ell(\mu)$, such that $\lambda_i = \sum_{j<n_{i-1}}^{n_i} \mu_j$ for all $i \in [r]$. Then the composition λ/μ is the composition of l whose parts are $(\lambda/\mu)_i = n_i - n_{i-1}$.

Lemma I.A.7. *Let $\lambda, \gamma < \mu$ be compositions of d, then we have $\lambda/\mu < \gamma/\mu$ if and only if $\lambda < \gamma$ and in this case we have that $\lambda/\gamma = \frac{\lambda/\mu}{\gamma/\mu}$.*

Proof. We continue with the notation above and similarly as for λ we have that if γ is of length k, then there is an increasing sequence of integers $m_0 < \cdots < m_k$ with $m_0 = 0$ and $m_k = l$ such that $\gamma_i = \sum_{j=m_{i-1}+1}^{m_i} \mu_j$, $\forall\ i \in [k]$. So $\lambda/\mu = (n_1 - n_0, \ldots, n_r - n_{r-1})$ and $\gamma/\mu = (m_1 - m_0, \ldots, m_k - m_{k-1})$ are two compositions of l.

If $\lambda < \gamma$, there is an increasing sequence of integers $z_0 < \cdots < z_r$ with $z_0 = 0$ and $z_r = k$ such that $\lambda_i = \sum_{j=z_{i-1}+1}^{z_i} \gamma_j$ $\forall\ i \in [r]$. Thus

$$\sum_{j=n_{i-1}+1}^{n_i} \mu_j = \lambda_i = \sum_{j=z_{i-1}+1}^{z_i} \left(\sum_{y=m_{j-1}+1}^{m_j} \mu_y \right) = \sum_{y=m_{z_{i-1}}+1}^{m_{z_i}} \mu_y,\ \forall\ i \in [r],$$

and since $m_0 = n_0$ we have $m_{z_i} = n_i$ and $m_{z_{i-1}} = n_{i-1}$. Thus

$$(\lambda/\mu)_i = n_i - n_{i-1} = m_{z_i} - m_{z_{i-1}} =$$

$$m_{z_i} - m_{z_i-1} + m_{z_i-1} - m_{z_i-2} + \cdots + m_{z_{i-1}+1} - m_{z_{i-1}} = \sum_{j=z_{i-1}+1}^{z_i} (\gamma/\mu)_j$$

and so $\lambda/\mu < \gamma/\mu$.

Conversely, if $\lambda/\mu < \gamma/\mu$, then there is an increasing sequence of integers $y_0 < \cdots < y_r$ with $y_0 = 0$ and $y_r = k$ such that

$$(\lambda/\mu)_i = \sum_{j=y_{i-1}+1}^{y_i} (\gamma/\mu)_j,\ \forall\ i \in [r].$$

Thus we have

$$n_i - n_{i-1} = (\lambda/\mu)_i = \sum_{j=y_{i-1}+1}^{y_i} (m_j - m_{j-1}) = m_{y_i} - m_{y_{i-1}},$$

and since $n_0 = 0 = m_0 = m_{z_0}$, we have $n_i = m_{y_i}$ $\forall\ i \in [r]$. Thus

$$\lambda_i = \sum_{j=n_{i-1}+1}^{n_i} \mu_j = \sum_{j=m_{y_{i-1}}+1}^{m_{y_i}} \mu_j = \sum_{j=y_{i-1}+1}^{y_i} \gamma_j\ \forall\ i \in [r]$$

and so $\lambda < \gamma$.

Lastly, since $n_i = m_{y_i}$ and $n_i = m_{z_i}$, we have $m_{y_i} = m_{z_i}$. Since the indices $m_0, \dots, m_k$ are distinct we have $y_i = z_i$. Thus

$$\left(\frac{\lambda/\mu}{\gamma/\mu}\right)_i = y_i - y_{i-1} = z_i - z_{i-1} = (\lambda/\gamma)_i \ \forall\ i \in [r]$$

and so we have $\lambda/\gamma = \frac{\lambda/\mu}{\gamma/\mu}$. ■

Next we need to look closer at the projection introduced in the beginning of the appendix. It should be noted that the following discussion and lemma is analogous to the approach in [Kos89], where the image of the power sums are studied instead of the elementary symmetric polynomials.

By Lemma I.A.1, $\mathcal{H}_s^\mu(F)$ is homeomorphic to $P^{n-l}(\mathcal{H}_s^\mu(F)) \subset \mathbb{R}^{l-s}$ and thus by Proposition I.2.9, $M := P^{n-l}(\mathcal{H}_s^\mu(F))$ is full-dimensional when $\mathcal{H}_s^\mu(F)$ is neither empty nor a single polynomial. Let $\pi : M \to \mathbb{R}^{l-s-1}$ be the projection given by $(x_1, \dots, x_{l-s}) \mapsto (x_1, \dots, x_{l-s-1})$, then for $H \in \mathcal{H}_s^\mu(F)$, the fibre $\pi^{-1}(\pi(P^{n-l}(H)))$ equals $P^{n-l}(\mathcal{H}_{l-1}^\mu(H))$. This fibre is by Proposition I.2.9, either the point $P^{n-l}(H)$, in which case it must lie on the boundary of M, or it is an interval. And if it is an interval, then its endpoints must lie on the boundary of M and its relative interior lies in the interior of M.

Thus the boundary of M can be written as the union of a “lower" and an “upper" part, $L \cup U$, where

$$L = \{(x_1, \dots, x_{l-s}) \in M \mid x_{l-s} \le y_{l-s} \ \forall\ (y_1, \dots, y_{l-s}) \in \pi^{-1}(\pi(x))\},$$

and

$$U = \{(x_1, \dots, x_{l-s}) \in M \mid x_{l-s} \ge y_{l-s} \ \forall\ (y_1, \dots, y_{l-s}) \in \pi^{-1}(\pi(x))\}.$$

Lemma I.A.8. *The sets L and U are closed.*

Proof. We just show that U is closed since the proof for L is analogous. So suppose $P^{n-l}(Q)$ is in the closure of U but not in U. By Lemma I.A.1, the boundary of $P^{n-l}(\mathcal{H}_s^\mu(F))$ is closed and thus $P^{n-l}(Q) \in L$. Thus $\pi^{-1}(\pi(P^{n-l}(Q)))$ is an interval whose relative interior lies in the interior of $P^{n-l}(\mathcal{H}_s^\mu(F))$. Let $P^{n-l}(G)$ be one of those relative interior points and let $\epsilon > 0$ be such that $B_\epsilon(P^{n-l}(G)) \subset P^{n-l}(\mathcal{H}_s^\mu(F))$.

For any $P^{n-l}(H) \in B_\epsilon(P^{n-l}(G))$, the point $\pi^{-1}(\pi(P^{n-l}(H))) \cap L$ lies below $B_\epsilon(P^{n-l}(G))$. Thus the distance between $P^{n-l}(Q)$ and any point in U is at least as large as $\epsilon/2$. Thus $P^{n-l}(Q)$ cannot be in the closure of U which is a contradiction and so $P^{n-l}(Q)$ must lie in U. ■

Lemma I.2.13. *Let $l \ge s + 2$, then the polynomial $H \in H_s^u(f)$ is minimal (resp. maximal) if and only if it is minimal (resp. maximal) for all strata that contain H and that are strictly contained in $\mathcal{H}_s^\mu(F)$.*

Proof. One implication is clear, so we just have to show that if for all compositions ν, with $H \in \mathcal{H}_s^\nu(F) \subsetneq \mathcal{H}_s^\mu(F)$, we have that H is minimal in $\mathcal{H}_s^\nu(F)$, then H is minimal in $\mathcal{H}_s^\mu(F)$. We assume $\mathcal{H}_s^\mu(F)$ is $(l-s)$-dimensional since the statement is clear when it is just a point. Also, the argument for maximal polynomials is analogous so we just prove it for minimal polynomials.

Suppose H is not minimal in $\mathcal{H}_s^\mu(F)$, then by Lemma I.A.4 it is not locally minimal. So for any $i \in \mathbb{N}$, $B_{1/i}(H) \cap \mathcal{H}_s^\mu(F)$ contains a polynomial G_i whose first free coefficient is smaller than the first free coefficient of H.

Without loss of generality assume $P^{n-l}(H)$ lies in the upper part of the boundary of M. Then for each fibre $\pi^{-1}(\pi(P^{n-l}(G_i)))$, let $P^{n-l}(Q_i)$ be the point in the upper part of the boundary of M. Since the upper part is compact by Lemma I.2.7 and Lemma I.A.1, $(P^{n-l}(Q_i))$ converges to a point in the upper part which is by design $P^{n-l}(H)$.

As there are finitely many compositions, there is an infinite subsequence of $(P^{n-l}(Q_i))$, where all the Q_i's have the same composition $\lambda \neq \mu$, that converges to $P^{n-l}(H)$. By Proposition I.2.9 and Lemma I.A.1, the image $P^{n-l}(\mathcal{H}_s^\lambda(F))$ is the closure of its relative interior which consists of the images of the polynomials with composition λ. Thus $H \in \mathcal{H}_s^\lambda(F)$ and it is by construction not the minimal polynomial. This is a contradiction and so H must be minimal in $\mathcal{H}_s^\mu(F)$. ■

Lemma I.A.9. *Let $l = \ell(\mu) \geq s+2$ and let $H \in \mathcal{H}_s^\mu(F)$ have s distinct roots. Then there are two polynomials with distinct compositions, γ and ν, in $\mathcal{H}_s^\mu(F)$ of length $\ell(\mu)-1$ and with $c(H) < \gamma, \nu$.*

Proof. Let $\lambda = c(H)$, then since $l \geq s+2$, $\ell(\lambda) = s$ and $\lambda < \mu$ one must replace at least two of the commas in μ with plus signs to obtain λ. So let $j \neq i$ be two indices such that

$$\gamma = (\mu_1, \dots, \mu_{j-1}, \mu_j + \mu_{j+1}, \mu_{j+2}, \dots, \mu_l)$$

and

$$\nu = (\mu_1, \dots, \mu_{i-1}, \mu_i + \mu_{i+1}, \mu_{i+2}, \dots, \mu_l)$$

are both greater than λ. By Proposition I.2.9 both of these compositions must occur in $\mathcal{H}_s^\mu(F)$. ■

Proposition I.A.10. *Let $H_s^\mu(F)$ be of $(l-s)$-dimensional and generic. Then $H \in \mathcal{H}_s^\mu(F)$ is the minimal (resp. maximal) polynomial if and only if $\ell(c(H)) = s$ and $c(H)/\mu$ is alternate odd (resp. even).*

Proof. We prove this by induction in the poset of strata of $\mathcal{H}_s^\mu(F)$. The initial step is when $l = s+1$ and is covered by Proposition I.A.6. Next, we assume the statement is true for the strata of dimension $l-s-1 \geq 1$ and we show that it is true when the stratum is $(l-s)$-dimensional. We will just show the proof for minimal polynomials as the proof for maximal polynomials is analogous.

Let $\lambda = c(H)$ and suppose λ/μ is alternate odd and that $\ell(\lambda) = s$. Let γ be any composition with $\lambda < \gamma < \mu$ such that $\mathcal{H}_s^\gamma(F)$ is at least one-dimensional. By Lemma I.A.7 we have that $\lambda/\gamma = \frac{\lambda/\mu}{\gamma/\mu}$. Note that the i-th part of λ/γ is equal to the i-th part of λ/μ minus some integer, thus λ/γ is alternate odd since λ/μ is. So by the induction hypothesis, H is the minimal polynomial of $\mathcal{H}_s^\gamma(F)$. And so by Lemma I.2.13, H is the minimal polynomial of $\mathcal{H}_s^\mu(F)$.

For the reverse statement, let H be the minimal polynomial. Then by Lemma I.A.3, H has s distinct roots. Since $H_s^\mu(f)$ is at least two-dimensional, then by Lemma I.A.9, there occurs at least two distinct compositions, γ and ν in $\mathcal{H}_s^\mu(F)$, of length $l-1$ and where $\lambda < \gamma, \nu$. By Proposition I.2.9, the strata $\mathcal{H}_s^\gamma(F)$ and $\mathcal{H}_s^\nu(F)$ are $(l-s-1)$-dimensional.

By Lemma I.2.13 and the induction hypothesis this means that λ/γ and λ/ν are alternate odd compositions. Since γ and ν are of length $l-1$, there are two indices $j \neq i$ such that

$$\lambda = (\mu_1, \ldots, \mu_{j-1}, \mu_j + \mu_{j+1}, \mu_{j+2}, \ldots, \mu_l)$$

and

$$\nu = (\mu_1, \ldots, \mu_{i-1}, \mu_i + \mu_{i+1}, \mu_{i+2}, \ldots, \mu_l).$$

Thus $\gamma/\mu = (1, \ldots, 1, 2, 1, \ldots, 1)$, where the index 2 is in the j-th position and $\nu/u = (1, \ldots, 1, 2, 1, \ldots, 1)$, where the index 2 is in the i-th position.

Since $\lambda/\gamma = \frac{\lambda/\mu}{\gamma/\mu}$ and $\lambda/\nu = \frac{\lambda/\mu}{\nu/\mu}$, we have that

$$\lambda/\gamma = ((\lambda/\mu)_1, \ldots, (\lambda/\mu)_{j-1}, (\lambda/\mu)_j - 1, (\lambda/\mu)_{j+1}, \ldots, (\lambda/\mu)_s)$$

and that

$$\lambda/\nu = ((\lambda/\mu)_1, \ldots, (\lambda/\mu)_{i-1}, (\lambda/\mu)_i - 1, (\lambda/\mu)_{i+1}, \ldots, (\lambda/\mu)_s).$$

Since $j \neq i$ then $\lambda/\gamma \neq \lambda/\nu$ and since both compositions are alternate odd then so must λ/μ be. ∎

Now that we have established the second part of Theorem I.2.12 for the generic case we will extend it to the non-generic case. Firstly we need the following lemma, although note that this just corresponds to the first half of Proposition I.3.12 but we include it here to ensure that there are no circular arguments.

Lemma I.A.11. *If F has no repeated roots and $\mathcal{H}_s^\mu(F)$ is $(l-s)$-dimensional, where $l > s$, then there is a $\delta > 0$ such that for any ϵ, with $0 < \epsilon < \delta$, $\mathcal{H}_s^\mu(F + \epsilon T^{n-s})$ is nonempty and generic.*

Proof. By Proposition I.2.9, there is an $H \in \mathcal{H}_s^\mu(F)$ with composition μ and $\mathcal{H}_{s-1}^\mu(F)$ is of dimension $l-s+1 > 0$. Thus H is in the relative interior of $\mathcal{H}_s^\mu(F)$ and by Lemma I.A.1, H is therefore in the interior of $P^{n-l}(\mathcal{H}_s^\mu(F))$. Thus we can choose an $\delta > 0$ such that $B_\delta(P^{n-l}(H)) \subset P^{n-l}(\mathcal{H}_{s-1}^\mu(F))$.

Since there are finitely many polynomials in $\mathcal{H}_{s-1}(F)$ with at most $s-1$ distinct roots we can choose δ such that for any ϵ, with $0 < \epsilon < \delta$, $B_\delta(P^{n-l}(H))$ contains only points $P^{n-l}(G)$ such that $\mathcal{H}_s^\mu(G)$ contains only polynomials with at least s distinct roots. By Proposition I.2.9, G has composition μ and we can choose G such that its s-th coefficient equals ϵ, thus $\mathcal{H}_s^\mu(G) = \mathcal{H}_s^\mu(F + \epsilon T^{n-s})$ is generic and clearly nonempty. ∎

Lemma I.A.12. *If $H \in \mathcal{H}_s^\mu(F)$ and $c(H) < \gamma$ for some γ, of length s, such that γ/μ is alternate odd (resp. even), then H is minimal (resp. maximal).*

Proof. Again, we just show the statement for minimal polynomials. If $\mathcal{H}_s^\mu(F)$ is just a point, the statement is clear so by Proposition I.2.9, we may assume it is $(l-s)$-dimensional, where $l > s$. Thus we may also assume F has no repeated roots. Suppose H is not minimal, then by Lemma I.A.4, H is not locally minimal.

Thus for any $\delta > 0$, $B_\delta(H) \cap \mathcal{H}_s^\mu(F)$ contains a polynomial contains a polynomial Q whose first free coefficient is $r \in \mathbb{R}_{>0}$ smaller than the first free coefficient of H. By Proposition I.2.9, $\mathcal{H}_s^\mu(F)$ is the closure of its relative interior, so we may assume $c(Q) = \mu$. So by Lemma I.A.1, $P^{n-l}(Q)$ is an interior point of $P^{n-l}(\mathcal{H}_0^\mu(F))$, and there is therefore an ϵ with $0 < \epsilon < r/2$ such that $B_\epsilon(P^{n-l}(Q)) \subset P^{n-l}(\mathcal{H}_0^\mu(F))$.

All compositions occur in $\mathcal{H}_0(F)$ and since $\mathcal{H}_0^\gamma(F)$ is the closure of its relative interior then $B_\epsilon(P^{n-l}(H)) \cap P^{n-l}(\mathcal{H}_0^\gamma(F))$ contains a point, $P^{n-l}(G)$, where $c(G) = \gamma$. The intersection $P^{n-l}(\mathcal{H}_s^\mu(G)) \cap B_\epsilon(P^{n-l}(Q))$ is nonempty since the first $s+1$ coefficients of Q equals the first $s+1$ coefficients of H and $P^{n-l}(G) \in B_\epsilon(P^{n-l}(H))$. Thus there is a polynomial from $B_\epsilon(Q)$ in $\mathcal{H}_s^\mu(G)$.

By Lemma I.A.11, we may assume $\mathcal{H}_s^\mu(G)$ is generic which, by Proposition I.A.10, means that G must be the minimal polynomial of $\mathcal{H}_s^\mu(G)$. However the first free coefficient of any polynomial from $B_\epsilon(Q)$ is smaller than the first free coefficient of H minus $r/2$ and the first free coefficient of G is at least as large as the first free coefficient of H minus $r/2$. This is a contradiction and so H must be minimal in $\mathcal{H}_s^\mu(F)$. ■

Lemma I.A.13. *If $H \in \mathcal{H}_s^\mu(F)$ and $c(H) \not\leq \nu$ for any ν, of length s, such that ν/μ is alternate odd (resp. even), then H is not minimal (resp. maximal).*

Proof. Again, we just show the statement for alternate odd compositions. If $s = 2$, then by the main theorem in [Meg92] either F has only one distinct root and $\mathcal{H}_2(F) = \{F\}$ or $\mathcal{H}_2(F)$ contains no polynomials with strictly less than two distinct roots. By assumption we are not in the former case and so $\mathcal{H}_2(F)$ is generic and thus the statement follows from Proposition I.A.10.

Next we treat the cases when $s \geq 3$ and by the previous paragraph we have that $H_2(F)$ is generic and all but the composition (n) occurs. By Proposition I.2.9, $\mathcal{H}_2^\mu(F)$ is the closure of its relative interior, so for any integer $i \geq 1$ there is a polynomial $G_i \in B_{1/i}(H) \cap \mathcal{H}_2^\mu(F)$ with composition μ. Due to Lemma I.A.11, we may assume $\mathcal{H}_s^\mu(G_i)$ is generic. Thus, by Proposition I.A.10, the composition, ν, of the minimal polynomial in $\mathcal{H}_s^\mu(G_i)$ is such that ν/μ is alternate odd.

Since there are finitely many compositions with this property, there is one such ν such that for infinitely many i, the minimal polynomial of $\mathcal{H}_s^\mu(G_i)$ has composition ν. So we may assume that for all $i \geq 1$, the minimal polynomial, Q_i, of $\mathcal{H}_s^\mu(G_i)$ has the same composition ν. Since $(1/i)_{i\geq 1}$ converges to zero and $\mathcal{H}_2^\mu(F)$ is compact, the sequence $(G_i)_{i\geq 1}$ converges. Similarly, since $\mathcal{H}_2^\mu(F)$ is sequentially compact, an infinite subsequence of $(Q_i)_{i\geq 1}$ converges and so for notations sake we will assume this is the sequence $(Q_i)_{i\geq 1}$.

The limit of $(G_i)_{i\geq 1}$ is H and since the first $s+1$ coefficients of Q_i is equal to the first coefficients of G_i, the limit, Q, of $(Q_i)_{i\geq 1}$ also lies in $\mathcal{H}_s^\mu(F)$. Since $\mathcal{H}_2^\nu(F)$ is the closure of its relative interior and $c(Q_i)=\nu$ for all i, then $c(Q)\leq\nu$ and thus by Lemma I.A.12, Q is the minimal polynomial of $\mathcal{H}_s^\mu(F)$. Since $c(H)$ is not smaller than a composition γ such that γ/μ is alternate odd, then $c(H)\not<\nu$ and thus $H\neq Q$. So H is not the minimal polynomial of $\mathcal{H}_s^\mu(F)$. ■

Proposition I.A.10 proves the second part of Theorem I.2.12 in the generic case and the combination of Lemma I.A.12 and Lemma I.A.13 proves it for the non-generic case. And since we proved the first part of Theorem I.2.12 for all cases in Section I.2, our work is done.

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Paper II

Linear slices of Hyperbolic polynomials and positivity of symmetric polynomial functions

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II

Abstract

A real univariate polynomial of degree n is called hyperbolic if all of its n roots are on the real line. Such polynomials appear quite naturally in different applications, for example, in combinatorics and optimization. The focus of this article is on families of hyperbolic polynomials which are determined through k linear conditions on the coefficients. The coefficients corresponding to such a family of hyperbolic polynomials form a semi-algebraic set which we call a **hyperbolic slice**. We initiate here the study of the geometry of these objects in more detail. The set of hyperbolic polynomials is naturally stratified with respect to the multiplicities of the real zeros and this stratification induces also a stratification on the hyperbolic slices. Our main focus here is on the **local extreme points** of hyperbolic slices, i.e., the local extreme points of linear functionals, and we show that these correspond precisely to those hyperbolic polynomials in the hyperbolic slice which have at most k distinct roots and we can show that generically the convex hull of such a family is a polyhedron. Building on these results, we give consequences of our results to the study of symmetric real varieties and symmetric semi-algebraic sets. Here, we show that sets defined by symmetric polynomials which can be expressed sparsely in terms of elementary symmetric polynomials can be sampled on points with few distinct coordinates. This in turn allows for algorithmic simplifications, for example, to verify that such polynomials are non-negative or that a semi-algebraic set defined by such polynomials is empty.

II.1 Introduction

A monic real univariate polynomial f which has only real roots is classically called a hyperbolic polynomial. Such polynomials and their multivariate relatives appear naturally in various mathematical contexts from differential equations to combinatorics, real algebraic geometry, and optimization (see for example [Brä11; Gül97; Gur06; MSS15]). By identifying monic polynomials of degree n with the list of coefficients, one can describe hyperbolic polynomials of degree n as a semi-algebraic subset of $\mathbb{R}^n$. We consider linear slices, i.e., intersections with linear subspaces, of this semi-algebraic set, which is in fact the closure of one connected component of the complement of the discriminant variety. The study of these **hyperbolic slices** is inspired by the works of Arnold who considered families of hyperbolic polynomials where the first k coefficients were fixed. Arnold [Arn86] and Givental [Giv87] showed that these sets are topologically contractible (see also [Meg91; Meg92]) and have a rich geometric structure as was shown by Kostov [Kos89] (see also [Kos07; KS02] for more related results). In a similar spirit to the works of Arnold and Meguerditchian we study the local extreme points of these sets (see Definition II.2.5). In analogy to their result, we show in Theorem II.2.8 that these points correspond to hyperbolic polynomials with few distinct roots. Furthermore, we show in Theorem II.2.14 that a generic hyperbolic slice only has finitely many local extreme points. This signifies in particular that the convex hull of each of its connected components is in fact a polyhedron. In contrast to the case considered by Arnold, our slices are in general not contractible and not compact. However, we are able to give some sufficient conditions to decide if a hyperbolic slice is compact or has at least a local extreme point.

One of our main interests in the study of these hyperbolic slices stems from an application to symmetric real polynomial functions, i.e., polynomial functions that are left invariant by any permutation of the variables. Real symmetric functions are related to hyperbolic polynomials via the so-called **Vieta map**: Recall that for $1 \leq i \leq n$ the i-th elementary symmetric polynomial in n variables is defined by

$$e_i := \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} X_{j_1} \cdots X_{j_i}.$$

By Vieta's formula the coefficients of a univariate monic polynomial of degree n are given by evaluating these elementary symmetric polynomials at the corresponding roots. Conversely, it is also classical that the roots depend continuously on the coefficients and the natural action of S_n permuting the roots does not affect the coefficients. Therefore, the polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by the above connection effectuates a homeomorphism from $\mathbb{R}^n/S_n$ to its image called the **Vieta map**. Since it is classically known that every symmetric polynomial can be uniquely written as a polynomial in the elementary symmetric polynomials one can view real symmetric polynomial functions as functions on the image of the Vieta map. This connection between univariate monic polynomials and symmetric polynomials in n variables gives rise to an application of our results on hyperbolic slices in the context of symmetric polynomial functions: We are interested in the question to what extent the global behavior of symmetric functions is determined by its behavior of symmetrical points or points with a large stabilizer. For example, several authors (e.g. [Kei67; Wat83]) have studied families of symmetric polynomials which attain their minimal values on symmetric points, i.e., points where all coordinates are equal. More generally, it has been shown that symmetric polynomial functions of a given degree $2d$ assume only non-negative values if and only if they have this property on point with at most d distinct coordinates [Rie12; Tim03]. To further this line of ideas, we introduce the notion of k-complete symmetric polynomial functions. Those are polynomial functions whose set of values is already obtained by evaluation only on points that have at most k distinct coordinates (see Definition II.3.1). Using the geometry of hyperbolic slices we are able to identify a new class of k-complete functions in Theorem II.3.8 which is given by functions that are constant or linear along a hyperbolic slice (see Definition II.3.5 for the technical definition). The results we give here also include the mentioned findings of [Rie12; Tim03] which can be interpreted by saying that every symmetric polynomial of degree $d \geq 4$ is $\lfloor \frac{d}{2} \rfloor$-complete.

The class of k-complete symmetric functions allows for significant algorithmic simplifications in several algorithmic tasks related to polynomial functions. For example, it is known (see [MK87]) that checking if a real multivariate polynomial f is non-negative is in general NP-hard, already in the case of polynomials of degree 4. However, as we discuss in this article, the complexity of verifying non-negativity for a k-complete symmetric polynomial can be drastically reduced if $k < n$, since the set of points that need to be considered is of dimension k. We highlight this and several related results in the second part of the article.

Outline: In Section II.2 we introduce the notion of hyperbolic slices as families of hyperbolic polynomials defined by linear conditions on the coefficients. Our main result in this section is that the local extreme points of such slices correspond to hyperbolic polynomials with few distinct roots (Theorem II.2.8) and that generically there are only finitely many such local extreme points (Theorem II.2.14). Finally, we give sufficient criteria for the existence of such local extreme points in the cases when a slice is not compact. In Section II.3 we study symmetric polynomials which attain their minima on points with few distinct coordinates, i.e., on points with a non trivial and potentially large stabilizer. Our main results there (Theorem II.3.8 and Corollary II.3.10) provide a large class of such functions based on the results from Section II.2. We furthermore highlight how to efficiently verify that a given symmetric polynomial satisfies the conditions needed to apply these results. The following Section II.4 highlights the applicability of our results. We show that our findings allow for simple proofs for different symmetric inequalities and also recover the mentioned known results. Furthermore, we in particular highlight in Theorem II.4.6 a family of symmetric polynomials that attain their minimum on symmetric points. Finally, we close with some concluding remarks and outlooks in Section II.5.

Notation: Throughout the article, we fix $n \in \mathbb{N}$ and denote by $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$ the polynomial ring in n variables over $\mathbb{R}$.

II.2 Hyperbolic slices

In this section, we define and analyze the notion of a hyperbolic slice. To begin we formalize the notion of hyperbolic polynomials as used in the article.

Definition II.2.1. We will denote by

$$\mathcal{H} := \left\{ z \in \mathbb{R}^n \;\middle|\; T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n \text{ only has real roots} \right\}$$

the set of **hyperbolic polynomials** of degree n, and for $1 \leq m \leq n$ the m**-boundary** of $\mathcal{H}$

$$\mathcal{H}^m := \left\{ z \in \mathcal{H} \;\middle|\; T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n \text{ has at most } m \text{ distinct roots} \right\}.$$

As described above we are interested in families of univariate monic hyperbolic polynomials whose coefficients are restricted by linear conditions. In order to define this more concretely, we fix throughout this section an integer $1 \leq k \leq n$, a real point $a \in \mathbb{R}^k$, and a surjective linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^k$. This choice of a linear map and a point characterizes the linear conditions we aim to impose on hyperbolic polynomials and the **hyperbolic slices** corresponding to these choices can be defined as follows.

Definition II.2.2. With the notation introduced above, the **hyperbolic slice** associated to L and a is the affine linear slice

$$\mathcal{H}_L(a) := \mathcal{H} \cap L^{-1}(a).$$

Furthermore, for $1 \leq m \leq n$ we define by

$$\mathcal{H}_L^m(a) := \mathcal{H}^m \cap L^{-1}(a),$$

its restriction to the m-boundary.

We briefly discuss one possible connection of the above definition to polynomial interpolation for which our results might be interesting in their own rights: For $k \in \mathbb{N}$ consider $a_1, b_1, \ldots, a_k, b_k \in \mathbb{R}$. Then the space of polynomials f of degree n which satisfy $f(a_i) = b_i$ for $1 \leq i \leq k$ is called a polynomial interpolation space. Now, since evaluations at given points define linear maps, an interpolation problem for which one is interested only in hyperbolic polynomials constitutes one example of a hyperbolic slice defined above.

Clearly, the assumption that L is surjective is only for convenience in the notation. As mentioned above the set of hyperbolic polynomials is tightly connected to the Vieta map.

Remark II.2.3. The set $\mathcal{H}$ of hyperbolic polynomials is the image of the so-called Vieta map

$$\begin{array}{cccc} \Gamma : & \mathbb{R}^n & \longrightarrow & \mathcal{H} \\ & x = (x_1, \ldots, x_n) & \longmapsto & (e_1(x), \ldots, e_n(x)) \end{array},$$

and the restriction of Γ to the polyhedral cone

$$\mathcal{W} := \{ x \in \mathbb{R}^n \mid x_1 \leq x_2 \leq \ldots \leq x_n \}$$

is a homeomorphism. In particular, the roots of a univariate polynomial depend continuously on its coefficients. $\mathcal{H}$ is in fact a basic closed semi-algebraic subset of $\mathbb{R}^n$. Clearly, $\mathcal{H} = \mathcal{H}^n \supset \mathcal{H}^{n-1} \supseteq \cdots \supseteq \mathcal{H}^1$ and $\mathcal{H}^{n-1}$ is the topological boundary of $\mathcal{H}$. Furthermore, for $1 \leq m \leq n$ the m-boundary $\mathcal{H}^m$ is the image of the union of the m-faces of $\mathcal{W}$ under Γ and therefore of dimension m. For more details, we refer to [Whi72, Appendix V.4].

The next example shows one of the simplest situations of a hyperbolic slice obtained by fixing the first two coefficients of a monic polynomial of degree 4.

Example II.2.4. For $k \geq 2$ we can fix the first k coefficients of a monic polynomial. The set of hyperbolic polynomials in such a family defines a hyperbolic slice and this setup corresponds to the situation studied by Arnold [Arn86] and Kostov [Kos89]. For example, we can consider $\mathcal{H}_L(0, -6)$, where

$$L: \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ (z_1, z_2, z_3, z_4) & \longmapsto & (z_1, z_2) \end{array}.$$

This choice yields the hyperbolic slice in the plane shown in Figure II.1.

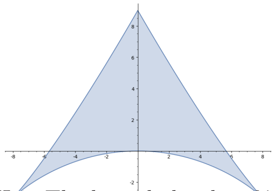


Figure II.1: The hyperbolic slice $\mathcal{H}_L(0, -6)$

As can be seen from the example above, a hyperbolic slice is not convex but bears some resemblance to a polytope. By the connection via the Vieta map, we have that $\mathcal{H}$ is homeomorphic to the polyhedral cone $\mathcal{W}$. Furthermore, one finds three extreme points/ vertices in the above picture. For convex sets in $\mathbb{R}^n$ the extreme points contain important information about the set. To generalize this notion to the sets defined above, we will be interested in the following local notion of extreme points.

Definition II.2.5. Let $A \subseteq \mathbb{R}^n$. We call $z \in A$ a **local extreme point** of A, if there is a neighborhood $U \subseteq \mathbb{R}^n$ of z such that z is an extreme point of $\operatorname{conv}(A \cap U)$. We denote the set of all local extreme points of A by $\operatorname{locextr}(A)$.

Classically, in convex optimization, the interest in extreme points stems from the fact that linear functions attain their minimum or maximum on these points. Similarly, the following holds for local extreme points.

Remark II.2.6. Let $A \subseteq \mathbb{R}^n$, and $\varphi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ and $z_\varphi \in A$ a (strict) local minimal point of φ in A. Then z_φ is also a local extreme point of A. Conversely, let $z \in A$ be a local extreme point of A, then there is $\varphi_z \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ such that z is a local minimal point of φ in A.

Example II.2.7. We more generally examine the local extreme points of the hyperbolic slices discussed above which are similar to the one in Figure II.1. We consider again the linear map

$$\begin{array}{cccc} L: & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ & (z_1,z_2,z_3,z_4) & \longmapsto & (z_1,z_2) \end{array},$$

and we examine local extreme points of the family of slices $\mathcal{H}_L(0,a)$, with $a \in \mathbb{R}$. Then we find that the local extreme points in this case are

$$\operatorname{locextr}(\mathcal{H}_L(0,a)) = \mathcal{H}_L^2(0,a) = \left\{ \left(0, a, \pm\left(\sqrt{-\frac{2a}{3}}\right)^3, -\frac{a^2}{12}\right), \left(0,a,0,\frac{a}{2}\right) \right\}.$$

By examining the resultants of the corresponding quartic polynomials and their second derivative, one finds that each of these local extreme points corresponds to hyperbolic polynomials with at most two distinct roots.

As a first result, we are now going to establish that the above example generalizes in the following sense. For a general hyperbolic slice, defined through k linear conditions, the local extreme points can be characterized as hyperbolic polynomials of the k-boundary. This generalizes Theorem [Rie12, Theorem 4.2] to general hyperbolic slices.

Theorem II.2.8. *The local extreme points of a hyperbolic slice are contained in the k-boundary, i.e.,*

$$\operatorname{locextr}(\mathcal{H}_L(a)) \subseteq \mathcal{H}_L^k(a).$$

Proof. Let $z \in \mathcal{H}_L(a)$ be a local extreme point, i.e., there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(\mathcal{H}_L(a) \cap U)$. We assume that $z \notin \mathcal{H}_L^k(a)$ and want to find a contradiction. To this end, we want to find $c \in \ker L$ non-zero such that $z \pm \varepsilon c \in \mathcal{H}_L(a)$ for all $\varepsilon > 0$ small enough. Consider $f := T^n - z_1T^{n-1} + \cdots + (-1)^n z_n$ with distinct roots $x_1, \ldots, x_m$ where $m > k$ and factor as follows:

$$f = \underbrace{\prod_{i=1}^{m}(T - x_i)}_{=:p} \cdot q,$$

where the set of zeros of q contains only elements from $\{x_1, \ldots, x_m\}$ and q is of degree $n-m$. Write $q = T^{n-m} + q_1T^{n-m-1} + \cdots + q_{n-m}$ and define $q_0 := 1$ and consider the linear map

$$\begin{array}{cccc} \chi: \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ y & \longmapsto & \left(\sum_{i+j=1} q_i y_j, \ldots, \sum_{i+j=n} q_i y_j\right) \end{array}.$$

Since $m > k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$. We define $h := b_1T^{m-1} + \cdots + b_m$ and $g := h \cdot q = c_1T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Now, because p has no multiple roots, $p \pm \varepsilon h$ is hyperbolic for $\varepsilon > 0$ small enough: the roots depend continuously on the coefficients and complex roots come as conjugated pairs (see Remark II.2.3). Hence

$$(p \pm \varepsilon h) \cdot q = f \pm \varepsilon h \cdot q = f \pm \varepsilon g$$

is hyperbolic for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \mathcal{H}_L(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\mathcal{H}_L(a) \cap U)$. ■

Remark II.2.9. If the map L is not surjective, one can obtain similar results by replacing k with $\operatorname{rank} L$.

In view of Remark II.2.6 we get the following.

Corollary II.2.10. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a linear or concave function and consider the optimization problem*

$$\min_{z \in \mathcal{H}_L(a)} g(z).$$

Let M denote the set of minimizers of this problem. If $\mathcal{H}_L(a)$ is non-empty and compact, then we have $M \cap H_L^k(a) \neq \emptyset$. In particular $H_L(a)$ contains a point $z \in \mathcal{H}_L^k(a)$.

Proof. Since $\mathcal{H}_L(a)$ is compact, there is a minimizer $z \in M$ such that z is an extreme point of the convex hull of $\mathcal{H}_L(a)$. In particular, z is a local extreme point of $\mathcal{H}_L(a)$ and therefore on the k-boundary of $\mathcal{H}_L(a)$ by Theorem II.2.8, i.e., $z \in M \cap \mathcal{H}_L^k(a)$. ■

As can be observed in the example shown in Figure II.1 connected components of hyperbolic slices appear to have a similarity to polytopes. They are not convex but appear to be "deflated" polytopes. To make this a bit more concrete we show that a generic hyperbolic slice has only finitely many local extreme points. This, in particular, implies that their convex hull, or in fact the convex hull of each of its connected components, is a polytope. The proof uses elementary properties of subdiscriminants. The relevance of subdiscriminants for counting roots of real univariate polynomials is explained in [BPR03, Chapter 4].

Definition II.2.11. Let $f \in \mathbb{R}[T]$ be a monic polynomial of degree n with roots $x_1, \ldots, x_n$ in $\mathbb{C}$. Then the $(n-m)$**-subdiscriminant**, $1 \leq m \leq n$, of f is defined as

$$\operatorname{sDisc}_{n-m}(f) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=m}} \prod_{\substack{i,j \in I \\ j>i}} (x_i - x_j)^2.$$

Remark II.2.12. Each $(n-m)$-subdiscriminant of f is defined above as a polynomial of degree $m(m-1)$ in terms of the roots of f. Noticing that each of the expressions is, in turn, symmetric in the roots, one immediately obtains that each subdiscriminant of f can be expressed in the elementary symmetric polynomials evaluated at the roots, i.e., in the coefficients of f. Indeed, the subdiscriminants of f can be obtained directly by minors of the Sylvester matrix - also called subresultants - of f and f'. So the degree of each $(n-m)$-subdiscriminant expressed in the coefficients is $2m-2$ [BPR03, Proposition 4.27].

Proposition II.2.13. *[BPR03, Remark 4.6 and Proposition 4.50] A monic polynomial $f \in \mathbb{R}[T]$ of degree n has exactly k distinct roots if and only if*

$$\mathrm{sDisc}_0(f) = \cdots = \mathrm{sDisc}_{n-k-1}(f) = 0,\ \mathrm{sDisc}_{n-k}(f) \neq 0.$$

Moreover, if and only if additionally

$$\mathrm{sDisc}_{n-k}(f) > 0, \ldots, \mathrm{sDisc}_{n-1}(f) > 0,$$

then f has only real roots.

Theorem II.2.14. *The k-boundary $\mathcal{H}_L^k(a)$ of a generic hyperbolic slice is finite. In particular, a generic hyperbolic slice has only finitely many local extreme points. The number of those points is bounded by*

$$\min\left\{2^{n-k}\frac{(n-1)!}{(k-1)!}, \binom{n}{k}\frac{(n-1)!}{(k-1)!}\right\}.$$

Proof. First, we establish that for a generic hyperbolic slice the k-boundary $\mathcal{H}_L^k(a)$ is finite. For this recall that the set of hyperbolic polynomials with at most k distinct roots, $\mathcal{H}^k$, is of dimension k by Remark II.2.3. Therefore, a generic $(n-k)$-dimensional affine linear subspace will intersect $\mathcal{H}^k$ in only finitely many points. Furthermore, in view of Proposition II.2.13 we see further that $\mathcal{H}^k$ is contained in the algebraic set defined by the vanishing of $n-k$ polynomials. On the one hand, each of the subdiscriminants describing this algebraic set is a homogeneous polynomial of degree $(2n-2), (2n-4), \ldots, (2k)$ expressed in the elementary symmetric polynomials by Remark II.2.12 and we can apply Bézout's Theorem to obtain the bound

$$2^{n-k}\frac{(n-1)!}{(k-1)!}.$$

On the other hand, we can apply the weighted Bézout's Theorem (see [Mon21, chapter VIII]): We assign to the i-th elementary symmetric polynomial e_i the weight i. Then each subdiscriminant is weighted homogeneous of degree $n(n-1), (n-1)(n-2), \ldots, (k+1)k$. Indeed, this is exactly the degree of the subdiscriminants expressed in the roots. Furthermore, we can bound the weighted degree of each of the k affine hyperplanes describing our slice by $n, n-1, \ldots, n-k+1$. So we obtain the bound

$$\frac{1}{n!}\frac{n!}{(n-k)!}\cdot\frac{n!(n-1)!}{k!(k-1)!} = \binom{n}{k}\frac{(n-1)!}{(k-1)!}.$$

■

Remark II.2.15. The second bound obtained in II.2.14 by the weighted Bézout's Theorem can even be refined when one considers the coefficients appearing in $L(z)$ for $z \in \mathcal{H}$. For example, if just the first coefficients are fixed, i.e., $L(z) = (z_1, \ldots, z_k)$, then $\binom{n}{k}$ can be replaced by 1.

Since the extreme points of the convex hull of a set are local extreme points, we can deduce the following.

Corollary II.2.16. *The convex hull of a generic hyperbolic slice is a polyhedron. The same applies to any of its connected components.*

Note that the proof of Theorem II.2.14 together with Proposition II.2.13 gives an explicit description of the k-boundary of a hyperbolic slice as a semi-algebraic set. The following example shows that the k-boundary of a hyperbolic slice can be infinite. But even in this case, there might only be finitely many local extreme points.

Example II.2.17. Consider $L:\ \mathbb{R}^4 \to \mathbb{R}^3,\ (z_1, z_2, z_3, z_4) \mapsto (z_1, z_3, z_4)$ and $a \in \mathbb{R}$. Then

$$\mathcal{H}_L(a,0,0) = \left\{ (a, z_2, 0, 0) \;\middle|\; z_2 \in \mathbb{R},\ z_2 \leq \frac{a^2}{4} \right\} = \mathcal{H}_L^3(a,0,0)$$

is not finite. But $\mathcal{H}_L(a,0,0)$ is obviously convex with only local extreme point

$$\left(a, \frac{a^2}{4}, 0, 0\right) \in \mathcal{H}_L^2(a,0,0).$$

Next, we will give sufficient conditions on L for the compactness of a hyperbolic slice and for the existence of local extreme points. For that, we will need the following definition.

Definition II.2.18. Let $f, g \in \mathbb{R}[T]$ be hyperbolic polynomials with real roots $\alpha_n \leq \cdots \leq \alpha_1$ and $\beta_m \leq \ldots \leq \beta_1$ respectively. We say that g **interlaces** f if $\alpha_n \leq \beta_m \leq \alpha_{n-1} \leq \ldots \leq \alpha_1$ or $\beta_m \leq \alpha_n \leq \beta_{m-1} \leq \ldots \leq \alpha_1$. Furthermore, we say f and g are **interlacing**, if f interlaces g or g interlaces f.

Remark II.2.19. If g interlaces f, then clearly f and g either have the same degree, i.e., $n = m$ or the degree of g is smaller by one, i.e., $m = n - 1$.

The following classical result (see [Ded92, Theorem 4.1.]) connects interlacing polynomials to linear pencils of hyperbolic polynomials.

Theorem II.2.20 (Dedieu)**.** *Let $f, g \in \mathbb{R}[T]$ be hyperbolic, non-zero polynomials of degree at most n. Then the following statements are equivalent:*

1. *f and g are interlacing.*
2. *$f + \xi \cdot g$ is hyperbolic for any $\xi \in \mathbb{R}$.*

From now on we express L in terms of k linearly independent linear forms $l_1, \ldots, l_k \in \mathbb{R}[Z_1, \ldots, Z_n]_1$ as $L:\ \mathbb{R}^n \to \mathbb{R}^k,\ z \mapsto (l_1(z), \ldots, l_k(z))$. We can use the results above to give a sufficient condition on $l_1, \ldots, l_k$ for the existence of local extreme points of a hyperbolic slice.

Lemma II.2.21. *If $Z_1 \in \operatorname{span}(l_1, \dots, l_k)$ and $\mathcal{H}_L(a) \neq \emptyset$, then $\mathcal{H}_L(a)$ has a local extreme point.*

Proof. Let $z \in \mathcal{H}_L(a)$ and write $Z_1 = \sum_{i=1}^k \lambda_i l_i$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. Furthermore, denote by $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the roots of

$$f_z := T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n.$$

Then $e_1(x) = z_1 = \sum_{i=1}^k \lambda_i l_i(z) = \sum_{i=1}^k \lambda_i a_i$ and hence

$$z_2 = e_2(x) = \frac{1}{2}\left(e_1(x)^2 - \sum_{i=1}^n x_i^2\right) \leq \frac{1}{2} e_1(x)^2 = \frac{1}{2}\left(\sum_{i=1}^k \lambda_i a_i\right)^2.$$

So the optimization problem

$$\max_{z \in \mathcal{H}_L(a)} z_2$$

has a non-empty set of maximizers M. Suppose $\mathcal{H}_L(a)$ has no local extreme point. Then M contains a line, i.e., there is a maximizer $m = (m_1, \dots, m_n) \in M$ and a $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ non-zero such that $y_1 = y_2 = 0$ and $m + \xi y \in \mathcal{H}$ for all $\xi \in \mathbb{R}$. This means $f := T^n - m_1 T^{n-1} + \cdots + (-1)^n m_n$ and $g := -y_3 T^{n-3} + \cdots + (-1)^n y_n$ are interlacing by II.2.20, which is not possible because of degree reasons. ■

We can use the existence of an extreme point, for example, to obtain the following result which connects to polynomial interpolation.

Corollary II.2.22. *Consider the set of polynomials of degree n, which solve a k-points interpolation problem. Then there exists a hyperbolic polynomial in this set if and only if there exists one with at most $k+1$ distinct roots.*

Proof. Under the conditions, the corresponding hyperbolic slice has at least one extreme point by Lemma II.2.21. ■

By prescribing not only the first but also the second-highest coefficient of a monic polynomial, one directly obtains a sufficient condition for the compactness of a hyperbolic slice.

Lemma II.2.23. *If $Z_1, Z_2 \in \operatorname{span}(l_1, \dots, l_k)$, then $\mathcal{H}_L(a)$ is compact.*

Proof. As the empty set is compact we can assume that there is $z \in \mathcal{H}_L(a)$. Furthermore we write $Z_1 = \sum_{i=1}^k \lambda_i l_i$ and $Z_2 = \sum_{i=1}^k \chi_i l_i$ for some $\lambda_1, \dots, \lambda_k, \chi_1, \dots, \chi_k \in \mathbb{R}$ and denote by $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the roots of

$$f_z := T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n.$$

Then $e_1(x) = z_1 = \sum_{i=1}^k \lambda_i l_i(x) = \sum_{i=1}^k \lambda_i a_i$ and $e_2(x) = \sum_{i=1}^k \chi_i a_i$ and hence

$$\sum_{i=1}^n x_i^2 = e_1(x)^2 - 2e_2(x) = \left(\sum_{i=1}^k \lambda_i a_i\right)^2 - \sum_{i=1}^k \chi_i a_i.$$

This shows that x is contained in a ball, thus $\mathcal{H}_L(a)$ is bounded. Furthermore, as the roots of a polynomial depend continuously on the coefficients it is clear that $H_L(a)$ is closed and therefore compact (see Remark II.2.6). ■

We close this section with a selection of examples of two-dimensional hyperbolic slices that highlight the various mentioned scenarios.

Example II.2.24. Consider $\mathcal{H}_L(a_2, a_4)$, where $a := (a_2, a_4) \in \mathbb{R}^2$ such that $a_2 < 0$ and $a_4 > 0$ and

$$L : \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ (z_1, z_2, z_3, z_4) & \longmapsto & (z_2, z_4) \end{array}.$$

Then, there are the following three possible situations.

a: If $a := (a_2, a_4)$ satisfy $a_2^2 - 4a_4 < 0$, the hyperbolic slice $\mathcal{H}_L(a)$ will contain two local extreme points. In particular, $\mathcal{H}_L^2(a) \neq \emptyset$. Furthermore, the local extreme points of $\mathcal{H}_L(a)$ are not global extreme points. Therefore, they are not extreme points of the convex hull of $\mathcal{H}_L(a)$. This is illustrated in Figure II.2a.

b: For all values $a := (a_2, a_4)$ with $a_2^2 - 4a_4 = 0$, $\mathcal{H}_L(a)$ will contain no local extreme points. But the 2-boundary of $\mathcal{H}_L(a)$ is non-empty. Indeed,

$$T^4 + a_2T^2 + a_4 = \left(T - \sqrt{\frac{-a_2}{2}}\right)^2 \left(T + \sqrt{\frac{-a_2}{2}}\right)^2,$$

and thus $(0, a_2, 0, a_4) \in \mathcal{H}_L^2(a)$. This situation is illustrated in Figure II.2b.

c: For the values $a := (a_2, a_4)$ with $a_2^2 - 4a_4 > 0$, $\mathcal{H}_L(a)$ will contain no local extreme point. Moreover, $\mathcal{H}_L^2(a)$ is empty in this case, while $\mathcal{H}_L(a) \neq \emptyset$. This is illustrated in Figure II.2c.
Indeed, the polynomial $f = T^4 + a_2T^2 + a_4$ is hyperbolic with the 4 distinct roots

$$x_{1,2,3,4} := \pm\sqrt{\frac{-a_2 \pm \sqrt{a_2^2 - 4a_4}}{2}}.$$

Therefore, the hyperbolic slice $\mathcal{H}_L(a)$ is non-empty. On the other hand, suppose that the 2- boundary $\mathcal{H}_L^2(a)$ is non-empty, i.e., that we can find $(a_1, a_2, a_3, a_4) \in \mathcal{H}_L^2(a)$. This in turn implies that there are $x, y \in \mathbb{R}$ such that the polynomial

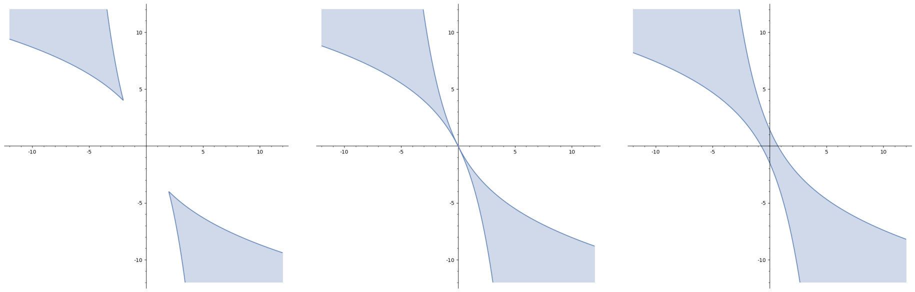
$$f_a := T^4 - a_1T^3 + a_2T^2 - a_3T + a_4$$

factors either as

$$f_a = (T - x)^3(T - y) \text{ or } f_a = (T - x)^2(T - y)^2.$$

In the first case a comparison of coefficients shows $a_2 = 3xy + 3x^2$ and a_4x^3y. Since $a_4 > 0$ we must have $x, y \neq 0$ and can solve $y = \frac{a_4}{x^3}$. This implies

$a_2 = \frac{3a_4}{x^2} + 3x^2$ and $3x^4 - a_2x^2 + 3a_4 = 0$. However, since $x \neq 0$, $a_2 < 0$ and $a_4 > 0$ we must have $3x^4 - a_2x^2 + 3a_4 > 0$, and thus have a contradiction. Analogously, for the second case, comparing coefficients shows $a_2 = 4xy + x^2 + y^2$ and $a_4 = x^2y^2$. We solve for y and get $y = \pm\frac{\sqrt{a_4}}{x}$ from which we find $a_2 = \frac{a_4}{x^2} + x^2 \pm 4\sqrt{a_4}$. But since $a_2 < 0$, $a_4 > 0$ and $a_2^2 - 4a_4 > 0$ the resulting polynomial equation $x^4 + (\pm 4\sqrt{a_4} - a_2)x^2 + a_4 = 0$ clearly has no real solution.



(a) $\mathcal{H}_L(-3, 4)$ contains the local extreme points $(\pm 2, -3, \mp 4, 4)$.

(b) $\mathcal{H}_L(-4, 4)$ has no local extreme point and $(0, -4, 0, 4) \in \mathcal{H}^2$.

(c) $\mathcal{H}_L(-5, 4)$ has no local extreme point and $\mathcal{H}_L^2(-5, 4) = \emptyset$.

Figure II.2: Illustrations of the 3 different situations occurring in Example II.2.24

II.3 Positivity of symmetric polynomial functions

In this section we will study real polynomial functions defined by symmetric polynomials. Since every symmetric polynomial can be written in a unique way as a polynomial in elementary symmetric polynomials, we can use the geometric description of hyperbolic slices obtained before to characterize the minimal points of a large class of symmetric polynomial functions which are sparse in an appropriate sense (see Definition II.3.5). Various authors had already observed that certain symmetric functions attain their minimal values on symmetric points (e.g. [For87; Kei67; KKR12]). Other authors found that symmetric polynomial functions of a bounded small enough degree attain their minima on points with few distinct coordinates (e.g. [Rie12; Tim03]). We generalize these results by considering symmetric polynomial functions which are completely characterized through their values on points with at most k distinct coordinates.

II.3.1 The notions of k-completeness and k-testability

Definition II.3.1. For $k \in \mathbb{N}$ we consider the set

$$\mathcal{A}_k := \left\{ x \in \mathbb{R}^n \;\middle|\; |\{x_1, \ldots, x_n\}| \leq k \right\}$$

of points with at most k different coordinates. Given a symmetric polynomial $f \in \mathbb{R}[\underline{X}]$ and $S \subseteq \mathbb{R}^n$ we say that f is

1. k-**complete** on S if
$$f(S) = f(S \cap \mathcal{A}_k).$$
2. k-**testable** on S if
$$\inf_{x \in S} f(x) = \inf_{x \in S \cap \mathcal{A}_k} f(x).$$

In case $S = \mathbb{R}^n$ we may omit it and just speak of k-testable and k-complete polynomials.

The two notions of k-complete and k-testable are very closely connected, but the first one is stronger, while the second one might be interesting in particular in the context of optimization. In order to motivate the study of this class, we exemplify first how algorithmic problems can be substantially simplified for k-complete and k-testable symmetric polynomials.

Definition II.3.2. A decreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ which sums up to n is called a **partition of n into k parts**. We will write $\lambda \vdash_k n$ to denote that λ is a partition of n into k parts. Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial. Then for $\lambda \vdash_k n$ we define
$$f^{\lambda} := f(\underbrace{X_1, \ldots, X_1}_{\lambda_1-\text{times}}, \ldots, \underbrace{X_k, \ldots, X_k}_{\lambda_k-\text{times}}) \in \mathbb{R}[X_1, \ldots, X_k].$$

Note that the number of partitions of n into k parts is at most $\binom{n+k}{k}$ and thus polynomial in n for a fixed k. Therefore the above notion allows reducing, for example, the question of whether a symmetric polynomial in n variables is non-negative to a polynomial number of such queries in k variables. It is, for example, known to be NP-hard to decide the non-negativity of a given polynomial of degree 4 (see e.g. [Blu+98] or [MK87]). Clearly, by applying the above procedure, one can obtain algorithmic simplifications that yield polynomial complexity for this kind of problem (see also [Fau+23] where this method is applied also for other algorithmic questions). We highlight in particular the following version of Artin's solution to Hilbert's 17th problem for k-complete symmetric polynomials, which is a direct consequence of the sketched procedure of identifying variables.

Proposition II.3.3 (Hilbert's 17th problem for k-complete polynomials)**.** *Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric k-testable polynomial. Then f attains only non-negative values on $\mathbb{R}^n$ if and only if for all $\lambda \vdash_k n$ we can find a sum of squares of polynomials $t \in \sum \mathbb{R}[X_1, \ldots, X_k]^2$ such that $t \cdot f^{\lambda}$ is also a sum of squares of polynomials.*

The main interest in the statements presented above is that the reduction of dimension also gives new complexity bounds for the degrees of the polynomials in question. For example, for Hilbert's 17th problem for k-complete polynomials, we can adapt the currently known complexity bounds.

Remark II.3.4. Let f be a n-variate k-complete polynomial of degree d. Then f is non-negative if and only if we can write each f^{λ} as a sum of at most 2^k

rational squares by [Pfi67]. We can also write each f^λ as a sum of squares of rational functions, where, following [LPR14], we obtain the following degree bounds for the numerators and denominators:

$$2^{2^{2^{d^{4^k}}}}.$$

II.3.2 Sufficient and quasi-sufficient polynomials

Now, we want to show that it is possible to produce a large class of k-complete symmetric polynomials based on the results on hyperbolic polynomials. Throughout this section we fix $1 \leq k \leq n$ and consider the k linearly independent linear forms $l_1, \ldots, l_k \in \mathbb{R}[Z_1, \ldots, Z_n]_1$ and the linear map $L: \mathbb{R}^n \to \mathbb{R}^k,\ z \mapsto (l_1(z), \ldots, l_k(z))$. Recall that a symmetric polynomial $f \in \mathbb{R}[\underline{X}]$ can be written uniquely in terms of the elementary symmetric polynomials, say $f = g(e_1, \ldots, e_n)$. Now evaluation of f in a point $x \in \mathbb{R}^n$ translates into the evaluation of g in a point $z \in \mathcal{H}$ and the evaluation of f on $\mathcal{A}_k$ translates into the evaluation of g on $\mathcal{H}^k$. By partitioning

$$\mathcal{H} = \bigcup_{a \in \mathbb{R}^k} \mathcal{H}_L(a) \text{ and } \mathcal{H}^k = \bigcup_{a \in \mathbb{R}^k} \mathcal{H}_L^k(a)$$

for the map L, we can use our previous results to show under some mild conditions that f is k-complete or K-testable if it allows for a special representation in terms of k linear forms of elementary symmetric polynomials. We define these representations in the following.

Definition II.3.5. Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial and write f in terms of elementary symmetric polynomials, say $f = g(e_1, \ldots, e_n)$ for some $g \in \mathbb{R}[Z_1, \ldots, Z_n]$.

1. We say that f is $(l_1, \ldots, l_k)$**-sufficient** if $g \in \mathbb{R}[l_1, \ldots, l_k]$.
2. We say that f is $(l_1, \ldots, l_k)$**-quasi-sufficient** if f admits a representation of the form
$$f = f_0 + f_1 e_1 + \cdots + f_n e_n$$
for some $(l_1, \ldots, l_k)$-sufficient polynomials $f_0, \ldots, f_n$.
3. Furthermore, we say that f is $(l_1, \ldots, l_k)$**-concave-sufficient** if g is concave on $H_L(a)$ for all $a \in \mathbb{R}^k$.

Moreover, we say that a symmetric semi-algebraic set $S \subseteq \mathbb{R}^n$ is $(l_1, \ldots, l_k)$**-sufficient**, if it can be described by $(l_1, \ldots, l_k)$-sufficient polynomials.

The following proposition is a direct consequence of the unique representation of a symmetric polynomial of degree d in terms of the elementary symmetric polynomials and may serve as a motivation for the definitions given above.

Proposition II.3.6. *Let $f \in \mathbb{R}[\underline{X}]$ be symmetric of degree d. Then f is $(Z_1, \ldots, Z_d)$-sufficient and $\left(Z_1, \ldots, Z_{\lfloor \frac{d}{2} \rfloor}\right)$-quasi-sufficient.*

Remark II.3.7. The notions defined above are increasingly strict in the following sense: Sufficiency (1) implies quasi-sufficiency (2), which in turn implies concave-sufficiency (3) of both f and $-f$.

The results on hyperbolic slices now translate to the following statements on symmetric real polynomial functions.

Theorem II.3.8. *Let $S \subseteq \mathbb{R}^n$ be a symmetric $(l_1, \ldots, l_k)$-sufficient semi-algebraic set and let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial.*

1. *If f is $(l_1, \ldots, l_k)$-sufficient and if every non-empty hyperbolic slice $\mathcal{H}_L(a)$ contains a local extreme point, then f is k-complete on S.*

2. *If f is $(l_1, \ldots, l_k)$-concave-sufficient and $\mathcal{H}_L(a)$ is compact for all $a \in \mathbb{R}^k$, then f is k-testable on S.*

3. *If f is $(l_1, \ldots, l_k)$-quasi-sufficient and $\mathcal{H}_L(a)$ is compact for all $a \in \mathbb{R}^k$ and $S \cap \mathcal{A}_k$ is connected, then f is k-complete on S.*

4. *If f is $(l_1, \ldots, l_k)$-concave-sufficient and not $(l_1, \ldots, l_k)$-sufficient and*
$$\inf_{x \in S} f(x) > -\infty,$$
then f is k-testable on S.

Proof. (1): Let $g \in \mathbb{R}[Z_1, \ldots, Z_n]$ such that $f = g(e_1, \ldots, e_n)$. Let $x \in S$ and consider $z := \Gamma(x)$ and $a := L(z)$. There is $\tilde{z} \in \mathcal{H}_L^k(a)$ by Theorem II.2.8 since $\mathcal{H}_L(a)$ admits a local extreme point. So there is $\tilde{x} \in \mathcal{A}_k$ with $\Gamma(\tilde{x}) = \tilde{z}$. Then $f(x) = f(\tilde{x})$ and $\tilde{x} \in S$ since f and S are $(l_1, \ldots, l_k)$-sufficient.

(2): Let $g \in \mathbb{R}[Z_1, \ldots, Z_n]$ such that $f = g(e_1, \ldots, e_n)$. Let $x \in S$ and consider $z := \Gamma(x)$ and $a := L(z)$. Since g is concave on $L^{-1}(a)$ by the concave-sufficiency of f and $\mathcal{H}_L(a)$ is compact we can apply Corollary II.2.10 and get that
$$\min_{y \in \mathcal{H}_L(a)} g(y) = \min_{y \in \mathcal{H}_L^k(a)} g(y),$$
i.e., there is $\tilde{z} \in \mathcal{H}_L^k(a)$ with $g(\tilde{z}) \leq g(z)$. Let $\tilde{x} \in \mathcal{A}_k$ with $\Gamma(\tilde{x}) = \tilde{z}$. Then $f(\tilde{x}) \leq f(x)$ and $\tilde{x} \in S$ since S is $(l_1, \ldots, l_k)$-sufficient and we can conclude that f is k-testable on S.

(3): Let $x_0 \in S$. We can apply (2) since f and $-f$ are both $(l_1, \ldots, l_k)$-concave-sufficient by Remark II.3.7 and get that
$$\inf_{x \in S} f(x) = \inf_{x \in S \cap \mathcal{A}_k} f(x) \quad \text{and} \quad \sup_{x \in S} f(x) = \sup_{x \in S \cap \mathcal{A}_k} f(x),$$
so there are $x_1, x_2 \in S \cap \mathcal{A}_k$ with $f(x_1) \leq f(x_0)$ and $f(x_2) \geq f(x_0)$. Since $S \cap \mathcal{A}_k$ is connected there is $\tilde{x} \in S \cap \mathcal{A}_k$ with $f(\tilde{x}) = f(x_0)$ by the intermediate value theorem.

(4): Let $g \in \mathbb{R}[Z_1, \ldots, Z_n]$ such that $f = g(e_1, \ldots, e_n)$. There is $x_0 \in S$ with
$$\inf_{x \in S} f(x) = f(x_0)$$

consider $z_0 := \Gamma(x_0)$ and $a := L(z)$. Since g is concave and not constant on $\mathcal{H}_L(a)$, g attains its minimum on an extreme point of $\mathcal{H}_L(a)$, i.e., we can assume that $z_0 \in \mathcal{H}_L^k(a)$ and therefore $x_0 \in \mathcal{A}_k$. ∎

The existence of local extreme points in Theorem II.3.8 (1) is indeed necessary, as in cases without local extreme points it is possible to construct situations where the statement will not hold. We showcase this in the following.

Example II.3.9. Let $K(h) = \mathbb{R}^4$, $l_1 := Z_2$, $l_2 := Z_4$ and $L:\ \mathbb{R}^4 \to \mathbb{R}^2,\ z \mapsto (l_1(z), l_2(z))$ and consider the (l_1,l_2)-sufficient symmetric polynomial

$$f = (e_2+5)^2 + (e_4-4)^2 \in \mathbb{R}[X_1,X_2,X_3,X_4].$$

The 2-boundary $\mathcal{H}_L^2(-5,4)$ is empty by Example II.2.24 (3). So $f(x) > 0$ for all $x \in \mathcal{A}_2$, but $f(1,-1,2,-2) = 0$.

One can in fact prove that the polynomial f in Example II.3.9 is still 3-complete. Indeed, the necessity of the existence of an extreme point in every hyperbolic slice seems to restrict the applications of Theorem II.3.8. However, by applying Lemma II.2.21 and Lemma II.2.23 we can obtain the following version of Theorem II.3.8 which avoids this issue at the price of a slightly weaker conclusion.

Corollary II.3.10. *Let $S \subseteq \mathbb{R}^n$ be a symmetric $(l_1,\dots,l_k)$-sufficient semi-algebraic set and let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial.*

1. *If f is $(l_1,\dots,l_k)$-sufficient, then f is $(k+1)$-complete on S.*
2. *If f is $(l_1,\dots,l_k)$-concave-sufficient, then f is $(k+2)$-testable on S.*
3. *If $f \in \mathbb{R}[\underline{X}]$ is $(l_1,\dots,l_k)$-quasi-sufficient and $S \cap \mathcal{A}_k$ is connected, then f is $(k+2)$-complete on S.*

Moreover if $Z_1 \in \operatorname{span}(l_1,\dots,l_k)$, then $(k+1)$ in (1) can be replaced by k-complete. If $Z_1, Z_2 \in \operatorname{span}(l_1,\dots,l_k)$, then $(k+2)$ in (2) and (3) can be replaced by k.

The results in this section were given entirely for symmetric functions. To conclude this section we remark the following direct translation of the results to even symmetric polynomials or equivalently copositive symmetric polynomials.

Remark II.3.11. The results on symmetric polynomials translate directly to even symmetric polynomials, i.e., polynomials invariant by the natural action of the Hyperoctahedral group $S_2 \wr S_n$. Denote by

$$\begin{aligned}\mathcal{E} :&= \left\{ z \in \mathbb{R}^n \;\middle|\; T^{2n} - z_1 T^{2(n-1)} + \dots + (-1)^n z_n \text{ is hyperbolic} \right\} \\ &= \left\{ z \in \mathcal{H} \;\middle|\; T^n - z_1 T^{n-1} + \dots + (-1)^n z_n \text{ has only non-negative roots} \right\}\end{aligned}$$

the set of even hyperbolic polynomials. Furthermore, we define

$$\mathcal{E}^k := \left\{ z \in \mathcal{E} \;\middle|\; T^n - z_1 T^{n-1} + \dots + (-1)^n z_n \text{ has at most } k \text{ positive roots} \right\}$$

and $\mathcal{E}_L(a) := \mathcal{E} \cap L^{-1}(a)$ and $\mathcal{E}_L^k(a)$ accordingly. Then the proof of Theorem II.2.8 translates to $\operatorname{locextr}(\mathcal{E}_L(a)) \subseteq \mathcal{E}_L^k(a)$ and both sets are generically finite. By replacing $\mathcal{A}_k$ by

$$\mathcal{B}_k := \left\{x \in \mathbb{R}^n \;\middle|\; |\{x_1^2, \dots, x_n^2\} \setminus \{0\}| \leq k\right\}$$

we can transfer the statements of Theorem II.3.8 and Corollary II.3.10 about k-completeness and k-testability of (quasi-)sufficient symmetric polynomials to (quasi-)sufficient even symmetric polynomials f, i.e., polynomials that admit a representation of the form

$$f = g(e_1(X_1^2, \dots, X_n^2), \dots, e_n(X_1^2, \dots, X_n^2))$$

with $g \in \mathbb{R}[l_1, \dots, l_k]$. Note that in this case, it suffices already to fix the first coefficient to obtain compactness, so one can replace $(k+2)$ in Corollary II.3.10 (2) and (3) by $(k+1)$.

II.3.3 Deciding sufficiency

Generally, the definition of sufficient and quasi-sufficient given above can appear to be not directly verifiable. Especially since most often one is given a symmetric polynomial without its representation in terms of linear combinations of elementary symmetric polynomials. Therefore, we want to briefly present how to algorithmically approach the question if a given symmetric polynomial is sufficient or quasi-sufficient. In order to decide if a symmetric polynomial $f \in \mathbb{R}[\underline{X}]$ is sufficient for some collection of linear forms $l_1, \dots, l_k$ one has principle two task:

1. Finding a representation of $f = g(e_1, \dots, e_n)$ in terms of elementary symmetric polynomials: This can be achieved, for example, by using the Gröbner basis $G := \{g_1, \dots, g_k\}$, where

$$g_k = \sum_{\substack{\alpha \in \mathbb{N}_0^{n-k+1} \\ |\alpha| = k}} X_k^{\alpha_1} \cdots X_n^{\alpha_{n-k+1}} + \sum_{i=1}^{k} (-1)^i Y_i \sum_{\substack{\alpha \in \mathbb{N}_0^{n-k+1} \\ |\alpha| = k-i}} X_k^{\alpha_1} \cdots X_n^{\alpha_{n-k+1}}$$

   of the ideal $I = (e_1 - Y_1, \dots, e_n - Y_n) \subseteq \mathbb{R}[\underline{X}, Y_1, \dots, Y_n]$ which is independent from f and then by computing the remainder g of f on division by G. One obtains now $f = g(e_1, \dots, e_n)$ (see Proposition 4 and Proposition 5 in §1 of Chapter 7 in [Cox+94] for details). Alternatively one can use the algorithm presented in [Vu22].

2. Once $g \in \mathbb{R}[e_1, \dots, e_n]$ is obtained, one has to decide if there exist $k < n$ linear combinations $l_1, \dots, l_k$ of the $e_1, \dots, e_n$ such that $g \in \mathbb{R}[l_1, \dots, l_k]$. Also, this can be accomplished quite concretely, for example, by using the approach outlined by Carlini [Car06]. As described there, the smallest

number k of linear forms $l_1, \dots, l_k$ needed such that $g \in \mathbb{R}[l_1, \dots, l_k]$ is obtained by computing the rank of the Catalectican matrix of g. This matrix is obtained by the coefficients of the partial derivatives of g. More concretely, one can also explicitly construct these linear forms by computing a basis for the vector space of the $(d-1)$-th partial derivatives of g.

The steps described above rely mostly on linear algebra and can be efficiently implemented for larger numbers of variables.

Remark II.3.12. In the special case when one wants to decide if a symmetric polynomial f is $e_{i_1}, \dots, e_{i_m}$-quasi-sufficient (where $1 \leq i_1 \leq \dots \leq i_n \leq k$) one can actually proceed with the following examination of the gradient of f without going through the steps above: As a symmetric polynomial f can be written as $f = g(e_1, \dots, e_n)$ we have

$$\nabla f = \nabla g J_{e_1, \dots, e_n}.$$

Noting that $J_{e_1, \dots, e_n}$ is invertible over $\mathbb{R}(X_1, \dots, X_n)$ we get

$$\nabla f J^{-1}_{e_1, \dots, e_n} = \nabla g.$$

Now, if for $I \subseteq \{1, \dots, n\}$ the corresponding entries in ∇g are constants, then f is $(e_i)_{\{1,\dots,n\} \setminus I}$-quasi-sufficient.

We give a short example to illustrate the algorithmic approach.

Example II.3.13. We consider the following toy example of a symmetric polynomial in three variables to showcase the methods described above

$$f = \sum_{\sigma \in S_3} \sigma \left(\frac{1}{2} X_1^3 + X_1^2 X_2^2 + 3X_1^2 X_2 + X_1^3 X_2 + X_1 X_2 X_3 - X_1^2 X_2^2 X_3^2 \right.$$
$$\left. + \frac{1}{2} X_1^3 X_2^3 X_3^2 - 2X_1^3 X_2^2 X_3 - X_1^3 X_2 X_3 - 2X_1^2 X_2^2 X_3 + \frac{5}{2} X_1^2 X_2 X_3 \right),$$

where S_3 acts on $\mathbb{R}[X_1, X_2, X_3]$ by permutation of variables.

The Gröbner basis corresponding to the ideal

$$I := \langle e_1 - Y_1, e_2 - Y_2, e_3 - Y_3 \rangle$$

is given by

$$G = \{X_1 + X_2 + X_3 - Y_1, X_2^2 + X_2 X_3 - X_2 Y_1 + X_3^2 - X_3 Y_1 + Y_2, X_3^3 - X_3^2 Y_1 + X_3 Y_2 - Y_3\}.$$

By computing the remainder of f on division by G one obtains

$$g = Y_1^3 + Y_1^2 Y_2 - 2Y_1^2 Y_3 - 2Y_1 Y_2 Y_3 + Y_1 Y_3^2 + Y_2 Y_3^2 \in \mathbb{R}[Y_1, Y_2, Y_3]$$

with $f = g(e_1, e_2, e_3)$. In order to compute the Catalactican of g, we fix a monomial basis

$$M = \{M_1, \dots, M_6\} = \{Y_1^2, Y_1 Y_2, Y_1 Y_3, Y_2^2, Y_2 Y_3, Y_3^2\}$$

for the ternary forms of degree $2 = \deg(g) - 1$. Calculating the partial derivatives

$$\partial_i g = c_{i1} M_1 + \cdots + c_{i6} M_6$$

we obtain he Catalactican C_g of g defined as $(C_g)_{ij} = c_{ij}$, i.e.

$$C_g = \begin{pmatrix} 3 & 2 & -4 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 & 0 & 1 \\ -2 & -2 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

The number of linear forms needed to express g is then equal to $\operatorname{rank}(C_g) = 2$. In order to find linear forms needed to express g, it suffices to compute a basis for the span of the second partial derivatives of g, we obtain

$$\{Y_1 - Y_3, Y_2 + Y_3\}$$

and indeed

$$g = (Y_2 + Y_3)(Y_1 - Y_3)^2 + (Y_1 - Y_3)^3,$$

i.e. f is $(Y_2 + Y_3, Y_1 - Y_3)$-sufficient and $(Y_1 - Y_3)$-quasi-sufficient.

II.4 Applications and examples

We will now show some applications of the theory developed here and use it on some concrete examples to underline the potential of the results presented. We begin with examining the following polynomial which was given by Robinson [Rob69] as an example of a non-negative form which is not a sum of squares. Note that this example could also be obtained by a variant of the half-degree principle to even symmetric polynomials.

Example II.4.1 (Robinson Polynomial)**.** The non-negativity of the Robinson polynomial

$$R = X^6 + Y^6 + Z^6 - \left(X^4Y^2 + X^2Y^4 + X^4Z^2 + X^2Z^4 + Y^4Z^2 + Y^2Z^4\right) + 3X^2Y^2Z^2$$

can be easily verified using Remark II.3.11. Indeed,

$$R = e_1(X^2, Y^2, Z^2)^3 - 4e_1(X^2, Y^2, Z^2)e_2(X^2, Y^2, Z^2) + 9e_3(X^2, Y^2, Z^2)$$

is a Z_1-quasi-sufficient even symmetric polynomial. Therefore, we only need to examine R on the set

$$\mathcal{B}_1 := \left\{x \in \mathbb{R}^3 \;\middle|\; |\{x_1^2, x_2^2, x_3^2\} \setminus \{0\}| \leq 1\right\}.$$

Since we easily find that the two (dehomogenized) univariate polynomials

$$R_1 = R(1, T, T) = T^4 - 2T^2 + 1 = (T-1)^2(T+1)^2$$
$$R_2 = R(1, T, 0) = T^6 - T^4 - T^2 + 1 = (T^2+1)(T-1)^2(T+1)^2.$$

are non-negative, R is indeed non-negative. Moreover, we directly also see that R has at least the 10 projective zeros

$$(1, \pm 1, \pm 1), (0, \pm 1, \pm 1), (\pm 1, 0, \pm 1), (\pm 1, \pm 1, 0)$$

which constitute the orbits of $(1, 1, 1)$ and $(1, 1, 0)$. One easily checks that these zeros are isolated. From this observation one immediately also obtains that R cannot be a sum of squares. Indeed, since a zero of a sum of squares also has to be a zero of every summand, a sextic which is a sum of squares can have at most 9 isolated zeros.

Furthermore, we will show how our results can be used to verify symmetric inequalities rather easily.

Example II.4.2 (AM–GM inequality)**.** The inequality of arithmetic and geometric means is a standard inequality from analysis, stating that for all $x \in \mathbb{R}^n_{\geq 0}$ we have

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n},$$

or equivalently

$$e_1^n - n^n e_n \geq 0 \text{ on } \mathbb{R}^n_{\geq 0}.$$

By squaring the variables this is equivalent to

$$F = e_1(X_1^2, \ldots, X_n^2)^n - n^n e_n(X_1^2, \ldots, X_n^2)$$

is non-negative, which can be proven by applying again Remark II.3.11 similarly to the previous example.

Example II.4.3 (Maclaurin's inequality)**.** More generally we have

$$\sqrt[i]{\frac{e_i(x)}{\binom{n}{i}}} \geq \sqrt[j]{\frac{e_j(x)}{\binom{n}{j}}}$$

for all $x \in \mathbb{R}^n_{\geq 0}$ and $i \leq j$ which is equivalent to

$$F = \binom{n}{j}^{2i} e_i(X_1^2, \ldots, X_n^2)^{2j} - \binom{n}{i}^{2j} e_j(X_1^2, \ldots, X_n^2)^{2i}$$

is non-negative. F is (Z_i)-concave-sufficient and even symmetric. First we show that $\inf_{x \in \mathbb{R}^k} f > -\infty$. Since F is in particular (Z_1, Z_i)-concave-sufficient, it suffices to show that

$$F_\lambda := F(\underbrace{X, \ldots, X}_{\lambda_1-\text{times}}, \underbrace{Y, \ldots, Y}_{\lambda_2-\text{times}}, \underbrace{0, \ldots, 0}_{\lambda_3-\text{times}})$$

is bounded from below for all partitions $\lambda_1 + \lambda_2 + \lambda_3 = n$. Since F_λ is homogeneous it suffices to show that the dehomogenization

$$\tilde{F_\lambda} = F_\lambda(X, 1)$$

has positive leading coefficient. It has leading coefficient

$$\binom{n}{j}^{2i}\binom{\lambda_1}{i}^{2j} - \binom{n}{i}^{2j}\binom{\lambda_1}{j}^{2i} > 0$$

for $i \leq \lambda_1 < n$ (this can be easily shown by induction on λ_1) and $\tilde{F_\lambda} = 0$ for $\lambda_1 = n$ and for $\lambda_1 < i$. Now we can use Theorem II.3.8 (4) and Remark II.3.11, so it suffices to check that

$$F_\mu := F(\underbrace{X, \dots, X}_{\mu-\text{times}}, \underbrace{0, \dots, 0}_{(n-\mu)-\text{times}})$$

is non-negative for all partitions $\mu + n - \mu = n$. Since F_μ is homogeneous it suffices to show that the dehomogenization

$$\tilde{F_\mu} = F_\mu(1) = \begin{cases} \binom{n}{j}^{2i}\binom{\mu}{i}^{2j} - \binom{n}{i}^{2j}\binom{\mu}{j}^{2i}, & \text{for } i \leq \mu < n \\ 0, & \text{else} \end{cases}$$

is non-negative.

It is interesting to notice that the idea of certifying symmetric inequalities in the way sketched has been done albeit not as general. For example, the main Lemma [Mit03, Lemma 2.4] used to prove some new inequalities between elementary symmetric polynomials can be seen as a special case of Remark II.3.11 for Z_1-quasi-sufficient even symmetric polynomials. Moreover, our setup also recovers as a special instance of Corollary II.3.10 together with Proposition II.3.6 the so-called Degree and Half-Degree Principle shown in [Tim03].

Corollary II.4.4 (Degree Principle)**.** *Let $S \subseteq \mathbb{R}^n$ be a symmetric semi-algebraic set, which can be described by symmetric polynomials of degree at most d. Then S is empty, if and only if $S \cap \mathcal{A}_d$ is empty.*

Corollary II.4.5 (Half-Degree Principle)**.** *Let $f \in \mathbb{R}[\underline{X}]$ be symmetric of degree d. Then f is k-complete, where $k := \max\left\{2, \left\lfloor \frac{d}{2} \right\rfloor\right\}$.*

We remark that it is known to be NP-hard already for quartics to decide non-negativity (see e.g. [Blu+98] or [MK87]). However, for univariate polynomials, non-negativity can be certified via a sums of squares decomposition. Such a decomposition can be efficiently obtained via semi-definite programming. The feasible region of a semi-definite program is given by a linear matrix inequality (LMI), i.e., an inequality of the form $A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0$, where $A_0, \ldots, A_n$ are real symmetric matrices all of the same size and $x_1, \ldots, x_n$ are supposed to be real scalars. Now for a symmetric 1-complete polynomial of degree $2d$ we have that f is non-negative if and only if the univariate polynomial $\tilde{f} := f(T, T, \ldots, T)$ of same degree is non-negative. This in turn is the case, if and only if there exists a symmetric matrix $A \in \mathbb{R}^{(d+1)\times(d+1)}$ which is non-negative and for which we have $\tilde{f} = (1, T, T^2, \ldots, T^n) \cdot A \cdot (1, T, T^2, \ldots, T^n)^t$. Therefore, non-negativity of a 1-complete symmetric polynomial can be decided with semi-definite programming. This motivates the following sufficient criterion for 1-complete polynomials.

Theorem II.4.6. *Let $l \in \mathbb{R}[Z_1, \dots, Z_n]_1$ be linear and homogeneous, say $l = \lambda_1 Z_1 + \dots + \lambda_n Z_n$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Let f be a l-sufficient symmetric polynomial. Let m denote the largest index i of the non-zero λ_i, i.e., $m := \max\{i \in \{1, \dots, n\} \mid \lambda_i \neq 0\}$. If m is odd, then f is 1-complete.*

Proof. Write f as $f := g(l(e_1 \dots, e_n))$ for some univariate polynomial g. Let $x \in \mathbb{R}^n$ and define $a := l(e_1(x), \dots, e_n(x)) \in \mathbb{R}$. We will show that $\mathcal{H}_l^1(a) \neq \emptyset$. Consider the univariate polynomial

$$p := \sum_{i=1}^{m} \lambda_i \binom{n}{i} T^i - a \in \mathbb{R}[T].$$

Since m is odd, p has a real zero $y \in \mathbb{R}$. Consider now $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ defined by $z_i := \binom{n}{i} y^i$. Then $z \in \mathcal{H}_l^1(a)$ by construction. Now

$$f(x) = g(a) = g(l(z_1, \dots, z_n)) = f(y, \dots, y).$$

■

Convex sets for which membership can be described via semi-definite programming, i.e., which are projections of feasibility regions of semi-definite programs are called **spectrahedral shadows**. Recently, Scheiderer [Sch18] was able to show that in general, the cone of positive semi-definite forms is not, in general, a spectrahedral shadow. Using Corollary II.3.10 and Remark II.3.11 we can identify families of convex cones of (even-)symmetric positive semi-definite forms which are spectrahedral shadows, generalizing Theorem 4.29 in [DR20].

Proposition II.4.7. *Let $\mathcal{P}_{2d}$ denote the convex cone of positive semi-definite n-ary forms of degree $2d$ and $2 \leq j \leq n$. Then, the subcones of all (Z_1, Z_j)-sufficient and (Z_1, Z_2)-quasi-sufficient symmetric forms are spectrahedral shadows. Similarly, the subcone of all (Z_1, Z_j)-quasi-sufficient even-symmetric forms is a spectrahedral shadow.*

Proof. All forms in the mentioned subcones are 2-complete by Corollary II.3.10 and Remark II.3.11. Therefore non-negativity can be decided by restricting to $\mathcal{A}_2$, respectively $\mathcal{B}_2$. Dehomogenizing the resulting binary forms we obtain univariate polynomials, which are non-negative if and only if they are sums of squares. ■

II.5 Conclusion and open questions

We have defined the notion of hyperbolic slices and showed that the local extreme points of such slices correspond to hyperbolic polynomials with few distinct roots. We show that generically these hyperbolic slices contain at most finitely many local extreme points. We expect that this holds generally, i.e., also in those cases when the k-boundary is not finite. In particular, we expect that the convex hull of each connected component of any hyperbolic slice is a polyhedron.

Arnold and Giventhal [Arn86; Giv87] had shown that the hyperbolic slices which are obtained by fixing the first k coefficients are contractible. Our examples show that hyperbolic slices are in general neither connected nor compact and therefore in particular not contractible. It would be very interesting to study the topological properties of these sets. Similarly to the results in [BR21], an understanding of the topology of these slices might allow for new efficient algorithms to compute the homology of symmetric semi-algebraic sets defined by k-complete polynomials. Furthermore, the definition of hyperbolic slices naturally involved elementary symmetric polynomials. From the viewpoint of symmetric polynomials, it seems interesting to study analogous sets for different choices of n symmetric polynomials which generate all symmetric polynomials. For example, the first author observed in [Rie16] that symmetric polynomials defined by any k Newton sums are at least $(2k+1)$-complete. Finally, a natural question is to explore the connections to invariant polynomials of other groups, most notably finite reflection groups. In [AV16; FRS18] the authors showed that the image of polynomial functions invariant by a finite reflection group can be described by the points on flats in the hyperplane arrangement if the degree is sufficiently small. We expect that the notions and techniques presented here can be transferred also to this more general setup.

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Paper III

Stable and Hurwitz slices, a degree principle and a generalized Grace-Walsh-Szegő theorem

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Abstract

Univariate polynomials are called stable with respect to a circular region $\mathcal{A}$, if all of their roots are in $\mathcal{A}$. We consider the special case where $\mathcal{A}$ is a half-plane and investigate affine slices of the set of stable polynomials. In this setup, we show that an affine slice of codimension k always contains a stable polynomial that possesses at most $2(k+2)$ distinct roots on the boundary and at most $(k+2)$ distinct roots in the interior of $\mathcal{A}$. This result also extends to affine slices of weakly Hurwitz polynomials, i.e. real, univariate, left half-plane stable polynomials. Subsequently, we apply these results to symmetric polynomials and varieties. Here we show that a variety described by polynomials in few multiaffine polynomials has no root in $\mathcal{A}^n$, if and only if it has no root in $\mathcal{A}^n$ with few distinct coordinates. This is at the same time a generalization of the degree principle to stable polynomials and a generalization of Grace-Walsh-Szegő's coincidence theorem.

III

III.1 Introduction

The study of univariate polynomials whose roots are restricted to a subset of $\mathbb{C}$ is a central topic in mathematics. For instance, a univariate real polynomial is called **hyperbolic** if it is real rooted. Given a **circular region** $\mathcal{A}$ a univariate complex polynomial is said to be **$\mathcal{A}$-stable** if all its roots lie in $\mathcal{A}$. Since the roots of real polynomials come in conjugated pairs, hyperbolic polynomials are thus exactly real stable polynomials relative to the upper half-plane. Well-known examples of stable polynomials are **Hurwitz stable** polynomials, which are real open left half-plane stable polynomials, and **Schur stable** polynomials, which are unit disk stable polynomials. In particular, stable polynomials have been extensively leveraged to gain insights into combinatorial objects (see e.g. [Brä07;

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DZ21; Fis08; HL72]), and Hurwitz polynomials are at the heart of control theory and are used for asymptotic stability for linear continuous-time systems (see e.g., [Max68] or [Eng15, p. 75]).

Studying the roots of univariate polynomials is deeply related to studying multivariate symmetric polynomials by the **Vieta formula**

$$\prod_{i=1}^{n}(T - x_i) = T^n - e_1(x)T^{n-1} + e_2(x)T^{n-2} + \ldots + (-1)^n e_n(x)$$

where $e_k = \sum_{I\subset[n],|I|=k}\prod_{i\in I} X_i$ denotes the k-th **elementary symmetric polynomial**. In the paper we associate points $z \in \mathbb{C}^n$ with monic polynomials

$$f_z = T^n - z_1T^{n-1} + z_2T^{n-2} - \ldots + (-1)^n z_n.$$

In particular, monic hyperbolic polynomials are described by the image of $\mathbb{R}^n$ under the Vieta map, i.e., the image under the evaluation of the n elementary symmetric polynomials. Similarly to this hyperbolic picture, monic $\mathcal{A}$-stable polynomials can be identified with the image of $\mathcal{A}^n$ under the Vieta map.

Sets of hyperbolic polynomials obtained by fixing the first k coefficients have been considered by various authors, beginning with the work of Arnold [Arn86; Giv87; Kos89; Meg92] and recently [LS; Rie12]. In the domain of the Vieta map, such sets are called **Vandermonde varieties**, whereas the corresponding sets in the image of the Vieta map are called **hyperbolic slices**. More generally, this notion has been introduced in [RS24] to sets of hyperbolic polynomials that are cut out by a $(n-k)$-dimensional affine subspace. A remarkable property of such hyperbolic slices concerns their local extreme points: It turns out that these local extreme points of linear functionals can be characterized as polynomials with at most k distinct roots. Similarly to this hyperbolic situation, we study affine slices of the set of upper half-plane stable polynomials defined by k linear combinations of coefficients and show in Theorem III.2.4 that the local extreme points of such **stable slices** have at most k non-real roots and at most $2k$ distinct real roots.

One of our main motivations for this result is provided by a natural connection to the classical **Grace-Walsh-Szegő's coincidence theorem**. This beautiful result states that for a symmetric multiaffine polynomial $f \in \mathbb{C}[\underline{X}]$ evaluated on a circular region $\mathcal{A} \subset \mathbb{C}$ there exists for all $(\zeta_1, \ldots, \zeta_n) \in \mathcal{A}^n$ some $\zeta \in \mathcal{A}$ with the property that $f(\zeta_1, \ldots, \zeta_n) = f(\zeta, \ldots, \zeta)$, under the assumption that the degree of f is n or $\mathcal{A}$ is convex. The coincidence theorem has several applications in stability testing since it allows reduction of the question of verifying multivariate stability to univariate polynomials. However, the assumptions of the theorem are relatively strict. It was proven by Brändén and Wagner [BW09] that no analogous result can be applied to any multiaffine polynomials invariant under a fixed proper permutation subgroup of S_n. We use our results on stable slices and the connection with symmetric polynomials to prove in Theorem III.4.6 and Corollary III.4.11 a Grace-Walsh-Szegő-like theorem for multivariate polynomials which can be written as a polynomial in few multiaffine symmetric polynomials when $\mathcal{A}$ is a half-plane. We show that for any point $\zeta \in \mathcal{A}^n$, we can find a point with few distinct coordinates and the same evaluation. Furthermore, in a similar spirit, we prove a **double-degree principle** for stable varieties in Corollary III.4.8 and also a **half-degree principle** for the upper half-plane in Theorem III.4.13. Our results on stable slices do not transfer directly to **Hurwitz slices** since the coefficients of those polynomials are real. However, we prove that if we fix k linear combinations of coefficients of a weakly Hurwitz polynomial, then there is a weakly Hurwitz polynomial satisfying the same relations and having only k roots with negative real part and $2k$ distinct roots with real part equal to zero (see Theorem III.3.3).

Structure of the article

In Section III.2 we study stable slices of univariate polynomials and show in particular that local extreme points of stable slices correspond to polynomials with few distinct roots (Theorem III.2.4). In Section III.3 we study Hurwitz slices and their boundary by root multiplicities. In Section III.4 we apply our results from Section III.2 to multivariate symmetric polynomials and formulate a double-degree principle for stable polynomials and our generalization of Grace-Walsh-Szegő's coincidence theorem (Theorem III.4.6, Corollaries III.4.8 and III.4.11). Finally, we formulate open questions.

III.2 Stable slices

Throughout the article we denote by $\mathbb{C}[T]$ and $\mathbb{R}[T]$ the rings of univariate complex and real polynomials and $k \leq n$ be fixed positive integers. For a complex number x we write $\Re(x)$ and $\Im(x)$ for its real and imaginary parts. Furthermore, we commonly identify the set of monic univariate polynomials with $\mathbb{C}^n$ via the bijection

$$(z_1, \ldots, z_n) \longmapsto T^n - z_1 T^{n-1} + z_2 T^{n-2} - \cdots + (-1)^n z_n.$$

In this section, we study univariate **stable polynomials**, i.e. polynomials that have all their roots lying in a half-plane. In particular, we are interested in intersections of the set of stable polynomials with affine subspaces of $\mathbb{C}^n$. As multiplication with units in $\mathbb{C}$ does not change the roots of a polynomial, we restrict to monic stable polynomials. We denote the closed upper half-plane by $\mathbb{H}_+$, i.e.

$$\mathbb{H}_+ = \{x \in \mathbb{C} \mid \Im(x) \geq 0\}$$

Definition III.2.1. Let $\mathbb{H}$ be a closed half-plane. We denote by

$$\S_{\mathbb{H}} := \left\{ z \in \mathbb{C}^n \;\middle|\; T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n \text{ has all roots in } \mathbb{H} \right\}$$

the set of monic **$\mathbb{H}$-stable polynomials** of degree n. Furthermore, we define

$$\mathbb{H}_{k,m} := \left\{ x \in \mathbb{H}^n \;\middle|\; |\{x_1 \ldots, x_n\} \cap \partial\mathbb{H}| \leq k \text{ and } |\{i \in \{1, \ldots, n\} \mid x_i \in \mathring{\mathbb{H}}\}| \leq m \right\}$$

the set of points with at most k distinct coordinates on the boundary of $\mathcal{A}$ and at most m coordinates in the interior of $\mathbb{H}$. The set of all polynomials with all roots in $\mathbb{H}_{k,m}$ is

$$\S_{\mathbb{H}}^{k,m} := \left\{ z \in \S_{\mathbb{H}} \;\middle|\; T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n \text{ has roots } (x_1, \ldots, x_n) \in \mathbb{H}_{k,m} \right\}.$$

For $a = (a_1, \ldots, a_k) \in \mathbb{C}^k$ and a surjective linear map $L: \mathbb{C}^n \to \mathbb{C}^k$ we define the **affine slice**

$$\S_{\mathbb{H}} \cap L^{-1}(a) = \{ z \in \S_{\mathbb{H}} \mid L(z) = a \} \text{ and } S_{\mathbb{H}}^{k,m} \cap L^{-1}(a) = \left\{ z \in \S_{\mathbb{H}}^{k,m} \;\middle|\; L(z) = a \right\}.$$

A set of the form $\S_{\mathbb{H}} \cap L^{-1}(a)$ is called a $\mathbb{H}$**-stable slice**. If $\mathbb{H} = \mathbb{H}_+$ is the upper half-plane, we write $\S$ for $\S_{\mathbb{H}_+}$.

Remark III.2.2. Notice that the set $\S_{\mathbb{H}}$ can be identified with a semi-algebraic set in $\mathbb{R}^{2n}$. In contrast to the set of hyperbolic polynomials, where an explicit description of the set of hyperbolic polynomials in terms of the coefficients can be obtained via Sturm's Theorem, it seems in general not easy to give an explicit description of $\S_{\mathbb{H}}$. However, in the case of polynomials with real coefficients, this is possible and we will present this case in Section III.3.

The assumption that L is surjective is only for convenience in the notation (see Remark III.2.5). It suffices to study stable slices of a fixed half-plane. This follows since translations and rotations are linear isomorphisms. Let $\phi : \mathbb{H} \to \mathbb{G}$ be a linear bijection between half-planes and let $\psi = \phi^{-1}$ be its inverse. Then $f_z \in \S_{\mathbb{H}}$ if and only if $f_z \circ \psi \in \S_{\mathbb{G}}$. In particular, we can restrict to $\mathbb{H}_+$-stable slices.

Definition III.2.3. Let $A \subset \mathbb{C}^n$ and let $z \in A$. We say that z is a **local extreme point** of A if there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(A \cap U)$.

Like the set of extreme points of a set A is the set of global minima of linear functions, the set of local extreme points of A is the set of local minima of linear functions.
The following theorem which is a generalization of [Rie12, Theorem 4.2] and [RS24, Theorem 2.8], is our main result on stable slices characterizing local extreme points. As a corollary, we obtain a result for arbitrary stable slices in Corollary III.2.10.

Theorem III.2.4. *The local extreme points of an $\mathbb{H}_+$-stable slice $\S \cap L^{-1}(a)$ correspond to polynomials that have at most k roots in $\mathbb{H}_+ \setminus \mathbb{R}$ and at most $2k$ distinct real roots.*

In other words, any local extreme point of the $\mathbb{H}_+-$stable slice $\S \cap L^{-1}(a)$ is contained in the set $\S^{2k,k}$. In the proof, we investigate the multiplicity of the roots of polynomials in the stable slice.

Proof. Let $z \in \S \cap L^{-1}(a)$ be a local extreme point, i.e., there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(\S \cap L^{-1}(a) \cap U)$. Consider

$f := T^n - z_1T^{n-1} + \cdots + (-1)^n z_n$ and factor $f = p \cdot r$, where p has only roots in $\mathbb{H}_+ \setminus \mathbb{R}$ and r has only real roots.

1. We show first that p has at most k roots, i.e., $\deg p \leq k$. We assume that $\deg p := m > k$ and want to find a contradiction. Write $r = T^{n-m} + r_1T^{n-m-1} + \cdots + r_{n-m}$ and define $r_0 := 1$ and consider the linear map

$$\begin{array}{rcl}\chi : \mathbb{C}^m & \longrightarrow & \mathbb{C}^n \\ y & \longmapsto & \left(\sum_{i+j=1} r_i y_j, \ldots, \sum_{i+j=n} r_i y_j\right)\end{array}.$$

Since $m > k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$. We define $h := b_1T^{m-1} + \cdots + b_m$ and $g := h \cdot r = c_1T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Now, because p has only roots in $\mathbb{C} \setminus \mathbb{R}$, $p \pm \varepsilon h$ is stable for $\varepsilon > 0$ small enough, since the roots depend continuously on the coefficients [HM87]. Hence

$$(p \pm \varepsilon h) \cdot r = f \pm \varepsilon h \cdot r = f \pm \varepsilon g$$

is stable for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \S \cap L^{-1}(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\S \cap L^{-1}(a) \cap U)$.

2. Now we show that r has at most $2k$ distinct roots. We assume r has distinct roots $x_1, \ldots, x_m$ where $m > 2k$ and want to find a contradiction. We factor f as follows:

$$f = \underbrace{\prod_{i=1}^{m}(T - x_i)}_{=:q} \cdot s,$$

where s is of degree $n - m$. Write $s = T^{n-m} + s_1T^{n-m-1} + \cdots + s_{n-m}$ and define $s_0 := 1$ and consider the linear map

$$\begin{array}{rcl}\chi : \mathbb{R}^m & \longrightarrow & \mathbb{C}^n \\ y & \longmapsto & \left(\sum_{i+j=1} s_i y_j, \ldots, \sum_{i+j=n} s_i y_j\right)\end{array}.$$

Since $m > 2k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$. We define $h := b_1T^{m-1} + \cdots + b_m$ and $g := h \cdot s = c_1T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Now, because q has only single roots in $\mathbb{R}$, $q \pm \varepsilon h$ is stable for $\varepsilon > 0$ small enough, since the roots depend continuously on the coefficients and complex roots come as conjugated pairs (see e.g. [HM87]). Hence

$$(q \pm \varepsilon h) \cdot s = f \pm \varepsilon \cdot g$$

is stable for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \S \cap L^{-1}(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\S \cap L^{-1}(a) \cap U)$.

■

Remark III.2.5. The assumption that L is surjective is only for convenience. In particular, if L is not surjective one obtains the same result as in Theorem III.2.4, where k can be replaced by $\operatorname{rank} L$.

We point out that the converse of Theorem III.2.4 is not true, i.e. not every point $z \in \S^{2k,k} \cap L^{-1}(a)$ is a local extreme point.

Example III.2.6. Let $n = 3$, $k = 1$ and

$$L : \begin{array}{ccc} \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\ (z_1, z_2, z_3) & \longmapsto & z_3 \end{array}.$$

Then $(i, 0, 0) \in \S^{2k,k} \cap L^{-1}(0)$, but

$$(i, 0, 0) = \frac{(0, 0, 0) + (2i, 0, 0)}{2}$$

is not a local extreme point since $(0, 0, 0), (2i, 0, 0) \in \S \cap L^{-1}(0)$.

We consider the set of stable polynomials of degree n with fixed first coefficients which is an instance of a stable slice.

Definition III.2.7. For an integer $k \geq 1$ and a point $a = (a_1, \dots, a_k) \in \mathbb{C}^k$ we define $\S(a) = \S \cap \{z \in \mathbb{C}^n | z_1 = a_1, \dots, z_k = a_k\}$ as the set of all monic $\mathbb{H}_+$-stable polynomials of degree n whose first k non-trivial coefficients are determined by the point a.

With our previous notation we have $\S(a) = S \cap L^{-1}(a)$ where $L : \mathbb{C}^n \to \mathbb{C}^k$ denotes the projection to the first k coordinates.

Lemma III.2.8. *For an integer $k \geq 2$ the stable slice $\S(a)$ is compact.*

Proof. As the empty set is compact we can assume that there is $z \in \S(a)$. Furthermore we denote by $x = (x_1, \dots, x_n) \in \mathbb{H}_+$ the roots of the polynomial

$$f_z := T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n.$$

Then, if e_1 and e_2 denote the first and second elementary symmetric polynomial

$$\sum_{i=1}^{n} x_i = e_1(x) = a_1$$

and hence the imaginary part of the $x_i's$ is contained in $[0, \Im(a_1)]$. Furthermore

$$\sum_{i=1}^{n} x_i^2 = e_1(x)^2 - 2e_2(x) = a_1^2 - 2a_2$$

and hence

$$\sum_{i=1}^{n} \Re(x_i)^2 = \sum_{i=1}^{n} \Re(x_i^2) + \Im(x_i)^2 \leq \sum_{i=1}^{n} \Re(x_i^2) + \Im(a_1)^2 = \Re\left(\sum_{i=1}^{n} x_i^2\right) + n\Im(a_1)^2 \ .$$

Since $\sum_{i=1}^{n} x_i^2 = a_1^2 - 2a_2$ we have

$$\sum_{i=1}^{n} \Re(x_i)^2 \leq \Re(a_1^2 - 2a_2) + n\Im(a_1)^2 \ .$$

This shows that also the real part of the $x_i's$ is bounded. Thus the set $\S(a)$ is bounded. Furthermore, as the roots of a polynomial depend continuously on the coefficients it is clear that $\S(a)$ is closed and therefore compact. ■

Remark III.2.9. For a surjective linear map $L: \mathbb{C}^n \to \mathbb{C}^k$ and $a \in \mathbb{C}^k$ we can have that the set $\S \cap L^{-1}(a)$ is unbounded. Then we consider the linear map $\widetilde{L}: \mathbb{C}^n \to \mathbb{C}^{k+2}$, where $\widetilde{L}(z) = (L(z), z_1, z_2)$. The set $\S_{\mathbb{H}_+} \cap \widetilde{L}^{-1}(b)$ is compact for any point $b \in \mathbb{C}^{k+2}$, by a similar argument as in the proof of Lemma III.2.8. Moreover, if one or both of the first two unit vectors are in the row span of a matrix representation of L, then we can consider $\widehat{L}(z) = (L(z), z_i)$ for $i \in \{1, 2\}$ instead of L or the original stable slice was already compact.

We are now ready to present our main result on general half-plane stable slices.

Corollary III.2.10. *Let $\mathbb{H}$ be a closed half-plane. Any non-empty $\mathbb{H}$-stable slice $\S_{\mathbb{H}} \cap L^{-1}(a) \neq \emptyset$ contains a point that corresponds to a polynomial with at most $k+2$ roots in $\mathring{\mathbb{H}}$ and at most $2(k+2)$ distinct roots in $\partial\mathbb{H}$, i.e.*

$$\S_{\mathbb{H}}^{2(k+2),k+2} \cap L^{-1}(a) \neq \emptyset.$$

Proof. Since $\mathbb{H}$ can be bijectively mapped to $\mathbb{H}_+$ under a linear isomorphism it suffices to show the theorem for $\mathbb{H} = \mathbb{H}_+$. Now the claim follows from Theorem III.2.4, Lemma III.2.8 and Remark III.2.9. ■

Corollary III.2.10 says that stable slices do always contain a point with few distinct zeros. Moreover, we can characterize the maximal number of distinct roots on the boundary of the half-plane and the number of distinct roots in the interior. We point out that the result is independent of the degree n and is more import if the degree is large. In particular, we observe a stabilization in the structure of local extreme points of stable slices if there are at least $n \geq 3k$ variables.

Remark III.2.11. In the case that L is the projection to the first $k < n$ coordinates, we can replace $2k$ by k in Theorem III.2.4 since $(0, \dots, 0, 1) \in \ker(L \circ \chi)$. This follows because one can choose $h = 1$ in the proof in this case. Furthermore, if $k \geq 2$ the stable slice is compact in this case by Lemma III.2.8. So we can exchange $\S_{\mathbb{H}}^{2(k+2),k+2} \cap L^{-1}(a)$ by $\S_{\mathbb{H}}^{k,k} \cap L^{-1}(a)$ in Corollary III.2.10.

One could hope that every stable slice contains also points that correspond to polynomials with k distinct roots in $\mathbb{H}_+$, analogous to the case of compact hyperbolic slices, mentioned in [RS24, Theorem 2.8]. The next example shows that this is not true in general even when L is the projection to the first k coordinates.

Example III.2.12. We consider $\S \cap L^{-1}(a)$, where

$$a := (-23i, -463, 8461i) \quad \text{and} \quad \begin{array}{rccc} L: & \mathbb{C}^4 & \longrightarrow & \mathbb{C}^3 \\ & (z_1, z_2, z_3, z_4) & \longmapsto & (z_1, z_2, z_3) \end{array}$$

is the projection to the first 3 coordinates. Then $\S \cap L^{-1}(a)$ is non-empty, since

$$(-23i, -463, 8461i, 8020) \in \S \cap L^{-1}(a).$$

The coefficient vector corresponds to a polynomial with roots $-20+i, i, 20+i$ and $20i$. Furthermore, $\S \cap L^{-1}(a)$ contains no point corresponding to a polynomial with at most 3 distinct roots.

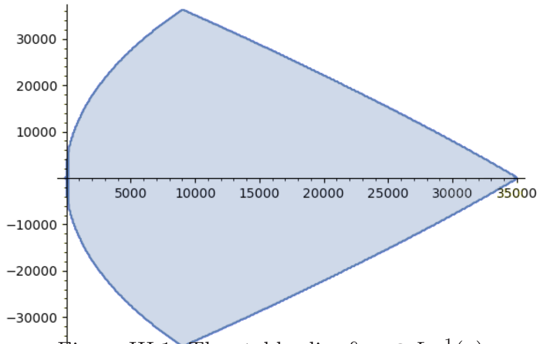


Figure III.1: The stable slice $\S_{\mathbb{H}_+} \cap L^{-1}(a)$

III.3 Hurwitz slices

In this section we consider **Hurwitz polynomials**, i.e. real univariate polynomials with all roots in the left half-plane. Moreover, polynomials with all roots having nonpositive real part are called **weakly Hurwitz**. We show in Theorem III.3.3 that the local extreme points of affine slices of the set of monic Hurwitz polynomials have few distinct roots and study a partial order on the set of monic Hurwitz polynomials in Subsection III.3.2.

Like for stable polynomials we identify monic weakly Hurwitz polynomials with their coefficients. Any monic weakly Hurwitz polynomial has nonnegative coefficients.
Similary to hyperbolic polynomials, monic Hurwitz polynomials can be characterized as poynomials with a positive definite **finite Hurwitz matrix** [Hur95]. While the finite Hurwitz matrix of any weakly Hurwitz polynomial is positive semidefinite, its converse is not true (see [Asn70]). Kemperman [Kem82] showed that weakly Hurwitz polynomials can be characterized in a similar way by their **infinite Hurwitz matrix** (see also [AGT18, Thm. 4.9] for another characterization).

III.3.1 Hurwitz slices and their local extreme points.

In contrast to the study of stable polynomials in Section III.2 where we considered surjective linear maps $\mathbb{C}^n \to \mathbb{C}^k$ over the field $\mathbb{C}$, we restrict to real linear maps over $\mathbb{R}$. However, since the roots of weakly Hurwitz polynomials can be complex, we cannot directly apply any result about hyperbolic polynomials.

Definition III.3.1. We write $\mathbb{H}_{\text{left}}$ for the left half-plane in $\mathbb{C}$, i.e.

$$\mathbb{H}_{\text{left}} := \{x \in \mathbb{C}^n \mid \Re(x) \leq 0\}$$

The set of monic **weakly Hurwitz** polynomials is defined by

$$\mathcal{HW} := \S_{\mathbb{H}_{\text{left}}} \cap \mathbb{R}^n := \{z \in \mathbb{R}^n \mid f_z \text{ has all roots in } \mathbb{H}_{\text{left}}\}.$$

Moreover, for a linear map $L: \mathbb{R}^n \to \mathbb{R}^k$ we call the set $\mathcal{HW} \cap L^{-1}(a)$ a **Hurwitz slice**.

We have the following connection between Hurwitz polynomials and stable polynomials.

Remark III.3.2. The set of monic weakly Hurwitz polynomials $\mathcal{HW}$ can be embedded in $\S$ in the following way: If $f(T) \in \mathcal{HW}$ is Hurwitz then the monic polynomial

$$\tilde{f}(T) = (-i)^n \cdot f(i \cdot T)) = T^n + \sum_{k=1}^{n} i^k z_k T^{n-k}$$

is upper half-plane stable with coefficients alternating from the sets $\mathbb{R}$ or $i \cdot \mathbb{R}$. The map $\tilde{}\,: \mathcal{HW} \to \S$ is linear, injective, not surjective, and its inverse is $g(T) \mapsto i^n g(-i \cdot T)$.
For instance, the polynomial

$$f = (T-2)(T-1+i)(T-1-i) = -4 + 6T - 4T^2 + T^3$$

is Hurwitz and

$$\tilde{f} = i^9 f(iT) = -4i - 6T + 4iT^2 + T^3$$

is $\mathbb{H}_+$-stable with alternating real and purely complex coefficients.
We get the same results about multiplicities of the roots of local extreme points of Hurwitz slices as for stable slices in Theorem III.2.4.

Theorem III.3.3. *Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map. The local extreme points of a Hurwitz slice $\mathcal{HW} \cap L^{-1}(a)$ correspond to polynomials that have at most k roots with negative real part and at most $2k$ distinct roots with real part equal to zero.*

Proof. Let $z \in \mathcal{HW} \cap L^{-1}(a)$ be a local extreme point, i.e., there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(\mathcal{HW} \cap L^{-1}(a) \cap U)$. Consider $f := f_z = T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n$ and factor $f = p \cdot r$, where p has only roots with negative real part and r has only roots with real part equal to zero. Note that since f has real coefficients, the roots of f come in complex conjugated pairs, so p and r have also real coefficients.

1. We show first that p has at most k roots, i.e., $\deg p \leq k$. We assume that $\deg p := m > k$ and want to find a contradiction. Write $r = T^{n-m} + r_1 T^{n-m-1} + \cdots + r_{n-m}$ and define $r_0 := 1$ and consider the linear map
$$\begin{array}{rcc} \chi : \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ y & \longmapsto & \left(\sum_{i+j=1} r_i y_j, \ldots, \sum_{i+j=n} r_i y_j \right) \end{array}.$$
Since $m > k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$. We define $h := b_1 T^{m-1} + \cdots + b_m$ and $g := h \cdot r = c_1 T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Now, because p has only roots with negative roots, $p \pm \varepsilon h$ is weakly Hurwitz for $\varepsilon > 0$ small enough, since the roots depend continuously on the coefficients (see e.g. [HM87]). Hence
$$(p \pm \varepsilon h) \cdot r = f \pm \varepsilon h \cdot r = f \pm \varepsilon g$$
is weakly Hurwitz for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \mathcal{HW} \cap L^{-1}(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then
$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$
a contradiction to z being an extreme point of $\operatorname{conv}(\mathcal{HW} \cap L^{-1}(a) \cap U)$.

2. Now we show that r has at most $2k$ distinct roots. We assume that all the distinct roots of r are $x_1, \ldots, x_m$ where $m > 2k$ and we want to find a contradiction. We factor f as follows:
$$f = \underbrace{\prod_{i=1}^{m} (T - x_i)}_{=:q} \cdot s,$$
where s is of degree $n - m$. Note that f and therefore q and s have real coefficients. Write $s = T^{n-m} + s_1 T^{n-m-1} + \cdots + s_{n-m}$ and define $s_0 := 1$ and consider the linear map
$$\begin{array}{rcc} \chi : \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ y & \longmapsto & \left(\sum_{i+j=1} s_i y_j, \ldots, \sum_{i+j=n} s_i y_j \right) \end{array}.$$

Since $m > 2k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$ with $b_{2i-1} = 0$ for all $i \in \left\{1, \ldots, \lfloor \frac{m}{2} \rfloor\right\}$. We define $h := b_1 T^{m-1} + \cdots + b_m$ and $g := h \cdot s = c_1 T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Note that q corresponds to a hyperbolic polynomial $\tilde{q}$ via the embedding stated in Remark III.3.2 where the degree is m instead of n. The same transformation maps h to a hyperbolic polynomial $\tilde{h}$. Now, because $\tilde{q}$ has only distinct roots, $\tilde{q} \pm \varepsilon \tilde{h}$ is hyperbolic for $\varepsilon > 0$ small enough since the roots depend continuously on the coefficients and complex roots come as conjugated pairs (see e.g. [HM87]). Moreover, we have $\tilde{q} \pm \varepsilon \tilde{h} = T^m + w_2 T^{m-2} + w_4 T^{m-4} + \ldots$ for some real numbers w_{2i}. Thus, $\tilde{q} \pm \varepsilon \tilde{h}$ lies in the image of the map $\tilde{}$ and we can apply the inverse of the transformation $\tilde{}$ from Remark III.3.2 which is also linear. So $q \pm \varepsilon h$ is weakly Hurwitz for $\varepsilon > 0$ small enough. Hence

$$(q \pm \varepsilon h) \cdot s = f \pm \varepsilon \cdot g$$

is Hurwitz for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \mathcal{HW} \cap L^{-1}(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\mathcal{HW} \cap L^{-1}(a) \cap U)$.

■

Note that although the result is the same as in Theorem III.2.4, the proof of (2) is different. In the case where L is the projection to the first k coordinates, one can again replace $2k$ by k in the proof of Theorem III.3.3. Furthermore, since closed subsets of compact sets are compact, we get from Lemma III.2.8 and Remark III.3.2 also that $\mathcal{HW} \cap L^{-1}(a)$ is compact if L is the projection to the first k coordinates. More generally, Remark III.2.9 translates in the same way.

III.3.2 Geometry and combinatorics of Hurwitz slices

In this subsection, we briefly discuss the interplay of the geometry and combinatorics of the set of weakly Hurwitz polynomials and Hurwitz slices. This is inspired by the rich geometry and combinatorics of linear slices of the set of monic univariate hyperbolic polynomials and should be seen as a starting point for further investigations.

The boundary of the set $\mathcal{HW}$ consists of polynomials of the form $f = p \cdot q$ where $p, q \in \mathbb{R}[T]$ are monic, $\deg(p) + \deg(q) = n$, p is Hurwitz of even degree $r < n$ and for any root z of q, we have $\Re(z) = 0$. In a neighborhood of p, one can perturb all coefficients but the leading coefficient of p and the obtained polynomial is again a monic Hurwitz polynomial. All imaginary roots $\pm ib_1, \ldots, \pm ib_r$ of q come in complex conjugated pairs. We assume $0 \leq |b_1| < \ldots < |b_r|$ and we have

$$q = T^s \prod_{i=1}^{l} ((T - ib_i)(T + ib_i))^{\mu_i} = T^s \prod_{i=1}^{l} (T^2 + b_i^2)^{\mu_i}$$

with $s \in \{0,1\}$. Note that s is uniquely determined by the degree of q. We have $s = 0$ if $\deg q$ is even and $s = 1$ otherwise. The real polynomial

$$q_e = T^s \prod_{i=1}^{l} (T + b_i^2)^{\mu_i}$$

has only real roots $0 \geq -b_1^2 > \ldots > -b_l^2$ with multiplicities $s, \mu_1, \ldots, \mu_l$. For a monic polynomial $f \in \mathbb{R}[T]$ whose roots are all of the form ib with $b \in \mathbb{R}$, we call f_e its associated **even polynomial**. We have a $1:1$ correspondence between monic polynomials $q \in \mathbb{R}[T]$ for which all of its roots have real part 0 and hyperbolic polynomials with only nonpositive roots. Let $\mu = (\mu_1, \ldots, \mu_l)$ be the **composition**, i.e. the sequence of positive integers, associated with the ordered roots of q_e. We call the tuple (s, μ) the **root multiplicity** of the even polynomial q_e.
For a weakly Hurwitz polynomial $f = p \cdot q$ we call the triple (r, s, μ) the **multiplicity** of f which we denote by $\operatorname{mult}(f)$. For instance, we have $\operatorname{mult}(T^5) = (0, 1, (2))$ and $(T+1)(T-i)(T+i)(T-2i)(T+2i)$ has multiplicity $(1, 0, (1, 1))$. Moreover, we define the set

$$\mathcal{HW}_{(r,s,\mu)} = \{f \in \mathcal{HW} \mid \operatorname{mult}(f) = (r, s, \mu)\}$$

of monic Hurwitz polynomials of degree n with r roots in the interior and the roots on the boundary are encoded by the root multiplicity (s, μ). For different multiplicity triples, the associated sets are disjoint. Note that $\mathcal{HW}_{(r,s,\mu)} \neq \emptyset$ if and only if $r + s + 2\sum_{i=1}^{l} \mu_i = n$, since one can find for any multiplicity (r, s, μ) a monic Hurwitz polynomial with $\operatorname{mult}(f) = (r, s, \mu)$. From the definition of $\mathcal{HW}_{(r,s,\mu)}$ we can say which sets $\mathcal{HW}_{(r',s',\mu')}$ are contained in $\operatorname{cl} \mathcal{HW}_{(r,s,\mu)}$. To do so, we define a partial order.

Definition III.3.4. Let $\mathcal{C}$ be the set of all triples (r, s, μ) where $r \leq n$ is a positive integer and, if $n - r$ is even then $s = 0$ and $\mu = (\mu_1, \ldots, \mu_l)$ is a composition of $\frac{n-r}{2}$, and otherwise $s = 1$ and $\mu = (\mu_1, \ldots, \mu_l)$ is a composition of $\frac{n-r-1}{2}$. We define the partial order $\trianglelefteq$ on $\mathcal{C}$ as the transitive and reflexive closure of the following relations. We say $(r, s, \mu) \trianglelefteq (r, s, \lambda)$ if μ can be obtained from λ by replacing some of the commas in the composition λ by the plus operation. We define $(r-1, 1, \mu) \trianglelefteq (r, 0, \mu)$ and $(r-1, 0, (1, \mu_1, \ldots, \mu_l)) \trianglelefteq (r, 1, \mu)$.

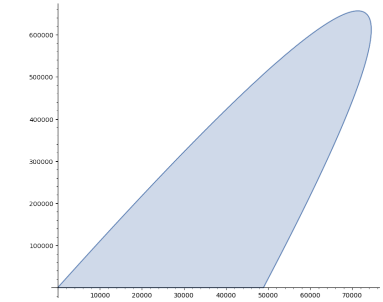
For instance, we have

$$(3, 0, (2)) \trianglelefteq (3, 0, (1, 1)), (2, 1, (1, 1)) \trianglelefteq (3, 0, (1, 1)) \text{ and } (2, 0, (1, 1, 1)) \trianglelefteq (3, 1, (1, 1)).$$

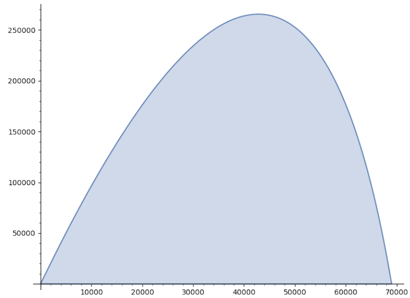
If μ' is a composition that can be obtained from μ by replacing some of the commas in μ plus signs, this means that we can continuously collapse a conjugated pair of roots of a polynomial in $\mathcal{HW}_{(r,s,\mu)}$ to obtain a polynomial in $\mathcal{HW}_{(r,s,\mu')}$. We have

$$\bigcup_{(r',s',\mu') \trianglelefteq (r,s,\mu)} \mathcal{HW}_{(r',s',\mu')} \subset \operatorname{cl} \mathcal{HW}_{(r,s,\mu)}.$$

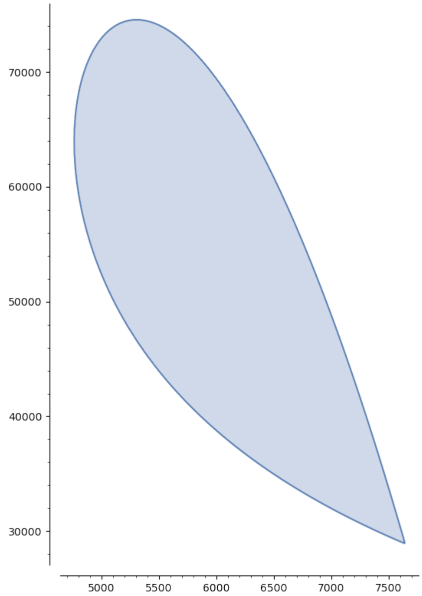
For fixed r the partial order $\trianglelefteq$ is the partial order considered to study the geometry of hyperbolic slices in [Lie23; LS]. There are many open questions about the interplay of the geometry of $\mathcal{HW}$ and the poset $(\mathcal{C}, \trianglelefteq)$. Can one use the understanding of the geometry of hyperbolic slices to understand Hurwitz slices? Is the set $\mathcal{HW}_{(r,s,\mu)}$ contractible? Is the geometry of the set $\mathcal{HW}$ completely described by the poset $(\mathcal{C}, \trianglelefteq)$, i.e. is the set $\mathcal{HW}$ a stratified manifold with a stratification indexed by the poset? Is the partial order $(\mathcal{C}, \trianglelefteq)$ a lattice?



(a) $\mathcal{HW} \cap \{z_1 = -14, z_2 = 546, z_3 = -6064\}$



(b) $\mathcal{HW} \cap \{z_1 = -32, z_2 = 561, z_3 = -5830\}$



(c) $\mathcal{HW} \cap \{z_1 = -14, z_2 = 546, z_4 = 405010\}$

Figure III.2: Hurwitz slices for $n = 5$ where $(|z_4|, |z_5|)$ resp. $(|z_3|, |z_5|)$ are displayed

In Figure III.2 we present three examples of Hurwitz slices for $n = 5$. The multiplicity of any polynomial on the upper arc in (A) is $(3, 0, (1))$ at all points but the two endpoints. At the left endpoint the multiplicity is $(3, 0, (1))$ with a double root at 0 and $(2, 1, (1))$ at the right endpoint with a root at $\approx \pm 21i$. The bottom line corresponds to the multiplicity $(4, 1, (0))$. The same multiplicities are true for the arcs in (B). In (C) any boundary point has multiplicity structure $(5, 0, (0))$.
In general, in a Hurwitz slice not every multiplicity occurs. It is an open question to classify which multiplicities do occur in Hurwitz slices. Is a Hurwitz slice where the first coefficients are fixed always connected? We do not expect connectivity for other slices. By Theorem III.3.3 for $k < \frac{n}{3}$ the Hurwitz slice can at least not be strictly convex. Adm, Garloff and Tyaglov classified [AGT18, Thm. 4.9] the

subset of weakly Hurwitz polynomials with r roots in the interior of the left half-plane. They showed that a monic polynomial $f(T) = p_0(T^2) + Tp_1(T^2) \in \mathbb{R}[T]$ is weakly Hurwitz with r roots in the open left half-plane if and only if the first r principal minors of the finite Hurwitz matrix are negative and the remaining ones are 0 and if the polynomial $\gcd(p_0, p_1)$ has only negative roots. Do the roots of $\gcd(p_0, p_1)$ correspond to the root multiplicity (s, μ)? Finally, one could study the combinatorics and geometry of general stable slices.

III.4 A Grace-Walsh-Szegő like theorem for symmetric polynomials in few multiaffine polynomials

Throughout this section, let $\mathbb{H}$ be a closed half-plane and let $\underline{X} = (X_1, \ldots, X_n)$ be a tuple of n variables.
The main result of this section is a generalization of the well-known Grace-Walsh-Szegő coincidence theorem and a generalization of the degree principle in Theorem III.4.6, Corollary III.4.11 and Corollary III.4.8. We refer to [RS02, p. 107] for background on the coincidence theorem. The main tool in this section will be our results on root multiplicities of local extreme points of stable slices from Section III.2.

Theorem III.4.1 (Grace-Walsh-Szegő coincidence theorem)**.** *Let $\mathcal{A}$ be a closed circular region and let $f \in \mathbb{C}[\underline{X}]$ be a multiaffine symmetric polynomial. If* $\deg(f) = n$ *or if $\mathcal{A}$ is convex, then for any $(x_1, \ldots, x_n) \in \mathcal{A}^n$ there exists a $y \in \mathcal{A}$ with $f(x_1, \ldots, x_n) = f(y, \ldots, y)$.*

We address a generalization to the case where the symmetric polynomial f is no longer assumed to be multiaffine but can be written as a polynomial in k multiaffine symmetric polynomials. However, we cannot expect a diagonal point in $\mathcal{A}$ any longer.

Definition III.4.2. Let $V \subseteq \mathbb{C}^n$ be a variety. We say V is $\mathbb{H}$**-stable** if $V \cap \mathbb{H}^n = \emptyset$. Moreover, we say a polynomial $f \in \mathbb{C}[\underline{X}]$ is $\mathbb{H}$**-stable** if the variety $V(f)$ is $\mathbb{H}$-stable.

Remark III.4.3. In Definition III.4.2 we follow the standard terminology for stability of multivariate polynomials which is in contrast to the definition of stability of univariate polynomials in Definition III.2.1. We say that a multivariate polynomial is $\mathbb{H}$-stable if there is no zero in $\mathbb{H}^n$, while any root of a univariate polynomial has to be contained in $\mathbb{H}$ if it is $\mathbb{H}$-stable. Since the complement of $\mathbb{H}$ is an open half-plane $\mathbb{H}^c$ one can see that for univariate polynomials $\mathbb{H}$-stability in Definition III.2.1 is the same as $\mathbb{H}^c$-stability in Definition III.4.2.

Recall that any n-variate symmetric polynomial can uniquely be written as a polynomial in the first n elementary symmetric polynomials by the fundamental theorem of symmetric polynomials. We are interested in symmetric polynomials, which can be written as polynomials in few linear combinations of elementary symmetric polynomials, which generalizes the notion of multiaffine symmetric polynomials.

Definition III.4.4. Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric polynomial and write f in terms of elementary symmetric polynomials, say $f = g(e_1, \ldots, e_n)$.

1. We say that f is $(l_1, \ldots, l_k)$**-sufficient** if $g \in \mathbb{C}[l_1, \ldots, l_k]$ where $l_1, \ldots, l_k$ are linear forms.

2. Moreover, we say that a symmetric variety $V \subseteq \mathbb{C}^n$ is $(l_1, \ldots, l_k)$**-sufficient**, if it can be described by $(l_1, \ldots, l_k)$-sufficient polynomials.

Remark III.4.5. A polynomial f is $(l_1, \ldots, l_k)$**-sufficient** for some linear forms $l_1, \ldots, l_k$, if and only if f can be written as a polynomial in k symmetric and multiaffine polynomials. In particular, every symmetric and multiaffine polynomial is l_1-sufficient for some linear form l_1.

For instance, the polynomial $e_1^2 + e_2 + 2e_3$ is (l_1, l_2) sufficient for $l_1(y) = y_1$ and $l_2(y) = y_2 + 2y_3$. For checking sufficiency of polynomials and more on the notion of sufficiency we refer to Subsection 3.3 in [RS24].
The following Theorem is our main result of this section and can be seen at the same time as some kind of **degree principle** for checking stability and as some kind of generalization of Grace-Walsh-Szegő's coincidence theorem.

Theorem III.4.6. *Let $V \subseteq \mathbb{C}^n$ be a symmetric $(l_1, \ldots, l_k)$-sufficient variety. Then V is $\mathbb{H}$-stable, if and only if $V \cap \mathbb{H}_{2(k+2),k+2} = \emptyset$.*

Proof. The forward implication follows from the definition. To prove the converse implication we suppose that V is not $\mathbb{H}$-stable and we want to show that

$$V \cap \mathbb{H}_{2(k+2),k+2} \neq \emptyset.$$

So let $x \in V \cap \mathbb{H}^n$ and consider $z := (e_1(x), \ldots, e_n(x)) \in \S_{\mathbb{H}} \cap L^{-1}(a)$, where

$$\begin{array}{rcc} L : \mathbb{C}^n & \longrightarrow & \mathbb{C}^k \\ y & \longmapsto & (l_1(y), \ldots, l_k(y)) \end{array} \quad \text{and} \quad a := L\left(z\right) \in \mathbb{C}^k.$$

Then by Corollary III.2.10 we find $\tilde{z} \in \S_{\mathbb{H}}^{2(k+2),k+2} \cap L^{-1}(a)$, i.e. there is $\tilde{x} \in \mathbb{H}_{2(k+2),k+2}$ with $\tilde{z} = (e_1(\tilde{x}), \ldots, e_n(\tilde{x}))$ and $L(\tilde{z}) = a = L(z)$. This means that $\tilde{x} \in V$, since V is $(l_1, \ldots, l_k)$-sufficient. ∎

The following proposition is a direct consequence of the unique representation of a symmetric polynomial of degree d in terms of the elementary symmetric polynomials and may serve as a motivation for Definition III.4.4. We consider new variables $\underline{Z} = (Z_1, \ldots, Z_n)$. For a symmetric polynomial in $\mathbb{R}[\underline{X}]$ there is a unique polynomial $g \in \mathbb{R}[\underline{Z}]$ with $f(\underline{X}) = g(e_1(\underline{X}), \ldots, e_n(\underline{X}))$.

Proposition III.4.7. *Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial of degree d. Then f is $(Z_1, \ldots, Z_d)$-sufficient, i.e. f can be written as $f = g(e_1, \ldots, e_d)$ for some $g \in \mathbb{C}[Z_1, \ldots, Z_d]$. Moreover, g is linear in $Z_{\lfloor \frac{d}{2} \rfloor + 1}, \ldots, Z_d$.*

Proof. See Proposition 2.3 in [Rie12]. ∎

From Theorem III.4.6 and Proposition III.4.7, we obtain immediately the following **double-degree principle**.

Corollary III.4.8 (Double-degree principle)**.** *Let $f_1, \ldots, f_m \in \mathbb{C}[\underline{X}]$ be symmetric polynomials of degree at most d. Then*

$$V(f_1, \ldots, f_m) \cap \mathbb{H}^n = \emptyset \iff V(f_1, \ldots, f_m) \cap \mathbb{H}_{2(d+2),d+2} = \emptyset.$$

Remark III.4.9*.* In the case that $\mathbb{H}$ is a rotation of the upper half-plane we can replace $\mathbb{H}_{2(d+2),d+2}$ by $\mathbb{H}_{d,d}$ in Corollary III.4.8. This follows immediately from Remark III.2.11 for $d = 2$ and the case $d = 1$ is trivial.

Although one might hope for a stronger degree principle, the next example shows that stability of a variety defined by symmetric polynomials of degree $\leq d$ cannot always be checked by testing points with at most d many distinct coordinates.

Example III.4.10. Let $n = 4$ and consider $f_1 := e_1 - 23i$, $f_2 := e_2 - 463i$ and $f_3 := e_3 - 8461i$. Then

$$V(f_1, f_2, f_3) \cap \mathbb{H}_+^4 \neq \emptyset \text{ and } V(f_1, f_2, f_3) \cap \{x \in \mathbb{H}_+^4 \mid |\{x_1, \ldots, x_4\}| \leq 3\} = \emptyset,$$

which can either be computed directly using Gröbner basis or concluded by using Example III.2.12.

From Remark III.4.5 and Theorem III.4.6, we get immediately the following generalization of Grace-Walsh-Szegő's coincidence Theorem.

Corollary III.4.11. *Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric polynomial that can be written as a polynomial in k symmetric and multiaffine polynomials. Furthermore, let $x \in \mathbb{H}^n$. Then there is $\tilde{x} \in \mathbb{H}_{2(k+2),k+2}$ with $f(x) = f(\tilde{x})$.*

Note that different from Grace-Walsh-Szegő's coincidence theorem we do not require f to be multiaffine. But our result is weaker in the following sense: We do not consider closed inner or outer circle. Moreover, if f is symmetric of degree $d \geq 2$ and multiaffine and $x \in \mathcal{A}^n$, then we can find $\tilde{x} \in \mathbb{H}_{d,d}^n$ with

$$f(x) = f(\tilde{x}),$$

while one can find $y \in \mathbb{H}$ with Grace-Walsh-Szegő's coincidence Theorem such that

$$f(x) = f(y, \ldots, y).$$

Remark III.4.12*.* The results of this section translate to open half-planes in the following way: Let $\mathbb{G}$ be an open circular region and $x \in \mathbb{G}^n$. Then $x \in \mathbb{H}^n$ for some closed half-plane $\mathbb{H} \subset \mathbb{G}$. So $\mathbb{G}_{2(k+2),k+2}$ can be replaced by $\mathbb{G}_{0,3(k+2)}$ in Theorem III.4.6 and Corollary III.4.11 and $\mathbb{G}_{2(d+2),d+2}$ can be replaced by $\mathbb{G}_{0,3(d+2)}$ in Corollary III.4.8.

If $\mathbb{H} = \mathbb{H}_+$ is the upper half-plane, one can also formulate a generalization of the half-degree principle.

Theorem III.4.13 (Half-degree principle for the upper half-plane)**.** *Let $f \in \mathbb{C}[\underline{X}]$ be a symmetric polynomial of degree $d \leq n$ and $\lambda, \mu \in \mathbb{R}$. Then*

$$\inf_{x\in\mathbb{H}_+^n} \lambda\Re(f(x)) + \mu\Im(f(x)) = \inf_{x\in\mathbb{H}_{+\,k,k}} \lambda\Re(f(x)) + \mu\Im(f(x)),$$

where $k = \max\{\lfloor \frac{d}{2} \rfloor, 2\}$.

Proof. Write $f = g(e_1, \ldots, e_d)$ for some $g \in \mathbb{R}[Z_1, \ldots, Z_d]$ and note that g is linear in $Z_{\lfloor \frac{d}{2} \rfloor+1}, \ldots, Z_d$ by Proposition III.4.7. Let now $x \in \mathbb{H}_+^n$ and consider $z := (e_1(x), \ldots, e_n(x)) \in \S(a)$, where $a := (e_1(x), \ldots, e_k(x))$. Since $\S(a)$ is compact and g is linear on $\S(a)$, the minimum of g on $\S(a)$ is taken on an extreme point of the convex hull of $\S(a)$, i.e. on a point $\tilde{z} \in \S^{k,k}$ by Remark III.2.11. ■

III.5 Conclusion and open questions

In our paper, we restrict to half-plane stable polynomials. However, the notion of stable polynomials can be formulated for any circular region, i.e. any open or closed subset of $\mathbb{C}$ that is bounded by a circle or by a line. It is well known that a Möbius transformation maps circular regions to circular regions and testing stability of a polynomial can always be reduced to testing whether an associated polynomial of possibly smaller degree is $\mathbb{H}_+$-stable. Let $\mathcal{A}$ be a circular region and let $\phi(z) = \frac{az+b}{cz+d}$ be a Möbius transformation mapping $\mathbb{H}_+$ to $\mathcal{A}$. Then a monic polynomial $f \in \mathbb{C}[T]$ is $\mathcal{A}$-stable if and only if the polynomial $(cT+d)^{\deg(f)} f\left(\frac{aT+b}{cT+d}\right)$ is $\mathbb{H}_+$-stable. The roots of the associated polynomial are contained in the image of the roots of f under ϕ^{-1}. However, the obtained polynomial must not necessarily be monic or can have fewer roots. This happens if one of the roots is a pole point of ϕ^{-1}. For instance, if $f = p \cdot (T-1)$ is $\{x \in \mathbb{C} : |x| \leq 1\}$-stable and p has only roots different from 1, then

$$(T+i)^{\deg(p)} p\left(\frac{T-i}{T+i}\right)(T+i)\left(\frac{T-i}{T+i} - 1\right) = (T+i)^{\deg(p)} p\left(\frac{T-i}{T+i}\right) \cdot (-2i)$$

is a non-monic $\mathbb{H}_+$-stable polynomial of degree $\deg(f) - 1$. Thus our proofs of Theorems III.2.4 and III.4.6 do not transfer to circular regions which are bounded by a circle. Nevertheless, the following questions seem worth to be asked.

Question III.5.1. *Can Theorem III.2.4 and Theorem III.4.6 be adapted to arbitrary circular regions? If not, can our generalization of the coincidence theorem be extended to a closed domain bounded by a circle?*

Question III.5.2. *Can our double-degree principle in Corollary III.4.8 be improved?*

Finally, we gave a possible combinatorial encoding for subsets of the set of weakly Hurwitz polynomials. For hyperbolic polynomials, there is the rich interplay between geometry and combinatorics of its roots. Hyperbolic slices with fixed first k coefficients and their strata are known to be contractible. Moreover, Lien [Lie23] showed that in this case, one can reconstruct the compositions of the stratification from the compositions of its 0-dimensional strata, and Schabert and Lien [LS] showed that this poset has a structure similar to polytopes, giving the same bounds on its number of i-dimensional strata. We ask if similar results hold for Hurwitz slices with fixed first k coefficients.

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Paper IV

Constructively describing orbit spaces of finite groups by few inequalities

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Abstract

Let G be a finite group acting linearly on $\mathbb{R}^n$. A celebrated Theorem of Procesi and Schwarz gives an explicit description of the orbit space $\mathbb{R}^n//G$ as a basic closed semi-algebraic set. We give a new proof of this statement and another description as a basic closed semi-algebraic set using elementary tools from real algebraic geometry. Bröcker was able to show that the number of inequalities needed to describe the orbit space generically depends only on the group G. Here, we construct such inequalities explicitly for abelian groups and in the case where only one inequality is needed. Furthermore, we answer an open question raised by Bröcker concerning the genericity of his result.

IV.1 Introduction

A set $S \subset \mathbb{K}^n$ defined as the intersection of finitely many polynomial inequalities is called a **basic semi-algebraic set**. Sets obtained as finite unions or complements of such basic sets are known as **semi-algebraic** sets. A fundamental statement in real algebraic geometry, attributed to Tarski and Seidenberg, asserts that this class of sets is closed under polynomial maps. However, obtaining an explicit description of the image for a given semi-algebraic set and a specific polynomial map is far from trivial. Furthermore, although the image of a semi-algebraic set under a polynomial map is also semi-algebraic, it is generally not true that the image of a basic semi-algebraic set remains basic.
In this article, we investigate a special class of polynomial maps which map basic semi-algebraic sets to basic ones. Let $\mathbb{K}$ denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Let G be a group which we fix for this article to be a finite group and assume that G acts linearly on $\mathbb{K}^n$. Hilbert observed that, in this case, the ring of invariant polynomials is a finitely generated $\mathbb{K}$-algebra, say

$$\mathbb{K}[X_1, \ldots, X_n]^G = \mathbb{K}[\pi_1, \ldots, \pi_m] \subset \mathbb{K}[\underline{X}] := \mathbb{K}[X_1, \ldots, X_n].$$

IV

This inclusion induces the **Hilbert map** $\Pi : \mathbb{K}^n \to V_{\mathbb{K}}(I_\Pi)$, where $V_{\mathbb{K}}(I_\Pi)$ is the variety in $\mathbb{K}^m$ defined by the algebraic relations (syzygies) of the generators.
In the algebraically closed case, i.e., when $\mathbb{K} = \mathbb{C}$, this map is surjective and affords a homeomorphism between the orbit space $\mathbb{C}^n/G$ and the variety $V_{\mathbb{C}}(I_\Pi)$. Indeed, the image of the polynomial map corresponds to the categorical quotient $\mathbb{K}^n//G = \operatorname{Spec}(\mathbb{C}[\underline{X}]^G)$. However, in general, if $\mathbb{K}$ is not algebraically closed, the map fails to be surjective. In this case, the real categorical quotient $\mathbb{K}^n//G$ is, by the Tarski-Seidenberg Theorem, a **semi-algebraic set** and can be written as a union of intersections of solution sets of polynomial inequalities.
Remarkably, it turns out that in this case the situation is more favorable: Even though, in general, the image of a polynomial map is not basic and obtaining explicit polynomial descriptions can be challenging, it was shown by Procesi and Schwarz [PS85] that the image of the Hilbert map is a basic closed semi-algebraic set. Moreover, in the case of compact Lie groups, these inequalities can be obtained directly from the chosen fundamental invariants.
For a polynomial p, we consider the differential dp defined by

$$dp = \sum_{j=1}^{n} \frac{\partial p}{\partial x_j} dx_j.$$

For finite (compact) G, we have a G-invariant inner product $\langle \cdot, \cdot \rangle$, which, when applied to the differentials, yields

$$\langle dp, dq \rangle = \sum_{j=1}^{n} \frac{\partial p}{\partial x_j} \cdot \frac{\partial q}{\partial x_j}.$$

Since differentials of G-invariant polynomials are G-equivariant, the inner products $\langle d\pi_i, d\pi_j \rangle$ $(i, j \in \{1, \dots, m\})$ are G-invariant, and hence every entry of the symmetric matrix polynomial

$$M_\Pi = (\langle d\pi_i, d\pi_j \rangle)_{1 \le i,j \le m}$$

is an invariant polynomial. With a slight misuse of notation, we can thus represent it as a matrix polynomial in $\pi_1, \dots, \pi_m$. Using this construction, Procesi and Schwarz [PS85] have shown the following.

Theorem IV.1.1 (Procesi and Schwarz)**.** *Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a finite group, and let $\Pi = (\pi_1, \dots, \pi_m)$ be fundamental invariants of G. Then the orbit space is given by polynomial inequalities,*

$$\Pi(\mathbb{K}^n) = \{z \in V(I_\Pi) \subseteq \mathbb{K}^m \mid M_\Pi(z) \text{ is positive semi-definite}\},$$

where $I_\Pi \subseteq \mathbb{K}[z_1, \dots, z_m]$ is the ideal of relations of $\pi_1, \dots, \pi_m$.

This theorem has many applications in various areas, including differential geometry, dynamical systems, and mathematical physics (see, for example, [Dub98; Fie07; Hui99]). Since the description of the semi-algebraic set, in combination with Artin's solution to Hilbert's 17th problem, gives rise to an equivariant Positivstellensatz, it can also be applied in polynomial optimization [MRV23; Rie+13]. The goal of this article is twofold. In the first part of the paper, we aim to demonstrate that this remarkable result can be established with elementary results in real algebraic geometry in the case of finite groups. Specifically, we show in Theorem IV.2.5 that the fact that the set is basic can be directly obtained by combining sums of squares with basic invariant theoretic results. This follows from the well-known fact that the polynomial ring is a finite module over the invariant ring. Additionally, we provide a short proof of the description by Procesi and Schwarz. Note that after their original paper, Procesi and Schwarz also obtained a rather elementary argument for their statement for finite groups [PS88]. Furthermore, using standard arguments in invariant theory, like Luna's slice theorem, the finite case can be transferred to the compact case. Thus, any elementary proof for the finite case is essentially generalizable. However, the motivation for our proof is not only that it is elementary but it also serves as a stepping stone for more efficient descriptions of orbit spaces, which is the question we focus on in the second part.

Given any basic semi-algebraic set, it is natural to ask about the minimal number of inequalities needed to describe it. A famous result by Bröcker and Scheiderer [Brö91; Sch89] shows that any closed basic semi-algebraic set in $\mathbb{K}^n$ can be described by $n(n-1)/2$ inequalities. On the other hand, the description by Procesi and Schwarz is better, as it yields m inequalities, where m is the number of fundamental invariants, which is the dimension the orbit space is embedded into. However, as observed by Procesi and Schwarz, if the order $|G|$ of G is odd, the Hilbert map is surjective and one does not need any inequality, although their construction still produces m inequalities. This raises the natural question of how the number of inequalities is related to the structure of G. Bröcker [Brö98] answered this question completely: The number k of inequalities needed to describe the orbit space generically, i.e. up to some lower dimensional set T, is exactly the maximal number for which G contains an elementary abelian subgroup of order 2^k. Although this completely answers the question, Bröcker's proof is not constructive. In the second part of the article, we turn to a first class of non-trivial examples, where we constructively build a description with the least number of inequalities as predicted by Bröcker's theorem. Furthermore, we answer a question raised by Bröcker: We give an example for which the generic description obtained from Bröcker is not a complete description of the orbit space, i.e. an example where one needs to add some lower dimension set T.

This article is structured as follows: In the following section, we provide elementary arguments to establish that the orbit space is a basic semi-algebraic set and give a new proof for the description due to Procesi and Schwarz, which is obtained by going through subgroup chains. The third section provides the construction of orbit spaces with the least number of inequalities, and we conclude with some open questions.

IV.2 Real orbit spaces for finite groups

We now aim to provide firstly a constructive argument for the remarkable fact that $\Pi(\mathbb{R}^n)$ is a basic semi-algebraic set and then secondly give an elementary proof for Theorem IV.1.1. Throughout this section, G is a finite group and we use the notations of Theorem IV.1.1.

IV.2.1 Notations

Let G act linearly on $\mathbb{K}^n$, then we get an action on the ring of polynomials defined by

$$h^\sigma = h(\sigma^{-1}(x)), \text{ for all } \sigma \in G.$$

Furthermore, we will use that the polynomial ring $\mathbb{K}[X]$ is a $\mathbb{K}[X]^G$ module. Notice that $\mathbb{K}[\underline{X}]$ is integral over $\mathbb{K}[\underline{X}]^G$. Indeed, for any $f \in \mathbb{K}[\underline{X}]$, we have a monic characteristic polynomial

$$\chi_f(T) := \prod_{\sigma \in G} (T - f^\sigma) \in \mathbb{K}[\underline{X}]^G[T].$$

Therefore, $\mathbb{K}[\underline{X}]$ is a finitely generated $\mathbb{K}[\underline{X}]^G$-module. Moreover, we can define a simple projection operator, called the **Reynolds operator** which gives a projection from $\mathbb{K}[X]$ to $\mathbb{K}[X]^G$. It is defined by

$$\mathcal{R}_G : \ h \mapsto \frac{1}{|G|} \sum_{\sigma \in G} h^\sigma.$$

To work in the algebraic setting of ring extensions, it is practical to identify the points in $\mathbb{K}^n$ with ring homomorphisms from $\mathbb{K}[\underline{X}]$ to $\mathbb{K}$. In this way we identify $V_{\mathbb{K}}(I_\Pi)$ with $\hom(\mathbb{K}[\underline{X}]^G, \mathbb{K})$ and every $z \in V_{\mathbb{K}}(I_\Pi)$ is identified with the ring homomorphism ϕ_z defined to be the evaluation in z. On the other hand, since every ring homomorphism is determined uniquely by the image of $X_1, \dots, X_n$, we can equivalently identify every $\phi \in \hom(\mathbb{K}[\underline{X}]^G, \mathbb{K})$ with a unique point $z_\phi \in V_{\mathbb{K}}(I_\Pi)$.

Proposition IV.2.1. *With the above identification a point $z \in V_{\mathbb{K}}(I_\Pi)$ is in the image of Π if and only if ϕ_z can be extended to a ring homomorphism from $\mathbb{K}[\underline{X}]$ to $\mathbb{K}$.*

One of the basic notions in real algebraic geometry is the notion of sums of squares and our proofs will rely on the set of invariant sums of squares.

Definition IV.2.2. A polynomial $f \in \mathbb{R}[\underline{X}]$ is called a sum of squares if it can be decomposed into the form $f = f_1^2 + \ldots + f_\ell^2$ for some polynomials $f_1, \dots, f_\ell \in \mathbb{R}[\underline{X}]$. We will write $\Sigma\mathbb{R}[\underline{X}]^2$ for the set of all these polynomials. Furthermore, we set $\left(\Sigma\mathbb{R}[\underline{X}]^2\right)^G = \Sigma\mathbb{R}[\underline{X}]^2 \cap \mathbb{R}[X]^G$. More generally, we call a symmetric matrix polynomial $A \in \mathbb{R}[\underline{X}]^{k\times k}$ a sums of squares matrix polynomial, if $A = L^t L$ for some $L \in \mathbb{R}[\underline{X}]^{k\times \ell}$ and we say that a sums of squares matrix polynomial is G invariant if all of its entries are G-invariant polynomials.

Notice that the set of invariant sums of squares $\left(\Sigma\mathbb{R}[\underline{X}]^2\right)^G$ defines a quadratic module in the ring $\mathbb{R}[\underline{X}]^G$, which in general is not finitely generated (see [CKS09, Example 5.3]). However, using the fact that $\mathbb{R}[X]$ is a finitely generated module it can be conveniently represented using sums of squares matrices. Indeed, let $b_1, \ldots, b_l \in \mathbb{R}[\underline{X}]$ be generators of $\mathbb{R}[\underline{X}]$ over $\mathbb{R}[\underline{X}]^G$ and define

$$B \in (\mathbb{R}[\underline{X}]^G)^{l\times l} \text{ by } B_{ij} := \mathcal{R}_G(b_i b_j),$$

then we have the following characterization (see [BR21; GP04; MRV23] for details):

Proposition IV.2.3. *Let $f \in \mathbb{R}[X]^G$. Then $f \in \Sigma(\mathbb{R}[\underline{X}]^G)^2$ if and only if there exists a G-invariant sums of invariant squares matrix polynomial $A \in (\mathbb{R}[\underline{X}]^G)^{t\times t}$ with a factorization $A = L^t L$ from some $L \in (\mathbb{R}[\underline{X}]^G)^{k\times \ell}$ such that*

$$f = \operatorname{Tr}(A \cdot B)$$

Proof. We sketch the proof for the convenience of the reader. We only consider the case when f is the sum of the orbit of one square - and the general case follows directly in the same way. Let $g \in \mathbb{R}[\underline{X}]$ with $f = \mathcal{R}_G(g^2)$ and write $g = \sum_{i=1}^{l} a_i b_i$ for some $a_1, \ldots, a_l \in \mathbb{R}[\underline{X}]^G$. Then

$$f = \mathcal{R}_G(g^2) = \mathcal{R}_G(\sum_{i,j=1}^{l} a_i a_j b_i b_j) = \sum_{i,j=1}^{l} a_i a_j \mathcal{R}_G(b_i b_j) = a^T B a = Tr(aa^t B)$$

where $a := (a_1, \ldots, a_l)^T$. ∎

IV.2.2 Orbit spaces as basic semi-algebraic sets

Based on the previous discussions it is almost directly clear that the semi-algebraic set $\Pi(\mathbb{R}^n)$ is basic. Indeed, it will be a consequence of the following simple observation.

Proposition IV.2.4. *Let $z \in V(I_\Pi)$ such that $\Pi^{-1}(z) \notin \mathbb{R}^n$, then there exists $f \in \left(\Sigma\mathbb{R}[\underline{X}]^2\right)^G$ such that $\phi_z(f) < 0$.*

Proof. Set $\xi = \Pi^{-1}(z) \in \mathbb{C}^n \setminus \mathbb{R}^n$, let $1 \le i \le n$ be such that $\operatorname{Im}(\xi_i) \neq 0$, and consider the polynomial $h := (X_i - \operatorname{Re}(\xi_i))^2$. Clearl, $h(\xi) < 0$. Let $\mathcal{O}_G(h)$ denote the orbit of h under G. We now construct the univariate polynomial $p \in \mathbb{R}[\underline{X}]^G[T]$ by

$$p(T) = \prod_{h' \in \mathcal{O}_G(h)} (T + h').$$

Each of the coefficients of p is in $(\Sigma\mathbb{R}[\underline{X}]^2)^G$. We evaluate these coefficients in z and observe that the resulting univariate polynomial will have at least one negative coefficient by Descartes' rule of signs. Thus, for some $1 \le k \le l = |\mathcal{O}_G(h)|$ we have an elementary symmetric polynomial $f := e_k(h^{\sigma_1}, \ldots, h^{\sigma_l}) \in \left(\Sigma\mathbb{R}[\underline{X}]^2\right)^G$ such that $\phi_z(f) = f(\xi) < 0$. ∎

Combining this observation with Proposition IV.2.3, we immediately get a description of $\Pi(\mathbb{R}^n)$ as basic semi-algebraic set.

Theorem IV.2.5. *The semi-algebraic set* $\Pi(\mathbb{R}^n)$ *is basic closed, it can be represented as*

$$\Pi(\mathbb{K}^n) = \{z \in V(I_\Pi) \subseteq \mathbb{K}^m \mid \phi_z(B) \textit{ is positive semi-definite}\}.$$

Proof. Clearly, the condition that $B(z)$ is positive semi-definite is necessary. Indeed, for $z \in \Pi(\mathbb{R}^n)$ there exists by definition $x \in \mathbb{R}^n$ such that $\Pi(x) = z$, and therefore $\phi_z(B) = B(x)$. Let $v \in \mathbb{R}^m$. Then $v^T B v \in (\Sigma\mathbb{R}[\underline{X}]^2)^G$, so $v^T \phi_z(B) v = (v^T B v)(x) \geq 0$. On the other hand, assume that $\phi_z(B)$ is positive semi-definite but that ϕ_z cannot be extended. By Proposition IV.2.4 we know that there exists an invariant sums of squares polynomial f with $\phi_z(f) < 0$. Furthermore, by Proposition IV.2.3 we have

$$f = \mathrm{Tr}(L(x)L(x)^t B(x)).$$

However, since $\phi_z(L(x))\phi_z(L(x)^t)$ and $\phi_z(B(x))$ are positive semi-definite matrices, we have a contradiction. ■

Remark IV.2.6. This result is quite constructive: It suffices to find generators of $\mathbb{R}[\underline{X}]$ over $\mathbb{R}[\underline{X}]^G$. If G is of order l, $\mathbb{R}[\underline{X}]$ is generated by

$$X_1^{\alpha_1} \cdots X_n^{\alpha_n} \qquad \alpha \in \{0, \dots, l-1\}^n$$

over $\mathbb{R}[\underline{X}]^G$ (use [Ati18] Proposition 2.16, Proposition 5.1 and Corollary 5.2). Furthermore, in the case of finite reflection groups these generators can be found very directly and with combinatorial methods [DR23; HR22]. Moreover, one can also use the G-harmonic polynomials, which can be obtained as the partial derivatives of the determinant of the Jacobian of the Hilbert map (see [Hel84] chapter III 1.1, 3.6 and 3.7).

IV.2.3 An elementary proof for Theorem IV.1.1

We now want to show an elementary proof for the concrete description given in Theorem IV.1.1. To begin, we reformulate the statement in terms of homomorphisms.

Proposition IV.2.7. *The following are equivalent:*

(i) $\Pi(\mathbb{R}^n) = \{z \in V_{\mathbb{R}}(I_\Pi) \mid \phi_z(M_\Pi) \textit{ is positive semi-definite}\}$.

(ii) *Let* $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ *be a ring homomorphism. Then* ϕ *can be extended to a ring homomorphism* $\tilde{\phi} : \mathbb{R}[\underline{X}] \to \mathbb{R}$ *if and only if*

$$\phi(\langle df, df \rangle) \geq 0 \textit{ for all } f \in \mathbb{R}[\underline{X}]^G.$$

Proof. Let $z \in V_{\mathbb{R}}(I_\Pi)$. Note that $z \in \Pi(\mathbb{R}^n)$ if and only if ϕ_z can be extended to a ring homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}] \to \mathbb{R}$. Furthermore, since $\pi_1, \ldots, \pi_m$ generate the ring of invariants we have $\phi_z(M_\Pi)$ is positive semi-definite if and only if $\phi_z(\langle df, df \rangle) \geq 0$ for all $f \in \mathbb{R}[\underline{X}]^G$. ∎

Note that $\phi_z(M_\Pi)$ being positive semi-definite in (i) is clearly necessary for z being in the image of the real Hilbert map. Equivalently, $\phi(\langle df, df \rangle) \geq 0$ for all $f \in \mathbb{R}[\underline{X}]^G$ is necessary in order to extend a homomorphism $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ to a homomorphism $\mathbb{R}[\underline{X}] \to \mathbb{R}$. So we will show that this is also sufficient, and our strategy consists in extending homomorphisms in steps, first to the ring of invariant polynomials for some subgroup H of G. In order to do this we have to make sure that our positivity condition in (ii) extends also to $\mathbb{R}[\underline{X}]^H$ in this case. The following is in fact the core of the argument.

Proposition IV.2.8. *Let H be a subgroup of G and let $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ be a homomorphism. Suppose*

$$\phi(\langle df, df \rangle) \geq 0 \text{ for all } f \in \mathbb{R}[\underline{X}]^G,$$

and ϕ can be extended to a homomorphism $\phi_H : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$. Then

$$\phi_H(\langle df, df \rangle) \geq 0 \text{ for all } f \in \mathbb{R}[\underline{X}]^H.$$

Proof. The ring of invariants $\mathbb{R}[\underline{X}]^H$ is a finitely generated $\mathbb{R}$-algebra, say $\mathbb{R}[\underline{X}]^H = \mathbb{R}[p_1, \ldots, p_k]$ and write $\pi_i = q_i(p_1, \ldots, p_k)$ for some $q_i \in \mathbb{R}[Y_1, \ldots, Y_k]$. Consider

$$\begin{aligned}
\Pi : \mathbb{C}^n \to V_{\mathbb{C}}(I_\Pi),\ & x \mapsto (\pi_1(x), \ldots, \pi_m(x)), \\
P : \mathbb{C}^n \to V_{\mathbb{C}}(I_P),\ & x \mapsto (p_1(x), \ldots, p_k(x)) \text{ and} \\
Q : V_{\mathbb{C}}(I_P) \to V_{\mathbb{C}}(I_\Pi),\ & x \mapsto (q_1(x), \ldots, q_k(x))
\end{aligned}$$

and furthermore, define the points

$$z := z_\phi := (\phi(\pi_1), \ldots, \phi(\pi_m)) \in V_{\mathbb{R}}(I_\Pi) \text{ and } y := y_{\phi_H} := (\phi_H(p_1), \ldots, \phi_H(p_k)) \in V_{\mathbb{R}}(I_P)$$

corresponding to ϕ and ϕ_H. Since P is surjective, there is $x \in \mathbb{C}^n$ with $P(x) = y$, so $\Pi(x) = z$. Denote the corresponding total derivatives of Π, P and Q in x, respectively in y, by

$$D_Q^y : \mathbb{C}^m \to \mathbb{C}^k,\ D_P^x : \mathbb{C}^k \to \mathbb{C}^n \text{ and } D_\Pi^x : \mathbb{C}^m \to \mathbb{C}^n.$$

We can assume furthermore that the stabilizer G_x of x in G is trivial and therefore included in H (otherwise we can perturbate y and therefore x and z a little bit by continuity). So $\dim(\operatorname{im}(D_P^x)) = \dim(\operatorname{im}(D_\Pi^x))$ (see [AS83]) and therefore $\operatorname{im}(D_P^x) = \operatorname{im}(D_\Pi^x)$, since $\operatorname{im}(D_P^x) \supseteq \operatorname{im}(D_\Pi^x)$ is trivial. So for every $v \in \mathbb{C}^k$ there is some $u \in \mathbb{C}^m$ with $D_\Pi^x(u) = D_P^x(v)$ and so the identity

$$v = D_Q^x(u) + (v - D_Q^x(u)) \in \operatorname{im} D_Q^y + \ker D_P^x$$

shows that $\operatorname{im} D_Q^y + \ker D_P^x = \mathbb{C}^k$.
Let now $f \in \mathbb{R}[\underline{X}]^H$, say $f = g(p_1, \dots, p_k)$. Then there is $u \in \ker D_P^x$ and $v \in \operatorname{im} D_Q^y$ with $a := (dg)(P(x)) = u + v$, i.e. $D_P^x(u) = 0$ and there is $w \in \mathbb{C}^m$ with $v = D_Q^y(w)$. Because a and y are real, we can assume that $w \in \mathbb{R}^m$. Then

$$\begin{aligned}
\phi_H(\langle df, df \rangle) &= \langle (df)(x), (df)(x) \rangle \\
&= \langle D_P^x(a), D_P^x(a) \rangle \\
&= \langle D_P^x(u+v), D_P^x(u+v) \rangle \\
&= \langle D_P^x(u) + D_P^x(v), D_P^x(u) + D_P^x(v) \rangle \\
&= \langle D_P^x(D_Q^y(w)), D_P^x(D_Q^y(w)) \rangle \\
&= \langle D_\Pi^x(w), D_\Pi^x(w) \rangle \\
&= \phi(\langle dh, dh \rangle) \geq 0,
\end{aligned}$$

where $h := w_1\pi_1 + \dots + w_m\pi_m \in \mathbb{R}[\underline{X}]^G$. ■

Now we will collect some facts about invariants of groups of order two because we will extend our homomorphism first to the ring of invariant polynomials for some order two subgroup H of G. This order two subgroup will correspond to a complex point and its complex conjugated point.

Lemma IV.2.9. *Let H be a subgroup of G.*

(a) $\mathbb{R}[\underline{X}]^H = \mathbb{R}[\underline{X}]^G \oplus \ker\left(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H}\right)$.

(b) *If $|G/H| = 2$, then* $\ker\left(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H}\right)^2 \subseteq \mathbb{R}[\underline{X}]^G$.

Proof.

(a) Let $g \in \ker \mathcal{R}_G \cap \mathbb{R}[\underline{X}]^G$. Then $g = \mathcal{R}_G(g) = 0$, so $\ker \mathcal{R}_G \cap \mathbb{R}[\underline{X}]^G = \{0\}$. Now let $f \in \mathbb{R}[\underline{X}]^H$. Then

$$f = \mathcal{R}_G(f) + f - \mathcal{R}_G(f) \in \mathbb{R}[\underline{X}]^G \oplus \ker\left(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H}\right)$$

and

$$\mathbb{R}[\underline{X}]^G \oplus \ker\left(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H}\right) \subseteq \mathbb{R}[\underline{X}]^H$$

is clear.

(b) Denote $G/H = \{H, \sigma H\}$ and let $r \in \ker\left(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H}\right)^2$. Then

$$0 = \mathcal{R}_G(r) = \frac{1}{2}(r + \sigma \cdot r),$$

so $\sigma r = -r$ and therefore $\sigma r^2 = r^2$, i.e. r^2 is G-invariant.

■

From these elementary considerations we now can give a proof of Theorem IV.1.1.

Proof of Theorem IV.1.1. We will show property (ii) in Proposition IV.2.7: "$\Rightarrow$" is clear.
"$\Leftarrow$" by contradiction: Let $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ be a ring homomorphism such that

$$\phi(\langle df, df\rangle) \geq 0 \text{ for all } f \in \mathbb{R}[\underline{X}]^G$$

and ϕ cannot be extended to $\mathbb{R}[\underline{X}]$, i.e. $z := (\phi(\pi_1), \ldots, \phi(\pi_m)) \notin \Pi(\mathbb{R}^n)$. There is $x \in \mathbb{C}^n$ such that $\Pi(x) = z$. Since $\Pi(\bar{x}) = z$, there is $\sigma \in G$ such that $\sigma \cdot x = \bar{x}$. Then σ has even order. Now consider the subgroups $C_\sigma = \langle\sigma\rangle$ and $C_{\sigma^2} = \langle\sigma^2\rangle$ of G. Since

$$f(x) = \sigma f(x) = f(\bar{x}) = \overline{f(x)}$$

for all $f \in \mathbb{R}[\underline{X}]^{C_\sigma}$, we can extend ϕ_z to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^{C_\sigma} \to \mathbb{R}$.
By Proposition IV.2.8, $\tilde{\phi}(\langle df, df\rangle) \geq 0$ for all $f \in \mathbb{R}[\underline{X}]^{C_\sigma}$.
Since $x \notin \mathbb{R}^n$, there is a linear $l \in \mathbb{R}[\underline{X}]$ with $l(x) = i$. Consider now

$$f = R_{C_{\sigma^2}}(l) - R_{C_\sigma}(l) \in \ker R_{C_\sigma}.$$

Note that $\langle df, df\rangle \in \mathbb{R}_{>0}$, because f is linear. Since C_{σ^2} acts trivially on x and $\sigma x = \bar{x}$ we get that $f(x) = i$ and therefore $\phi(f^2) = f^2(x) = -1 < 0$ with $f^2 \in \mathbb{R}[\underline{X}]^{C_\sigma}$ by Lemma IV.2.9.
Now

$$\tilde{\phi}(\langle df^2, df^2\rangle) = \tilde{\phi}(\langle 2f df, 2f df\rangle) = \tilde{\phi}(4f^2)\langle df, df\rangle < 0,$$

which is a contradiction to $\tilde{\phi}(\langle dg, dg\rangle) \geq 0$ for all $g \in \mathbb{R}[\underline{X}]^{C_\sigma}$. ∎

Theorem IV.1.1 can be extended to compact Lie groups by Luna's slice theorem, similar to what is done in [PS85]. We will end this section by briefly sketching an alternative purely algebraic approach. If the group G is not finite, then $\mathbb{R}(\underline{X})^G$ will not be algebraic over $\mathbb{R}(\underline{X})$, i.e. there are transcendental elements $T_1, \ldots, T_k \in \mathbb{R}(\underline{X})$ over $\mathbb{R}(\underline{X})^G$ such that $\mathbb{R}(\underline{X})|\mathbb{R}(\underline{X})^G[T_1, \ldots, T_k]$ is algebraic and $\mathbb{R}(\underline{X})^G[T_1, \ldots, T_k]|\mathbb{R}(\underline{X})^G$ is purely transcendental. We can assume without loss of generality, that $T_i = X_i$. Furthermore, $\mathbb{R}(\underline{X})^G[X_1, \ldots, X_k] = \mathbb{R}(\underline{X})^H$, where $H = \operatorname{stab}(X_1, \ldots, X_k)$. Note that H is finite since $\mathbb{R}(\underline{X})|\mathbb{R}(\underline{X})^H$ is algebraic. We can now extend ϕ to $\mathbb{R}[\underline{X}]^H$ by the following proposition.

Proposition IV.2.10. *Let H be a subgroup of G such that $\mathbb{R}(\underline{X})^H$ is purely transcendental over $\mathbb{R}(\underline{X})^G$, say $\mathbb{R}(\underline{X})^H = \mathbb{R}(\underline{X})^G[T_1, \ldots, T_k]$ for some $T_1, \ldots, T_k \in \mathbb{R}[\underline{X}]$ transcendental over $\mathbb{R}(\underline{X})^G$. Furthermore, let $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ be a ring homomorphism. Then for any $t_1, \ldots, t_k \in \mathbb{R}$, we get an extension $\phi_H : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$ of ϕ by setting $\phi_H(T_i) = t_i$ and $\phi_H(f) = \phi(f)$ for all $f \in \mathbb{R}[\underline{X}]^G$.*

Proof. The homomorphism ϕ_H is well defined because $T_1, \ldots, T_k$ are transcendental over $\mathbb{R}(\underline{X})^G$. ∎

Since we can choose $t_1, \ldots, t_k \in \mathbb{R}$ arbitrarily, it should be possible to extend ϕ to $\mathbb{R}[\underline{X}]^H$ in such a way that the *Procesi-Schwarz matrix* $M_{X_1,\ldots,X_k,\Pi}$ corresponding to H is positive semi-definite, although we were not able to prove this. This would reduce then to the finite case.

Remark IV.2.11. Alternatively to the view point of ring homomorphisms one could work also in the spectral setting, i.e. replacing $\mathbb{R}^n$ and $V_{\mathbb{R}}(I_\Pi)$ by the real spectrum of $\mathbb{R}[\underline{X}]$ and $\mathbb{R}[\underline{X}]^G$. In this setup one considers extensions of orders instead of extensions of homomorphisms. This approaches should be essentially equivalent and we decided not to take the spectral point of view here in order to keep our results more accessible.

IV.3 Describing orbit spaces with few inequalities

In the previous section, we obtained inequalities describing the orbit space as a basic closed semi-algebraic set. The aim of this section is to find descriptions of the orbit space involving fewer inequalities. To this end, we fix again a finite group G and introduce some notation.

Definition IV.3.1. We say that the orbit space $\Pi(\mathbb{R}^n)$ (or $\mathbb{R}^n/G$) is **described by** $f_1, \ldots, f_k \in \mathbb{R}[\underline{X}]^G$ if

$$\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid \phi_z(f_1) \geq 0, \ldots, \phi_z(f_k) \geq 0\}.$$

Furthermore, we will say that $\Pi(\mathbb{R}^n)$ is **generically described by** $f_1, \ldots, f_k$ if

$$\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid \phi_z(f_1) > 0, \ldots, \phi_z(f_k) > 0\} \cup T,$$

where $\dim(T) < \dim(\Pi(\mathbb{R}^n))$.

It was already observed that if G has odd order, then one needs no inequalities to describe the orbit space $\mathbb{R}^n/G$, i.e. the Hilbert map is surjective. More generally, Bröcker ([Brö98][Proposition 5.6.]) proved the following about the number of inequalities needed to generically describe the orbit space.

Theorem IV.3.2 (Bröcker)**.** *Let k be the maximal number for which G contains an elementary abelian subgroup of order 2^k. Then the orbit space $\Pi(\mathbb{R}^n)$ is generically described by some $f_1, \ldots, f_k \in \mathbb{R}[\underline{X}]^G$.*

Note that the proof of Bröcker's Theorem is non-constructive. The goal of this section is to construct these few inequalities for the case where $k = 1$ and for abelian groups. We answer first an open question raised by Bröcker [Brö98] and point out a small mistake in his Theorem: Bröcker writes that one needs in all examples he knows no T. He also states Theorem IV.3.2 with non-strict inequalities. The following example shows that sometimes some lower-dimension T is needed and that the strict inequalities in Theorem IV.3.2 can not be replaced by non-strict inequalities.

Example IV.3.3. Consider $C_4 = \langle (1, 2, 3, 4) \rangle$ acting by permutation on $\mathbb{R}^4$. Denote by $\pi_1, \ldots, \pi_k$ the fundamental invariants and by

$$\Pi : \mathbb{C}^4 \to \mathbb{C}^k$$

the Hilbert map as usual. Suppose $\Pi(\mathbb{R}^4)$ is described by one inequality $f \geq 0$, where $f \in \mathbb{R}[X_1, \ldots, X_4]^{C_4}$. Since

$$g(a, b, \bar{a}, \bar{b}) = (13)(24)g(a, b, \bar{a}, \bar{b}) = g(\bar{a}, \bar{b}, a, b) = \overline{g(a, b, \bar{a}, \bar{b})}$$

and
$$g(a,\bar{a},a,\bar{a}) = (1234)g(a,\bar{a},a,\bar{a}) = g(\bar{a},a,\bar{a},a) = \overline{g(a,\bar{a},a,\bar{a})}$$
for all $a,b \in \mathbb{C}$ and for all $g \in \mathbb{R}[X_1,\ldots,X_4]^{C_4}$, we get that
$$\Pi(a,b,\bar{a},\bar{b}) \in \mathbb{R}^k \quad \text{and} \quad \Pi(a,\bar{a},a,\bar{a}) \in \mathbb{R}^k.$$
for all $a,b \in \mathbb{C}$. Therefore
$$f(a,b,\bar{a},\bar{b}) \leq 0$$
for all $a,b, \in \mathbb{C}$, where the inequality is strict for a or b are not real. So we get that
$$f(a,b,a,b) = 0$$
for all $a,b \in \mathbb{R}$. This means that the 2-variate polynomial
$$g := f(X_1,X_2,X_1,X_2)$$
vanishes on $\mathbb{R}^2$, so $g = 0$ and thus $f(i,-i,i,-i) = g(i,-i) = 0$, contradicting $(i,-i,i,-i) \notin \Pi(\mathbb{R}^4)$.
In a similar way one can show that the strict inequalities in Theorem IV.3.2 can not be replaced by non-strict inequalities. Assume
$$\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid \phi_z(f) \geq 0\} \cup T,$$
where $\dim(T) < \dim(\Pi(\mathbb{R}^n))$ for some G-invariant f. We get again
$$f(a,b,\bar{a},\bar{b}) \leq 0$$
for all $a,b \in \mathbb{C}$. If
$$f(a,b,a,b) = 0$$
for all $a,b \in \mathbb{R}$, then we get the same contradiction as before. So
$$f(a,b,a,b) < 0$$
for some $a,b \in \mathbb{R}$. So
$$f < 0 \text{ on some neighbourhood of } (a,b,a,b) \text{ in } \mathbb{R}^4,$$
which is a contradiction to $\dim(T) < \dim(\Pi(\mathbb{R}^n))$.

IV.3.1 Strategy for constructing few inequalities

In this subsection, we introduce a strategy and some tools to prove a constructive version of Bröcker's Theorem. This is easy if G is already an elementary abelian 2-subgroup. Therefore, one might be tempted to try the following approach:

1. First we try to find an elementary abelian 2-subgroup H that corresponds to all complex conjugations that might appear. If such a subgroup exists, then we can extend every homomorphism $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$ (see Lemma IV.3.5).

2. Now one gets H-invariant polynomials $h_1, \ldots, h_k$ describing the orbit space $\mathbb{R}^n/H$.

3. Then one tries to symmetrize these polynomials. If one finds G-invariant polynomials $g_1, \ldots, g_k$ describing the orbit space $\mathbb{R}^n/H$, then these will also generically describe the orbit space $\mathbb{R}^n/G$.

The first step is in particular fulfilled if G contains a so-called broad subgroup. In general, the first step fails, as shown in Example IV.3.9.

Definition IV.3.4. An elementary abelian 2-subgroup H of G is called **broad**, if every involution of G is conjugated to an element of H.

Lemma IV.3.5. *Let H be a broad subgroup of G. Then every homomorphism $\phi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ of principal orbit type can be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$.*

Proof. Let $\varphi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ be a homomorphism. There is an extension $\psi : \mathbb{R}[\underline{X}] \to \mathbb{C}$ of φ. Then $\overline{\psi}$ is also an extension of φ, i.e. there is $\sigma \in G$ such that

$$\sigma\psi = \overline{\psi}.$$

Now $\sigma^2\psi = \psi$ and therefore σ is an involution, since ψ has principal orbit type. Since H is broad, there is $h \in H$ and $g \in G$ such that $hg = g\sigma$. Consider now the extension $\tilde{\psi} := g\psi : \mathbb{R}[\underline{X}] \to \mathbb{C}$ of φ. Then

$$h\tilde{\psi} = hg\psi = g\sigma\psi = g\overline{\psi} = \overline{\tilde{\psi}}$$

and therefore

$$\tilde{\psi}(f) = \tilde{\psi}(hf) = h\tilde{\psi}(f) = \overline{\tilde{\psi}}(f)$$

for all $f \in \mathbb{R}[\underline{X}]^H$. So $\tilde{\varphi}|_{\mathbb{R}[\underline{X}]^H} : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$ is a real extension of φ to $\mathbb{R}[\underline{X}]^H$. ■

There are at least three types of groups that contain broad subgroups. The first two results are given by [GR22].

Theorem IV.3.6 (Guralnick and Robinson)**.** *Let G be quasi-simple. Then G contains a broad subgroup.*

Remark IV.3.7. As mentioned in [GR22], S_n contains also a broad subgroup. Indeed

$$H := \left\langle (1,2), (3,4), \ldots, \left(2\left\lfloor\frac{n}{2}\right\rfloor - 1, 2\left\lfloor\frac{n}{2}\right\rfloor\right)\right\rangle$$

is a broad subgroup of S_n.

Lemma IV.3.8. *Assume all maximal elementary abelian 2-subgroups of G are of order 2. Then every involution of G generates a broad subgroup of G.*

Proof. This follows from the fact that two involutions are either conjugated or commute with a third involution: Let $\sigma, \tau \in G$ be involutions. Then

$$(\sigma\tau)^k = 1$$

for some $k \in \mathbb{N}$. If k is even, then σ and τ commute with the involution $(\sigma\tau)^{\frac{k}{2}}$. If k is odd then

$$(\sigma\tau)^{\frac{k-1}{2}} \sigma \left((\sigma\tau)^{\frac{k-1}{2}}\right)^{-1} = \tau.$$

■

In general G does not always contain a broad subgroup.

Example IV.3.9. Consider the dihedral group

$$D_4 := \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

with corresponding invariant ring

$$\mathbb{R}[X_1, X_2]^{D_4} = \mathbb{R}[\underbrace{X_1^2 + X_2^2}_{\pi_1}, \underbrace{X_1^4 + X_2^4}_{\pi_2}].$$

It is easy to check that D_4 has no broad subgroup. Furthermore, there is no elementary abelian 2-subgroup H such that every homomorphism of principal orbit type $\phi : \mathbb{R}[\underline{X}]^{D_4} \to \mathbb{R}$ can be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$. To show this it suffices to show this for all maximal elementary abelian 2-subgroups:

case 1:

$$H := \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

Consider the homomorphism

$$\phi : \mathbb{R}[\underline{X}]^{D_4} \to \mathbb{R}, g(\pi_1, \pi_2) \mapsto g(0, 2).$$

Now $f := X_1X_2 \in \mathbb{R}[X_1, X_2]^H$ with $f^2 \in \mathbb{R}[X_1, X_2]^G$ and

$$\phi(f^2) = \phi\left(\frac{\pi_1^2 - \pi_2}{2}\right) = -1 < 0,$$

so ϕ cannot be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$. Furthermore, it is easy to check that ϕ has principal orbit type, since it corresponds to the complex preimage $(1, i)$.

case 2:

$$H := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

Consider the homomorphism

$$\phi : \mathbb{R}[\underline{X}]^{D_4} \to \mathbb{R}, g(\pi_1, \pi_2) \mapsto g(0, -8).$$

Now $f := X_1^2 - X_2^2 \in \mathbb{R}[X_1, X_2]^H$ with $f^2 \in \mathbb{R}[X_1, X_2]^G$ and

$$\phi(f^2) = \phi\left(2\pi_2 - \pi_1^2\right) = -16 < 0,$$

so ϕ can not be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$. Furthermore, it is easy to check, that ϕ has principal orbit type, since it corresponds to the complex preimage $(1 + i, 1 - i)$.

IV.3.2 Bröcker's Theorem for $k = 1$

In this subsection, we will show how to describe the orbit space by one inequality if $k = 1$ in Bröcker's Theorem IV.3.2. To this end, we need to understand first the orbit space for groups of order two.

Lemma IV.3.10. *Let G be a group of order two. The orbit space $\Pi(\mathbb{R}^n)$ is described by*

$$f = \sum_{i=1}^{n} f_i^2$$

where $f_i := X_i - \mathcal{R}_G(X_i)$.

Proof. It suffices to show that a homomorphism $\varphi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \to \mathbb{R}$ if and only if $\varphi(f) \geq 0$. First note that f is G-invariant by Lemma IV.2.9, so $\varphi(f) \in \mathbb{R}$ and one implication is clear because f is a sum of squares. For the other implication, let $\varphi : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ be a homomorphism such that $\varphi(f) \geq 0$. The homomorphism φ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \to \mathbb{C}$. Note that

$$\mathbb{R}[\underline{X}] = \mathbb{R}[\underline{X}]^G[f_1, \ldots, f_n]$$

<u>Case 1:</u> $\varphi(f_i^2) = 0$ for all $i \in \{1, \ldots, n\}$. Then $\tilde{\varphi}(f_i) = 0$ for all $i \in \{1, \ldots, n\}$ and therefore $\operatorname{im} \tilde{\varphi} \subseteq \mathbb{R}$.
<u>Case 2:</u> $\varphi(f_i^2) > 0$ for some $i \in \{1, \ldots, n\}$. Then $\tilde{\varphi}(f_i) \in \mathbb{R} \setminus \{0\}$. Now $\tilde{\varphi}(f_j) \in \mathbb{R}$ for all $j \in \{1, \ldots, n\}$ since $\tilde{\varphi}(f_i)\tilde{\varphi}(f_j) = \varphi(f_i f_j) \in \mathbb{R}$ by Lemma IV.2.9. So again $\operatorname{im} \tilde{\varphi} \subseteq \mathbb{R}$. ∎

Theorem IV.3.11. *Let G be a group such that all maximal elementary abelian 2-subgroups of G are of order 2. Then $\Pi(\mathbb{R}^n)$ is generically described by some $g \in \mathbb{R}[\underline{X}]^G$.*
More precisely, if $|G| = q2^l$ with q odd and

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_l \subseteq G$$

are nested subgroups with $|H_i| = 2^i$, then one can choose

$$g = \prod_{\sigma H_l \in G/H_l} \left(\sigma R_{H_l} \left(\sum_{i=1}^{n} \left(X_i - R_{H_1}\left(X_i\right)\right)^2 \right) \right).$$

Proof. It suffices to show that there is some $g \in \mathbb{R}[\underline{X}]^G$, such that for every homomorphism $\varphi: \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ of principal orbit type, that $\varphi(g) \geq 0$ if and only if φ can extended to a homomorphism $\mathbb{R}[\underline{X}] \to \mathbb{R}$.
By Lemma IV.3.8, H_1 is a broad subgroup of G. So by Lemma IV.3.5 every homomorphism of principal orbit type $\varphi: \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ can be extended to a homomorphism $\varphi_H: \mathbb{R}[\underline{X}]^{H_1} \to \mathbb{R}$.
Since H_1 is of order 2, there is a H_1-invariant sum of squares

$$f_1 = \sum_{i=1}^{n} \left(X_i - R_{H_1}(X_i)\right)^2$$

such that φ_H can be extended to a homomorphism $\tilde{\varphi}: \mathbb{R}[\underline{X}] \to \mathbb{R}$ if and only if $\varphi_H(f_1) \geq 0$ by Lemma IV.3.10. The argument now follows from the following two lemmas.

Lemma IV.3.12. *For every $1 \leq i \leq l$ there exists a sum of squares $f_i \in \mathbb{R}[\underline{X}]^{H_i}$ such that φ can be extended to a homomorphism $\tilde{\varphi}: \mathbb{R}[\underline{X}] \to \mathbb{R}$ if and only if $\varphi_H(f_i) \geq 0$.*

Proof. We go by induction on i. The base case $i = 1$ is done above. For the induction step from i to $i+1$ we suppose the claim holds for some $1 \leq i \leq l-1$, i.e. there is $f_i \in \mathbb{R}[\underline{X}]^{H_i}$ such that φ can be extended if and only if $\varphi_H(f_i) \geq 0$. Either f_i is already H_{i+1}-invariant and we are done or

$$\mathbb{R}(\underline{X})^{H_i} = \mathbb{R}(\underline{X})^{H_{i+1}}(f_i) = \mathbb{R}(\underline{X})^{H_{i+1}}(\sigma f_i).$$

Consider the sum of squares $f_{i+1} := f_i + \sigma f_i \in \mathbb{R}[\underline{X}]^H_{i+1}$, where H_{i+1}/H_i is generated by $\sigma \in G$. If φ can be extended to a homomorphism $\tilde{\varphi}: \mathbb{R}[\underline{X}] \to \mathbb{R}$, then $\varphi_H(f_{i+1}) \geq 0$, since $\tilde{\varphi}$ is real and f_{i+1} is a sum of squares.
If $\varphi_H(f_{i+1}) \geq 0$, then $\varphi_H(f_i) \geq 0$ or $\varphi_H(\sigma f_i) \geq 0$, because σf_i is H_i-invariant. In the first case we are done by the induction hypothesis and in the second case we consider the homomorphism $\sigma^{-1}\varphi_H$ instead of φ_H: The homomorphism $\sigma^{-1}\varphi_H$ is a real extension of φ and $\sigma^{-1}\varphi_H(f_i) \geq 0$, so we can use the induction hypothesis. ∎

By Lemma IV.3.12 there is a H_l-invariant sum of squares f_l such that φ can be extended to a homomorphism $\tilde{\varphi}: \mathbb{R}[\underline{X}] \to \mathbb{R}$ if and only if $\varphi_H(f_l) \geq 0$.

Lemma IV.3.13. *φ can be extended to a homomorphism $\tilde{\varphi}: \mathbb{R}[\underline{X}] \to \mathbb{R}$ if and only if*

$$\varphi\Big(\prod_{\sigma H_l \in G/H_l} \sigma f_l\Big) \geq 0.$$

Proof. The if case is clear, since $f := \prod_{\sigma H_l \in G/H_l} \sigma f_l$ is a sum of squares. For the other direction assume that $\varphi(f) \geq 0$ and consider the complex extension $\tilde{\varphi}: \mathbb{R}[\underline{X}] \to \mathbb{C}$ of φ. Now

$$0 \leq \varphi(f) = \prod_{\sigma H_l \in G/H_l} \tilde{\varphi}(\sigma f_l) = \prod_{\sigma H_l \in G/H_l} \sigma\tilde{\varphi}|_{\mathbb{R}[\underline{X}]^H}(f_l).$$

If the image of $\sigma\tilde{\varphi}|_{\mathbb{R}[\underline{X}]^H}$ is not real for some σ, then also its complex conjugated appears in the product, so their product is positive. Since $|G/H_l|$ is odd, there has to be $\sigma H_l \in G/H_l$, such that $\sigma\tilde{\varphi}(f_l) \geq 0$ and $\sigma\tilde{\varphi}|_{\mathbb{R}[\underline{X}]^H}$ is a real homomorphism. Now we apply Claim 1 to $\sigma\varphi|_H$ instead of $\varphi|_H$. ■

The poof now follows directly from Lemmas IV.3.12 and IV.3.13. ■

Note that Theorem IV.3.11 is constructive. We give some examples.

Example IV.3.14.

1. Consider S_3 with the standard action on $\mathbb{R}[X_1, X_2, X_3]$. Following the proof of Theorem IV.3.11 we consider first some order two subgroup of S_3, say $S_2 = \langle (12) \rangle$. We obtain using Lemma IV.3.10 the S_2-invariant polynomial

   $$f = (X_1 - R_{S_2}(X_1))^2 + (X_2 - R_{S_2}(X_2))^2 + (X_3 - R_{S_2}(X_3))^2 = (X_1 - X_2)^2$$

   in Claim 1. This polynomial is symmetrized in Claim 2 and we get, that the orbit space $\mathbb{R}^3/S_3$ is generically described by the discriminant

   $$g = \prod_{i=0}^{2} (123)^i f = \prod_{1 \leq i < j \leq 3} (X_i - X_j)^2.$$

2. Consider the Quaternion group

   $$Q_8 = \left\langle \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

   with the canonical action on $\mathbb{R}[X_1, X_2, X_3, X_4]$. Consider again first the unique order two subgroup $H = \langle -I_4 \rangle$. In Theorem IV.3.11 Claim 1 we obtain the H-invariant polynomial

   $$f = X_1^2 + X_2^2 + X_3^2 + X_4^2$$

   using Lemma IV.3.10. Since f is already Q_8-invariant, we don't need to apply the second step. So the orbit space $\mathbb{R}^4/Q_8$ is already generically described by f.

3. Consider the Dihedral group

   $$D_5 = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

with the canonical action on $\mathbb{R}[X_1, X_2, X_3, X_4, X_5]$. Theorem IV.3.11 yields that the orbit space $\mathbb{R}^5/D_5$ is generically described by

$$\big((X_1 - X_5)^2 + (X_2 - X_4)^2\big)\big((X_2 - X_1)^2 + (X_3 - X_5)^2\big)\big((X_3 - X_2)^2 + (X_4 - X_1)^2\big)$$
$$\big((X_4 - X_3)^2 + (X_5 - X_2)^2\big)\big((X_5 - X_4)^2 + (X_1 - X_3)^2\big).$$

IV.3.3 Bröcker's Theorem for abelian groups

In this subsection, we want to give a constructive version of Bröcker's Theorem IV.3.2 for abelian groups. To this end, we need to understand first what happens if we have a normal subgroup of G.

Lemma IV.3.15. *Let H be a normal subgroup of G. Assume that:*

1. *There are $g_1, \ldots, g_k \in \mathbb{R}[\underline{X}]^G$ such that every homomorphism $\varphi_G : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ can be extended to a homomorphism $\varphi_H : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$, if and only if $\varphi_G(g_1) \geq 0, \ldots, \varphi_G(g_k) \geq 0$.*
2. *There is $h \in \mathbb{R}[\underline{X}]^H$ such that every homomorphism $\varphi_H : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$ can be extended to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \to \mathbb{R}$, if and only if $\varphi_H(h) \geq 0$.*

Then a homomorphism $\varphi_G : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ can be extended to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \to \mathbb{R}$, if and only if $\varphi_G(\mathcal{R}_G(h)) \geq 0, \varphi_G(g_1) \geq 0, \ldots, \varphi_G(g_k) \geq 0$.

Proof. Let $\varphi_G : \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ be a homomorphism.
"$\Rightarrow$:" If φ_G can be extended to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \to \mathbb{R}$, then $\varphi_G(g_1) \geq 0, \ldots, \varphi_G(g_k) \geq 0$ by (1). For every $\sigma \in G$ we have that $\sigma\varphi|_{\mathbb{R}[\underline{X}]^H}$ is an extension of φ_G, so $\varphi(\sigma h) = \sigma\varphi|_{\mathbb{R}[\underline{X}]^H}(h) \geq 0$ by (2) and therefore $\varphi_G(\mathcal{R}_G(h))) \geq 0$.
"$\Leftarrow$:" Suppose $\varphi_G(\mathcal{R}_G(h)) \geq 0, \varphi_G(g_1) \geq 0, \ldots, \varphi_G(g_k) \geq 0$. Then φ_G can be extended to a homomorphism $\varphi_H : \mathbb{R}[\underline{X}]^H \to \mathbb{R}$ by (1). Now σh is H-invariant for all $\sigma \in G$, since H is a normal subgroup of G. Now

$$\sum_{\sigma \in G} \varphi_H(\sigma h) = |G|\varphi_H(\mathcal{R}_G(h)) = |G|\varphi_G(\mathcal{R}_G(h)) \geq 0$$

and since all the terms are real, $\varphi_H(\sigma_0 h) \geq 0$ for some $\sigma_0 \in G$. Then $\sigma_0\varphi_H(h) = \varphi_H(\sigma_0 h) \geq 0$ and $\sigma_0\varphi_H \in \hom(\mathbb{R}[\underline{X}]^H, \mathbb{R})$, so we can extend $\sigma_0\varphi_H$ to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \to \mathbb{R}$ by (2). ∎

Note that Theorem IV.3.11 includes in particular cyclic groups of even order. The following example shows that Lemma IV.3.10 together with Lemma IV.3.15 give a polynomial of lower degree describing generically the orbit space in this case. Moreover, this example generalizes Example IV.3.3 above.

Example IV.3.16. Let $G = C_m$ be a cyclic group. If m is odd, the Hilbert map is surjective, so we need no inequality. If m is even, write $m = 2^k q$, where q is odd. Furthermore, consider the cyclic subgroups

$$\{\mathrm{id}\} = C_1 \subset C_2 \subset C_4 \subset \cdots \subset C_{2^k} \subset C_m$$

and the C_m-invariant polynomials

$$f_i := R_{C_m}\left(\sum_{j=1}^{n}\left(R_{C_{2^{i-1}}}(X_j) - R_{C_{2^i}}(X_j)\right)^2\right)$$

Then from Lemma IV.3.10 and Lemma IV.3.15 we obtain that the orbit space $\mathbb{R}^n/C_m$ is generically described by f_1. Furthermore, $\mathbb{R}^n/C_m$ is described by $f_1,\ldots,f_k$: For every non-real preimage $x \in \mathbb{C}^n \setminus \mathbb{R}^n$, there is σ of order 2^i with $\sigma x = \bar{x}$ (replace σ by σ^r for some odd r). Now it is easy to check that $f_i(x) < 0$:

$$\begin{aligned}
f_i(x) &= R_{C_m}\left(\sum_{j=1}^{n}\left(R_{C_{2^{i-1}}}(x_j) - R_{C_{2^i}}(x_j)\right)^2\right)\\
&= R_{C_m}\left(\sum_{j=1}^{n}\left(\frac{1}{|C_{2^{i-1}}|}\sum_{l=1}^{2^{i-1}}\sigma^{2l}(x_j) - \frac{1}{|C_{2^i}|}\sum_{l=1}^{2^i}\sigma^{l}(x_j)\right)^2\right)\\
&= \frac{1}{|C_{2^i}|}R_{C_m}\left(\sum_{j=1}^{n}\left(2\sum_{l=1}^{2^{i-1}}\sigma^{2l}(x_j) - \sum_{l=1}^{2^{i-1}}\sigma^{2l}(x_j) + \sigma^{2l+1}(x_j)\right)^2\right)\\
&= \frac{1}{2^i}R_{C_m}\left(\sum_{j=1}^{n}\left(\sum_{l=1}^{2^{i-1}}\sigma^{2l}(x_j) - \sigma^{2l+1}(x_j)\right)^2\right)\\
&= \frac{1}{2^i m}\sum_{\tau\in C_m}\tau\left(\sum_{j=1}^{n}\left(\sum_{l=1}^{2^{i-1}}\sigma^{2l}(x_j - \overline{x_j})\right)^2\right)\\
&= \frac{1}{2m}\sum_{\tau\in C_m}\left(\sum_{j=1}^{n}(\tau x_j - \overline{\tau x_j})^2\right) < 0.
\end{aligned}$$

In general, $\mathbb{R}^n/C_m$ can not be described by less than k polynomials, which can be shown in a similar way as in Example IV.3.3.

We can now prove Bröcker's Theorem for abelian groups.

Theorem IV.3.17. *Let G be abelian and choose $k \in \mathbb{N}_0$ such that the maximal abelian 2-subgroups of G are of order 2^k. Then $\mathbb{R}^n/G$ is generically described by $f_1,\ldots,f_k \in \mathbb{R}[\underline{X}]^G$.*

Proof. Since G is abelian, it is isomorphic to the direct product of cyclic groups, say $G \cong H_1 \times \cdots \times H_k \times H_{k+1} \times \cdots \times H_m$, where $H_1,\ldots,H_k$ are of even order and $H_{k+1},\ldots,H_m$ are of odd order.
By applying iteratively Remark IV.3.16 and Lemma IV.3.15 we find $g_1,\ldots,g_i \in \mathbb{R}[\underline{X}]^G$ $(1 \le i \le k)$ such that every homomorphism $\varphi_G :\ \mathbb{R}[\underline{X}]^G \to \mathbb{R}$ of principal

orbit type can be extended to a homomorphism $\varphi_i:\ \mathbb{R}[\underline{X}]^{H_{i+1}\times\cdots\times H_m}\to\mathbb{R}$, if and only if $\varphi_G(g_1)\geq 0,\ldots,\varphi_G(g_i)\geq 0$. Since $H_{k+1}\times\cdots\times H_m$ is odd, we can extend every homomorphism $\varphi_k:\ \mathbb{R}[\underline{X}]^{H_{k+1}\times\cdots\times H_m}\to\mathbb{R}$ to a homomorphism $\varphi:\ \mathbb{R}[\underline{X}]\to\mathbb{R}$. ■

The following example shows how to use the proof of Theorem IV.3.17 to construct inequalities describing generically the orbit space of abelian groups.

Example IV.3.18. Consider the abelian group

$$G := \left\langle \begin{pmatrix} 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix} \right\rangle \cong C_4\times C_2$$

and the subgroup $H := \left\langle \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix} \right\rangle \cong C_2$ of H. The orbit space $\mathbb{R}^4/H$ is generically described by the polynomial

$$h = X_1^2+X_2^2+X_3^2+X_4^2$$

by Example IV.3.16. Furthermore, we can also apply Example IV.3.16 to the ring extension

$$\mathbb{R}[\underline{X}]^G\subseteq\mathbb{R}[\underline{X}]^H$$

and get that a homomorphism $\phi:\ \mathbb{R}[\underline{X}]^G\to\mathbb{R}$ of principal orbit type can be extended to a homomorphism $\tilde{\phi}:\ \mathbb{R}[\underline{X}]^H\to\mathbb{R}$ if and only if $\phi(g)\geq 0$, where

$$g=(X_1^2-X_3^2)^2+(X_2^2-X_4^2)^2+(X_1X_2-X_3X_4)^2+(X_1X_4-X_2X_3)^2.$$

Now by Lemma IV.3.15 the orbit space $\mathbb{R}^4/G$ is generically described by the polynomials g and
$mathcal R_G(h)=h$.
By considering first

$$H_2 = \left\langle \begin{pmatrix} 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle$$

instead of H, one obtains that $\mathbb{R}^4/G$ is also generically described by

$$h_2=(X_1-X_3)^2+(X_2-X_4)^2 \text{ and } g_2=(\sum_{i=1}^4 X_i)^2+(\sum_{i=1}^4 X_i^3)^2+(X_1X_2^2+X_2X_3^2+X_3X_4^2+X_4X_1^2)^2.$$

IV.4 Conclusion and open questions

We conclude our article with some open questions and points for further inquiry. We were able to show Bröcker's result constructively for some classes of groups. However, our techniques developed here do not seem to directly apply to other interesting groups. For example, note that the symmetric S_n contains the broad subgroup

$$H_n := \left\langle (1,2),(3,4),\ldots,\left(2\left\lfloor\frac{n}{2}\right\rfloor - 1, 2\left\lfloor\frac{n}{2}\right\rfloor\right)\right\rangle$$

for all n by Remark IV.3.7. Still, our techniques fail to give a generic description of the orbit space $\mathbb{R}^n/S_n$ with $\lfloor \frac{n}{2} \rfloor$ inequalities for $n \geq 4$. By Theorem IV.3.17 we get that $\mathbb{R}^n/H_n$ is described by the $\lfloor \frac{n}{2} \rfloor$ polynomials

$$f_1 := (X_1 - X_2)^2, f_2 := (X_3 - X_4)^2, \ldots, f_{\lfloor \frac{n}{2} \rfloor} := (X_{2\lfloor \frac{n}{2} \rfloor - 1} - X_{2\lfloor \frac{n}{2} \rfloor})^2$$

and since H_n is broad we can extend every homomorphism in $\hom(\mathbb{R}[\underline{X}]^{S_n}, \mathbb{R})$ to a homomorphism in $\hom(\mathbb{R}[\underline{X}]^{H_n}, \mathbb{R})$ by Lemma IV.3.5. A natural approach might be to symmetrize the polynomials $f_1, \ldots, f_{\lfloor \frac{n}{2} \rfloor}$ in such a way that the set they describe remains the same and we could in this way obtain a description of $\mathbb{R}^n/H_n$ in terms of S_n-invariant polynomials. Then, these polynomials would also describe $\mathbb{R}^n/S_n$. However, we currently are not able to produce such a symmetrization. Scheiderer mentioned to us a way to generically describe $\mathbb{R}^4/S_4$ with two inequalities if S_4 acts by permutation. We include this example with his permission.

Example IV.4.1. The invariant ring $\mathbb{R}[\underline{X}]^{S_4}$ is generated by the elementary symmetric polynomials e_1, e_2, e_3, e_4. Denote the corresponding Hilbert-map by E. Then $z \in E(\mathbb{R}^4)$, if and only if the univariate polynomial

$$f = T^4 - z_1 T^3 + z_2 T^2 - z_3 T + z_4$$

has only real roots. Furthermore, it is well known that the nature of the roots of f_z is given by the eigenvalues of its Hermite matrix H_f. The number of real roots of f_z is equal to the signature of H_f, which is generically equal to the number of sign changes in the series p_1, p_2, p_3, p_4 of the leading principal minors of H_f. So the signature of H_f is non-negative. Now, since $p_1 = 4$ is constant, H_f is positive definite, if and only if $p_2 p_4 > 0$ and $p_3 > 0$. I.e., the orbit space $\mathbb{R}^4/S_4$ is generically described by $p_2 p_4$ and p_3.

This example can be generalized to get $n-2$ inequalities that generically describe the orbit space $\mathbb{R}^n/S_n$, but it seems hopeful that a similar approach might give the expected number of $\lfloor \frac{n}{2} \rfloor$ inequalities. This setup seems particularly interesting, since the orbit space $\mathbb{R}^n/S_n$ can be viewed as the space of hyperbolic univariate polynomials.

A further question consists in the relationship of our sums of squares approach in section 2 and the general description. In [PS85, Theorem 4.9] a description of the orbit space using covariants is given. It seems that the description we

obtain by Theorem IV.2.5 naturally contains this description, - as covariants are a natural subset of the ring extension we used to represent the sums of squares - but our description additionally gives redundant inequalities. It would therefore. be very interesting to see if there is a simple way to remove these redundant inequalities to directly arrive at a more compact description.
Finally, the description of the orbit space in terms of the matrix polynomial M_Π gives a natural finitely generated quadratic module in the invariant ring:

$$\{\mathrm{Tr}\,(A \cdot M_\Pi) \mid A \text{ is a sums of invariant squares matrix polynomial}\} \subseteq \mathbb{R}[\underline{X}]^G$$

This quadratic module naturally is contained in the invariant sums of squares polynomial, but might be substantially smaller. It is unclear how big the difference is, and this is a question which might be interesting, in particular, from the viewpoint of applications.

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