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Teaching Linear Algebra to Engineering Students in Two Different Educational Settings: Design and Learning Outcomes

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ABSTRACT: The present paper has a two-part perspective on a new teaching format in a linear algebra course given to engineering students. The design resembles the flipped classroom, but with some adjustments. The first part highlights the basis that the design was built on, both research results and practical issues, but also the implementation of the teaching. It did not go quite as expected. The second part focuses on the outcome. It concentrates on students' feedback on learning in the course. To inform this part, answers to a questionnaire on learning from students in the new teaching design are included. These are both related to corresponding feedback from students in a previous linear algebra course that was taught traditionally and to the students' interpretation of own achievements. As a result of paper's data analysis and discussion based on other research results, is a proposal for an adapted teaching format to better facilitate learning. This contributes to knowledge about how research-based teaching arrangements may develop.

Keywords: Engineering students, flipped classroom, achievement, learning, linear algebra

1 INTRODUCTION

Linear algebra is a field of mathematics that is considered difficult by many students. This is mainly due to its nature and of the type of thinking needed to understand its content (Dorier, 2000). Love et al have shown that with a flipped classroom teaching style in a linear algebra course, social comfort among students increases significantly (Love, Hodge, Grandgenett, & Swift, 2014). This is an important prerequisite for learning. A dominant mode of teaching in mathematics at university is nevertheless lectures, often because the classes have so many students (Iannone & Miller, 2019).

The present paper is inspired by the research setting in (Love et al., 2014), with a number of modifications: The focus is on engineering students, the flipped classroom design is modified, and a discussion of learning in linear algebra is added. To inform the study, a combination of own experiences as teacher in the course and students' feedback given in an anonymous questionnaire is drawn upon. Data from two groups of engineering students are included, one class in 2019 before and one class in 2022 after the closure of the society due to Covid. This means that the social contexts for the two classes were similar and normal, the content of the linear algebra course was the same and the same teacher taught the course both years. The main difference was the teaching designs, with traditional lectures in 2019 and an adjusted flipped classroom style in 2022. This paper has a two-fold aim. One is to outline and discuss the basis upon which the adjusted flipped design was built. The other is to capture the students' learning experiences with the new format. The latter is done by highlighting the students' statements about learning in 2022 as

- related to corresponding feedback from students in 2019
- related to students' perception of own achievements in linear algebra.

The outline of the paper is as follows: Initially, the theoretical background for the study is presented. Then follows a description of the linear algebra course and the research method. The results are presented in two parts guided by the paper's two aims. In the first part, teaching conditions and course implementation are presented. The second part shows results about learning. The discussion focuses on challenges with the teaching format informed by answers to an open question in the questionnaire. This leads to a conclusion that suggests how to adjust the mode of teaching.

2 THEORETICAL BACKGROUND

A main focus of the paper is how different teaching formats can facilitate learning, thus a literature review of this is needed. Initially, however, a brief introduction to the teaching of linear algebra and engineering students' learning of the subject is given.

2.1 Linear Algebra, Learning Strategies and Engineering Students in Short

Dorier and Sierpinska (2001) highlight three sources of difficulties: The mathematical language and theoretical concepts that students find difficult; what Dorier called the "obstacle of formalism" (Dorier, 1997). The pedagogical approach because proofs is perceived as challenging among students. Finally, "the cognitive flexibility" that is required in linear algebra because one has to switch between different languages, both practical and theoretical representations (Dorier & Sierpinska, 2001). The cognitive flexibility comes with mathematical maturity, but engineering students may not reach this stage. These students rarely study mathematics for its own sake but because it can be used in the more engineering specific subjects (Khiat, 2010). Engineering students may not be willing to invest the effort in linear algebra necessary to reach the needed maturity.

Learning mathematics is a prerequisite for understanding the subject, and there are a number of research works that study such learning. Often, approaches are divide into two, one being of a more instrumental and rule-based nature and one being more focused on relationships. Already Skemp in 1979 defined what he called relational understanding in mathematics as "relating a task to an appropriate schema" (Skemp, 1979, p. 259) where "schema" concerns the conceptual structure of a student's cognitive map that links mathematical concepts together. Skemp contrasted this with instrumental understanding where memorizing and use of rules are prominent. Another classical and related distinction is between procedural and conceptual knowledge, given by Hiebert and Lefevre (1986). But there are numerous such pairings (Herheim, 2023). Herheim distinguishes five characteristics of instrumental and relational understanding in order to facilitate for operationalization of the concepts, and thus make them usable in an analytical framework. Zakariya et al show that instrumentalism is linked to a surface approach to learning. If previous mathematical experience mainly focus on such approaches, it mediates a direct effect on performance in university mathematics (Zakariya , Nilsen , Bjørkestøl, & Goodchild, 2021). Therefore, background experience is vital. Especially for engineering students, Liebendörfer et al show that prior mathematical knowledge is one of the most important predictors of mathematical performance (Liebendörfer et al., 2022). They conclude that practicing mathematics in a non-repetitive manner and resisting frustration predicts these students' mathematical ability to gain knowledge (2022). Engineering students seeks applications of mathematical knowledge as a first priority (Harris et al., 2015), and may find abstractions difficult. Teaching linear algebra for engineering students should therefore be devoted to applications at the expense of some of the abstract arguments (Rensaa, Hogstad, & Monaghan, 2020).

2.2 The Teaching Styles

Lecturing is often referred to as the traditional way of teaching mathematics. Mesa, Celis and Lande (2014), defines this teaching style as a transmissionist approach. It focuses on the process of transferring the mathematical content to the students and on how this must be structured to fit within a period of time. This is supported by many researchers (e.g. Alsina, 2001; Jaworski et al., 2017; Mesa et al., 2014). A usual organization is for the lecturer to deliver a spoken-written presentation for the students at the blackboard in a chalk-talk manner (Artemeva & Fox, 2011;. Students are normally familiar and comfortable with the lecturing style of teaching and appreciate a classroom organized for lectures (Love et al., 2014). The quality of a mathematics lecture depends heavily on the lecturer. Bergsten provides a triangular model to capture the quality of a lecture, some factors being more personal than others (Bergsten, 2007). His list of ten aspect that will make quality of a lecture are information delivery as trying to communicate the main core of the course, illumination of connections between different parts, gestures to make ideas visible, inspiration and personalization (2007).

However, the lecture format is debated (e.g. Alsina, 2001; Pritchard, 2010). While Pritchard speaks in favour of the format, and stresses that lectures can be effective both when it comes to communicating information, in the sense of modelling mathematical reasoning and in motivating students to do mathematics, Alsina finds it problematic. Among Alsina's list of objections is "the perfect theory presentation" (Alsina, 2001, p. p. 5), also highlighted by Weber (2004). Lectures are most often linearly well-ordered presentations of readymade mathematical theory, and thus do not illuminate how mathematics can be a human social activity that involves struggle and creativity (Alsina, 2001; Weber, 2004). Students become passive listeners rather than participating in learner activities that include interaction with others (Fritze & Nordkvelle, 2004).

An increasingly used pedagogical framework in mathematics is a flipped classroom (FC). The format receives appreciations for being tailored to students' active engagement with the course content (Lo, Hew, & Chen, 2017). Its typical design is students preparing before coming to class and the in-class part is facilitated for collaboration between students to explore topics in greater depth (Bergmann & Sams, 2012; Bishop & Verleger, 2013). The review by Lo et al (2017) refers to a number of benefits with this format, ranging from the out-of-class sessions where students have access to on-demand video lectures and are encouraged to prepare for class, to valuable in-class time including face-to-face time with the teacher and collaborative and inquiry-based activities with peers. The out-of-class activities require the students to have access to teaching material, often in the form of videos (Bishop & Verleger, 2013) and the design of good videos has engaged researchers. According to Mayer's cognitive theory of multimedia learning, videos ought to be short to reduce students' cognitive load, help them to stay focused and facilitate learning (Mayer & Fiorella, 2022). This is in line with Guo et al who stress that short videos engage students the most (Guo, Kim, & Rubin, 2014). Their set of recommendations for the video format emphasizes the value of short videos, but also, for example having informal talking-heads and tablet drawings. When it comes to the in-class part of the FC setting, the literature review in Fredriksen's PhD thesis (2020) shows that collaboration between students is highlighted as an important in-class component (Fredriksen, 2020). This also appears in his own research work, like in his investigation of affordances for mathematical learning in a flipped classroom interpreted in an activity theoretical perspective. Results show a number of affordances at different levels of the activity system, and at the collective level Fredriksen concludes that "..facilitating collaboration and students' participation in groups and whole-class discussions seem to be an important structural component for meaningful activity in class." (2020, p. 209). Also teaching of linear algebra in particular may benefit from adopting a FC instructional approach. Love et al show that in such a course the student collaboration improved and the students thought they learned a lot from the provided video components (Love et al., 2014). Indeed, Love et al found that by explaining an idea or a problem to a peer, students could gain a deeper understanding of linear algebra. For engineering students, a FC approach can promote students' choice of which parts of the mathematics they want to work on, promote better alignment between instruction and assessment, and promote supportive learning communities and open communication (Kerrigan & Prendergast, 2022). Kerrigan and Prendergast discuss how this pedagogical framework can be implemented in large classes; a situation that is often the case in mathematics courses in engineering educations. This includes the possibility of utilizing the framework in online teaching services: "With an increasing number of classes moving to a fully online format, it might even be possible to repurpose some of the time ULAs spend in class to offer additional short sessions outside of class where they can coach students on how to learn mathematics in a flipped classroom"(Kerrigan & Prendergast, 2022, p. 1121). Online tutoring was a main component in the present project, offered through the Zoom platform (https://zoom.us/). The practitioners in Gilbert et al's study (2023) address a number of advantages and disadvantages of such online support. They highlight benefits such as flexibility, convenience and intimacy. One practitioner also accentuates that students attend such online support sessions more prepared (Gilbert et al., 2023). But several disadvantages of the format are also pointed out, such as being less personal, less engaging for students, less spontaneous. The disadvantage that most practitioners identify, though, is the challenge of interactions and collaboration between students, both socially and professionally (Gilbert et al., 2023).

The FC framework also has its challenges. Lo et al (2017) organize these challenges in three categories that are retrieved from (Betihavas, Bridgman, Kornhaber, & Cross, 2016); student-related challenges, faculty-related challenges and operational challenges. A student-related challenge is, for example, when students do not complete the out-of-class activities, thus meet unprepared for the in-class work. This leads to a number of problems since these students are unable to accomplish tasks meant for in-class sessions. An example of such friction involving gender can be found in (Rensaa & Fredriksen, 2022).

Typical faculty-related challenges are increased workload and increased time used both to prepare and carrying out the teaching. Finally, operational challenges concern, for instance, a lack of IT skills and IT resources (Lo et al., 2017). Trenholm et al (2022) lift forward challenges of relevance for all these categories, focusing on the use of recorded videos in replacement for traditional lessons. Recurring use of recorded lectured videos reduces students' learning (Trenholm, 2022).

3 THE EDUCATIONAL SETTING AND METHOD

3.1 The Linear Algebra Course

Both classes of students in the present study were experienced, starting their two-year master's program in engineering. They had completed their bachelor's degree in engineering either at a Norwegian or foreign university, and courses were taught in English. As part of their previous degree, students should have completed an introductory course in linear algebra. This course is typically of a more procedural nature. Theoretical and abstract parts are postponed until the beginning of the two-year master's programs; thus the current course had a higher level of abstraction. Students often find this challenging, which is also well known from the research literature Dorier & Sierpinska, 2001; Harel, 2017).

The time table for the master's engineering courses at our university is intensive. This means that students study one subject a time for two non-consecutive weeks. During the linear algebra weeks, all days are devoted to this subject. The linear algebra course is the first in the autumn semester. Thus, the timetable is new to the students, they are often new to the university and they do not know their fellow students. Students from different engineering programs take this course together, and in the first week of classes it is usually very quiet since the students neither know each other nor who is in the same engineering program as they are.

The teaching format in 2019 was more traditional, with four consecutive 45 minutes lectures in a lecture hall before lunch and task solving sessions in assigned classrooms after lunch. During the task solving sessions I went as a teacher between classrooms and helped the students. This arrangement was repeated every day throughout the week. The only new thing in this was that all lectures were recorded and made available to the students both as simultaneous streaming and as recordings afterwards. In 2020 and 2021 the society was closed due to Covid and teaching was carried out in special ways. Experiences from this lockdown period, however, inspired to change the traditional form of teaching in linear algebra. Therefore, when the society reopened in 2022, a new format was introduced.

3.2 How Data was Collected

Data was collected by distributing an anonymous questionnaire to all students both in 2019 and 2022. Anonymity was ensured by running the surveys via an external server (nettskjema.no). The number of students in 2019 was 53, the number of active students in 2022 was 57. The response rates on the questionnaire were 85% in 2019 (45 students) and 93% (53 students) in 2022. The question dealing with learning was identical in 2019 and 2022. It contained a list of statements about learning with which students were asked to rate their agreement on a Likert scale. Statements had been developed in a previous research project, originating from students' explanations of their own learning (Rensaa, 2017). In addition, the students in 2022 were asked to give their interpretation of their own achievement level and were categorized accordingly. The percentages here were 29% High-achieving students, 58% Middle-achieving, and 13% Low-achieving students. AS the teaching format was new in 2022, this year an additional open question was given, where students were asked to state their positive and negative opinions about the format. Since the questionnaire was anonymous, it is reasonable to assume that the students answered the questions honestly.

4 RESULTS

4.1 The New Teaching Format

4.1.1 Conditions That the New Teaching Format Needed to Meet

A new mode of teaching had to be adapted to a number of requirements, both research-based pedagogical objectives and physical frames, designed to:

- Encourage students to take an active part in the learning process as opposed to being passive listeners in a lecture hall (Kerrigan & Prendergast, 2022). This could be done by initiating a type of FC arrangement.
- Incorporate that students during lockdown had become used to flexible teaching styles and getting access to recorded lecture videos as out-of-class sessions (Bishop & Verleger, 2013; Lo, Hew, & Chen, 2017; Love et al., 2014). Such teaching formats have grown rapidly, and started already before the pandemic (Trenholm, 2022).
- Utilize research results that say that educational videos should preferably be shorter than ten minutes both to engage students (Guo, Kim, & Rubin, 2014) and reduce cognitive load (Mayer & Fiorella, 2022).
- Initiate collaboration between students during in-class time, which is an important part of a FC design (Fredriksen, 2020; Kerrigan & Prendergast, 2022; Love et al., 2014). This meant facilitating for the students to get to know others from the same engineering program. Therefore, mixing students in big lecture halls should be avoided.
- Have a assistance system where the teacher is available for support on request. Previous experience was that circulating between classrooms during problem-solving sessions sometimes disrupted mathematical discussions between student. Also, when students are stuck on a task, they should be able to contact the teacher directly instead of waiting for her to come on her next "round". This could be facilitated through an online "assistance on request"-support where students may have more prepared questions (Gilbert, Schürmann, Liebendörfer, Lawson, & Hodds, 2023).
- Fit into an intensive-based teaching schedule.

4.1.2 Course Implementation

In line with an activity-based teaching style, a variant of a FC design was introduced in 2022. Before the linear algebra course started, several shorter teaching videos were made, with content corresponding to that presented in lectures in 2019. In addition, teaching schemes were made for each day. These schemes consisted of links to each of the short videos interspersed with tasks to practice the content of the videos. Most of the tasks coincided with those given for task solving sessions in 2019. Preparing such a scheme was rather time-consuming, which Lo et al point to as a challenge for flipped classrooms (2017).

The intensive based teaching schedule made it difficult to expect students to have watched all the videos before coming to class each morning. If they were to do this, it would involve very extensive work in advance. Therefore, the students were asked to make each day's teaching plan during class. This was to take place in designated classrooms booked in the name of each program. The idea was to motivate students from the same program to meet, get to know each other and work together.

For the support part of the schedule, I as the teacher, was available on Zoom all day. This cloud-based service had been used to teach online during the pandemic. Thus, it was expected that the students were familiar with the service. The experience was that Zoom had worked well during the closure since task solutions and documents could be shared and commented on by the teacher using a document camera. Zoom has the possibility of generating a waiting room so that students can be admitted one at a time, which is an advantage when students are shy.

The implementation of the teaching seemed to proceed in accordance with the intentions. Students watched videos, worked with tasks and visited the Zoom meeting when they had questions. Some visited Zoom with a list of their own questions. Some indicated having discussed tasks with fellow students first. Some came to ask to clarify things. It appeared that the students participated in the teaching and that the scheme generated activity and self-paced learning. Some students visited the Zoom meeting often, some came more rarely, and some did not come at all. But this also happened in previous years' task solving sessions – some students could ask about relatively small details, while others did not want to ask and preferred to consult classmates if necessary.

4.2 Students' Interpretation of Learning

4.2.1 Learning in Different Teaching Formats

Table 1 shows how the students' feedback on the statements about learning was distributed in 2022 and 2019. In 2022, all students gave feedback on almost all the statements, as only one student gave a blank on the statement about simplifying an expression (statement 9). In 2019, two students submitted a blank on all statements and one student blank on three statements. Since blanking is also a type of feedback, the total number of students are included as denominator in the calculation of the percentages in the table. The result is that not all rows add up to 100%. In the table, abbreviations are SA=Strongly Agree, A=Agree, U=Unsure, D=Disagree, SD=Strongly Disagree.

Table 1: Agreement to statements about learning in 2022 and 2019

Statement	Year	SA	A	U	D	SD
1. Gain knowledge about applications of linear algebra in engineering fields; both engineering problems and problems in other engineering subjects	2022	47,2%	47,2%	3,8%	1,9%	0%
	2019	26,7%	42,2%	17,8%	4,4%	2,2%
2. Gain understanding of logical thinking and why we are doing things the way we are	2022	39,6%	41,5%	17,0%	0%	1,9%
	2019	33,3%	42,2%	13,3%	4,4%	2,2%
3. Utilize theory and theoretical aspects	2022	24,5%	56,6%	17,0%	1,9%	0%
	2019	24,4%	40,0%	26,7%	4,4%	0%
4. Grasp the meaning of symbols and how to formulate myself about linear algebra	2022	26,4%	50,9%	18,7%	3,8%	0%
	2019	17,8%	55,6%	17,8%	2,2%	0%
5. Isolate myself with learning materials and study until I gain wanted knowledge	2022	13,2%	34,0%	26,4%	22,6%	3,8%
	2019	13,3%	24,4%	23,3%	31,1%	4,4%
6. Solve tasks with focus on giving mathematical arguments, not on obtaining correct answers	2022	26,4%	32,0%	30,2%	9,4%	1,9%
	2019	6,7%	53,3%	15,6%	20,0%	0%
7. Be able to relate new knowledge to what I know from earlier	2022	37,7%	58,5%	1,9%	1,9%	0%
	2019	20,0%	64,4%	11,1%	0%	0%

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8. Know routines for how to solve tasks and obtain correct answers	2022	34,0%	50,9%	13,2%	1,9%	0%
	2019	20,0%	53,3%	13,3%	6,7%	0%
9. Know how to simplify an expression or calculations.	2022	30,2%	52,8%	13,2%	1,9%	0%
	2019	15,6%	62,2%	11,1%	6,7%	0%
10. Discuss with fellow students and the teacher until I gain wanted knowledge	2022	41,5%	34,0%	11,3%	9,4%	3,8%
	2019	15,6%	51,1%	11,1%	15,6%	2,2%

In Table 1 it is worth noticing that the statement with which most students agreed was about being able to relate with previous knowledge (statement 7). In this statement, the sum of Strongly Agree and Agree rates in 2022 was 96% and in 2019 was 84%. The statement with which the second most students agreed was about applications (statement 1), with corresponding percentages 94% and 69% respectively. This was also the statement with the biggest difference between the years, 25%. Opposite, the statement with which the fewest students agreed was that of isolating with the learning materials (statement 5). In this statement, the sum of Strongly Agree and Agree rates in 2022 was 47% while in 2019 it was 38%. In contrast, the statement about discussion with fellow students (statement 10) had 76% agreement in 2022 but only 67% in 2019.

4.2.2 Learning Related to Own Interpretation of Achievement.

For the FC arrangement in 2022, statements about learning are related to one's own interpretation of achievements as student given as shown in Figures 1a and 1b.

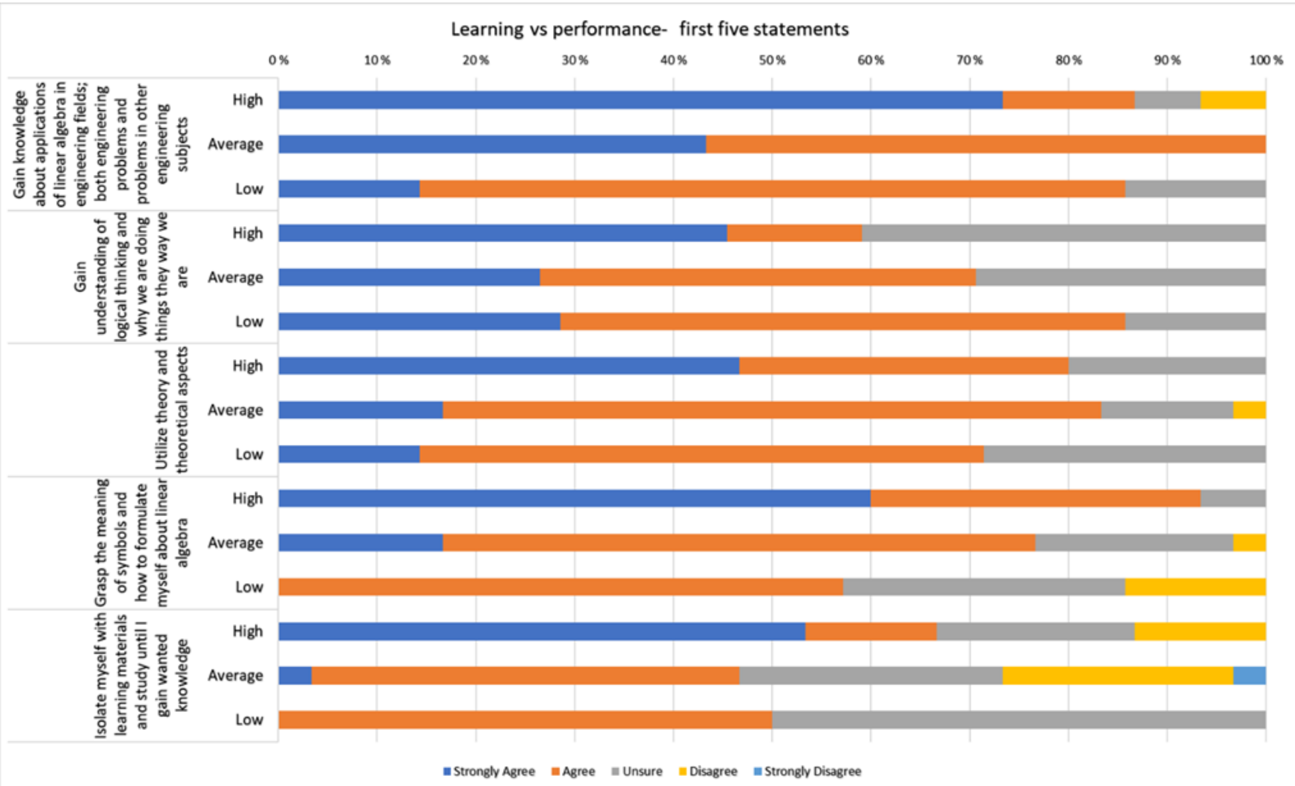

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Fig. 1a. Statements 1-5 in 2022 arranged by level of achievement

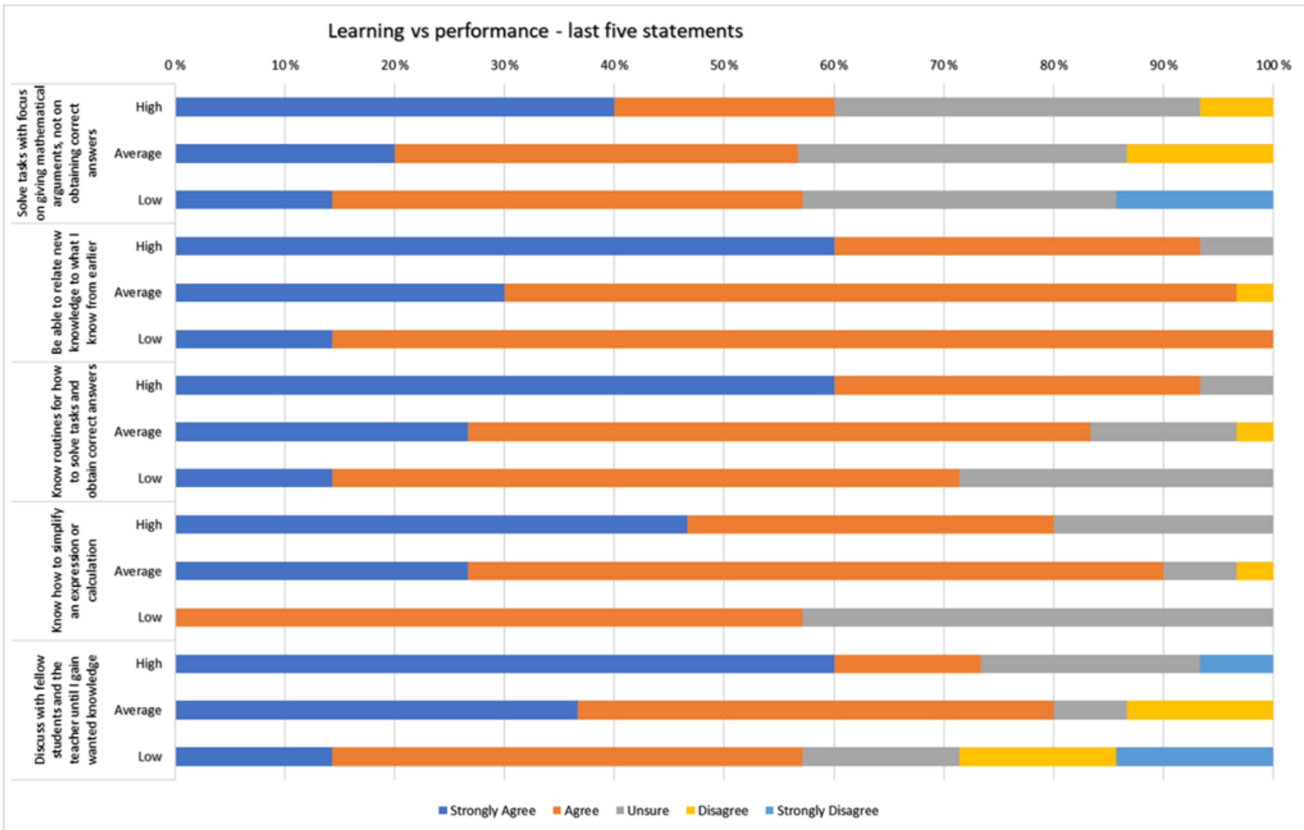

Fig. 1b. Statements 6-10 in 2022 arranged by level of achievement

For the statement about the importance of being able to relate to previous knowledge that most students agreed with, Fig. 1b (second statement), shows that all the Low-achieving students agreed with this. For High- and Average-achieving students, all but one student in each of these categories agreed. The second statement that received a full score on agreement was applications (*Fig. 1a*, first statement). Here, all the Average-achieving students agreed or strongly agreed with the statement. The agreement levels for the other students who responded to this statement were approximately 85%. Nevertheless, there was one High-achieving student who disagreed that learning linear algebra is about gaining knowledge about applications.

The statements with which the fewest students grouped by achievement level agreed were those about isolating (*Fig. 1a*, last statement) and about discussing with others (*Fig. 1b*, last statement). For the statement about isolation with learning materials, 47% of the Average-achieving students agreed or strongly agreed with this. For the High-achieving students, the percentage was 67. For the Low-achieving students, half of those who responded agreed with the statement. For this statement, however, there were three Low-achieving students who submitted blank. For the contrasting statement about discussions with fellow students as a learning arena, 57% of the Low-achieving students agreed or strongly agreed with this. Corresponding percentages for High- and Average-achieving students were 73% and 80% respectively.

5 DISCUSSION

The discussion is divided into two sections, one for each of the research aims.

5.1 Experiences With the New Teaching Format

What initially seemed like an expected implementation of the course changed when the thematic analysis of the students' reflections on teaching and learning in the new format was carried out. This emerged in explanations from individual students. Out of the 53 who gave feedback to an open question about the mode of teaching in 2022, 24 can be classified as positive and 29 as negative. Included in this were 8 statements which contained both pros and cons but were classified as negative since they had some objections to the format. The following is an example: “I like it because it’s flexible in terms of time scheduling. I don't like it, because I can't pay as much attention compared to in-person classes” (S3). Among the positive responses, on-demand Zoom support and having videos available were the most highlighted arguments. Lo et al’s review of the benefits of flipped classrooms includes access to videos when needed (Lo et al., 2017). Still, since improvements follow from criticism, statements from students who did not appreciate the scheme are discussed here. Some of the arguments can be objected to by reference to literature, as the statement “..it will be better to make longer videos by mixing shorts because we have to open many tabs while reading” (S30). According to research, students are more engaged and learn better when videos are short (Guo et al., 2014). Other statements concerned effectiveness in the learning process like “Shorter videos and Zoom support were a good alternative. But I would prefer a standard lecture in class, I feel that I learn more that way” (S45) and “Great that we can manage our time as we want, but not so effective as live lectures would be” (S13). This concerns learning in different modes of teaching and the students highlights lectures as more educational and effective. This is countered by researchers who emphasize that lectures represent a passive and transmissive teaching style that promotes superficial learning (Alsina, 2001; Fritze & Nordkvelle, 2004; Jaworski et al., 2017; Mesa et al., 2014). Pritchard (2010), on the other hand, points to the effectiveness of lectures in terms of the ability to communicate information, model mathematical reasoning and motivate deeper learning. The learning potential depends on the delivery of the lectures (Bergsten, 2007).

There were, however, two other student arguments in disfavour of the form of teaching in 2022 that were considered so problematic that it could not continue in the form it had. They concerned Zoom support and collaboration between students. For Zoom, it was assumed that students would not mind asking for help via this platform since it had been frequently used during the pandemic. This proved not to be the case for everyone. Some students felt insecure or uncomfortable with the software, as illustrated by the following statement: “I have never used zoom and I don't really know how it works” (S21). Reluctance to use an online support platform like Zoom can be categorized as an operational challenge (Betihavas et al., 2016). Some of the practitioners in Gilbert et al’s study stressed that a higher level of proactivity is necessary if the students are to take contact online support (Gilbert et al., 2023). Other students found that the Zoom arrangement did not work as intended: “I tend to often forget my questions by the time I am done watching the videos/doing an exercise, if I don't write them down. It is also some kind of threshold to go into the zoom-meeting just to ask small questions about something that was said in a video” (S31). The plan when the scheme with Zoom support was designed was that simple comments or questions could be discussed among the students in the class first but consult the teacher if necessary. An explanation of why this did not happen can be found in a statement from another student: “[The teaching format does not] encourage cooperation with other students since everyone will be sitting with headphones on while working to watch the lecture videos” (S38). A main advantage of a FC arrangement is the self-paced learning part where students adjust their work according to their needs, in line with “the degree of personalization” as denoted by Kerrigan and Pendergast (2022, p. 1117). But according to student S38 this was what prevented collaboration in the present design. The students were working in different places in the progress plan, thus if they consulted their neighbour, they would disturb more than initiate a constructive discussion. In Table 1, statement 5 is in line with this. It is about learning as working in isolation with a focus on acquiring knowledge. More students in 2022 than in 2019 agreed with this.

5.2 Students’ Learning in the New Teaching Format

When it comes to learning in the new teaching format, three statements about learning singled out as particularly interesting – relevant both for the 2019/2022-alignment and when keeping learning opportunities up against achievement for the 2022 students. Two of these were statements with which the students showed the highest degree of agreement, one was where the degree of agreement was relatively low.

Regardless of mode of teaching and levels of achievement, the 2022-students emphasized that relating new knowledge to previous knowledge is most important for their learning (Table 1 – statement 7, *Fig.* 1b – statement 2). This is in accordance with a number of research results focusing on the importance of relational understanding (Herheim, 2023; Hiebert & Lefevre, 1986; Skemp, 1979). For engineering students, the correlation between prior and present performance in mathematics is high (Liebendörfer et al., 2022) and the effect of background experiences on performance, especially when these are of an instrumental type, is described in Zakariya et al (2021). The students in the present investigation knew some basic concepts and procedures from an introductory course in linear algebra in their bachelor's studies. The result suggests that connecting new to previous knowledge is most important for Low achieving students, as every one of them agreed or strongly agreed with this statement (*Fig.* 1b – statement 2). When students stress relation to previous knowledge as one of the most important aspects of learning, they focus on a fundamental aspect of learning linear algebra.

The second statement with which a large number of students agreed is about the relevance of applications of linear algebra (Table 1 – statement 1, *Fig.* 1a - statement 1). Considerably more students in 2022 than in 2019 found this important. In the flipped classroom, 94% of the students agreed with the statement, while the corresponding percentage for the traditionally taught students is 69%. This statement has the biggest difference between 2022 and 2019 data, which could have several reasons. A teaching-related reason could be that although the linear algebra content was the same both years, the time spent on the various parts may have been different. During live lectures there are more digressions than in videos due to questions and supplementary information. Artemeva and Fox (2011) point to the possibility of talking to and with students during lectures. Videos are more straight to the point. When students in live lectures focus less on applications of linear algebra, it may be because questions and digressions often address concepts and theory that they find difficult. In linear algebra these are often of an abstract nature (Dorier & Sierpinska, 2001; Harel, 2017). If more time is spent on abstract parts, it may signalize that these are more important. In the flipped classroom, all Average-achieving students agreed or strongly agreed with the statement about applications, while 85% of both Low- and High-achieving students did the same. A reason may be that the Average-achieving students were qualified enough to seek more than just basic knowledge in linear algebra and learning about applications can be motivating. High-achieving students have prerequisites to understand and enjoy the more abstract parts of the subject. Engineering students want to know the “applicability” of mathematics (Harris et al., 2015). Still, as notified by Dorier; “Although linear algebra is used in many fields, the abstractness of concepts of vector space theory cannot be justified by only a few applications” (Dorier, 1995, p. 185). This is something to keep in mind regardless of which form of teaching is offered.

The statement that most students both years most disagreed with, was that learning is to “Isolate myself with learning materials and study until I gain wanted knowledge”. The biggest disagreement was for the traditionally taught students in 2019, where 36% disagreed. In the flipped classroom class, 26% disagreed. This is the opposite of what was expected. The aim of introducing FC was to encourage students to work together to learn the subject better, in accordance with the fact that this learning arena is based on collaborative activities (Bergmann & Sams, 2012; Bishop & Verleger, 2013; Lo et al., 2017). The unexpected result was a consequence of students studying at their own pace, and thus found themselves in different places in the curriculum. It made collaboration difficult.

6 CONCLUSION

The present paper focuses on students' learning perspectives in two different teaching environments. To measure the "success" of each design, it could be argued that the exam results for the linear algebra courses in 2019 and 2022 should be compared. Grade distributions for the two years is given in Figure 2.

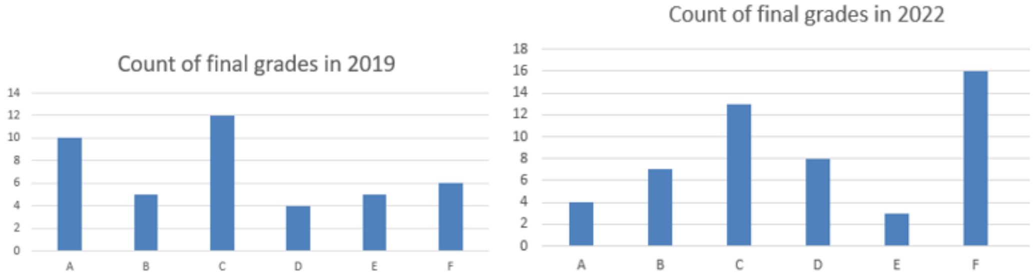

Fig. 2. Grade distributions for the exams in 2019 and 2022

From Figure 2 it is seen that in 2019 there was a fairly high number of A's and a fairly low number of F's. In 2022 there was a more Gaussian distribution of the grades and quite many F’s. An immediate conclusion would then be that the outcome for the traditional teaching method used in 2019 was better than in 2022 and that the new teaching format was a failure. However, a direct comparison of exam results between different exam years is not easy. The students were different and the exam tasks were different. It could be that in 2019 there happened to be a class with particularly good students, while the students in 2022 were more average. It may also be that the exam in 2019 was somewhat easier, while the exam in 2022 had a more correct degree of difficulty. Another issue is that students' experiences of taking proctored exams differed as 2022 was the first regular assessment semester after the Covid lockdown. Thus, there can be several reasons why the exam results vary and it cannot be immediately concluded that the teaching scheme in 2022 was a failure. Nevertheless, the discussion in the previous section shows that adjustments to this new teaching format is necessary. On-demand access to videos and support from the teacher on Zoom were appreciated. Both, however, had problematic disadvantages: Students who did not collaborate and students who felt uncomfortable with Zoom support. Thus, adjustments are necessary:

a) Each day should be introduced by a live lecture. This will
    - meet the need some students have to attend live lectures
    - bring students together
    - ensure a certain progression
b) Following each day’s lecture, the remaining linear algebra themes of the day should be studied in a FC approach. Students should be asked to form groups, preferably with students from the same engineering program. Here they can watch videos together, which enables them students to discuss the content and collaborate.
c) To reduce the threshold of attending Zoom, an initial lecture should be given in a Zoom meeting. Its announcement should be that important information about the subject will be given. Students who have used the platform say that it is not icky once they have tried it.
d) Some task solving sessions should be with the teacher present in class, circulating and helping students in a traditional in-person way.

Combining traditional lectures with task-solving sessions and videos with Zoom supported activities is valuable as it represents a greater variety in teaching styles, meeting students who have different teaching preferences. Both methods have advantages that merit them. A redesign must, however, take particular account of facilitation for collaboration, as this appears to be lacking and is vital in the students' learning process. In a typical flipped classroom this is taken care of as all problem-solving sessions are in the classroom – and all physical live teaching takes place in these sessions. Thus, if the students are to experience live teaching, they must participate in task-solving sessions where the main learning activity is usually working on tasks in groups. Collaboration between students is highlighted as an important component of the in-class part of FC (Fredriksen, 2020). Therefore, the amount of physical task-solving sessions (point d) above) should be given particular consideration.

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