

# Binary forms of suprageneric rank and the multiple root loci

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## Abstract

We state the relation between the variety of binary forms of given rank and the dual of the multiple root loci. This is a new result for the suprageneric rank that appears as a continuation of the cited work by Buczyński, Han, Mella and Teitler. We describe the strata of these varieties and explore their singular loci.

## 1 Introduction

Let  $V$  be a vector space of dimension  $m + 1$  over an algebraically closed field  $K$  of characteristic zero. Let  $f \in S_d V^\vee$  be a homogeneous form of degree  $d$ . The rank of  $f$ , also called the Waring rank, is defined to be the smallest integer  $r$  such that

$$f = l_1^d + \cdots + l_r^d,$$

where  $l_i$ ,  $i = 1, \dots, r$  are linear forms.

The general rank  $g$  of a form  $f$ , where the general rank means the rank that a general  $f \in S_d V^\vee$  has, is a well known result and it is given by

$$g = \left\lceil \frac{\binom{m+d}{d}}{m+1} \right\rceil,$$

with exception of a finite number of cases, see [1] and [2].

Let  $S_{d,r} = \{f \in S_d V^\vee \mid \text{rank } f = r\}$  be the set of forms of rank  $r$ . Let  $X \subset \mathbb{P}(S_d)$  be the Veronese variety. Then the variety obtained from the Zariski closure of  $S_{d,r}$  coincides with the  $r$ -secant variety of the Veronese variety,

$$\sigma_r(X) = \overline{S_{d,r}}$$

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for every  $r \leq g$ . On the other hand, since  $\sigma_g(X)$  fulfills the ambient space,  $\overline{S_{d,r}}$  cannot be expressed as a secant variety if  $r > g$ . The subgeneric rank has been vastly studied, however the suprageneric rank is still a challenge to understand, even examples in higher dimensions are difficult to find, however it has been an active area of research in past years. For instance [3] works on the suprageneric rank loci for several examples, and [4] works on the question of the maximal rank. More recently [12, Corollary 3.4] shows that the highest orbit of cubic surfaces with infinitely many singularities consists of cubics of suprageneric rank 6. Furthermore, [12, Theorem 3.3] shows the existence of smaller orbits consisting of cubics of rank 7.

In this article, we look at the case of binary forms (i.e.,  $m=1$ ) of suprageneric rank, in the light of the work developed in [6] on the strata of binary forms of rank at most generic, where the following description is obtained.

**Theorem** (Comas-Seiguer). Let  $0 \leq k < \lceil \frac{d+1}{2} \rceil$  be an integer, then

$$\overline{S_{d,k+1}} = (\cup_{i=1}^{k+1} S_{d,i}) \cup (\cup_{i=0}^k S_{d,d-i+1}),$$

where  $\overline{S_{d,0}} = \overline{S_{d,d+1}} = \emptyset$ . Furthermore, we have that

$$\overline{S_{d,k+1}} \setminus \overline{S_{d,k}} = S_{d,k+1} \cup S_{d,d-k+1}.$$

We prove a similar result for the suprageneric case, notice that the second union has a shift on the indices.

**Theorem 1.** Let  $k$  be an integer and suppose that  $d \geq d - k > \lceil (\frac{d+1}{2}) \rceil$ , then  $\overline{S_{d,d-k}}$  is the union

$$\overline{S_{d,d-k}} = (\cup_{i=1}^{k+1} S_{d,i}) \cup (\cup_{i=0}^k S_{d,d-i}).$$

In particular  $\overline{S_{d,d-k}} \setminus \overline{S_{d,d-k+1}} = S_{d,k+1} \cup S_{d,d-k}$ .

In order to prove these results, we show the relation between binary forms of fixed rank and the variety of multiple root loci. For an integer  $0 \leq r \leq \lceil \frac{d+1}{2} \rceil$ , it is known that  $\overline{S_{d,r}} = \Delta_{2r,1^{d-2r}}^\vee$ .

We find in [3, Proposition 4.3] a proof of the following nested inclusions of irreducible varieties

$$\overline{S_{d,k+1}} \subseteq \overline{S_{d,d-k}} = \overline{S_{d,k}} + \tau(X) \subseteq \overline{S_{d,k+2}} \subseteq \dots,$$

where  $\tau(X) = \{l^{d-1}g \mid l, g \text{ are linear forms}\}$  is the tangential variety of the Veronese variety and in [3] the notation translates to  $W_k = \overline{S_{d,d-k}}$ . Moreover, each variety in the inclusion chain has codimension 1 in the next variety, thus we deduce the dimension of  $\overline{S_{d,d-k}}$  as  $\dim(\overline{S_{d,d-k}}) = \dim(\overline{S_{d,k+2}}) - 1 = (2(k+2) - 1) - 1 = 2k+2$ . Using this we find that an analogous relation with the dual of the multiple root loci also exists, and obtain the following proposition, where  $\Delta_\lambda^\vee$  is the dual variety of the multiple root locus  $\Delta_\lambda$  associated to a partition  $\lambda$  as we define at the beginning of the following section.

**Proposition 2.** *Let  $k$  be an integer and suppose that  $d \geq d - k > \lceil \frac{d+1}{2} \rceil$ , then*

$$\overline{S_{d,d-k}} = \Delta_{3,2^k,1^{d-2k-3}}^\vee.$$

This proposition also allows us to prove that forms of rank different from  $d - k$  in  $\overline{S_{d,d-k}}$  are singular points of this variety. More precisely

**Theorem 3.** *Let  $d - k > \lceil \frac{d+1}{2} \rceil$ , then the singular locus of  $\overline{S_{d,d-k}}$  contains the subvariety  $\overline{S_{d,k+1}} \cup \overline{S_{d,d-k+1}}$ .*

**Acknowledgements.** The authors would like to thank the anonymous referees for their valuable comments that helped in improving the quality of this work. We are especially grateful for their suggestions on the proof of Theorem 1. The authors also would like to thank Giorgio Ottaviani for proposing this work and for the valuable discussions, suggestions, and encouragement. This work has been kindly and generously supported by European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions, grant agreement 813211 (POEMA).

The Version of Record of this manuscript has been published and is available in *Communications in Algebra*, published online: 11 August 2023, [doi.org/10.1080/00927872.2023.2242970](https://doi.org/10.1080/00927872.2023.2242970).

## 2 Preliminaries

### 2.1 The Multiple Root Loci of Binary Forms

We follow basically the notation in [10]. Given an integer  $d$ , we say that a vector  $\lambda = (\lambda_1, \dots, \lambda_n)$  is a partition of  $d$  with  $n$  parts if

$$\lambda_1 \geq \dots \geq \lambda_n > 0$$

and  $|\lambda| := \lambda_1 + \dots + \lambda_n = d$ . Apart from this notation, we may also write a partition as a multiset  $\lambda = \{1^{m_1}, \dots, p^{m_p}\}$ , where  $m_i \geq 0$  is an integer for  $i = 1, \dots, p$ , and represents that there are  $m_i$  elements in the partition that are equal to  $i$ .

The set of homogeneous binary forms of degree  $d$  corresponds to a variety on  $\mathbb{P}^d$  associating the points to the coefficient of each monomial in the polynomial expansion. The multiple root locus  $\Delta_\lambda$  associated to a partition

$$\lambda = (\lambda_1, \dots, \lambda_n)$$

of  $d$  is a subvariety of  $\mathbb{P}^d$  associated to the polynomials that have  $n$  roots with multiplicity  $\lambda_1, \dots, \lambda_n$ . The dimension of this variety is  $\dim(\Delta_\lambda) = n$  and its singular locus is a subset of the union

$$\bigcup_{\lambda \text{ properly refines } \mu} \Delta_\mu,$$

as described in [9, Section 3], and in [5].

We are particularly interested in the dual varieties  $\Delta_\lambda^\vee$ . These are studied in [11] and [10]. In particular, Hilbert found that the degree of  $\Delta_\lambda$  is  $\deg(\Delta_\lambda) = \frac{n}{m_1! \cdots m_p!} \lambda_1 \cdots \lambda_n$  and, when the dual  $\Delta_\lambda^\vee$  is a hypersurface (i.e.,  $m_1 = 0$ ), [11, Theorem 5.3] establishes that its degree is  $\deg(\Delta_\lambda^\vee) = \frac{(n+1)!}{m_2! \cdots m_p!} (\lambda_1 - 1) \cdots (\lambda_n - 1)$ .

Notice that, given a partition  $\lambda$  as above, we have another definition for  $\Delta_\lambda$ , it also is the image of

$$(\mathbb{P}^1)^n \longrightarrow \mathbb{P}^d, (l_1, \dots, l_n) \longmapsto l_1^{\lambda_1} \dots l_n^{\lambda_n}.$$

It follows that the dimension of  $\Delta_\lambda$  is  $n$  and its smooth points are those in which all the linear forms  $l_i$  are pairwise different.

From the discussion after [10, Lemma 2.2] the conormal variety of  $\Delta_\lambda$  is given by the closure of the set

$$\{(f, g) \mid f \in \Delta_\lambda \text{ is a smooth point and } g \perp T_f \Delta_\lambda\}.$$

This leads to a parametrization of the conormal variety: it can be seen as the set of points  $(f, g)$  of the form

$$f(x, y) = \prod_{i=1}^n (t_i x - s_i y)^{\lambda_i}, \quad g(u, v) = \sum_{i=1, \lambda_i \neq 1}^n (s_i u + t_i v)^{d - \lambda_i + 2} g_i(u, v),$$

where  $g_i(u, v)$  are binary forms of degree  $\lambda_i - 2$ , and  $(s_i, t_i) \in \mathbb{P}^1$ . The dual variety  $\Delta_\lambda^\vee$  is the image of the projection of the conormal variety onto the second factor. It is a irreducible variety, and its dimension, using [8, Corollary 7.3], is given by

$$\dim \Delta_\lambda^\vee = d - m_1 - 1.$$

The inclusions between multiple root loci can be characterized in terms of refinements of the partitions that define them. Hence we have that  $\Delta_\lambda \subset \Delta_\mu$  if and only if  $\mu$  refines  $\lambda$ .

In addition, for a partition  $\lambda = \{1^{m_1}, \dots, p^{m_p}\}$ , we denote its derived partition  $\lambda' := (1^{m_2}, \dots, (p-1)^{m_p})$ , and this is a partition of  $d - n$ , where  $n = \sum m_i$  is the number of parts. The next proposition gives a result similar to the one in the previous paragraph for inclusions between dual varieties. These inclusions are also characterized via refinements of partitions although it is not as direct as the previous one: the equivalent condition for the inclusion of duals involves refinements of derived partitions. Expressing this new condition requires thus the related partitions that we have just introduced.

**Proposition 4.** [10, Proposition 3.4] *Given two partitions  $\lambda, \mu$  of  $d$ , then  $\Delta_\lambda^\vee \subset \Delta_\mu^\vee$  holds if and only if  $|\lambda'| \leq |\mu'|$  and, by adding to the parts,  $\lambda'$  can be transformed into a partition  $\tilde{\lambda}$  that is refined by  $\mu'$ .*

Relations between the ranks of binary forms and the multiple root loci have already appeared in the literature. In particular, the reviewers made us aware of the following result.

**Theorem 5** ([13, Theorem 2.1]). *If a binary form  $f$  of degree  $d$  has a root with multiplicity  $m$ , then  $\text{rank}(f) \geq m + 1$ .*

For suprageneric rank, i.e.  $m > \lceil \frac{d+1}{2} \rceil$ , this implies  $\Delta_{m,1^{d-m}} \subset \overline{S_{d,m+1}}$ .

## 2.2 Apolarity Lemma

In this section we introduce the apolar ideal, which will be fundamental to associating the multiple root loci and varieties generated by forms of certain fixed rank. The apolar ideal can be seen as the ideal formed by all the forms perpendicular to  $f$  with respect to the scalar product by differentiation through dual variables.

**Definition 6.** Let  $f$  be a form of degree  $d$ , the apolar ideal of  $f$ , denoted  $(f)^\perp$ , is the ideal of elements  $g \in SV^\vee$  such that  $g \cdot f = 0$ , where  $\cdot$  represents the contraction (by differentiation) of  $f$  by  $g$ .

After this definition, we remember the next well-known result.

**Lemma 7** (Apolarity Lemma). *Let  $f \in S_d$ . Then  $f = l_1^d + \dots + l_s^d$ , where the summands  $l_i \in S_1$  are pairwise non-proportional linear forms, if and only if  $(f)^\perp \supseteq I$ , where  $I$  is the ideal of the set  $X = \{[l_1], \dots, [l_s]\} \subseteq \mathbb{P}S_1V^\vee$  of  $s$  different points formed by all the  $s$  pairwise non-proportional linear forms in the previous expression of  $f$  as a sum of  $d$ -th powers of linear forms.*

A final important remark concerns the good description of the apolar ideal that we have in the case of a binary form.

**Remark 8.** If  $f$  is a binary form of degree  $d$ , then  $(f)^\perp = (g_1, g_2)$  with  $\deg(g_1) + \deg(g_2) = d + 2$ . In addition, if  $\deg(g_1) \leq \deg(g_2)$ , then  $\text{rank}(f) = \deg(g_1)$  if  $g_1$  is squarefree and  $\text{rank}(f) = \deg(g_2)$  otherwise.

## 3 Binary forms of suprageneric rank

### 3.1 The variety of rank $k$ forms and the multiple root loci

The relation between the variety  $\overline{S_{d,k}}$  was well known for degrees smaller than 6. So the first interesting example is the case where the degree is  $d = 6$ . We explore this case for ranks bigger than the generic rank  $r = 4$ . In the particular case of  $f \in S_{6,6}$ , we have that the ideal  $(f)^\perp = (g_1, g_2)$  with  $d_1 + d_2 = 8$ , where  $d_1$  and  $d_2$  are the respective degrees. Since the rank of  $f$  is 6, we must have  $d_1 = 2, d_2 = 6$ , and  $g_1$  has a double root. Therefore the only possibility is that  $g_1 = l^2$ , where  $l$  is a linear form. In such case, by an immediate application of [10, Corollary 2.3], we know that  $f \in \Delta_{3,1^3}^\vee$ . The other inclusion follows from

dimensional count. We can use such idea to compute any  $S_{d,r}$ . For example, proceeding similarly for the rank 5 we have that  $d_1 = 3, d_2 = 5$  and therefore we have that  $g_1$  has two possible cases: either  $l_1^3$  or  $l_1^2 l_2$ . In such case,  $f \in \Delta_{4,1^2}^\vee$  or  $f \in \Delta_{3,2,1}^\vee$ , respectively. We can see that the first is contained in the second, and therefore  $f \in \Delta_{3,2,1}^\vee$ . The other side follows again by dimensional count.

In [3, Proposition 4.3] it was obtained that the dimension of  $\overline{S_{d,r}}$  for  $r$  bigger than the generic rank is given by

$$\dim \overline{S_{d,r}} = 2(d - r + 1).$$

Using this fact together with the preceding idea developed in the example, we obtain the following argument.

*Proof of Proposition 2.* Let  $f \in S_{d,d-k}$  be a homogeneous polynomial of degree  $d$  and rank  $d - k$ . We know that the apolar ideal  $(f)^\perp$  is generated by  $(g_1, g_2)$ , such that  $d_1 + d_2 = d + 2$ , with  $d_1 \leq d_2$  the respective degrees, and  $\text{rank}(f) = d_2$ , if  $g_1$  is not squarefree, or  $\text{rank}(f) = d_1$  otherwise. So we may assume that  $d_2 = d - k, d_1 = k + 2$  and  $g_1$  has a double root. Thence  $g_1$  has the following form  $l_0^2 l_1 \dots l_k$  and  $f \in \Delta_{3,2^k,1^{d-2k-3}}^\vee$ . (Notice that all other possibilities for  $g_1$ , that is, with more than a single double root, lead to a different partition  $\lambda$  but all of those are such that  $\lambda'$  is refined by  $(2, 1^k)$  and therefore we have  $\Delta_\lambda^\vee \subset \Delta_{3,2^k,1^{d-2k-3}}^\vee$  in such case.) It follows that  $S_{d,d-k} \subseteq \Delta_{3,2^k,1^{d-2k-3}}^\vee$ .

From the proof of [3, Proposition 4.3], we have that  $\dim \overline{S_{d,d-k}} = \dim \overline{S_{d,k+2}} - 1 = (2k+3) - 1 = 2k+2$ , and  $\dim \Delta_{3,2^k,1^{d-2k-3}}^\vee = d - m_1 - 1 = 2k+2$ , so they have same dimension. We conclude  $\overline{S_{d,d-k}} = \overline{\Delta_{3,2^k,1^{d-2k-3}}^\vee}$ , because  $\Delta_{3,2^k,1^{d-2k-3}}^\vee$  is irreducible.  $\square$

Following [6, Theorem 2], we obtain a similar result for the varieties of rank  $r$  bigger than the generic rank. Furthermore, we also give another description for  $\overline{S_{d,d-k}}$ .

*Proof of Theorem 1.* From [3, Proposition 4.3] we have  $\overline{S_{d,d-k}} = \tau(X) + \overline{S_{d,k}} = \overline{\Delta_{3,2^k,1^{d-2k-3}}^\vee}$ . Notice that  $X \subset \tau(X)$ , thus  $\overline{S_{d,k+1}} \subset \overline{S_{d,d-k}}$ , in particular  $\overline{S_{d,i}} \subset \overline{S_{d,d-k}}$  for all  $1 \leq i \leq k + 1$ . Also, for  $j \leq k$ ,

$$\overline{S_{d,d-j}} = \overline{\Delta_{3,2^j,1^{d-2j-3}}^\vee} \subset \overline{\Delta_{3,2^k,1^{d-2k-3}}^\vee} = \overline{S_{d,d-k}}.$$

This shows  $\overline{S_{d,d-k}} \supseteq (\cup_{i=1}^{k+1} S_{d,i}) \cup (\cup_{i=0}^k S_{d,d-i})$ .

For the other inclusion, suppose that  $f \in \Delta_{3,2^k,1^{d-2k-3}}^\vee$ , then  $f = l_0^{d-1} g + l_1^d + \dots + l_k^d$  for some linear forms  $l_0, \dots, l_k, g$ . We analyse two cases. If  $g = l_0$ , it is clear that  $\text{rank}(f) \leq k + 1$ . Otherwise, suppose that  $g \neq l_0$ , then  $l_0^{d-1} g$  has rank  $d$ . Since all the other summands are power of linear forms, each of them can either increase or decrease the rank by 1, or leave the rank unchanged, in any case we have that  $\text{rank}(f) \geq d - k$ , hence we have the equality.  $\square$

*Proof of Theorem 3.* Let  $f = l_0^{d-1}g + l_1^d + \dots + l_k^d$  be a point of  $\overline{S_{d,d-k}}$ . We compute the tangent cone at  $f$ , by considering  $l_i = a_i x + b_i y$  and  $g = \alpha x + \beta y$ . We can consider a curve

$$f(t) = \sum_{i=1}^k (a_i(t)x + b_i(t)y)^d + (a_0(t)x + b_0(t)y)^{d-1}(\alpha(t)x + \beta(t)y),$$

with  $f(0) = f$ , then taking the derivatives on the  $a_i, b_i, \alpha, \beta$  we have that the tangent cone is generated by

$$\text{TC}_f \overline{S_{d,d-k}} = \langle y l_i^{d-1}, x l_i^{d-1}, x l_0^{d-2} g, y l_0^{d-2} g, x l_0^{d-1}, y l_0^{d-1} \rangle, \quad i = 1, \dots, k.$$

The tangent cone at  $f$  has  $2k+4$  generators, but we notice that the last four of them span a 3-dimensional space, so it has projective dimension  $2k+2$  in a general point, as expected. We consider two cases now, first if  $g$  is equal to  $l_0$ , in other words, the case that  $f$  is a general element of  $\overline{S_{d,k+1}}$ . We notice that instead of a 3-dimensional space, the last four elements on the span generate a 2-dimensional space, this means that the projective dimension of  $\text{TC}_f \overline{S_{d,d-k}}$  is at most  $2k+1$ , therefore  $f$  is a singular point. Now instead, assume that  $l_i = l_j$  for some  $i, j \neq 0$  and  $i \neq j$ , then  $f$  is a general element of  $\overline{S_{d,d-k+1}}$  and the dimension of  $\text{TC}_f \overline{S_{d,d-k}}$  is less than  $2k+2$ , again this gives that  $f$  is a singular element of  $\overline{S_{d,d-k}}$ .  $\square$

### 3.2 The hypersurface $\overline{S_{2k+1,k+2}}$

Let  $f \in \overline{S_{2k+1,k+2}}$ , the maximal catalecticant matrix  $C_f$  associated to  $f$  has size  $(k+1) \times (k+2)$ . In [10, Theorem 4.1] it is proven that this hypersurface has degree  $2k(k+1)$  and its equation is computed, namely, the defining polynomial is the discriminant of

$$q(u, v) = \det \begin{bmatrix} u^{k+1} & u^k v & \dots & u v^k & v^{k+1} \\ a_0 & a_1 & \dots & a_k & a_{k+1} \\ a_1 & a_2 & \dots & a_{k+1} & a_{k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_k & a_{k+1} & \dots & a_{2k} & a_{2k+1} \end{bmatrix}.$$

With this description we can obtain the following result.

**Theorem 9.**  $\overline{S_{2k+1,k}}$  is an irreducible component of  $\overline{\text{Sing}(S_{2k+1,k+2})}$ .

*Proof.* Let  $(a_0, \dots, a_d)$  be the coefficients of polynomials in  $S_d$ . The equation of  $\overline{S_{2k+1,k+2}}$  has degree  $2k$  in the  $(k+1)$ -minors  $b_j$  (for  $j = 0, \dots, k$ ) of the maximal catalecticant matrix of size  $(k+1) \times (k+2)$ . Each  $b_j$  is a homogeneous polynomial of degree  $(k+1)$  in the  $a_i$ . Let  $b_0^{\alpha_0} \dots b_k^{\alpha_k}$  be a monomial with  $|\alpha| = 2k$ . The derivative with respect to  $a_i$  of such monomial is  $\sum_j \alpha_j b_0^{\alpha_0} \dots b_j^{\alpha_j-1} \dots b_k^{\alpha_k} \frac{\partial b_j}{\partial a_i}$ . Evaluated at a point  $(a_0, \dots, a_d)$  where all  $b_j$  vanishes (this is a point in  $\overline{S_{2k+1,k}}$ ) this monomial vanishes. This proves that  $\overline{S_{2k+1,k}}$  is contained in the singular

locus of the hypersurface  $\overline{S_{2k+1,k+2}}$ . Finally, as  $\overline{S_{2k+1,k}}$  is irreducible and with codimension 2, we get that it is in fact an irreducible component of this singular locus, which concludes the proof.  $\square$

The case  $k = 2$  was studied before in [7] by Comon and Ottaviani, it is known as the *apple invariant*, in such case the singular locus has two irreducible components, one is  $\overline{S_{5,2}}$ , that comes from the minors of the catalecticant, and the other comes from the pullback from the locus of cubics with a triple root  $\Delta_{3,1,1}$ , that is the dual of the tangent variety  $\tau(\overline{S_{5,5}}) = \overline{S_{5,4}}$ . For  $k \geq 3$ ,  $\text{Sing}(\overline{S_{2k+1,k+2}})$  has at least three irreducible components, one is  $\overline{S_{2k+1,k}}$ , that is obtained from the minors of the catalecticant, the other two components arrive from the two irreducible components of the singular locus of the discriminant of  $\sum_{i=0}^{k+1} a_i t^i$ , it comes as the pullback from the locus of degree  $k+1$  polynomials with two double roots and with a triple root. For  $k = 3$  the components can be computed in Macaulay2, one is  $\overline{S_{7,3}}$ , that has codimension 2 and degree 10. The other two components have codimension 2 and degree respectively 24 (8 generators of degree 7, it comes as pullback from locus of quartics with two double roots) and 36 (55 generators of degree among 8 and 12, it comes as pullback from locus of quartics with a triple root), this case was named as the *big apple invariant* in [10].

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