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Martingale Hardy Spaces and Some New Weighted Maximal Operators of Fejér Means of Walsh–Fourier Series

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Abstract

In this paper, we introduce some new weighted maximal operators of the Fejér means of the Walsh–Fourier series. We prove that for some "optimal" weights, these new operators indeed are bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$, for $0 < p < 1/2$. Moreover, we also prove sharpness of this result. As a consequence, we obtain some new and well-known results.

Keywords Walsh system · Fejér means · Martingale Hardy space · Maximal operators · Weighted maximal operators

Mathematics Subject Classification 42C10

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1 Introduction

All symbols used in this introduction can be found in Sect. 2.

In the one-dimensional case, the weak (1,1)-type inequality for the maximal operator σ^* of Fejér means σ_n with respect to the Walsh system

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$$

can be found in Schipp [21] and Pál, Simon [16] (see also [2]). Fujii [7] and Simon [23] proved that σ^* is bounded from H_1 to L_1. Weisz [29] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [22] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [10]. Moreover, in [11] (see also [19]) he proved that there exists a martingale $F \in H_p$ $(0 < p \leq 1/2)$, such that

$$\sup_{n\in\mathbb{N}} \|\sigma_n F\|_p = +\infty.$$

Weisz [32] proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$.

For $0 < p < 1/2$ in [26] the weighted maximal operator $\overset{\sim}{\sigma}^{*,p}$, defined by

$$\overset{\sim}{\sigma}^{*,p} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{(n+1)^{1/p-2}}, \tag{1}$$

was investigated, and it was proved that the following estimate holds:

$$\left\| \overset{\sim}{\sigma}^{*,p} F \right\|_p \leq c_p \|F\|_{H_p}. \tag{2}$$

Moreover, it was proved that the rate of sequence $(n_k+1)^{1/p-2}$ given in the denominator of (1) cannot be improved. In the case $p = 1/2$ analogical results for the maximal operator $\overset{\sim}{\sigma}^{*}$ defined by

$$\overset{\sim}{\sigma}^{*} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{\log^2(n+1)}$$

were proved in [25].

To study the convergence of subsequences of Fejér means and their restricted maximal operators on the martingale Hardy spaces $H_p(G)$ for $0 < p \leq 1/2$, the central role is played by the fact that any natural number $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_j 2^j, \quad n_j \in Z_2 \ (j \in \mathbb{N}),$$

where only a finite number of n_j differs from zero and their important characters $[n]$, $|n|$, $\rho(n)$, and $V(n)$ are defined by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) = |n| - [n]$$

and

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \text{ for all } n \in \mathbb{N}$$

Weisz [31] (see also [30]) also proved that for any $F \in H_p(G)$ $(p > 0)$, the maximal operator $\sup_{n\in\mathbb{N}} |\sigma_{2^n} F|$ is bounded from the Hardy space H_p to the Lebesgue space L_p. Persson and Tephnadze [18] generalized this result and proved that if $0 < p \leq 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive integers, such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) \leq c < \infty, \tag{3}$$

then the maximal operator $\widetilde{\sigma}^{*,\triangle}$, defined by

$$\widetilde{\sigma}^{*,\triangle} F := \sup_{k\in\mathbb{N}} \left|\sigma_{n_k} F\right|, \tag{4}$$

is bounded from the Hardy space $H_p(G)$ to the space $L_p(G)$. Moreover, if $0 < p < 1/2$ and $\{n_k : k \geq 0\}$ is a sequence of positive numbers, such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) = \infty, \tag{5}$$

then there exists a martingale $F \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \left\|\sigma_{n_k} F\right\|_p = \infty.$$

From these facts, it follows that if $0 < p < 1/2$, $f \in H_p$, and $\{n_k : k \geq 0\}$ is any sequence of positive numbers, then the maximal operator defined by (4) is bounded from the Hardy space H_p to the Lebesgue space L_p if and only if the condition (3) is fulfilled.

In [27], it was proved that if $F \in H_{1/2}$, then there exists an absolute constant c, such that

$$\|\sigma_n F\|_{H_{1/2}} \leq c V^2(n) \|F\|_{H_{1/2}}.$$

Moreover, the rate of sequence $V^2(n)$ cannot be improved.

In [27], it was also proved that if $0 < p < 1/2$ and $F \in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\|\sigma_n F\|_{H_p} \leq c_p 2^{\rho(n)(1/p-2)} \|F\|_{H_p}.$$

Moreover, if $0 < p < 1/2$ and $\{\Phi_n\}$ is any nondecreasing sequence, such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) = \infty, \quad \overline{\lim_{k\to\infty}} \frac{2^{\rho(n_k)(1/p-2)}}{\Phi_{n_k}} = \infty,$$

then there exists a martingale $F \in H_p$, such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{\sigma_{n_k} F}{\Phi_{n_k}} \right\|_{\text{weak}-L_p} = \infty.$$

Convergence and summability of Fejér means of Walsh–Fourier series can be found in [1], [3], [4], [5], [6], [8], [9], [14], [15], [17], [28], and [29].

One main aim of this paper is to generalize the estimate (2) for $f \in H_p(G)$, $0 < p < 1/2$. Our main idea is to investigate much more general maximal operators by replacing the weights $(n+1)^{1/p-2}$ in (1) by more general "optimal" weights $2^{\rho(n)(1/p-2)}(\varphi(\rho(n)))$, where $\varphi : \mathbb{N}_+ \to \mathbb{R}_+$ is any nonnegative and nondecreasing function satisfying the condition

$$\sum_{n=1}^{\infty} 1/\varphi^p(n) < c < \infty$$

and prove that it is bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$, for $0 < p < 1/2$. As a consequence, we obtain some new and well-known results. In particular, we prove that the maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla,\varepsilon} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{2^{\rho(n)(1/p-1)}\left((\rho(n))\right)^{(1+\varepsilon)/p}}, \quad \text{where } 0 < p < 1/2, \quad \varepsilon \geq 0,$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for any $\varepsilon > 0$ but is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ when $\varepsilon = 0$.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Sect. 2. The main results and some of their consequences can be found in Sect. 3. For the proofs of the main results, we need some auxiliary lemmas, which are presented in Sect. 4. Detailed proofs are given in Sect. 5.

2 Definitions and Notations

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is, $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is $1/2$.

Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2. The elements of G are represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots), \quad \text{where} \quad x_k = 0 \vee 1.$$

It is easy to give a base for the neighborhood of $x \in G$:

$$I_0(x) := G, I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}(n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \backslash I_n$ and

$$e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G, \quad \text{for} \quad n \in \mathbb{N}.$$

Then it is easy to show that

$$\overline{I_M} = \bigcup_{i=0}^{M-1} I_i \backslash I_{i+1} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l) \right) \bigcup \left(\bigcup_{k=0}^{M-1} I_M(e_k) \right). \tag{6}$$

The norms (or quasi-norms) of the spaces $L_p(G)$ and $weak-L_p(G)$, $\left(0 < p < \infty\right)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu$$

and

$$\|f\|_{weak-L_p(G)}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda) < +\infty.$$

The k-th Rademacher function $r_k(x)$ is defined by

$$r_k(x) := (-1)^{x_k} \qquad (x \in G, \ k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as follows:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \qquad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [13] and [20]).

If $f \in L_1(G)$, we can define the Fourier coefficients, partial sums of Fourier series, Fejér means, and Dirichlet and Fejér kernels in the usual manner:

$$\widehat{f}(n) := \int_G f w_n d\mu, \ \ (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \ (n \in \mathbb{N}_+, \ S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f,$$

$$D_n := \sum_{k=0}^{n-1} w_k,$$

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \ (n \in \mathbb{N}_+).$$

Recall that (see [13] and [20]) for any $t, n \in \mathbb{N}$,

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases} \tag{7}$$

and

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, \text{if } x \in I_n(e_t), n > t, x \in I_t \backslash I_{t+1}, \\ (2^n + 1)/2, \text{if } x \in I_n, \\ 0, \ \text{otherwise}. \end{cases} \tag{8}$$

Let

$$n = \sum_{i=1}^{r} 2^{n^i}, \quad n^1 > n^2 > \cdots > n^r \geq 0$$

and

$$n^{(k)} := 2^{n^{k+1}} + 2^{n^{k+2}} + \cdots + 2^{n^r}.$$

Then (see [13] and [20]), for any $n \in \mathbb{N}$,

$$nK_n = \sum_{A=1}^{r} \left(\prod_{j=1}^{A-1} w_{2^{n^j}} \right) \left(2^{n^A} K_{2^{n^A}} + n^{(A)} D_{2^{n^A}} \right). \tag{9}$$

The σ-algebra, generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n $(n \in \mathbb{N})$. Denote by $F = (F_n, n \in \mathbb{N})$ a martingale with respect to ζ_n $(n \in \mathbb{N})$ (see e.g., [30]).

The maximal function F^* of a martingale F is defined by

$$F^* := \sup_{n\in\mathbb{N}} |F_n|.$$

In the case $f \in L_1(G)$ the maximal function f^* is given by

$$f^*(x) := \sup_{n\in\mathbb{N}} \left(\frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right| \right).$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G)$ consist of all martingales for which

$$\|F\|_{H_p} := \left\|F^*\right\|_p < \infty.$$

A bounded measurable function a is called a p-atom if there exists a dyadic interval I such that

$$\operatorname{supp}(a) \subset I, \quad \int_I a\, d\mu = 0,\ \|a\|_\infty \le \mu(I)^{-1/p}.$$

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n\to\infty} \int_G F_n(x)\, w_k(x)\, d\mu(x)$$

exists, and it is called the k-th Walsh–Fourier coefficients of F.

If $F := (S_{2^n} f : n \in \mathbb{N})$ is a regular martingale, generated by $f \in L_1(G)$, then (see e.g., [19], [24], and [30])

$$\widehat{F}(k) = \widehat{f}(k), \ k \in \mathbb{N}.$$

3 The Main Results

Our first main result reads:

Theorem 1 *Let $0 < p < 1/2$, $f \in H_p(G)$, and $\varphi : \mathbb{N}_+ \to \mathbb{R}$ be any nonnegative and nondecreasing function satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi^p(n)} < c < \infty. \tag{10}$$

Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, *defined by*

$$\widetilde{\sigma}^{*,\nabla}F := \sup_{n\in\mathbb{N}}\frac{|\sigma_n F|}{2^{\rho(n)(1/p-2)}\varphi(\rho\,(n))},$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space* $L_p(G)$.

We also state and prove the sharpness of Theorem 1:

Theorem 2 *Let* $0 < p < 1/2$, $\{n_k : k \geq 0\}$ *be a sequence of positive numbers and* $\varphi : \mathbb{N}_+ \to \mathbb{R}$ *is any nonnegative and nondecreasing function satisfying the condition*

$$\sum_{n=1}^{\infty}\frac{1}{\varphi^p(n)} = \infty. \tag{11}$$

Then there exist p-atoms f_{n_k}, *such that*

$$\sup_{k\in\mathbb{N}}\frac{\left\|\sup_{n\in\mathbb{N}}\frac{|\sigma_n f_{n_k}|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))}\right\|_p}{\left\|f_{n_k}\right\|_{H_p}} = \infty.$$

As we will point out (see Remark 1) Theorem 1 can be of special interest even if we restrict it to subsequences.

Corollary 1 *Let* $0 < p < 1/2$, $f \in H_p\,(G)$, $\varphi : \mathbb{N}_+ \to \mathbb{R}$ *be any nonnegative and nondecreasing function satisfying the condition* (10), *and* $\{n_k : k \geq 0\}$ *be any sequence of positive numbers. Then the weighted maximal operator* $\widetilde{\sigma}^{*,\nabla}$, *defined by*

$$\widetilde{\sigma}^{*,\nabla}F := \sup_{k\in\mathbb{N}}\frac{\left|\sigma_{n_k} F\right|}{2^{\rho(n_k)(1/p-2)}\varphi(\rho\,(n_k))}, \tag{12}$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space* $L_p(G)$.

If we take $\varphi(n) = n^{(1+\varepsilon)/p}$, for any $\varepsilon > 0$, we get that the condition (10) is fulfilled. On the other hand, if we take $\varphi(n) = n^{1/p}$, then the condition (11) holds. Hence, Theorem 1 and Theorem 2 imply the following sharp result:

Corollary 2 *a) Let* $0 < p < 1/2$ *and* $f \in H_p\,(G)$. *Then the weighted maximal operator* $\widetilde{\sigma}^{*,\nabla,\varepsilon}$, *defined by*

$$\widetilde{\sigma}^{*,\nabla,\varepsilon}F := \sup_{n\in\mathbb{N}}\frac{|\sigma_n F|}{2^{\rho(n)(1/p-2)}\,(\rho\,(n))^{(1+\varepsilon)/p}}, \quad \varepsilon > 0,$$

is bounded from the Hardy space $H_p(G)$ *to the Lebesgue space* $L_p(G)$.

b) The weighted maximal operator $\widetilde{\sigma}^{*,\nabla,0}$, *defined by*

$$\widetilde{\sigma}^{*,\nabla,0}F := \sup_{n\in\mathbb{N}}\frac{|\sigma_n F|}{2^{\rho(n)(1/p-2)}\,(\rho\,(n))^{1/p}},$$

is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Remark 1 Suppose that $\{n_k : k \geq 0\}$ is a sequence of positive numbers, such that

$$\sup_{k\in\mathbb{N}} [n_k] < c < \infty.$$

Then

$$\sup_{k\in\mathbb{N}} \varphi([n_k]) < \varphi(c) < \infty,$$
$$2^{\rho(n_k)(1/p-2)} \sim 2^{|n_k|(1/p-2)} \sim n_k^{1/p-1} \sim (n_k+1)^{1/p-2}$$

and the maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by (12), can be estimated by

$$\widetilde{\sigma}^{*,\nabla} F \leq \sup_{k\in\mathbb{N}} \frac{|\sigma_{n_k} F|}{(n_k+1)^{1/p-2}}.$$

Let

$$\sup_{k\in\mathbb{N}} [n_k] = \infty.$$

Then we have the following estimation:

$$\sup_{k\in\mathbb{N}} \frac{|\sigma_{n_k} F|}{(n_k+1)^{1/p-1}} \leq \widetilde{\sigma}^{*,\nabla} F.$$

In particular, we find that from Theorem 1, Remark 1, and the theorem proved in [26] follows immediately the following result:

Corollary 3 *Let $0 < p < 1/2$, $f \in H_p(G)$, and $\varphi : \mathbb{N}_+ \to \mathbb{R}_+$ be any nonnegative and nondecreasing function satisfying the condition* (10)*. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by*

$$\widetilde{\sigma}^{*,\nabla} F := \sup_{n\in\mathbb{N}} \frac{|\sigma_n F|}{\min\{2^{\rho(n)(1/p-2)}\varphi(\rho(n)), (n+1)^{1/p-2}\}},$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

From Theorem 1 and Theorem 2 follows immediately the following result given in [18]:

Corollary 4 *a) Let $0 < p \leq 1/2$ and $(n_k, k \in \mathbb{N})$ be a subsequence of positive numbers such that condition* (3) *is fulfilled. Then the maximal operator $\widetilde{\sigma}^{*,\triangle}$, defined by* (4)*, is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

b) Let $0 < p < 1$ and $(n_k, k \in \mathbb{N})$ be a subsequence of positive numbers satisfying the condition (5)*. Then the maximal operator $\widetilde{\sigma}^{*,\triangle}$, defined by* (4)*, is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

4 Auxiliary Results

The dyadic Hardy martingale spaces H_p for $0 < p \leq 1$ have an atomic characterization. Namely, the following holds (see [19], [24], [30], and [31]):

Lemma 1 *A martingale $F = (F_n, n \in \mathbb{N})$ belongs to H_p $(0 < p \leq 1)$ if and only if there exists a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers, such that, for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{13}$$

Moreover,

$$\|F\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of F of the form (13).

From this result follows the following important lemma proved by Weisz [30]:

Lemma 2 *Suppose that an operator T is σ-sublinear and*

$$\int_{\overline{I}} |Ta|^p \, d\mu \leq c_p < \infty, \, (0 < p \leq 1)$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then, for $0 < p \leq 1$,

$$\|TF\|_p \leq c_p \|F\|_{H_p}.$$

The proof of the next lemma can be found in Persson and Tephnadze [18]:

Lemma 3 *Let $n \in \mathbb{N}$, $[n] \neq |n|$, and $x \in I_{[n]+1}\left(e_{[n]-1} + e_{[n]}\right)$. Then*

$$|nK_n(x)| = \left| \left(n - 2^{|n|} \right) K_{n-2^{|n|}}(x) \right| \geq \frac{2^{2[n]}}{4}.$$

We note that if $[n] = 0$, we have the set $I_2(e_0)$.

We also need the following lemma (see [12]):

Lemma 4 *Let $n \geq 2^M$ and $x \in I_M(e_k + e_l)$, $k = 0, \ldots, M-1$, $l = k+1, \ldots, M$. Then*

$$\int_{I_M} |K_n(x+t)| \, d\mu(t) \leq \frac{c2^{k+l}}{2^{2M}}.$$

5 Proofs of the Theorems

Proof of Theorem 1 Since σ_n is bounded from L_∞ to L_∞ by Lemma 2, the proof will be complete, if we prove that

$$\int_{\overline{I}} \left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} \right)^p d\mu \leq c_p < \infty, \tag{14}$$

for every p-atom a. We may assume that a be an arbitrary p-atom with support I, $\mu(I) = 2^{-M}$, and $I = I_M$. It is easy to see that

$$\sigma_n a(x) = 0, \quad \text{when} \;\; n < 2^M.$$

Therefore, we can suppose that $n \geq 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, we find that

$$\begin{aligned} &\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} \\ &\leq \frac{1}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} \|a\|_\infty \int_{I_M} |K_n(x+t)|\, d\mu(t) \\ &\leq \frac{1}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} 2^{M/p} \int_{I_M} |K_n(x+t)|\, d\mu(t). \end{aligned} \tag{15}$$

Let $x \in I_{l+1}(e_k + e_l)$, $0 \leq k < l < [n] \leq M$. Then $x + t \in I_{l+1}(e_k + e_l)$ and if we apply (7), (8), and (9), then we get that

$$K_n(x+t) = 0, \quad \text{for} \quad t \in I_M$$

and from (15) it follows that

$$\frac{1}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} |\sigma_n a(x)| = 0. \tag{16}$$

Let

$$x \in I_{l+1}(e_k + e_l), \; [n] \leq k < l < M \text{ or } k < [n] \leq l < M.$$

Since $|n| \geq M$ by using (15) and Lemma 4 we can conclude that

$$\begin{aligned} \frac{1}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} |\sigma_n a(x)| &\leq \frac{c2^{M(1/p-2)+k+l}}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} \\ &= \frac{c2^{[n](1/p-2)}2^{M(1/p-2)+k+l}}{2^{|n|(1/p-2)}\varphi(\rho(n))} \\ &\leq \frac{c2^{M(1/p-2)}}{2^{|n|(1/p-2)}} \frac{2^{[n](1/p-2)+k+l}}{\varphi(M-l)} \end{aligned}$$

$$\leq \frac{c2^{M(1/p-2)}}{2^{|n|(1/p-2)}}\frac{2^{k+l(1/p-1)}}{\varphi(M-l)}$$
$$\leq \frac{c2^{k+l(1/p-1)}}{\varphi(M-l)}. \tag{17}$$

By applying (16) and (17) for any $x \in I_{l+1}(e_k+e_l)$, $0 \leq k < l < M$ we find that

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} \leq \frac{c2^{k+l(1/p-1)}}{\varphi(M-l)}. \tag{18}$$

Let $x \in I_M(e_k)$, $0 \leq k < M$. By using again (15) and Lemma 4 for $k = l$ we can conclude that

$$\frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} \leq c2^{M/p}\int_{I_M}|K_n(x+t)|\,d\mu(t)$$
$$\leq c2^{M/p}\frac{2^k}{2^M} = c2^{k+M(1/p-1)}.$$

and

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))} \leq c2^{k+M(1/p-1)}. \tag{19}$$

By combining (6), (18), and (19), we obtain that

$$\int_{\overline{I_M}}\left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))}\right)^p d\mu(x)$$
$$= \sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\int_{I_{l+1}(e_k+e_l)}\left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))}\right)^p d\mu(x)$$
$$+ \sum_{k=0}^{M-1}\int_{I_M(e_k)}\left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))}\right)^p d\mu(x)$$
$$\leq c_p\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}\frac{2^{pk+l(1-p)}}{\varphi^p(M-l)} + c_p\sum_{k=0}^{M}\frac{1}{2^M}2^{pk+M(1-p)}$$
$$:= I + II. \tag{20}$$

Hence,

$$I \leq c_p\sum_{k=0}^{M-2}2^{pk}\sum_{l=k+1}^{M-1}\frac{1}{2^{pl}\varphi^p(M-l)} \tag{21}$$

$$= c_p \sum_{k=0}^{M-2} 2^{pk} \sum_{l=k+1}^{[(k+M)/2]} \frac{1}{2^{pl}\varphi^p(M-l)} + c_p \sum_{k=0}^{M-2} 2^{pk} \sum_{l=[(k+M)/2]+1}^{M-1} \frac{1}{2^{pl}\varphi^p(M-l)}$$
$$:= I_1 + I_2.$$

By using (10) for I_1 we get that

$$\begin{aligned} I_1 &\leq c_p \sum_{k=0}^{M-2} \frac{2^{pk}}{\varphi^p([(M-k)/2])} \sum_{l=k+1}^{[(k+M)/2]} \frac{1}{2^{pl}} \\ &\leq c_p \sum_{k=0}^{M-2} \frac{1}{\varphi^p([(M-k)/2])} < c_p < \infty. \end{aligned}$$

For I_2 we find that

$$\begin{aligned} I_2 &\leq c_p \sum_{k=0}^{M-2} 2^{pk} \sum_{l=[(k+M)/2]+1}^{M-1} \frac{1}{2^{pl}} \leq c_p \sum_{k=0}^{M-2} 2^{pk} \frac{1}{2^{p[(k+M)/2]}} \\ &\leq c_p \sum_{k=0}^{M-2} \frac{2^{pk/2}}{2^{pM/2}} < c_p < \infty. \end{aligned}$$

For II we can conclude that

$$II \leq c_p \sum_{k=0}^{M-2} \frac{2^{pk}}{2^{pM}} < c_p < \infty. \tag{22}$$

By combining (20)-(22) we conclude that (14) holds so the proof is complete. □

Proof of Theorem 2 In view of (11) we have that

$$\left(\sum_{s=1}^{n_k-1} \frac{1}{\varphi^p(s)} \right)^{1/p} \to \infty, \quad \text{as} \quad k \to \infty. \tag{23}$$

Set

$$f_{n_k}(x) = D_{2^{n_k+1}}(x) - D_{2^{n_k}}(x), \qquad n_k \geq 3.$$

It is evident that

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = 2^{n_k}, \ldots, 2^{n_k+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we easily can derive that

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{2^{n_k}}(x), \text{ if } i = 2^{n_k}, \ldots, 2^{n_k+1}-1, \\ f_{n_k}(x), \text{ if } i \geq 2^{n_k+1}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{24}$$

Since

$$D_{j+2^{n_k}}(x) - D_{2^{n_k}}(x) = w_{2^{n_k}} D_j(x), \qquad j = 1, 2, .., 2^{n_k}, \tag{25}$$

from (7) it follows that

$$\begin{aligned} \| f_{n_k} \|_{H_p} &= \left\| \sup_{n\in\mathbb{N}} S_{2^n} f_{n_k} \right\|_p = \| D_{2^{n_k+1}} - D_{2^{n_k}} \|_p \\ &= \| D_{2^{n_k}} \|_p = 2^{n_k(1-1/p)}. \end{aligned} \tag{26}$$

Let $q_{n_k}^s \in \mathbb{N}$ be such that $2^{n_k} \leq q_{n_k}^s \leq 2^{n_k+1}$ and $[q_{n_k}^s] = s$, where $0 \leq s < n_k$. By applying (24) we can conclude that

$$\begin{aligned} \left| \sigma_{q_{n_k}^s} f_{n_k}(x) \right| &= \frac{1}{q_{n_k}^s} \left| \sum_{j=1}^{q_{n_k}^s} S_j f_{n_k}(x) \right| \\ &= \frac{1}{q_{n_k}^s} \left| \sum_{j=2^{n_k}+1}^{q_{n_k}^s} S_j f_{n_k}(x) \right| \\ &= \frac{1}{q_{n_k}^s} \left| \sum_{j=2^{n_k}+1}^{q_{n_k}^s} \left(D_j(x) - D_{2^{n_k}}(x) \right) \right| \\ &= \frac{1}{q_{n_k}^s} \left| \sum_{j=1}^{q_{n_k}^s - 2^{n_k}} \left(D_{j+2^{n_k}}(x) - D_{2^{n_k}}(x) \right) \right|. \end{aligned}$$

According to (25) we obtain that

$$\begin{aligned} \left| \sigma_{q_{n_k}^s} f_{n_k}(x) \right| &= \frac{1}{q_{n_k}^s} \left| \sum_{j=0}^{q_{n_k}^s - 2^{n_k}} D_j(x) \right| \\ &= \frac{q_{n_k}^s - 2^{n_k}}{q_{n_k}^s} \left| K_{q_{n_k}^s - 2^{n_k}}(x) \right|. \end{aligned}$$

Let $x \in I_{s+1}(e_{s-1} + e_s)$. By using Lemma 3 we have that

$$\left| \sigma_{q_{n_k}^s} f_{n_k}(x) \right| \geq \frac{c 2^{2s}}{2^{n_k}}$$

and

$$\frac{\left|\sigma_{q_{n_k}^s} f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q_{n_k}^s\right)}\varphi\left(\rho\left(q_{n_k}^s\right)\right)} \geq \frac{c_p 2^{s/p}}{2^{n_k(1/p-1)}\varphi(n_k - s)}.$$

Hence,

$$\begin{aligned}
&\int_G \left(\sup_{k\in\mathbb{N}} \left|\frac{\left|\sigma_{q_{n_k}^s} f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q_{n_k}^s\right)}\varphi\left(\rho\left(q_{n_k}^s\right)\right)}\right|\right)^p d\mu(x) \\
&\geq \frac{1}{2}\sum_{s=0}^{n_k-1} \int_{I_{s+1}(e_{s-1}+e_s)} \left(\frac{\left|\sigma_{q_{n_k}^s} f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q_{n_k}^s\right)}\varphi\left(\rho\left(q_{n_k}^s\right)\right)}\right)^p d\mu(x) \\
&\geq c_p \sum_{s=0}^{n_k-1} \frac{1}{2^s} \frac{2^s}{2^{n_k(1-p)}\varphi^p(n_k - s)} \\
&\geq \frac{c_p}{2^{n_k(1-p)}} \sum_{s=1}^{n_k} \frac{1}{\varphi^p(s)}.
\end{aligned}$$

Finally, by using this estimate combined with (23) and (26) we find that

$$\begin{aligned}
&\frac{\left(\int_G \left(\sup_{k\in\mathbb{N}} \sup_{0\leq s<n_k} \left|\frac{\left|\sigma_{q_{n_k}^s} f_{n_k}(x)\right|}{2^{(1/p-2)\rho\left(q_{n_k}^s\right)}\varphi\left(\rho\left(q_{n_k}^s\right)\right)}\right|\right)^p d\mu(x)\right)^{1/p}}{\left\| f_{n_k}\right\|_{H_p}} \\
&\geq \frac{\left(\frac{c_p}{2^{n_k(1-p)}} \sum_{s=1}^{n_k} \frac{1}{\varphi^p(s)}\right)^{1/p}}{2^{n_k(1-1/p)}} \\
&\geq c_p \left(\sum_{s=1}^{n_k} \frac{1}{\varphi^p(s)}\right)^{1/p} \to \infty, \quad \text{as} \quad k \to \infty.
\end{aligned}$$

The proof is complete. □

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