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ON HARDY-TYPE INEQUALITIES AS AN INTELLECTUAL ADVENTURE FOR 100 YEARS

Lars-Erik Persson[1,2] · Natasha Samko[1]

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Abstract
We describe some chosen ideas and results for more than 100 years prehistory and history of the remarkable development concerning Hardy-type inequalities. In particular, we present a newer convexity approach, which we believe could partly have changed this development if Hardy had discovered it. In order to emphasize the current very active interest in this subject, we finalize by presenting some examples of the recent results, which we believe have potential not only to be of interest for a broad audience from a historical perspective, but also to be useful in various applications.

Keywords Integral inequalities · Hardy-type inequalities · Convexity · History and biography · Applications

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Introduction

We consider the following statements of the Hardy inequality: the discrete inequality asserts that if $\{a_n\}_1^\infty$ is a sequence of non-negative real numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \quad p>1, \tag{1.1}$$

The continuous inequality informs us that if f is a non-negative p-integrable function on $(0,\infty)$, then f is integrable over the interval $(0,x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p>1, \tag{1.2}$$

✉ Natasha Samko
Natasha.G.Samko@uit.no

1 UiT The Arctic University of Norway, P.O. Box 385, N-8505, Narvik, Norway

2 Karlstad University, 65188 Karlstad, Sweden

see [14]. The development of the famous Hardy inequality in both discrete and continuous forms during the period 1906 to 1928 has its own history or, as we call it, prehistory. Contributions of mathematicians other than G.H.Hardy, such as E.Landau, G.Pòlya, E.Schur, and M.Riesz, are important here. This dramatic prehistory was described in detail in [21]: In particular, the following is clear:

(a) Inequalities Eqs. (1.1) and (1.2) are the standard forms of the Hardy inequalities that can be found in many textbooks on Analysis and were highlighted first in the famous book [16] by Hardy, Littlewood and Pòlya.
(b) By restricting Eq. (1.2) to the class of step functions, one proves easily that Eq. (1.2) implies Eq. (1.1).
(c) The constant $(p/(p-1))^p$ in both Eqs. (1.1) and (1.2) is *sharp*: it cannot be replaced with a smaller number such that Eqs. (1.1) and (1.2) remain true for all relevant sequences and functions, respectively.
(d) The main motivation for Hardy to begin this dramatic history in 1915 was to find a simpler proof of the Hilbert inequality from 1906:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \le \pi\left(\sum_{m=1}^{\infty}a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2}. \tag{1.3}$$

(In Hilbert's version of Eq. (1.3), the constant 2π appears instead of the sharp one π.) We remark that nowadays, the following more general form of Eq. (1.3) is also sometimes referred in the literature as Hilbert's inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^{p'}\right)^{1/p'}, \tag{1.4}$$

where $p > 1$ and $p' = p/(p-1)$. However, Hilbert was not even close to consider this case (the l_p-spaces appeared only around 1910).
(e) A limit case of Eq. (1.1) (as $p \to \infty$) is *the Carleman inequality*

$$\sum_{n=1}^{\infty}\sqrt[n]{a_1 a_2 \cdots a_n} \le e\sum_{m=1}^{\infty}a_n,\ (a_n \ge 0).$$

(f) A limit case of Eq. (1.2) (as $p \to \infty$) is *the Pólya-Knopp inequality*

$$\int_0^{\infty}\exp\left(\frac{1}{x}\int_0^x \ln g(t)dt\right)dx \le e\int_0^{\infty}g(x)dx.$$

The constant e is sharp in both of these inequalities.
(g) The first weighted version of Eq. (1.2) was proved by Hardy himself in 1928 (see [15]):

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \le \left(\frac{p}{p-1-a}\right)^p\int_0^{\infty}f^p(x)x^a dx, \tag{1.5}$$

where f is a measurable and non-negative function on $(0,\infty)$ whenever $a < p-1, p > 1$ (see [15]).
In addition to Hardy's original proof of Eq. (1.2), today there exist more than 20 different proofs. In Sect. 2, we present one such proof, which is especially simple and which shows both the equivalence and convexity nature of several power weighted Hardy-type inequalities. In particular, this proof shows that Eqs. (1.2) and (1.5) are equivalent and in its fundamental form Eq. (2.1) it even holds for $p = 1$.
In Sect. 3, we shortly inform about the further almost unbelievable development, but of course, no details can be given in this limited space. However, just as examples of this development, we describe mostly some results, where we ourselves have been involved in this development up to 2017; for complementary details, see our book [24] and the references therein. Finally, in Sect. 4, we illustrate the still ongoing interest of this fascinating area by selecting some especially interesting results and ideas after 2017. We hope that these results are not only new and interesting, but that they can also serve as a source of inspiration for further research.

A convexity approach to investigate power weighted Hardy-type inequalities

The fact that the concept of convexity can be used to prove several inequalities, both classical and new ones, was of course known by Hardy himself. For example, in the famous book [16], this concept and the more or less equivalent Jensen inequality were frequently used. Hence, it may be surprising that Hardy himself never discovered that also his famous inequality in both original (see Eq. (1.2)) and power weighted form (see Eq. (1.5)) follow more or less directly as described below. Concerning convexity and its applications, e.g., to prove inequalities, we refer to the recent book [27], the papers [30–32], and the references therein.

A new look on the inequalities Eqs. (1.2) and (1.5)

Observation 2.1 We note that for $p > 1$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,$$

$$\Leftrightarrow$$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq 1 \cdot \int_0^\infty g^p(x)\frac{dx}{x}, \tag{2.1}$$

where $f(x) = g(x^{1-1/p})x^{-1/p}$.

This means that Hardy's inequality Eq. (1.2) is equivalent to Eq. (2.1) for $p > 1$ and, thus, that Hardy's inequality can be proved in the following simple way (see form Eq. (2.1)): By Jensen's inequality and Fubini's theorem, we have that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq \int_0^\infty \left(\frac{1}{x}\int_0^x g^p(y)dy\right)\frac{dx}{x} = \int_0^\infty g^p(y)\int_y^\infty \frac{dx}{x^2}dy = \int_0^\infty g^p(y)\frac{dy}{y}.$$

Instead, by making the substitution $f(t) = g(t^{\frac{p-1-a}{p}})t^{-\frac{1+a}{p}}$ in Eq. (2.1), we see that this inequality is also equivalent to Eq. (1.5). These facts imply especially the following:

(a) Hardy's inequalities Eqs. (1.2) and (1.5) hold also for $p < 0$ (because the function $\varphi(u) = u^p$ is convex also for $p < 0$) and hold in the reverse direction for $0 < p < 1$ (with sharp constants $\left(\frac{p}{1-p}\right)^p$ and $\left(\frac{p}{a+1-p}\right)^p$, $a > p-1$, respectively).

(b) The inequalities Eqs. (1.2) and (1.5) are equivalent, for $p > 1$.

(c) The inequality Eq. (2.1) also holds with equality for $p = 1$, which gives us a possibility to interpolate and get more information about the mapping properties of the Hardy operator. In particular, we can use interpolation theory to see that in fact the Hardy operator H maps each interpolation space B between $L_1\left((0,\infty), \frac{dx}{x}\right)$ and $L_\infty\left((0,\infty), \frac{dx}{x}\right)$ into B, i.e., that $\|Hf\|_B \leq C\|f\|_B$. We call Eq. (2.1) a fundamental form of Hardy's inequality.

(d) In our next subsection, we develop this idea further by presenting some equivalent scales of power weighted, equivalent and sharp Hardy-type inequalities in both direct and reversed forms, even when the interval of integration can be finite. In particular, this shows that Hardy's original inequalities can be replaced by infinitely many other Hardy-type inequalities. This means that we have reason to ask: What should have happened (with the further development of Hardy-type inequalities) if Hardy had discovered this? (see [31]).

Further consequences of the convexity approach

For the finite interval case, we need the following extension of our basic observation in Sect. 2.1. The proof is similar, and also the sharpness is easy to prove (see [31]).

Lemma 2.2 *Let g be a non-negative and measurable function on* $(0,\ell), 0<\ell\leq\infty$.

a) *If* $p<0$ *or* $p\geq 1$, *then*

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} \leq 1\cdot \int_0^\ell g^p(x)\Big(1-\frac{x}{\ell}\Big)\frac{dx}{x}. \tag{2.2}$$

(In the case $p<0$, we assume that $g(x)>0, 0<x\leq\ell$).

b) *If* $0<p\leq 1$, *then Eq.* (2.2) *holds in the reversed direction.*

c) *The constant* $C=1$ *is sharp in both a) and b).*

By using this Lemma and straightforward calculations, the following equivalence theorem can be proved (see [31]):

Theorem 2.3 *Let* $0<\ell\leq\infty$, *let* $p\in\mathbb{R}_+\backslash\{0\}$, *and let f be a non-negative function. Then*

a) *the inequality*

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)dy\right)^p x^a dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x)x^a\left[1-\left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right]dx \tag{2.3}$$

holds for all measurable functions *f*, each $\ell, 0<\ell\leq\infty$ and all a in the following cases:

$$(a_1)\qquad p\geq 1, a<p-1,$$

$$(a_2)\qquad p<0, a>p-1.$$

b) *For the case* $0<p<1, a<p-1$, *inequality Eq.* (2.3) *holds in the reversed direction under the conditions considered in a*).

c) *The inequality*

$$\int_\ell^\infty \left(\frac{1}{x}\int_x^\infty f(y)dy\right)^p x^{a_0} dx \leq \left(\frac{p}{a_0+1-p}\right)^p \int_\ell^\infty f^p(x)x^{a_0}\left[1-\left(\frac{\ell}{x}\right)^{\frac{a_0+1-p}{p}}\right]dx \tag{2.4}$$

holds for all measurable functions *f*, each $\ell, 0\leq\ell<\infty$ and all a in the following cases:

$$(c_1)\qquad p\geq 1, a_0>p-1,$$

$$(c_2)\qquad p<0, a_0<p-1.$$

d) *For the case* $0<p\leq 1,\ a_0>p-1$, *inequality Eq.* (2.4) *holds in the reversed direction under the conditions considered in c*).

e) *All inequalities above are sharp.*

f) *Let* $p\geq 1$ *or* $p<0$. *Then, the statements in a) and c) are equivalent for all permitted a and* a_0 *because they are in all cases equivalent to Eq.* (2.2) *via substitutions.*

g) *Let* $0<p<1$. *Then, the statements in b) and d) are equivalent for all permitted a and* a_0.

For the case $p < 0$, we just avoid trivial situations by assuming, e.g., that $\int_0^x f(y)dy > 0$ or $\int_x^\infty f(y)dy > 0$ for $x > 0$.

Remark 2.4 Note that in the theory of (weighted) Hardy-type inequalities, we usually have good estimates of the sharp constant (= the operator norm). However, in some cases as above, we can even find the sharp constant, and this is especially interesting and maybe regarded as an art of its own. For example, the constants in all inequalities in Theorem 2.3 are sharp. Later on, we present two more examples where the sharp constants nowadays are known.

Remark 2.5 Note that inequality Eq. (2.3) has no meaning when $a = p - 1$. However, by restricting to finite intervals and involving some suitable logarithms, C. Bennett [6] first succeeded to prove such a result when he developed his well-known theory for real interpolation between the (fairly close) spaces L and $LLogL$. This result reads:

Proposition 2.6 *Let $\alpha > 0, 1 \leq p < \infty$ and f be a non-negative and measurable function on* $[0, 1]$. *Then,*

$$\left(\int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x}\right)^{1/p} \leq \alpha^{-1}\left(\int_0^1 x^p[\log(e/x)]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}\right)^{1/p}. \tag{2.5}$$

A further development of Bennett's inequality involving two sharp constants

The following refined more general form of Eq. (2.5) was proved and discussed in [4]:.

Theorem 2.7 *Let $\alpha, p > 0$ and f be a non-negative and measurable function on* $[0, 1]$.

(a) *If $p > 1$, then*

$$\alpha^{p-1}\left(\int_0^1 f(x)dx\right)^p + \alpha^p \int_0^1 [\log(e/x)]^{\alpha p-1}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x} \leq$$

$$\leq \int_0^1 x^p[\log(e/x)]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}. \tag{2.6}$$

Both constants α^{p-1} and α^p in Eq. (2.6) are sharp. Equality is never attained unless f is identically zero.

(b) *If $0 < p < 1$, then Eq.* (2.6) *holds in the reverse direction and the constant is sharp. Equality is never attained unless f is identically zero.*

(c) *If $p = 1$, we have equality in Eq.* (2.6) *for any measurable function f and any $\alpha > 0$.*

Remark 2.8 In his original paper, [6] C. Bennett never discussed the sharpness of his inequality Eq. (2.5). On the contrary, both constants in our inequality Eq. (2.6) are sharp and, moreover, Eq. (2.6) holds in the reversed direction when $0 < p \leq 1$ with equality when $p = 1$. Of course, this can never happen without the additional term in Eq. (2.6) vis-á-vis Eq. (2.5).

Remark 2.9 Also, the crucial argument to prove Eq. (2.6) is to use a special convexity argument of other type. Inequalities of the type Eq. (2.6) are called refined Hardy-type inequalities. Moreover, we say that the "breaking point" is $p = 1$ since the inequality sign of the inequality changes at this point. Another recent idea is to use other concepts of convexity type to prove refined Hardy-type inequalities with other breaking points then $p = 1$. Maybe the first paper of this type was [28], where the concept "superquadraticity" was used to prove a refined Hardy-type inequality with breaking point $p = 2$. By using other alternatives of convexity and the corresponding Jensen-type inequalities, it is possible to prove refined Hardy-type inequalities with other breaking points (see, e.g., [1, 2] and the references therein).

On the further development of Hardy-type inequalities up to 2017

The history of Hardy-type inequalities up to 2017 can be found, e.g., in the books [20] and [24] and the references therein. It is of course impossible to mention here all types of interesting results described in these books and related papers so we will only give some examples, where we ourselves in most cases have been involved in one way or other. Finally, in the last section, we have selected some results after 2017, which we consider especially interesting and believe have potential to be a basis for further research in this fascinating area.

On the general weighted case

One important early question was the following:
For which weights u and v and parameters p and q does it hold that

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leq C\left(\int_0^b f^p(x)v(x)dx\right)^{1/p},$$

$0 < b \leq \infty$, for some finite constant C ?
For the last 80 years, there have been a lot of activities to answer this and more general questions concerning Hardy-type inequalities, and a lot of interesting results have been proved and applied.
Just as one example, we mention the following well-known result:

Theorem 3.1 *Let $1 < p \leq q < \infty$ and u and v be weight functions on $\mathbb{R}_+$. Then, each of the following conditions is necessary and sufficient for the inequality*

$$\left(\int_0^b \left(\int_0^x f(t)\,dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^b f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{3.1}$$

to hold for all positive and measurable functions on $\mathbb{R}_+$:

a) *The condition*

$$A := \sup_{x>0}\left(\int_x^b u(t)\,dt\right)^{\frac{1}{q}}\left(\int_0^x v(t)^{1-p'}\,dt\right)^{\frac{1}{p'}} < \infty, \tag{3.2}$$

with the estimation $C \in [A, \lambda A]$ for the best constant C in Eq. (3.1), where

$$\lambda = \min\{p^{1/q}(p')^{1/p'}, q^{1/q}(q')^{1/p'}\}.$$

b) *The condition*

$$B := \sup_{x>0} V(x)^{-\frac{1}{p}}\left(\int_0^x u(t)V(t)^q\,dt\right)^{\frac{1}{q}} < \infty, \quad V(x) := \int_0^x v(t)^{1-p'}\,dt, \tag{3.3}$$

with the estimation $C \in [B, p'B]$ for the best constant in Eq. (3.1).

Remark 3.2 For the dramatic history until Eq. (3.2) was derived, see our book [24] and the references therein. A simple proof of the characterization Eq. (3.2) was given by B. Muckenhoupt in 1972 for $p = q$ (see [26]) and by J.S.Bradley in 1978 for $p \leq q$ (see [8]). In 2002 L.E. Persson and V.D. Stepanov presented an elementary proof and application of the alternative condition Eq. (3.3) in connection to their investigation of the geometric mean operator (see [35]). In this connection, it should be mentioned some even earlier papers by G. Talenti (1969), see [43], and G. Tomaselli (1969), see [44], respectively, by G. Sinnamon and V. D. Stepanov (1996), see [41].

Remark 3.3 It has recently been discovered that also these two conditions to characterize Eq. (3.1) are not unique and can even be replaced by infinitely many equivalent conditions, in fact even by *scales of conditions*. In Sect. 3.4, we shortly present and discuss this remarkable fact. More complete information can be found in Sections 7.3.2 and 7.3.3 of the book [24] (see also the references therein).

The sharp constant for the power weighted case when $1 < p < q < \infty$

For the case $p = q$, we have already presented our results concerning sharpness of the constant in Hardy-type inequalities (see Theorems 2.3 and 2.7). In this subsection, we will present the current knowledge also for the case $1 < p < q < \infty$, but in order to clarify this result, we give some introductory information.
By applying the general results (see Theorem 3.1) and making straightforward calculations (see [34]) for the power weighted case, we get the following:

Lemma 3.4 *Let* $1 < p < q < \infty$. *The following statements* (*a*) *and* (*b*) *hold and are equivalent*:

(a) *The inequality*

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^\alpha dx\right)^{1/q} \leq C\left(\int_0^x f^p(x)x^\beta dx\right)^{1/p} \tag{3.4}$$

holds for all positive and measurable functions $f(t)$ on $(0, \infty)$ if and only if

$$\beta < p - 1 \text{ and } \frac{\alpha + 1}{q} = \frac{\beta + 1}{p} - 1. \tag{3.5}$$

(b) *The inequality*

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt\right)^q x^{\alpha_0} dx\right)^{1/q} \leq C\left(\int_0^\infty f^p(x)x^{\beta_0} dx\right)^{1/p} \tag{3.6}$$

holds for all positive and measurable functions $f(t)$ on $(0, \infty)$ if and only if

$$\beta_0 > p - 1, \frac{\alpha_0 + 1}{q} = \frac{\beta_0 + 1}{p} - 1. \tag{3.7}$$

Moreover, it yields that

(c) *the formal relation between the parameters* β *and* β_0 *is* $\beta_0 = -\beta - 2 + 2p$ *and in this case the best constants* C *in Eqs.* (3.4) *and* (3.6) *are the same*.

The next result was proved in 2015 by L.E. Persson and S. Samko (see [34]). For $\beta = 0$, c. f. also the result by G.A.Bliss from 1930 (see [7]).

Theorem 3.5 *Let* $1 < p < q < \infty$ *and the parameters* α *and* β *satisfy Eq.* (3.5). *Then, the sharp constant in Eq.* (3.4) *is* $C = C^*_{pq}$, *where*

$$C^*_{pq} = \left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{3.8}$$

Equality in Eq. (3.4) *occurs exactly when*

$$f(x) = \frac{cx^{-\frac{\beta}{p-1}}}{\left(dx^{\frac{p-1-\beta}{p-1}\cdot\left(\frac{q}{p}-1\right)}+1\right)^{\frac{q}{q-p}}}.$$

Moreover,

$$C^*_{pq} \to \frac{p}{p-1-\beta} \text{ as } q \to p. \tag{3.9}$$

Remark 3.6 By using Eq. (3.9), we see that we have the expected continuity in the sharp constants in Eqs. (1.5) and (3.4).

Remark 3.7 By using Theorem 3.5 and Lemma 3.4, we can also derive the sharp constant in Eq. (3.6) (see [34]).

Concerning the kernel operator case in Hardy-type inequalities

This means that we aim to characterize the following more general Hardy-type inequality

$$\|Tf\|_{q,u} \le C\|f\|_{p,v}, \tag{3.10}$$

where u and v are weight functions and

$$Tf(x) := \int_a^x k(x,y)f(y)dx,$$

$k(x, y)$ denote a positive kernel.
Some facts:

(a) Without restrictions on the kernel $k(x, y)$ the problem to characterize Eq. (3.10) is open.
(b) The solution of this problem is known for a number of special cases and parameters.

For the current main knowledge in this case, we refer to the book [24, Chapter 7.5] and the review paper [23]. In particular, in [23], the following result was proved:

Theorem 3.8 *Let* $1 < p \le q < \infty$, $a < b \le \infty$, *u and v be weights. Let* $k(x, y)$ *be a non-negative kernel.*

(a) *Then, Eq.* (3.10) *holds if*

$$A_s := \sup_{a<y<b} \left(\int_y^b k^q(x,y)u(x)V^{\left(\frac{q(p-s-1)}{p}\right)}(x)dx\right)^{1/q} V^{s/p}(y) < \infty, \tag{3.11}$$

for <u>any</u> $s < p - 1$.

(b) *The condition Eq.* (3.11) *can not be improved in general for* $s > 0$ *because for product kernels, it is even necessary and sufficient for Eq.* (3.10) *to hold.*
(c) *For the best constant* C *in Eq.* (3.10), *we have the following estimate*:

$$C \leq \inf_{s<p-1} \left(\frac{p}{p-s-1}\right)^{1/p'} A_s.$$

Here as usual

$$U(x) := \int_x^b u(y)dy, \qquad V(x) := \int_a^x v^{1-p'}(y)dy, \tag{3.12}$$

Remark 3.9 This result opens a possibility that the condition Eq. (3.11) can be a candidate to solve the open question we have pointed out above. At least it cannot be improved in general.

Some new scales of equivalent conditions to characterize the Hardy-type inequality Eq. (3.1)

We have already mentioned that the conditions $A < \infty$ and $B < \infty$ in Theorem 3.1 can be replaced by infinitely many equivalent conditions (see Remark 3.3). We discovered the first results around 2002 together with the PhD student A. Wedestig (see [46]). After that, this theory has been developed in an almost unbelievable way. For a description of this development and the most important results, we refer to the book [24, Chapter 7.3], the review paper [22], and the references therein. In particular, the following is known (see [9]):

Theorem 3.10 *Let* $1 < p \leq q < \infty$, $0 < s < \infty$, *and define, for the weight functions* u, v, *the functions* U *and* V *by Eq.* (3.12). *Then, Eq.* (3.1) *can be characterized by any of the conditions* $A_i(s) < \infty$, *where* $A_i(s), i = 1, 2, 3, 4$ *are defined by*

$$A_1(s) := \sup_{0<x<b} \left(\int_x^b u(t)V^{q(\frac{1}{p'}-s)}(t)dt\right)^{1/q} V^s(x);$$

$$A_2(s) := \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t)U^{p'(\frac{1}{q}-s)}(t)dt\right)^{1/p'} U^s(x);$$

$$A_3(s) := \sup_{0<x<b} \left(\int_0^x u(t)V^{q(\frac{1}{p'}+s)}(t)dt\right)^{1/q} V^{-s}(x);$$

$$A_4(s) := \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t)U^{p'(\frac{1}{q}+s)}(t)dt\right)^{1/p'} U^{-s}(x).$$

Remark 3.11 Note that the constants A and B in Theorem 3.1 are just points on these scales, namely,

$$A = A_1\left(\frac{1}{p'}\right), \quad B = A_3\left(\frac{1}{p}\right).$$

Remark 3.12 The scales of conditions in Theorem 3.10 can even (equivalently) be complemented with 10 more scales of condition. Very surprising. Also all other known alternative conditions are just points on these 14 scales of conditions. Moreover, it is also known that the B-condition can be replaced by a number of equivalent scales of conditions.

Some new results and ideas concerning Hardy-type inequalities after 2017

There are also some other possibilities for generalizations of Hardy-type inequalities, e.g., the following:

(i) *More general function spaces.*
(ii) *More general Hardy-type operators than in the kernel case.*
(iii) *More results in the multidimensional case.*
(iv) *New applications.*
(v) *New technics of proofs.*
(vi) *New Hardy-type inequalities on homogeneous groups.*
(vii) *New information concerning sharpness of Hardy-type inequalities.*

In this section, we give examples of some new results after 2017 concerning all of (i)–(vii), but first we give the following:

Remark 4.1 Concerning such generalizations as those in (i) up to 2017, see our book [24, Chapter 7.6]. Such results are known, e.g., for the following cases: Orlicz, Lorentz, rearrangement invariant, Morrey-type, Hölder-type and variable $L^{p}(\cdot)$ spaces. However, very little is known, e.g., in more general Banach function spaces or metric spaces.

First, we mention that in [3], some new results in both cases (i) and (ii) are proved and discussed. In particular, in this paper, the following results can be found:

Theorem 4.2 *Let* $0 < b \leq \infty, -\infty \leq a < c \leq \infty$, *let* Φ *be a positive and convex function on* (a, c) *and E be a Banach function space on* $[0, b)$. *If E has the Fatou property and* $a < f(x) < c$, *then*

$$\left\|\Phi\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)\right\|_E \leq \int_0^b \Phi(f(t))\left\|\frac{1}{x}\chi_{[t,b]}(x)\right\|_E \mathrm{d}t.$$

Example 4.3 Just as one simple example of application of Theorem 4.2, we get the following generalization of the Pólya-Knopp inequality:

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \log f(t)\mathrm{d}t\right)x^{\alpha}\mathrm{d}x \leq e^{1+\alpha}\int_0^b f(x)x^{\alpha}\left(1-\frac{x}{b}\right)\mathrm{d}x, \alpha > -1,$$

The classical Pólya-Knopp inequality is just the case $\alpha = 0, b = \infty$, which is just a limit case of Eq. (1.2) when $p \to \infty$. Let (Ω, Σ, μ) denote a $\sigma-$finite measure space.
Let σ_x denote a $\sigma-$finite positive measure on a measure space S such that $\sigma_x(S) < \infty$. Moreover, we suppose that σ_x is absolutely continuous with respect to a $\sigma-$finite positive measure σ on S. We define

$$Tf(x) = \frac{1}{\sigma_x(S)}\int_S f(t)\mathrm{d}\sigma_x(t),$$

where f is measurable defined on S with values in $(a, b), -\infty \leq a < b \leq \infty$, and $\sigma_x(S) = \int_S \mathrm{d}\sigma_x(t)$. In [3] also, the following result was proved:

Theorem 4.4 *Let* $-\infty \leq a < b \leq \infty$ *and let* Φ *be a positive and convex function on* (a, b), *where* $\Phi(f)$ *is measurable on S. Moreover, let E be a Banach function space on S with Fatou property. Then,*

$$\|\Phi(Tf(\cdot))\|_E \leq \int_S \Phi(f(y))\left\|\frac{1}{\sigma_x(S)}\frac{\mathrm{d}\sigma_x(y)}{\mathrm{d}\sigma(y)}\right\|_E \mathrm{d}\sigma(y).$$

Next, we consider case (iii) and remark that there is a much less developed theory for the multidimensional case then in the one-dimensional case. We also remark that many such results up to 2017 can be found in the book [24, Chapter 7.7]. Here, we just report on some new results in two possible cases which are possible to develop further:

Idea 1 (concerning the spherical Hardy operator). Just use polar coordinates and several Hardy-type inequalities in $\mathbb{R}^n$, $\mathbb{R}^n_+$ or even more general "cones in $\mathbb{R}^n$" can be obtained by using the corresponding one-dimensional results as one crucial tool.

This idea can be used even for more general Hardy-type operators, in particular for bilinear and iterated Hardy operators as shown, e.g., in the paper [17]. In particular, the following fairly general situation was considered in [17]: Let $H_2^N = H_2^N(f,g)$ be defined by

$$H_2^N(f,g)(x) := H^N f(x) \cdot H^N g(x) = \int_{B(0,|x|)} f(t)\,dt \int_{B(0,|x|)} g(t)\,dt.$$

Just as one typical example of result, we mention the following:

Theorem 4.5 *Let $0 < q < \infty$, $1 < p_1, p_2 \le q < \infty$ and w, v_1, v_2 be weight functions defined on $\mathbb{R}^N$, $N \in \mathbb{Z}_+$. Then, the inequality*

$$\left(\int_{\mathbb{R}^N} \left[H_2^N(f,g)(x)\right]^q w(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^N} f^{p_1}(x) v_1(x)\,dx\right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^N} g^{p_2}(x) v_2(x)\,dx\right)^{\frac{1}{p_2}}$$

holds if and only if

$$B_1^N := \sup_{0<\alpha<\infty} \left(\int_{|x|\ge\alpha} w(x)\,dx\right)^{\frac{1}{q}} \left(\int_{|x|\le\alpha} v_1^{1-p_1'}(x)\,dx\right)^{\frac{1}{p_1'}} \left(\int_{|x|\le\alpha} v_2^{1-p_2'}(x)\,dx\right)^{\frac{1}{p_2'}} < \infty.$$

Remark 4.6 In [17] the corresponding characterizations were derived for the following cases: (1) $1 < p_1 \le q \le p_2 < \infty$, (2) $1 < p_2 \le q \le p_1 < \infty$, (3) $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, (different characterizations for the cases $\frac{1}{q} \le \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$). For another new result of this type, see [12].

Remark 4.7 The idea above can be generalized to the case with "cones in $\mathbb{R}^n$" (even finite such cones), so we can in particular cover the cases $\mathbb{R}^n$ and $\mathbb{R}^n_+$. But a more important further development of this polar coordinate idea to be able to even handle some metric spaces has recently been presented and applied by M. Ruzhansky and collaborators. The basic idea is as follows:

Consider metric spaces $\mathbb{X}$ with a Borel measure dx allowing for the following polar decomposition at $a \in \mathbb{X}$: we assume that there is a locally integrable function $\lambda \in L^1_{loc}$ such that for all $f \in L^1(\mathbb{X})$, we have

$$\int_{\mathbb{X}} f(x)dx = \int_0^\infty \int_{\Sigma_r} f(r,\omega)d\omega dr,$$

for $\Sigma_r = \{x \in \mathbb{X} : d(x,a) = r\} \subset \mathbb{X}$ with a measure on it denoted by $d\omega$, and $r, \omega \to a$ as $r \to 0$.

For some recent results connected to Hardy-type inequalities with this idea in focus, we refer to [39] and, for the reversed case, [19] and the references therein. These results give also a new development of the case (i).

Remark 4.8 One reason why we judge this to be regarded as an important development is that it can give new applications far beyond the obvious ones in $\mathbb{R}^n$ and $\mathbb{R}^n_+$. Examples of such applications refer to homogeneous groups, hyperbolic spaces, and Cartan-Hadamard manifolds. Concerning Hardy-type inequalities on homogenous groups, see the book [38] and also Remark 4.25.

Idea 2 The case with "rectangular like" Hardy operators, e.g., $H_2 : H_2(f(x,y)) = \int_0^x \int_0^y f(s,t)dsdt$. The basic (at that time surprising and not so well-understood) result by E. Sawyer from 1985 reads (see [40]):

Theorem 4.9 *Let* $1 < p \le q < \infty$ *and* u *and* v *be weights on* $\mathbb{R}^2_+$. *Then, the inequality*

$$\left(\int_0^\infty \int_0^\infty \left(\int_0^{x_1} \int_0^{x_2} f(t_1,t_2)dt_1dt_2\right)^q u(x_1,x_2)\, dx_1dx_2\right)^{\frac{1}{q}} \le$$

$$\le C\left(\int_0^\infty \int_0^\infty f^p(x_1,x_2)v(x_1,x_2)\, dx_1dx_2\right)^{\frac{1}{p}} \tag{4.1}$$

holds for all non-negative and measurable functions on $\mathbb{R}^2_+$, *if and only if the following three conditions are satisfied*:

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+} \left(\int_{y_1}^\infty \int_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} \left(\int_0^{y_1} \int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p'}} < \infty, \tag{4.2}$$

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+} \frac{\left(\int_0^{y_1}\int_0^{y_2}\left(\int_0^{x_1}\int_0^{x_2} v(t_1,t_2)^{1-p'}dt_1dt_2\right)^q u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int_0^{y_1}\int_0^{y_2} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{p}}} < \infty, \tag{4.3}$$

and

$$\sup_{(y_1,y_2)\in\mathbb{R}^2_+} \frac{\left(\int_{y_1}^\infty\int_{y_2}^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty u(t_1,t_2)dt_1dt_2\right)^{p'} v(x_1,x_2)^{1-p'}dx_1dx_2\right)^{\frac{1}{q}}}{\left(\int_{y_1}^\infty\int_{y_2}^\infty u(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q'}}} < \infty. \tag{4.4}$$

Remark 4.10 Note that Eq. (4.2) corresponds to the Muckenhoupt-Bradley condition Eqs. (3.2), (4.3) corresponds to the condition Eqs. (3.3), and (4.4) corresponds to the dual condition of Eq. (3.3). According to Theorem 3.10 and Remark 3.11, all these conditions are equivalent in the one-dimensional case, but in view of [40], it is not so in the two-dimensional case.

Remark 4.11 In the PhD theses by A. Wedestig, see [46], and E. Ushakova, see [45] (and in related papers), it was proved that if the weight to the left or to the right in Eq. (4.1) is of product type, then the inequality can be characterized by using only *one* condition. Moreover, in this case, the results could be extended to a general n-dimensional setting.

Remark 4.12 In [40], E. Sawyer also proved that none of the conditions Eqs. (4.2), (4.3), or (4.4) could be removed. However, recently, it was proved in [36] that for the case $1 < p < q < \infty$, it is possible to replace Eqs. (4.2)–(4.4) by one condition, namely, Eq. (4.2), and still we obtain a characterization of Eq. (4.1). This was a very surprising result even for experts, and it opens new possibilities for further developments and understanding of the E. Sawyer result.

One main result for this case reads:

Theorem 4.13 *Let* $1 < p < q < \infty$. *Then, the inequality Eq.* (4.1) *holds if and only if Eq.* (4.2) *holds. Moreover,* $A \approx C$, *where A is the constant defined by Eq.* (4.2).

Remark 4.14 It was also proved in [36] that Theorem 4.13 in fact holds in a general n-dimensional setting. Moreover, in [36], a similar result was also proved for the case $1 < q < p < \infty$ (see also [42]).

Remark 4.15 One reason why Hardy-type inequalities have survived as an important area of research for more than 100 years is heavily depending on its importance for applications. Next, we present two such new applications:

The first one is the following new result concerning Fourier inequalities (see [25]):

Theorem 4.16 *Let* $1 < p \le q < \infty, 0 < s < \infty$ *and* u, v, h *be weight functions on* $\mathbb{R}^n$, $n \in \mathbb{Z}_+$. *Denote*

$$U(x) = \int_x^\infty \frac{u^*(1/t)}{t^2} dt, \ V(x) = \int_0^x \left[\left(\frac{1}{v}\right)^*(t)\right]^{p'-1} dt,$$

where u^* *and* $\left(\frac{1}{v}\right)^*$ *are the decreasing rearrangements of* u *and* $\frac{1}{v}$, *respectively.*

Define

$$\begin{aligned}
A_1(x,s) &:= \left(\int_x^\infty \tfrac{u^*(1/t)}{t^2} V^{q(\frac{1}{p'}-s)}(t)dt\right)^{1/q} V^s(x);\\
A_2(x,s) &:= \left(\int_0^x \left[\left(\tfrac{1}{v}\right)^*(t)\right]^{p'-1} U^{p'(\frac{1}{q}-s)}(t)dt\right)^{1/p'} U^s(x);\\
A_3(x,s) &:= \left(\int_0^x \tfrac{u^*(1/t)}{t^2} V^{q(\frac{1}{p'}+s)}(t)dt\right)^{1/q} V^{-s}(x);\\
A_4(x,s) &:= \left(\int_x^\infty \left[\left(\tfrac{1}{v}\right)^*(t)\right]^{p'-1} U^{p'(\frac{1}{q}+s)}(t)dt\right)^{1/p'} U^{-s}(x);
\end{aligned}$$

Then, the Fourier inequality

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\gamma)|^q u(\gamma) d\gamma\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p} \tag{4.5}$$

holds for all measurable functions $f \ge 0$ *if and only if any of the quantities* $A_i(s) = \sup_{x>0} A_i(x,s)$ *is finite for any* $i = 1,2,3,4$ *and any fixed* $s > 0$. *Moreover, for the best constant C in Eq.* (4.5), *we have* $C \approx \min_{i,s} A_i(s), \ i = 1,2,3,4, \ s > 0$.

Remark 4.17 Since $\int_0^{1/x} u^*(t)dt = \int_x^\infty \frac{u^*(1/t)}{t^2} dt$, we see that the condition $A_1(x, 1/p') < \infty$ coincides with the condition in the original paper by J. Benedetto and H.P. Heinig in 2003 (see [5]), which was essentially complemented in [37]. Moreover, in addition to the four fundamental scales in Theorem 4.16, it was proved in [25] that we can add further 10 independent characterizing scales of conditions. Very surprising.

Remark 4.18 Unlike the situation with Hardy-type inequalities, in this case, no alternative condition seems to be known to characterize the Fourier inequality Eq. (4.5) before 2021, see [24] and, e.g., the statement of Theorem 4.16. The crucial part is to use a general equivalence theorem from Hardy-type inequalities (see [23, Chapter 7.3]). Hence, it is not surprising that we can compare Theorem 4.16 with Theorem 3.10 combined with Remark 3.12.

Remark 4.19 The second recent application we want to mention is due to A. Kalybay, R. Oinarov, and Y.Sultanaev (see [18]), where they proved a new Hardy-type weighted differential inequality and applied it to describe the oscillatory properties of a fourth order differential equation in detail. A further development was recently derived in [29].

Remark 4.20 In this review paper, we have for simplicity mostly considered only the case $1 < p \leq q$. We remark that the corresponding results are also known for the case $p > q, p \geq 1, q > 0$, but historically, these cases have been treated separately, so, therefore, we judge that our next remark is of special interest and importance.

Remark 4.21 As indicated before (see (v)), finding new proofs of known results not only complements and refreshes the research in this area, but also leads to new directions and possibilities. In this connection, we judge that the new proof presented by A. Gogatishvili and L. Pick (see [10]) has all reasons to be such a crucial one. This proof is, in our opinion, very surprising, e.g., because

(a) Only Fubini's theorem, Hölder's inequality, Minkowki's integral inequality, and Hardy's lemma are used.

(b) It unifies the proofs of all the usual cases including the convex case $1 \leq p \leq q$ and the non-convex case $p > q,\ p \geq 1$ and $q > 0$. In the classical theory, the proofs of these main cases were very different.

Remark 4.22 It is easy to see that the continuous Hardy inequality Eq. (1.2) (via using step functions) implies the discrete version Eq. (1.1)). However, the corresponding implications for more general Hardy-type inequalities are much more delicate. As an example of a new discrete Hardy-type inequality proved in this way, we refer to [13].

Remark 4.23 Concerning (v), in addition to Remarks 4.21–4.22, we also want to pronounce the powerful discretization and antidiscretization techniques developed and applied by A. Gogatishvili and his collaborators. For a recent result where these techniques were used, see, e.g., [11].

Remark 4.24 Chapter 3 of the book [24] is devoted to Hardy-type inequalities with more general so-called Hardy-Steklov operators involved (the upper and lower bounds in the defining integrals are increasing functions instead of 0 and x). Some new impressive information for this case is published in a new book in Russian by D. Prokhorov, V. Stepanov, and E. Ushakova, but now, it is also translated (see [36]).

Remark 4.25 Concerning (vi), we pronounce another new book [38] by M. Ruzhansky and D. Suragan, where they present, develop, and apply an interesting branch on Hardy-type inequalities, namely, that on homogeneous groups. Moreover, much complementary information of the development of the 100 years of Hardy inequalities is given.

Remark 4.26 In the classical theory of Hardy-type inequalities, good estimates of the sharp constant (= the operator norm) are usually given. However, in more general situations, it is more difficult. To derive the really sharp constant is important for applications but also like an art in its own. Some important results of this type have been presented in this paper (see, e.g., Theorem 2.3, Theorem 2.7, Theorem 3.5, and Remark 3.7). We finalize this paper by presenting and applying some new results from the paper [33].

Inspired by the convexity approach presented in Sect. 2.2 concerning sharp constants, it seems to be natural to describe Hardy-type inequalities by using the Haar measure dx/x instead of dx. This is also natural in relation to many applications, e.g., to the theory of Lorentz spaces and Interpolation theory. First, we present the following reversed version of Eq. (2.3) on the cone of non-decreasing functions:

Theorem 4.27 *Let $p > 0, 0 < \alpha < p$ and let f be a measurable, non-negative, and non-increasing function on $(0, \ell)$, $0 < l \leq \infty$.*

a) If $p \geq 1$, then

$$\int_0^\ell \left(\int_0^x f(y)dy \right)^p x^{-\alpha} \frac{dx}{x} \geq \frac{p}{\alpha} \int_0^x (xf(x))^p \, x^{-\alpha} \left(1 - \left(\frac{x}{\ell}\right)^\alpha\right) \frac{dx}{x}. \tag{4.6}$$

b) If $0 < p \leq 1$, then Eq. (4.6) holds in the reversed direction.

c) The constant $C = p/\alpha$ is sharp in both a) and b), and equality appears for each function $f(x) = A\chi_{(0,c)}(x)$ for some $c \in (0, l)$ and $A > 0$.

By combining Theorem 4.27 and making some obvious substitutions in Theorem 2.3 a), b), and e), we obtain the following:

Corollary 4.28 *Let $p > 0, o < \alpha < p,\ 0 < \ell \leq \infty$ and let f be a measurable, non-negative, and non-increasing function on $(0, \ell)$.*

If $p > 1$, then

$$\left(\frac{p}{\alpha}\right)^{1/p} I_1 \leq I_2 \leq \frac{p}{\alpha} I_1, \tag{4.7}$$

where

$$I_1 = \left(\int_0^\ell (xf(x))^p x^{-\alpha} \left(1 - \left(\frac{x}{\ell}\right)^\alpha\right) \frac{dx}{x} \right)^{1/p}$$

and

$$I_2 = \int_0^\ell \left(\int_0^x f(y)dy \right)^p x^{-\alpha} \frac{dx}{x}.$$

If $0 < p \leq 1$, then both inequalities in Eq. (4.7) hold in the reversed direction. Moreover, both constants $(p/\alpha)^{1/p}$ and p/α are sharp for all $p > 0$.

Remark 4.29 *a)* This means that the equivalence $I_2 \approx I_1$ holds and the corresponding so called "optimal target function" is

$$g(x) = 1 - \left(\frac{x}{\ell}\right)^\alpha.$$

b) In the lower inequality of Eq. (4.7), we can even have equality, while in the upper inequality, the sharpness follows by choosing a sequence of non-increasing functions (a well-known fact from the theory of Hardy-type inequalities).

Remark 4.30 In [33], a similar sharp result was also proved for the case $\alpha > p$. In particular, for the case $\ell = \infty$, this inequality reads:

$$\int_0^\infty \left(\int_0^x f(y)dy \right)^p x^{-\alpha} \frac{dx}{x} \geq pB(p, \alpha - p + 1) \int_0^\infty (xf(x))^p x^{-\alpha} \frac{dx}{x},$$

and the constant $pB(p, \alpha - p + 1)$ is sharp (as usual B denotes the Beta-function).

We only point out an application related to Lorentz spaces.
Let f^* denote the non-increasing rearrangement of a function f on a measure space (Ω, μ). The Lorentz spaces $L^{p,q}$, $0 < p, q < \infty$ are defined by using the quasi-norm (norm when $p > 1, q \geq 1$)

$$\|f\|_{p,q}^* := \left(\int_0^\infty \left(f^*(t)t^{1/p}\right)^q \frac{dt}{t} \right)^{1/q}.$$

It is well-known that for the case $p > 1$, this quasi-norm is equivalent to the following one equipped with the usual Hardy operator:

$$\|f\|_{p,q}^{**} := \left(\int_0^\infty \left(\int_0^t f^*(u)du \right)^q t^{-q/p'} \frac{dt}{t} \right)^{1/q}.$$

By applying Corollary 4.28 with p replaced by q and α replaced by q/p', we obtain:

Example 4.31 With the notations and assumptions above, $p > 1$ and $0 < \ell \leq \infty$, we have that

$$\left(p'\right)^{1/q} I_\ell^* \leq I_\ell^{**} \leq p' I_\ell^*, \tag{4.8}$$

where $q > 1$,

$$I_\ell^* := \left(\int_0^\ell \left(f^*(t)t^{1/p}\right)^q \left(1 - \left(\frac{t}{\ell}\right)^{q/p'}\right) \frac{dt}{t} \right)^{1/q}$$

and

$$I_\ell^{**} := \left(\int_0^\ell \left(\int_0^t f^*(u)du \right)^q t^{-q/p'} \frac{dt}{t} \right)^{1/q}.$$

If $0 < q \leq 1$, then the inequalities in Eq. (4.8) hold in the reversed directions. Both constants $(p')^{1/q}$ and p' are sharp for all $q > 0$.

Remark 4.32 In particular, for the case $\ell = \infty$, Example 4.31 implies that the following well-known estimates for the quasi-norms in Lorentz spaces hold for $p > 1$

$$\left(p'\right)^{1/q} \|f\|_{p,q}^* \leq \|f\|_{p,q}^{**} \leq p' \|f\|_{p,q}^*, \tag{4.9}$$

whenever $q > 1$ and the reversed inequalities hold if $0 < q \leq 1$. Moreover, the result in Example 4.31 indicates that it could be reasonable to involve the optimal target function $1 - \left(\frac{t}{m(\Omega)}\right)^{q/p'}$ in the definition of Lorentz spaces when the underlying measure space has finite measure.

Remark 4.33 The paper [33] also contains similar results as these in Theorem 4.27 for all possible situations with monotone functions involved and with $\int_0^x$ replaced by $\int_x^\infty$. In particular, this gives similar two-sided sharp estimates (as those in Eq. (4.9)) for the corresponding Lorentz quasi-norms also in the case $0 < p < 1$.

Summing up: The fascinating theory and history concerning Hardy-type inequalities seems to continue, even in an increasing power, over to the next decennium and why not to the next century?

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